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A CLASS OF OPTIMIZATION PROBLEMS WITH APPLICATIONS IN CONTACT MECHANICS

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We consider a class of optimization problems in reflexive Banach spaces. The existence of minimizers is a direct consequence of the Weierstrass theorem. We study the dependence of the solution with respect to a perturbation of the cost functional as well as with respect to the set of constraints. We establish weak convergence results and, under additional assumptions, strong convergence results. Next, we introduce a mathematical model which describes the equilibrium of an elastic body in contact with a rigid-plastic obstacle. We derive the weak formulation of the model which is in the form of an optimization problem for the displacement field. Then, we use our abstract results in order to obtain the existence of a unique weak solution of the model as well as its continuous dependence with respect to the data.

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1. INTRODUCTION

Optimization methods represent a powerful mathematical tool used in various domains of Applied Mathematics, including the analysis and numerical approximation of various nonlinear boundary value problems. They are intensively used in mathematical economics, mathematical physics, biology, and aerospace, chemical, civil, electrical, and mechanical engineering. The Optimization Theory was developed based on arguments of monotonicity, semicontinuity, subdifferentiability, compactness and convexity, among others. Basic references in the field include the books [2, 5, 6, 9, 10].

Processes of contact between deformable bodies abound in industry and everyday life. A few simple examples are brake pads in contact with wheels, tires on roads, and pistons with skirts. Because of the importance of contact processes in structural and mechanical systems, considerable effort has been put into their modeling, analysis and numerical simulations. The literature in the field is extensive. It includes the books [1, 7, 8, 12, 14, 16–18], for instance. There, the analysis and numerical simulations of contact problems with elastic, viscoelastic and viscoplastic materials can be found. Excellent references in the mathematical theory of elasticity and plasticity are the monographs [3, 4] and [11, 20], respectively.

In this paper, we consider an abstract optimization problem with applications in the study of mathematical models of contact with linearly elastic materials. The problem is formulated as follows.

Problem. $\mathcal{P}$. Find $u \in K$ such that

$$J(u,\eta) = \min_{v \in K} J(v,\eta). \tag{1.1}$$

Here K represents a subset of a reflexive Banach space X and $J(\cdot,\eta)$ is a given function which depends on the parameter η. We state sufficient conditions which guarantee the existence of a solution to Problem $\mathcal{P}$. Then, for each $n \in \mathbb{N}$, we consider a perturbation of Problem $\mathcal{P}$ defined as follows.

Problem. $\mathcal{P}_n$. Find $u_n \in K$ such that

$$J(u_n,\eta_n) = \min_{v \in K} J(v,\eta_n). \tag{1.2}$$

In (1.2) and below η_n represents a perturbation of the parameter η. Besides the solvability of this problem we study the behavior of the solution when $n \to \infty$ and prove various convergence result. Finally, for $n \in \mathbb{N}$ we consider a second perturbation of Problem $\mathcal{P}$, denoted $\mathcal{P}_n^n$, defined as follows.

Problem. $\mathcal{P}_n^n$. Find $u_n \in K_n$ such that

$$J(u_n,\eta_n) = \min_{v \in K_n} J(v,\eta_n) \tag{1.3}$$

Here K_n denotes a perturbation of the set K and, again, η_n denotes a perturbation of η. For Problem $\mathcal{P}_n^n$ we prove existence, uniqueness and convergence results as the sequence $\{\eta_n\}$ converges weakly to η and the sequence $\{K_n\}$ converges to K in the sense of Mosco.

Besides the mathematical interest in the abstract results presented in this paper, they are important from mechanical point of view. Indeed, a large number of mathematical models which describe the equilibrium of linearly elastic materials in contact with a foundation lead to optimization problems of the form (1.1). There, u represents either the displacement or the stress field, K is the set of constraints and the function J represents the energy function, related to the constitutive law, the applied forces and the contact conditions. Thus, the abstract results we present in this paper can be used in the study of the corresponding contact problems. They provide the continuous dependence of the solution with the data and lead to interesting mechanical interpretations.

The rest of the paper is structured as follows. In Section 2, we state and prove our first result, Theorem 2.1. It states the existence of the solutions of problems $\mathcal{P}$ and $\mathcal{P}_n$, together with a weakly convergence result. Then, we reinforce the assumptions on the data and prove our second result, Theorem 2.2. It states the existence of a unique solution of Problems $\mathcal{P}$ and $\mathcal{P}_n$, respectively, together with a strongly convergence result. In Section 3, we obtain similar results, Theorems 3.2 and 3.3, for Problems $\mathcal{P}$ and $\mathcal{P}_n^n$. The proofs of the theorems are based on arguments of compactness and lower semicontinuity. In Section 4, we introduce a mathematical model which describes the equilibrium of an elastic body in contact with a rigid-plastic obstacle. We list the assumptions on the data and provide the variational formulation of the model. Then, in Section 5, we state and prove our main result in the study of this problem, Theorem 5.1. The proof is based on the abstract result provided by Theorem 3.3.

Note that the linear spaces considered in this paper including abstract normed spaces, Banach spaces and various function spaces are assumed to be real linear spaces. Notation $\|\cdot\|_Z$ will represent the norm of the normed space Z. In addition, we denote by $\to$ and $\rightharpoonup$ the strong and weak convergence in various normed spaces, which will be specified. We end this introduction by recalling the following version of the Weierstrass theorem.

THEOREM 1.1. *Let X be a reflexive Banach space space, K a nonempty weakly closed subset of X and $J : X \to \mathbb{R}$ a weakly lower semicontinuous function. In addition, assume that J is coercive, i.e., $J(v) \to \infty$ as $\|v\|_X \to \infty$. Then, the following statements hold.*

i) *There exists at least an element u such that*

$$u \in K, \qquad J(u) \le J(v) \qquad \forall\, v \in K. \tag{1.4}$$

ii) *If, moreover, K is a convex set and J is strictly convex function, then the solution of the optimization problem* (1.4) *is unique.*

Theorem 1.1 will be used in Sections 2 and 3 to prove the solvability and the unique solvability of Problems $\mathcal{P}$, $\mathcal{P}_n$ and $\mathcal{P}_n^n$, respectively. Its proof can be found in many books and surveys, see, for instance, [6, 13, 18].

2. THE STUDY OF PROBLEMS $\mathcal{P}$ AND $\mathcal{P}_n$

The functional framework in which we study Problems $\mathcal{P}$ and $\mathcal{P}_n$ is the following. First, X is a reflexive Banach space, Y a normed space, $K \subset X$, $\Lambda \subset Y$ and $J : X \times \Lambda \to \mathbb{R}$. Moreover, we consider the following assumptions.

(Λ) Λ is a nonempty weakly closed subset of Y.

(K) K is a nonempty weakly closed subset of X.

$$(J_1)\qquad \left\{\begin{array}{l}\text{For all sequences } \{u_k\} \subset X \text{ and } \{\eta_k\} \subset \Lambda \text{ such that}\\ u_k \rightharpoonup u \text{ in } X,\ \eta_k \rightharpoonup \eta \text{ in } Y \text{ and for all } v \in X,\\ \text{the inequality below holds:}\\ \limsup\limits_{k\to\infty} [J(v,\eta_k) - J(u_k,\eta_k)] \le J(v,\eta) - J(u,\eta).\end{array}\right.$$

$$(J_2)\qquad \left\{\begin{array}{l}\text{For all sequences } \{u_k\} \subset X \text{ and } \{\eta_k\} \subset \Lambda \text{ such that}\\ \|u_k\|_X \to \infty,\ \eta_k \rightharpoonup \eta \text{ in } Y \text{ one has } J(u_k,\eta_k) \to \infty.\end{array}\right.$$

$$(J_3)\qquad \left\{\begin{array}{l}\text{For all sequence } \{\eta_k\} \subset \Lambda \text{ such that } \eta_k \rightharpoonup \eta \text{ in } Y\\ \text{and all } \ v \in X \text{ one has } J(v,\eta_k) - J(v,\eta) \to 0.\end{array}\right.$$

Note that below in this section we shall use assumptions (J_1)–(J_3) together with assumption (Λ). This guarantees that, if $\{\eta_k\} \subset \Lambda$ and $\eta_n \rightharpoonup \eta$ in Y, then $\eta \in \Lambda$. For this reason, in the statement of assumptions (J_1)–(J_3) we do not indicate explicitly that $\eta \in \Lambda$, which represents a necessary condition to define the functional $J(\cdot,\eta)$.

Next, we consider an element $\eta \in \Lambda$ and, for each $n \in \mathbb{N}$, let $\eta_n \in \Lambda$. Moreover, we assume that

$$\eta_n \rightharpoonup \eta \quad \text{in} \quad Y. \tag{2.1}$$

Our first result in this section is the following.

THEOREM 2.1. *Assume (Λ), (K) and (J_1)–(J_3). Then the following statements hold.*

i) *Problem $\mathcal{P}$ has at least one solution and Problem $\mathcal{P}_n$ has at least one solution, for each $n \in \mathbb{N}$.*

ii) *If (2.1) holds and u_n is a solution of Problem $\mathcal{P}_n$, for each $n \in \mathbb{N}$, there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in K$, such that*

$$u_n \rightharpoonup u \quad \text{in} \quad X. \tag{2.2}$$

Moreover, u is a solution to Problem $\mathcal{P}$.

Proof. i) We take $\eta_k = \eta$ in (J_1) to see that for all sequences $\{u_k\} \subset X$ such that $u_k \rightharpoonup u$ and for all $v \in X$ we have

$$\limsup_{k\to\infty} \,[J(v,\eta) - J(u_k,\eta)] \le J(v,\eta) - J(u,\eta),$$

which implies that

$$\liminf_{k\to\infty} \, J(u_k,\eta) \ge J(u,\eta).$$

We conclude from here that the function $J(\cdot,\eta) : X \to \mathbb{R}$ is lower semicontinuous. Moreover, taking $\eta_k = \eta$ in (J_2) we deduce that $J(\cdot,\eta)$ is coercive. Recall also the assumption (K). The existence of at least one solution to Problem $\mathcal{P}$ is now a direct consequence of Theorem 1.1 i). The existence of at least one solution to Problem $\mathcal{P}_n$ follows by the same argument, applied to the function $J(\cdot,\eta_n) : X \to \mathbb{R}$, for each $n \in \mathbb{N}$.

ii) Assume (2.1). We claim that the sequence $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is not bounded then we can find a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, such that $\|u_n\|_X \to \infty$. Therefore, using assumptions (2.1) and (J_2) we deduce that

$$J(u_n,\eta_n) \to \infty. \tag{2.3}$$

On the other hand, using (J_3) it follows that

$$J(v,\eta_n) \to J(v,\eta), \tag{2.4}$$

v being an arbitrary element in K. Finally, since u_n is a solution of Problem $\mathcal{P}_n$ we obtain that

$$J(u_n,\eta_n) \le J(v,\eta_n). \tag{2.5}$$

We now pass to the limit in (2.5) and use the convergences (2.3) and (2.4) to deduce that $J(v,\eta) = \infty$ which represents a contradiction. We conclude from above that the sequence $\{u_n\}$ is bounded in X and, using a standard compactness argument, we deduce that there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in X$, such that (2.2) holds.

We now prove that u is a solution of Problem $\mathcal{P}$. To this end, we note that assumption (K) together with the convergence (2.2) guarantees that $u \in K$. Moreover, since u_n is the solution to Problem $\mathcal{P}_n$, we have

$$J(u_n,\eta_n) \le J(v,\eta_n) \qquad \forall\, v \in K$$

which implies that

$$0 \le \limsup_{n\to\infty} \,[J(v,\eta_n) - J(u_n,\eta_n)] \qquad \forall\, v \in K. \tag{2.6}$$

On the other hand, convergences (2.1) and (2.2) combined with assumption (J_1) yield

$$\limsup_{n\to\infty} \,[J(v,\eta_n) - J(u_n,\eta_n)] \le J(v,\eta) - J(u,\eta) \qquad \forall\, v \in K. \tag{2.7}$$

We now combine (2.6) and (2.7) to deduce that

$$J(u,\eta) \le J(v,\eta) \qquad \forall\, v \in K.$$

This implies that u is a solution of Problem $\mathcal{P}$, which concludes the proof. $\square$

We now reinforce the conditions on the data by considering the following assumptions.

$(\widetilde{K})$ $K \subset X$ is a nonempty closed convex subset.

$(\widetilde{J})$ $J(\cdot,\eta) : X \to \mathbb{R}$ is a strictly convex function, for each $\eta \in \Lambda$.

$$(J^*) \quad \begin{cases} \text{There exists } m > 0 \text{ such that} \\ (1-t)J(u,\eta) + tJ(v,\eta) - J((1-t)u + tv,\eta) \ge mt(1-t)\|u-v\|_X^2 \\ \text{for all } u,\, v \in X,\ \eta \in \Lambda,\ t \in [0,1]. \end{cases}$$

Note that assumption (J^*) implies assumption $(\widetilde{J})$. Moreover, assumption $(\widetilde{K})$ implies assumption (K).

Our second result in this section is the following.

THEOREM 2.2. *Assume* (Λ), $(\widetilde{K})$, (J_1)–(J_3) *and* $(\widetilde{J})$. *Then, the following statements hold.*

i) *Problem* $\mathcal{P}$ *has a unique solution* u *and Problem* $\mathcal{P}_n$ *has a unique solution* u_n, *for each* $n \in \mathbb{N}$.

ii) *If* (2.1) *holds, then sequence* $\{u_n\}$ *converges weakly to* u, *i.e.*, $u_n \rightharpoonup u$ *in* X.

iii) *If* (2.1) *and* (J^*) *hold, then the sequence* $\{u_n\}$ *converges strongly to* u, *i.e.*, $u_n \to u$ *in* X.

Proof. i) We use arguments similar to those used in the proof of Theorem 2.1 i). The difference arises in the fact that, since $(\widetilde{K})$ and $(\widetilde{J})$ hold, we are now in a position to apply Theorem 1.1 ii). In this way we deduce the existence of a unique minimizer for the functions $J(\cdot,\eta)$ and $J(\cdot,\eta_n)$ on the set K, which implies the unique solvability of Problems $\mathcal{P}$ and $\mathcal{P}_n$, for each $n \in \mathbb{N}$.

ii) Assume that (2.1) holds. Then, a careful analysis of the proof of Theorem 2.1 ii) reveals that the sequence $\{u_n\}$ is bounded and any weakly convergent sequence of $\{u_n\}$ converges to a solution of Problem $\mathcal{P}$. On the other hand, Problem $\mathcal{P}$ has a unique solution, denoted u, as proved in the first part of the theorem. The weak convergence of the whole sequence $\{u_n\}$ to u is now a consequence of a standard argument.

iii) Assume now that (2.1) holds and, in addition, (J^*) holds, too. Let

$n \in \mathbb{N}$. Then, using (J^*) with $t = \frac{1}{2}$ we find that

$$\frac{m}{4}\|u_n - u\|_X^2 \le \frac{1}{2}\Big[J(u_n,\eta_n) - J\Big(\frac{u_n+u}{2},\eta_n\Big)\Big] + \frac{1}{2}\Big[J(u,\eta_n) - J\Big(\frac{u_n+u}{2},\eta_n\Big)\Big]$$

and, since u_n is a minimizer for the function $J(\cdot,\eta_n)$ on K, we find that

$$\frac{m}{4}\|u_n - u\|_X^2 \le \frac{1}{2}\Big[J(u,\eta_n) - J\Big(\frac{u_n+u}{2},\eta_n\Big)\Big] \tag{2.8}$$

On the other hand, the part ii) shows that then the sequence $\{u_n\}$ converges weakly to u, i.e., $u_n \rightharpoonup u$ in X which implies that

$$\frac{u_n+u}{2} \rightharpoonup u \quad \text{in} \quad X. \tag{2.9}$$

We use the convergences (2.1), (2.9) and assumption (J_1) with $v = u$ to deduce that

$$\limsup_{n\to\infty}\Big[J(u,\eta_n) - J\Big(\frac{u_n+u}{2},\eta_n\Big)\Big] \le 0. \tag{2.10}$$

We now combine inequalities (2.8) and (2.10) to deduce that $u_n \to u$ in X which concludes the proof. $\square$

3. THE STUDY OF PROBLEMS $\mathcal{P}$ AND $\mathcal{P}_n^n$

We now move to the study of Problem $\mathcal{P}_n^n$ and the link of its solutions with the solutions of Problem $\mathcal{P}$. To this end, we need to complete the list of assumptions with the following ones.

(K_n) K_n is a nonempty weakly closed subset of X, for each $n \in \mathbb{N}$.

$(\widetilde{K}_n)$ $K_n \subset X$ is a nonempty closed convex subset, for each $n \in \mathbb{N}$.

(J_4) $\begin{cases} \text{For all sequences } \{v_k\} \subset X \text{ and } \{\eta_k\} \subset \Lambda \text{ such that} \\ v_k \to v \text{ in } X,\ \eta_k \rightharpoonup \eta \text{ in } Y \text{ one has} \\ J(v_k,\eta_k) - J(v,\eta_k) \to 0. \end{cases}$

Note that assumption $(\widetilde{K}_n)$ implies assumption (K_n). Next, we recall the following notion of Mosco convergence.

Definition 3.1. Let X be a normed space, $\{K_n\}$ a sequence of nonempty subsets of X and K a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K in the Mosco sense if the following conditions hold.

(M_1) $\begin{cases} \text{For each } v \in K \text{ there exists a sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \to v \text{ in } X. \end{cases}$

$$(M_2)\qquad \begin{cases} \text{For each sequence } \{v_n\} \text{ such that} \\ v_n \in K_n \text{ for each } n \in \mathbb{N} \text{ and } v_n \rightharpoonup v \text{ in } X \text{ we have } v \in K. \end{cases}$$

We denote in what follows the convergence in the Mosco sense by $K_n \xrightarrow{M} K$ in X and we recall that this convergence depends on the topology of the normed space X. More details on this topic could be found in [15]. Consider now the assumption

$$K_n \xrightarrow{M} K \quad \text{in} \quad X. \tag{3.1}$$

Our first result in this section is the following.

THEOREM 3.2. *Assume* (Λ), (K), (K_n), (J_1)–(J_4). *Then the following statements hold.*

i) *Problem* $\mathcal{P}$ *has at least one solution and Problem* $\mathcal{P}_n^n$ *has at least one solution, for each* $n \in \mathbb{N}$.

ii) *If* (2.1), (3.1) *hold and* u_n *is a solution of Problem* $\mathcal{P}_n^n$, *for each* $n \in \mathbb{N}$, *there exists a subsequence of the sequence* $\{u_n\}$, *again denoted* $\{u_n\}$, *and an element* $u \in K$, *such that*

$$u_n \rightharpoonup u \quad \text{in} \quad X. \tag{3.2}$$

Moreover, u *is a solution to Problem* $\mathcal{P}$.

Proof. i) The solvability of Problem $\mathcal{P}$ follows from Theorem 2.1 i). The solvability of Problem $\mathcal{P}_n^n$ follows from similar arguments, applied to function $J(\cdot, \eta_n)$ and the set K_n.

ii) Assume now that (2.1), (3.1) hold. We claim that the sequence $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is not bounded, then we can find a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, such that $\|u_n\|_X \to \infty$. Therefore, using assumptions (2.1) and (J_2) we deduce that

$$J(u_n, \eta_n) \to \infty. \tag{3.3}$$

Let v be a given element in K and note that assumption (3.1) implies that condition (M_1) holds. Thus, there exists a sequence $\{v_n\}$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and

$$v_n \to v \quad \text{in} \quad X. \tag{3.4}$$

Moreover, since u_n is a solution of Problem $\mathcal{P}_n^n$ we obtain that $J(u_n, \eta_n) \leq J(v_n, \eta_n)$ and, therefore,

$$J(u_n, \eta_n) \leq J(v_n, \eta_n) - J(v, \eta_n) + J(v, \eta_n) - J(v, \eta) + J(v, \eta) \qquad \forall\, n \in \mathbb{N}. \tag{3.5}$$

On the other hand, the convergences (3.4) and (2.1) allow us to use assumption (J_4) to find that $J(v_n, \eta_n) - J(v, \eta_n) \to 0$ and, in addition, assumption

(J_3) shows that $J(v,\eta_n) - J(v,\eta) \to 0$. Thus, inequality (3.5) implies that the sequence $\{J(u_n,\eta_n)\}$ is bounded, which contradicts (3.3). We conclude from above that the sequence $\{u_n\}$ is bounded in X and, therefore, there exists a subsequence of the sequence $\{u_n\}$, again denoted $\{u_n\}$, and an element $u \in X$, such that (3.2) holds.

We now prove that u is a solution of Problem $\mathcal{P}$. To this end, we use (3.2) and condition (M_2), guaranteed by assumption (3.1), to deduce that $u \in K$. Next, we consider an arbitrary element $v \in K$ and, using condition (M_1), we know that there exists a sequence $\{v_n\}$ such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and (3.4) holds. Since u_n is the solution to Problem $\mathcal{P}_n^n$ we have $J(u_n,\eta_n) \leq J(v_n,\eta_n)$ which implies that

$$0 \leq \ [J(v,\eta_n) - J(u_n,\eta_n)] + [J(v_n,\eta_n) - J(v,\eta_n)]. \tag{3.6}$$

We now use the convergences (3.4), (2.1) and assumptions (J_1), (J_4) to see that

$$\limsup_{n\to\infty}\,[J(v,\eta_n) - J(u_n,\eta_n)] \leq [J(v,\eta) - J(u,\eta), \tag{3.7}$$

$$J(v_n,\eta_n) - J(v,\eta_n) \to 0\,. \tag{3.8}$$

We now combine (3.6)–(3.8) to deduce that u is a solution of Problem $\mathcal{P}$, which concludes the proof. □

THEOREM 3.3. *Assume that* (Λ), $(\widetilde{K})$, $(\widetilde{K}_n)$, (J_1)–(J_4) *and* $(\widetilde{J})$ *hold. Then:*

i) *Problem* $\mathcal{P}$ *has a unique solution* u *and Problem* $\mathcal{P}_n^n$ *has a unique solution* u_n, *for each* $n \in \mathbb{N}$.

ii) *If* (2.1), (3.1) *hold, then sequence* $\{u_n\}$ *converges weakly to* u, *i.e.,* $u_n \rightharpoonup u$ *in* X.

iii) *If* (2.1), (3.1) *and* (J^*) *hold, then the sequence* $\{u_n\}$ *converges strongly to* u, *i.e.,* $u_n \to u$ *in* X.

Proof. i)–ii) We use arguments similar to those used in the proof of Theorem 2.2 i), ii). Since the modifications are straightforward, we skip the details.

iii) Assume now that (2.1), (3.1) and (J^*) hold and let $\{\widetilde{u}_n\}$ be a sequence such that $\widetilde{u}_n \in K_n$ for each $n \in \mathbb{N}$ and

$$\widetilde{u}_n \to u \quad \text{in} \quad X. \tag{3.9}$$

Recall that the existence of such sequence follows from assumption (M_1), guaranteed by condition (3.1). Then, using (J^*) with $t = \frac{1}{2}$ we find that

$$\frac{m}{4}\|\widetilde{u}_n - u_n\|_X^2 \leq \frac{1}{2}\,\Big[J(\widetilde{u}_n,\eta_n) - J\Big(\frac{\widetilde{u}_n + u_n}{2},\eta_n\Big)\Big]$$

$$+\frac{1}{2}\Big[J(u_n,\eta_n)-J\Big(\frac{\widetilde{u}_n+u_n}{2},\eta_n\Big)\Big]$$

and, since u_n is a minimizer for the function $J(\cdot,\eta_n)$ on K, we find that

$$\frac{m}{4}\|\widetilde{u}_n-u_n\|_X^2\leq\frac{1}{2}\Big[J(\widetilde{u}_n,\eta_n)-J\Big(\frac{\widetilde{u}_n+u_n}{2},\eta_n\Big)\Big]$$

which implies that

(3.10)

$$\frac{m}{4}\|\widetilde{u}_n-u_n\|_X^2\leq\frac{1}{2}\Big[J(u,\eta_n)-J\Big(\frac{\widetilde{u}_n+u_n}{2},\eta_n\Big)\Big]+\frac{1}{2}\Big[J(\widetilde{u}_n,\eta_n)-J(u,\eta_n)\Big].$$

We use the convergences (2.1), (2.2), (3.9) and assumption (J_1), (J_4) to deduce that

$$\limsup_{n\to\infty}\Big[J(u,\eta_n)-J\Big(\frac{u_n+u}{2},\eta_n\Big)\Big]\leq 0, \tag{3.11}$$

$$J(\widetilde{u}_n,\eta_n)-J(u,\eta_n)\to 0. \tag{3.12}$$

We now combine inequalities (3.10)–(3.12) to deduce that $u_n\to u$ in X which concludes the proof. □

4. THE MODEL

We consider an elastic body which occupies a bounded domain $\Omega\subset\mathbb{R}^d$ $(d=1,2,3)$ with a Lipschitz continuous boundary Γ, divided into three measurable disjoint parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1)>0$. The body is fixed on Γ_1, is acted upon by given body forces of density $\boldsymbol{f}_0$ and given surface tractions of density $\boldsymbol{f}_2$ on Γ_2. Moreover, it is in contact with an obstacle on Γ_3. To describe the equilibrium of the elastic body in the physical setting above we denote by $\boldsymbol{u}=(u_i)$ and $\boldsymbol{\sigma}=(\sigma_{ij})$ the displacement field and the stress field, respectively. The functions $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ depend on the spatial variable $\boldsymbol{x}=(x_i)\in\Omega\cup\Gamma$ and play the role of the unknowns of the problem. Here and below the indices i, j, k, l run between 1 and d and, unless stated otherwise, the summation convention over repeated indices is used. Moreover, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable $\boldsymbol{x}$, e.g. $u_{i,j}=\partial u_i/\partial x_j$. Also, $\boldsymbol{\varepsilon}$ and Div will represent the deformation and the divergence operators, respectively, *i.e.*,

$$\boldsymbol{\varepsilon}(\boldsymbol{u})=(\varepsilon_{ij}(\boldsymbol{u})),\quad \varepsilon_{ij}(\boldsymbol{u})=\frac{1}{2}\,(u_{i,j}+u_{j,i}),\quad \mathrm{Div}\,\boldsymbol{\sigma}=(\sigma_{ij,j})$$

and, therefore, $\boldsymbol{\varepsilon}(\boldsymbol{u})$ represents the linearized strain tensor. Note also that, in order to simplify the notation, we usually do not indicate explicitly the dependence of various functions on the spatial variable $\boldsymbol{x}$.

We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or, equivalently, the space of symmetric matrices of order d. The inner product and norm on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i\ , \qquad \|\boldsymbol{v}\| = (\boldsymbol{v}\cdot\boldsymbol{v})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{u} = (u_i),\ \boldsymbol{v} = (v_i) \in \mathbb{R}^d,$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}\ , \qquad \|\boldsymbol{\tau}\| = (\boldsymbol{\tau}\cdot\boldsymbol{\tau})^{\frac{1}{2}} \qquad \forall\, \boldsymbol{\sigma} = (\sigma_{ij}),\ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d,$$

and the zero element of these spaces will be denoted by $\boldsymbol{0}$. Also, we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ and u_ν, $\boldsymbol{u}_\tau$ will represent the normal and tangential components of $\boldsymbol{u}$ on Γ given by $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$, respectively. Finally, σ_ν and $\boldsymbol{\sigma}_\tau$ denote the normal and tangential stress on Γ, that is $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu})\cdot\boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$.

With these preliminaries, the classical formulation of the contact problem we consider in this section is as follows.

Problem. Q. Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \to \mathbb{S}^d$ such that

$$\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\mathrm{Div}\,\boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \quad \text{in} \quad \Omega, \tag{4.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_1, \tag{4.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \quad \text{on} \quad \Gamma_2, \tag{4.4}$$

$$u_\nu \le g, \qquad \left.\begin{array}{ll} \sigma_\nu = 0 & \text{if } \ u_\nu < 0 \\ -F \le \sigma_\nu \le 0 & \text{if } \ u_\nu = 0 \\ \sigma_\nu = -F & \text{if } \ 0 < u_\nu < g \\ \sigma_\nu \le -F & \text{if } \ u_\nu = g \end{array}\right\} \quad \text{on} \quad \Gamma_3, \tag{4.5}$$

$$\boldsymbol{\sigma}_\tau = \boldsymbol{0} \quad \text{on} \quad \Gamma_3. \tag{4.6}$$

We now provide a short description of the equations and boundary conditions in Problem Q. First, equation (4.1) represents the elastic constitutive law of the material in which $\mathcal{E}$ represents the fourth order elasticity tensor. Equation (4.2) is the equation of equilibrium and conditions (4.3), (4.4) represent the displacement and the traction boundary conditions, respectively. Condition (4.5) models the contact with a foundation made of a rigid body covered by a layer made of rigid-plastic material of thickness g, say asperities. It was used in a number of papers, including [19] and the references therein. The function F is assumed to be positive and could be interpreted as the yield limit of the foundation. Finally, condition (4.6) represents the frictionless contact condition.

In the study of Problem Q, we need to introduce further notation and preliminary material. Everywhere below we use the standard notation for Sobolev and Lebesgue spaces associated to Ω and Γ. In particular, we use the spaces $L^2(\Omega)^d$, $L^2(\Gamma_2)^d$, $L^2(\Gamma_3)$ and $H^1(\Omega)^d$, endowed with their canonical inner products and the associated norms. Moreover, we consider the spaces

$$X = \{\, \boldsymbol{v} \in H^1(\Omega)^d : \ \boldsymbol{v} = \boldsymbol{0} \ \text{ on } \ \Gamma_1 \,\}, \quad Y = L^2(\Gamma_3) \times L^2(\Omega)^d \times L^2(\Gamma_2)^d$$

which are real Hilbert spaces endowed with the canonical inner products given by

$$(\boldsymbol{u}, \boldsymbol{v})_X = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x,$$

$$(\boldsymbol{\eta}, \boldsymbol{\xi})_Y = \int_{\Gamma_3} FG \, \mathrm{d}a + \int_\Omega \boldsymbol{f}_0, \boldsymbol{g}_0 \, \mathrm{d}x + \int_{\Gamma_2} \boldsymbol{f}_2 \boldsymbol{g}_2 \, \mathrm{d}a$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$, $\boldsymbol{\eta} = (F, \boldsymbol{f}_0, \boldsymbol{f}_2)$, $\boldsymbol{\xi} = (G, \boldsymbol{g}_0, \boldsymbol{g}_2) \in Y$. The associated norms on these spaces are denoted by $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Also, recall that the completeness of the space X follows from the assumption $meas\,(\Gamma_1) > 0$ which allows the use of Korn's inequality.

For any element $\boldsymbol{v} \in X$ we denote by v_ν and $\boldsymbol{v}_\tau$ its normal and tangential components on Γ given by $v_\nu = \boldsymbol{v} \cdot \boldsymbol{\nu}$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \boldsymbol{\nu}$. Moreover, for a regular stress function $\boldsymbol{\sigma}$, the following Green's formula holds:

$$\int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \, \mathrm{d}x + \int_\Omega \mathrm{Div}\, \boldsymbol{\sigma} \cdot \boldsymbol{v} \, \mathrm{d}x = \int_\Gamma \boldsymbol{\sigma\nu} \cdot \boldsymbol{v} \, \mathrm{d}a \quad \text{for all } \boldsymbol{v} \in H^1(\Omega)^d. \tag{4.7}$$

We also recall that there exists $c_0 > 0$ which depends on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma)^d} \le c_0 \|\boldsymbol{v}\|_X \quad \text{for all} \ \ \boldsymbol{v} \in V. \tag{4.8}$$

Inequality (4.8) represents a consequence of the Sobolev trace theorem.

In the study of the contact problem (4.1)–(4.6) we assume that the elasticity tensor $\mathcal{E}$ satisfies the following conditions.

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E} = (\mathcal{E}_{ijkl}) : \Omega \times \mathbb{S}^d \to \mathbb{S}^d. \\ \text{(b) } \mathcal{E}_{ijkl} = \mathcal{E}_{klij} = \mathcal{E}_{jikl} \in L^\infty(\Omega), \ 1 \le i,j,k,l \le d. \\ \text{(c) There exists } m_\mathcal{E} > 0 \ \text{ such that} \\ \qquad \mathcal{E}\boldsymbol{\tau} \cdot \boldsymbol{\tau} \ge m_\mathcal{E} \|\boldsymbol{\tau}\|^2 \ \ \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \ \text{a.e. in } \Omega. \end{array} \right. \tag{4.9}$$

We also assume that the densities of body forces and tractions, the yield limit of the foundation and the bound of the normal displacement are such that

$$\boldsymbol{f}_0 \in L^2(\Omega)^d, \ \ \boldsymbol{f}_2 \in L^2(\Gamma_2)^d, \ \ F \in L^2(\Gamma_3), \ \ F(\boldsymbol{x}) \ge 0 \quad \text{a.e. } \boldsymbol{x} \in \Gamma_3, \tag{4.10}$$

(4.11) $g > 0$.

Under these assumptions we introduce the sets $K \subset X$, $\Lambda \subset Y$ the bilinear form $a : X \times X \to \mathbb{R}$ and the functions $j : X \times \Lambda \to \mathrm{I\!R}$, $J : X \times \Lambda \to \mathrm{I\!R}$ defined by

$$K = \{\, \boldsymbol{v} \in X : v_\nu \le g \ \text{ a.e. on } \Gamma_3 \,\}, \tag{4.12}$$

$$\Lambda = \{\, \boldsymbol{\eta} = (F, \boldsymbol{f}_0, \boldsymbol{f}_2) \in Y \ : \ F(\boldsymbol{x}) \ge 0 \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \,\}, \tag{4.13}$$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{v}) \,\mathrm{d}x, \tag{4.14}$$

$$j(\boldsymbol{v}, \boldsymbol{\eta}) = \int_{\Gamma_3} F v_\nu^+ \,\mathrm{d}a - \int_\Omega \boldsymbol{f}_0 \cdot \boldsymbol{v} \,\mathrm{d}x - \int_{\Gamma_2} \boldsymbol{f}_2 \cdot \boldsymbol{v} \,\mathrm{d}a, \tag{4.15}$$

$$J(\boldsymbol{v}, \boldsymbol{\eta}) = \frac{1}{2}\, a(\boldsymbol{v}, \boldsymbol{v}) + j(\boldsymbol{v}, \boldsymbol{\eta}) \tag{4.16}$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$ and $\boldsymbol{\eta} = (F, \boldsymbol{f}_0, \boldsymbol{f}_2) \in \Lambda$. Here and below, r^+ denotes the positive part of r, *i.e.*, $r = max\,\{0, r\}$.

We now derive the variational formulation of Problem Q and, to this end, we assume that $(\boldsymbol{u}, \boldsymbol{\sigma})$ are sufficiently regular functions which satisfy (4.1)–(4.6). Then, using (4.5) and (4.12) it follows that

$$\boldsymbol{u} \in K. \tag{4.17}$$

Let $\boldsymbol{v} \in K$ and denote $\boldsymbol{\eta} = (F, \boldsymbol{f}_0, \boldsymbol{f}_2)$ which, due to assumption (4.10), belongs to Λ. We use the definitions (4.14)–(4.16), the properties of the elasticity tensor $\mathcal{E}$ and the constitutive law $\boldsymbol{\sigma} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u})$ to see that

$$\begin{aligned}
J(\boldsymbol{v}, \boldsymbol{\eta}) - J(\boldsymbol{u}, \boldsymbol{\eta}) &= \frac{1}{2}\, a(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{\eta}) - j(\boldsymbol{u}, \boldsymbol{\eta}) \\
&\ge a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{v}, \boldsymbol{\eta}) - j(\boldsymbol{u}, \boldsymbol{\eta}) \\
&= \int_\Omega \boldsymbol{\sigma} \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{u})) \,\mathrm{d}x + \int_{\Gamma_3} F(v_\nu^+ - u_\nu^+) \,\mathrm{d}a \\
&\quad - \int_\Omega \boldsymbol{f}_0 \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}x - \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}a.
\end{aligned}$$

Therefore, using Green's formula (4.7) and equalities (4.2)–(4.4) we deduce that

$$J(\boldsymbol{v}, \boldsymbol{\eta}) - J(\boldsymbol{u}, \boldsymbol{\eta}) \ge \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}a + \int_{\Gamma_3} F(v_\nu^+ - u_\nu^+) \,\mathrm{d}a. \tag{4.18}$$

On the other hand, since

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) \,\mathrm{d}a = \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu) \,\mathrm{d}a + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \,\mathrm{d}a,$$

the frictionless condition (4.6) yields

$$\int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{v} - \boldsymbol{u})\, \mathrm{d}a = \int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, \mathrm{d}a. \tag{4.19}$$

We now use the contact condition (4.5) and the positivity of the function F to see that

$$\sigma_\nu (v_\nu - u_\nu) \geq F(u_\nu^+ - v_\nu^+) \quad \text{on } \Gamma_3.$$

Therefore,

$$\int_{\Gamma_3} \sigma_\nu (v_\nu - u_\nu)\, \mathrm{d}a + \int_{\Gamma_3} F(v_\nu^+ - u_\nu^+)\, \mathrm{d}a \geq 0. \tag{4.20}$$

Next, we combine relations (4.18)–(4.20) to find that

$$J(\boldsymbol{v}, \boldsymbol{\eta}) \geq J(\boldsymbol{u}, \boldsymbol{\eta}). \tag{4.21}$$

Finally, we combine inequality (4.21) with regularity (4.17) to deduce the following variational formulation of Problem Q.

Problem. $\mathcal{Q}$. Find a displacement field $\boldsymbol{u}$ such that

$$\boldsymbol{u} \in K, \qquad J(\boldsymbol{u}, \boldsymbol{\eta}) \leq J(\boldsymbol{v}, \boldsymbol{\eta}) \qquad \forall\, \boldsymbol{v} \in K. \tag{4.22}$$

The analysis of Problem $\mathcal{Q}$, including existence, uniqueness and various convergence results, will be provided in the next section. Here we restrict ourselves to mention that a couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ which satisfies (4.22) and (4.1) is called a weak solution of the elastic contact problem (4.1)–(4.6).

5. EXISTENCE, UNIQUENESS AND CONVERGENCE RESULTS

For each $n \in \mathbb{N}$ we consider a perturbation $\boldsymbol{f}_{0n}$, $\boldsymbol{f}_{2n}$, F_n, g_n of the data $\boldsymbol{f}_0$, $\boldsymbol{f}_n$, F, g and n, respectively, such that

(5.1) $\boldsymbol{f}_{0n} \in L^2(\Omega)^d$, $\boldsymbol{f}_{2n} \in L^2(\Gamma_2)^d$, $F_n \in L^2(\Gamma_3)$, $F_n(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.

(5.2) $g_n > 0$.

Denote $\boldsymbol{\eta}_n = (F_n, \boldsymbol{f}_{0n}, \boldsymbol{f}_{2n})$ which, clearly, belongs to Λ and let K_n be the set

$$K_n = \{\, \boldsymbol{v} \in X : v_\nu \leq g_n \ \text{ a.e. on } \Gamma_3 \,\}. \tag{5.3}$$

With these data we consider the following perturbation of Problem $\mathcal{Q}$.

Problem. $\mathcal{Q}_n^n$. Find a displacement field $\boldsymbol{u}$ such that

$$\boldsymbol{u}_n \in K_n, \qquad J(\boldsymbol{u}_n, \boldsymbol{\eta}_n) \leq J(\boldsymbol{v}, \boldsymbol{\eta}_n) \qquad \forall\, \boldsymbol{v} \in K_n. \tag{5.4}$$

We have the following existence, uniqueness and convergence result.

THEOREM 5.1. *Assume (4.9)–(4.11), (5.1)–(5.2). Then, the following statements hold.*

i) *Problem $\mathcal{Q}$ has a unique solution and, for each $n \in \mathbb{N}$, Problem $\mathcal{Q}_n$ has a unique solution u_n.*

ii) *If $\boldsymbol{f}_{0n} \rightharpoonup \boldsymbol{f}_0$ in $L^2(\Omega)^d$, $\boldsymbol{f}_{2n} \rightharpoonup \boldsymbol{f}_2$ in $L^2(\Gamma_2)^d$, $F_n \to F$ in $L^2(\Gamma_3)$ and $g_n \to g$, then the sequence $\{u_n\}$ converges strongly to $\boldsymbol{u}$, i.e., $\boldsymbol{u}_n \to \boldsymbol{u}$ in X.*

To provide the proof of Theorem 5.1 we need some preliminary results that we present in what follows. First, we note that assumption (4.9) on the elasticity tensor implies that the bilinear form a is symmetric, continuous and coercive with constant $m_{\mathcal{F}}$. Therefore,

$$\text{(5.5)} \qquad \boldsymbol{u}_k \to \boldsymbol{u} \ \text{ in } \ X, \quad \boldsymbol{v}_k \to \boldsymbol{v} \ \text{ in } \ X \quad \Longrightarrow \quad a(\boldsymbol{u}_k, \boldsymbol{v}_k) \to a(\boldsymbol{u}, \boldsymbol{v}).$$

$$\text{(5.6)} \qquad a(\boldsymbol{u}, \boldsymbol{v}) \le M \|\boldsymbol{u}\|_X \|\boldsymbol{v}\|_X \qquad \forall\, \boldsymbol{u}, \, \boldsymbol{v} \in X \ \text{ with } \ M > 0.$$

$$\text{(5.7)} \qquad a(\boldsymbol{v}, \boldsymbol{v}) \ge m_{\mathcal{F}} \|\boldsymbol{v}\|_X^2 \qquad \forall\, \boldsymbol{v} \in X.$$

This implies that the function $\boldsymbol{v} \mapsto a(\boldsymbol{v}, \boldsymbol{v})$ is weakly lower semicontinuous on X, *i.e.*,

$$\text{(5.8)} \qquad \boldsymbol{v}_k \rightharpoonup \boldsymbol{v} \ \text{ in } \ X \quad \Longrightarrow \quad \liminf_{k \to \infty} a(\boldsymbol{v}_k, \boldsymbol{v}_k) \ge a(\boldsymbol{v}, \boldsymbol{v}).$$

On the other hand, using the trace inequality (4.8) and the definition (4.13) of the set Λ, it is easy to see that the function j satisfies the following properties:

$$\text{(5.9)} \qquad j(\boldsymbol{v}, \boldsymbol{\eta}) \le (2c_0 + 1) \|\boldsymbol{v}\|_X \|\boldsymbol{\eta}\|_Y \qquad \forall\, \boldsymbol{v} \in X, \ \boldsymbol{\eta} \in \Lambda.$$

$$\text{(5.10)} \qquad j(\boldsymbol{v}, \boldsymbol{\eta}) \ge -(c_0 + 1) \|\boldsymbol{v}\|_X \|\boldsymbol{\eta}\|_Y \qquad \forall\, \boldsymbol{v} \in X, \ \boldsymbol{\eta} \in \Lambda.$$

Moreover, the compactness of the trace map $\gamma : X \to L^2(\Gamma)^d$ and the compactness of the embedding $X \subset L^2(\Omega)^d$ yield

$$\text{(5.11)} \qquad \boldsymbol{v}_k \rightharpoonup \boldsymbol{v} \ \text{ in } \ X, \quad \boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta} \ \text{ in } \ Y \quad \Longrightarrow \quad j(\boldsymbol{v}_k, \boldsymbol{\eta}_k) \to j(\boldsymbol{v}, \boldsymbol{\eta}).$$

We are now in a position to provide the proof of Theorem 5.1.

Proof. i) It is easy to see that the set Λ is a nonempty closed convex subset of Y which implies that condition (Λ) holds. On the other hand, K is a closed convex subset of X and, since $g > 0$, the zero element of X belongs to K. Therefore, condition $(\widetilde{K})$ is satisfied. Similar arguments show that condition $(\widetilde{K}_n)$ holds, too. Moreover, a simple calculation based on the definitions (4.14)–(4.15), the properties of the form a and the convexity of the function $r \to r^+$ shows that

$$(1-t)J(\boldsymbol{u}, \boldsymbol{\eta}) + tJ(\boldsymbol{v}, \boldsymbol{\eta}) - J((1-t)\boldsymbol{u} + t\boldsymbol{v}, \boldsymbol{\eta}) \ge \frac{t(1-t)}{2} a(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v})$$

for all $\boldsymbol{u}$, $\boldsymbol{v} \in X$, $\boldsymbol{\eta} \in \Lambda$, $t \in [0,1]$. We combine this inequality with inequality (5.7) to see that condition (J^*) holds.

Assume now that $\{\boldsymbol{u}_k\} \subset X$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$ are two sequences such that $\boldsymbol{u}_k \rightharpoonup \boldsymbol{u}$ in X, $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y and let $\boldsymbol{v} \in X$. We have

$$J(\boldsymbol{v},\boldsymbol{\eta}_k) - J(\boldsymbol{u}_k,\boldsymbol{\eta}_k) = \frac{1}{2}\,a(\boldsymbol{v},\boldsymbol{v}) - \frac{1}{2}\,a(\boldsymbol{u}_k,\boldsymbol{u}_k) + j(\boldsymbol{v},\boldsymbol{\eta}_k) - j(\boldsymbol{u}_k,\boldsymbol{\eta}_k)$$

and, using (5.8), (5.11), we deduce that

$$\begin{aligned}\limsup_{k\to\infty}\,&[J(\boldsymbol{v},\boldsymbol{\eta}_k) - J(\boldsymbol{u}_k,\boldsymbol{\eta}_k)]\\ &\le \frac{1}{2}\,a(\boldsymbol{v},\boldsymbol{v}) - \frac{1}{2}\,\liminf_{k\to\infty}\,a(\boldsymbol{u}_k,\boldsymbol{u}_k) + j(\boldsymbol{v},\boldsymbol{\eta}) - j(\boldsymbol{u},\boldsymbol{\eta})\\ &\le \frac{1}{2}\,a(\boldsymbol{v},\boldsymbol{v}) - \frac{1}{2}\,a(\boldsymbol{u},\boldsymbol{u}) + j(\boldsymbol{v},\boldsymbol{\eta}) - j(\boldsymbol{u},\boldsymbol{\eta})\\ &= J(\boldsymbol{v},\boldsymbol{\eta}) - J(\boldsymbol{u},\boldsymbol{\eta}).\end{aligned}$$

It follows from here that condition (J_1) is satisfied.

On the other hand, for any sequences $\{\boldsymbol{u}_k\} \subset X$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$, using inequalities (5.7) and (5.10) we have

$$\begin{aligned}J(\boldsymbol{u}_k,\boldsymbol{\eta}_k) &= \frac{1}{2}\,a(\boldsymbol{u}_k,\boldsymbol{u}_k) + j(\boldsymbol{u}_k,\boldsymbol{\eta}_k)\\ &\ge \frac{m_{\mathcal{F}}}{2}\|\boldsymbol{u}_k\|_X^2 - (c_0+1)\|\boldsymbol{u}_k\|_X\|\boldsymbol{\eta}_k\|_Y.\end{aligned} \tag{5.12}$$

Assume now that and $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y. Then $\{\boldsymbol{\eta}_k\}$ is bounded in Y and, if $\|\boldsymbol{u}_k\|_X \to \infty$, inequality (5.12) shows that $J(\boldsymbol{u}_k,\boldsymbol{\eta}_k) \to \infty$. We conclude from above that condition (J_2) is satisfied, too.

Let $\{\boldsymbol{\eta}_k\} \subset \Lambda$ be a sequence such $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y and let $\boldsymbol{v} \in X$. We have

$$J(\boldsymbol{v},\boldsymbol{\eta}_k) - J(\boldsymbol{v},\boldsymbol{\eta}) = j(\boldsymbol{v},\boldsymbol{\eta}_k) - j(\boldsymbol{v},\boldsymbol{\eta})$$

and, using (5.11) we obtain that $J(\boldsymbol{v},\boldsymbol{\eta}_k) - J(\boldsymbol{v},\boldsymbol{\eta}) \to 0$ which shows that condition (J_3) holds.

Assume now that $\{\boldsymbol{v}_k\} \subset X$ and $\{\boldsymbol{\eta}_k\} \subset \Lambda$ are two sequences such that $\boldsymbol{v}_k \to \boldsymbol{v}$ in X and $\boldsymbol{\eta}_k \rightharpoonup \boldsymbol{\eta}$ in Y. We have

$$J(\boldsymbol{v}_k,\boldsymbol{\eta}_k) - J(\boldsymbol{v},\boldsymbol{\eta}_k) = \frac{1}{2}\,a(\boldsymbol{v}_k,\boldsymbol{v}_k) - \frac{1}{2}\,a(\boldsymbol{v},\boldsymbol{v}) + j(\boldsymbol{v}_k,\boldsymbol{\eta}_k) - j(\boldsymbol{v},\boldsymbol{\eta}_k)$$

and, using the convergences (5.5), (5.11), we deduce that

$$J(\boldsymbol{v}_k,\boldsymbol{\eta}_k) - J(\boldsymbol{v},\boldsymbol{\eta}_k) \to 0.$$

which shows that condition (J_4) holds.

On the other hand, if $\boldsymbol{f}_{0n} \rightharpoonup \boldsymbol{f}_0$ in $L^2(\Omega)^d$, $\boldsymbol{f}_{2n} \rightharpoonup \boldsymbol{f}_2$ in $L^2(\Gamma_2)^d$, $F_n \to F$ in $L^2(\Gamma_3)$, then $\boldsymbol{\eta}_n = (F_n, \boldsymbol{f}_{0n}, \boldsymbol{f}_{2n}) \rightharpoonup \boldsymbol{\eta} = (F, \boldsymbol{f}_0, \boldsymbol{f}_2)$ in Y, which shows that condition (2.1) holds. Finally, definitions (4.12) and (5.3) combined with assumptions (4.11), (5.2) imply the equality $K_n = \frac{g_n}{g} K$, for each $n \in \mathbb{N}$. Therefore, using the compactness of the trace operator, it is easy to see that if $g_n \to g$ holds, then condition (3.1) is satisfied.

To conclude, it follows from above that conditions (Λ), $(\widetilde{K})$, $(\widetilde{K}_n)$, (J_1)–(J_4), (J^*), (2.1) and (3.1) hold. Theorem 5.1 is now a direct consequence of Theorem 3.3. □

Note that Theorem 5.1 provides the unique weak solvability of Problem $\mathcal{Q}$. Moreover, in addition to the mathematical interest in the convergence result in Theorem 5.1 ii), it is important from mechanical point of view, since it shows that the weak solution of the contact problem $\mathcal{P}$ depends continuously on the densities of the applied forces, the yield limit and the thickness of the rigid-plastic layer of the foundation.

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