# DECOMPOSING NORMALIZED UNITS IN COMMUTATIVE MODULAR GROUP RINGS

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Let $F$ be a perfect field of characteristic $p$ and let $G$ be an abelian $p$-group. For the normalized unit group $V(FG)$ of the group ring $FG$ we find a useful criterion only in terms of $F$ and $G$ for validity of the equalities $V(FG) = G(1+I^2(FG;G))$ and $V(FG) = G(1+I^p(FG;G))$ for $p > 2$, where $I(FG;G)$ is the augmentation ideal in $FG$.

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## 1. INTRODUCTION

Everywhere in the text of the present paper, suppose $F$ is a field of non-zero characteristic $p$ and $G$ is a multiplicative abelian group. As usual, $FG$ denotes the group ring of $G$ over $F$ with normalized group of units $V(FG)$, with augmentation ideal $I(FG;G)$ and with nil-radical $N(FG)$. For any ideal $I$ of $FG$, we set $I^n = \underbrace{I \dots I}_{n}$, where $n \in \mathbb{N}$ is a positive integer. Standardly, $C_p$ will denote the cyclic group of order $p$. Recall that a field $F$ is said to be *perfect* if $F = F^p = \{x^p \mid x \in F\}$, and the group $G$ is said to be *p-divisible* if $G = G^p = \{g^p \mid g \in G\}$. Likewise, our epimorphisms will always mean surjective homomorphisms. All other undefined and unstated explicitly notions and notations will follow essentially those from the monographs [15] and [16].

Some brief history on the recent progress concerning the decomposable properties in commutative modular group rings is as follows: in [7] it was obtained a result concerning the decomposition $V(RG) = G\times(1+N(R)G.I(RG;G))$, where $R$ is a commutative ring with identity of prime characteristic $p$ with nil-radical $N(R)$, and $G$ is an abelian group. A slight generalization of the preceding result was established in [10].

In [11] a result about the validity of the decomposition $V(RG)=GV(RG_0)(1 + N(RG).I(RG;G))$ was proved, where $R$ is an arbitrary commutative ring with identity, and $G$ is an arbitrary abelian group with maximal torsion subgroup $G_0$. In [12] it was shown that the general validity of the formula $V(RG) = GV(RG_0)(1 + N(R)G.I(RG; G))$ depends only on some minimal limitations on the commutative ring $R$ and the abelian group $G$. Note that the inclusion $N(R)G \subseteq N(RG)$ holds always (see, *e.g.*, [14]).

Some more results as well as a complete bibliography related to this subject can be found in the author's articles [1–6] plus [8] and [9]. Some other interesting things pertaining to this topic are nicely presented in [14].

In the case when $G$ is a $p$-primary group, it is easily seen that $V(FG) = 1+I(FG; G)$. The aim of this paper is to find a criterion only in terms associated with $F$ and $G$ when the equality $V(FG) = G(1 + I(FG; G).I(FG; G))$ holds, provided $G$ is a $p$-group. Iterating, it will be very useful for applications to the classical *Direct Factor Problem* for modular group rings from [16] to know when the equation

$$V(FG) = G(1 + I(FG; G).I(FG; G). \cdots .I(FG; G))$$

is fulfilled, where the number of times the ideal $I(RG; G)$ appears may vary. We shall restrict in the sequel our attention to the case when this number is a prime $p \geq 3$.

## 2. THE MAIN RESULT

We start here with a plain but helpful technicality.

LEMMA 2.1. *Let $R$ be a commutative ring with identity of prime characteristic $p$, $G$ an abelian group and $A$ an abelian $p$-group. If the map $G \to G/A$ is an epimorphism, then its element-wise extending map $V(RG) \to V(R(G/A))$ is also an epimorphism.*

*Proof.* Since $G \to G/A$ is a homomorphism, it is plainly verified that $V(RG) \to V(R(G/A))$ is also a homomorphism with kernel $1 + I(RG; A)$. But it is not too hard to check that $I(RG; A) = RG.I(RA; A)$ is a nil ideal, and thus $1 + I(RG; A)$ is obviously a $p$-group. Now the desired epimorphism can be readily detected. □

We are now ready to proceed by proving the following basic statement, which also appeared in [13] but for the sake of completeness and for the readers' convenience we provide a detailed proof.

THEOREM 2.2. *Suppose that $F$ is a perfect field of characteristic $p > 0$ and $G$ is an abelian $p$-group. Then the equality*

$$V(FG) = G(1 + I^2(FG; G))$$

*holds if, and only if, one of the following two conditions is true:*

(1) $$G = G^p$$

*or*

(2) $$G \neq G^p \text{ and } F = \mathbb{Z}_p.$$

*Proof.* “Necessity”. Assuming $G \neq G^p$, it follows that $G/G^p$ is a non-trivial and bounded by $p$ factor-group and thus, by [15], it must be a direct sum of cyclic $p$-groups of the same order $p$; say $C_p$ is one of the direct factors of this direct sum. So, one sees that there is a sequence of two epimorphisms $G \to G/G^p \to C_p$, where the first one is the canonical epimorphism while the second one is the canonical projection; thus there is an epimorphism $G \to C_p$. Utilizing Lemma 2.1 or directly by simple independent arguments, it can be extended to the epimorphism $V(FG) \to V(FC_p)$ which sends $I(FG; G)$ to $I(FC_p; C_p)$, because $V(FG) = 1 + I(FG; G)$ and $V(FC_p) = 1 + I(FC_p; C_p)$. Consequently, the relation $V(FG) = G(1 + I^2(FG; G))$ implies the relation $V(FC_p) = C_p(1 + I^2(FC_p; C_p))$. Clearly, $I(FC_p; C_p)$ is a linear space over $F$, and $F$ is an one-dimensional linear space over itself. We shall now construct a linear map $\Phi : I(FC_p; C_p) \to F$ such that $\Phi$ will send $I^2(FC_p; C_p)$ to $\{0\}$. To this purpose, we set $C_p = \langle a \rangle$ with $a^p = 1$. Define

$$\Phi(\sum_{1\leq i\leq p} f_i(a^i - 1)) = \sum_{1\leq i\leq p} if_i,$$

where $f_i \in F$. It is elementary to check that this is a correctly defined map between two linear spaces, because for any $d \in C_p$ we have $f_i d(a^i - 1) = f_i(da^i - 1) - f_i(d - 1)$ and, since $d = a^l$ for some positive integer $l$ with $1 \leq l \leq p$, we deduce that $f_i d(a^i - 1) = f_i(a^{i+l} - 1) - f_i(a^l - 1)$, so we are done. Moreover, because of the self-evident reduction formula, $(b-1)(c-1) = (bc-1) - (b-1) - (c-1)$ for some $b, c \in C_p$, say $b = a^j$ and $c = a^k$ with $j, k \in [1, p]$, it follows that $\Phi((b-1)(c-1)) = \Phi((a^{j+k} - 1) - (a^j - 1) - (a^k - 1)) = (j+k)\cdot 1 - j\cdot 1 - k\cdot 1 = j\cdot 1 + k\cdot 1 - j\cdot 1 - k\cdot 1 = 0$, where 1 is the identity element of $F$. Therefore, since products of the type $(b-1)(c-1)$ form a basis for $I^2(FC_p; C_p)$, one infers that $\Phi(I^2(FC_p; C_p)) = \{0\}$, as wanted.

Furthermore, for any $f \in F$, we consider the normalized unit $1 + f(a-1)$ which can be written like this:

$$1 + f(a - 1) = b(1 + z),$$

where $b \in C_p$ and $z \in I^2(FC_p; C_p)$.

Thus $f(a - 1) = (b - 1) + bz$ with $b = a^j$ for some $1 \leq j \leq p$. Since $bz$ lies in $I^2(FC_p; C_p)$, acting by $\Phi$ on both sides of this equality, we deduce that $f = j \cdot 1$, where $1 \in F$. Hence $F \cong \mathbb{Z}_p$, and we are finished.

"Sufficiency". Since $G$ is $p$-torsion, one observes that $V(FG) = 1 + I(FG;G)$. Firstly, if $G$ is divisible, then for every $g \in G$ we have that $g = h^p$ for some $h \in G$, so that $1 - g = 1 - h^p = (1-h)^p \in I^2(FG;G)$, because $p \geq 2$. Since the elements $1 - g$ of $FG$ form a natural basis for $I(FG;G)$, we deduce that $I(FG;G) = I^2(FG;G)$ and hence $V(FG) = 1 + I(FG;G) = 1 + I^2(FG;G) = G(1 + I^2(FG;G))$, so we are done.

Secondly, assume that $G$ is not $p$-divisible and that $F$ is the simple field of $p$ elements, that is, $F = \mathbb{Z}_p$. Given an arbitrary element $x \in V(FG)$, in view of the formula $V(FG) = 1 + I(FG;G)$ we can write with no harm in generality that

$$x = 1 + k_1 g_1(a_1 - 1) + \cdots + k_s g_s(a_s - 1),$$

where $1 \leq k_1, \cdots, k_s \leq p - 1$; $g_1, a_1, \cdots, g_s, a_s \in G$; $s \in \mathbb{N}$.

Since $k_i g_i(a_i - 1) = k_i(g_i - 1)(a_i - 1) + k_i(a_i - 1)$ for all $i \in [1, s]$, by repeating the summands we may without loss of generality assume that $k_1 = \cdots = k_s = 1$, and thus we need to consider only the element

$$y = 1 + (a_1 - 1) + \cdots + (a_s - 1).$$

Furthermore, because of the reduction formula $(a_i - 1) + (a_j - 1) = (a_i - 1)(1 - a_j) + (a_i a_j - 1)$ which decreases the number of summands of the basis type $w - 1$ for some $w \in G$ in the record, we may assume by induction that $s = 2$. Therefore, $y = 1 + (a_1 - 1) + (a_2 - 1) = a_1 a_2 + (a_1 - 1)(1 - a_2) = a_1 a_2(1 + a_1^{-1} a_2^{-1}(a_1 - 1)(1 - a_2)) \in G(1 + I^2(FG;G))$, as required. □

The next immediate consequence is somewhat rather surprising.

COROLLARY 2.3. *If $G$ is an abelian $p$-group, then the following equality is always true:*

$$V(\mathbb{Z}_p G) = G(1 + I^2(\mathbb{Z}_p G;G)).$$

The next comments shed some more light on the specification of the above explored equalities.

*Remark.* It is worthwhile noticing that if in Theorem 2.2 we have $p = 2$, then $I^2(FC_p;C_p) = \{0\}$. In fact, since $C_2 = \{1, c \mid c^2 = 1\}$, the basis elements for $I^2(FC_p;C_p)$ have to be of the form $(1-1)(1-c) = 0$ or $(1-1)(1-1) = 0$ or $(1-c)(1-c) = (1-c)^2 = 1 - c^2 = 0$, which substantiates our claim.

That is why, $V(FC_p) = C_p$ which readily leads to $|F| = |C_p| = 2$.

Contrasting with the exceptional case alluded to above when we may have $p = 2$, we are now in a position to prove the following somewhat curious assertion in which, whenever $p \geq 3$, the $p$-group has to be necessarily $p$-divisible and thus divisible.

Proposition 2.4. *Let $F$ be a perfect field of characteristic $p > 2$ and let $G$ be an abelian $p$-group. Then the equality $V(FG) = G(1 + I^p(FG; G))$ is true if, and only if, $G = G^p$.*

*Proof.* Considering the left-to-right implication, we will use the same idea that was used in the "Necessity" part of the proof of Theorem 2.2. Indeed, assuming to the contrary that $G$ is not $p$-divisible (i.e., it is *not* divisible), and using the same arguments as before, we will obtain the equality

$$V(FC_p) = C_p(1 + I^p(FC_p; C_p)).$$

However, we assert that $I^p(FC_p; C_p) = \{0\}$, because the basis for $I^p(FC_p; C_p)$ must be equal to zero. In fact, write explicitly $C_p = \{1, c, c^2, \cdots, c^{p-1} \mid c^p = 1\}$ for the generating element $c$ of $C_p$. Thus $(1-c^k)$ is a multiple of $1-c$ for all $k \in \mathbb{N}$ with $1 \leq k \leq p-1$. Since all the non-trivial variants (i.e., excluding the 1) of the existing basis for the product $I^p(FC_p; C_p)$ must contain $(1-c)^p = 1-c^p = 0$, our claim can now be easily verified. Consequently, we derive that $V(FC_p) = C_p$, which allows us to conclude as above that $|F| = |C_p| = 2$. Therefore $p = 2$, which contradicts our initial assumption that $p > 2$. Finally, $G = G^p$, as needed.

As for the right-to-left implication, since any element $g \in G$ must be of the form $g = a^p$ for some $a \in G$, we obtain that $1-g = 1-a^p = (1-a)^p \in I^p(FG; G)$ whence $V(FG) = 1 + I(FG; G) = 1 + I^p(FG; G) = G(1 + I^p(FG; G))$, as desired. □

We note here that we cannot generally have in the last proposition that $G \cap (1 + I^p(FG; G)) = \{1\}$, because if $G^p \neq 1$ and $g \in G \setminus \{1\}$, then one may have that $1 \neq g^p = 1 + g^p - 1 = 1 + (g-1)^p \in 1 + I^p(FG; G)$, as expected.

## 3. AN UNRESOLVED PROBLEM

Recall that a commutative ring of prime characteristic $p$ is said to be *perfect*, provided that $R = R^p = \{r^p \mid r \in R\}$.

We end our work with the following challenging question.

Problem. *Suppose $R$ is a commutative ring with identity and positive characteristic (in particular, a perfect ring of prime characteristic $p$) and suppose $G$ is an abelian group (in particular, a $p$-group). Find a necessary and sufficient condition for the truthfulness of the equality $V(RG) = G(1+I^n(RG; G))$, where $n \in \mathbb{N}$ is an arbitrary fixed natural number.*

We note here that the cases when $n = 2$ with an arbitrary prime $p$ and $n = p > 2$ were already settled above, provided $R$ is a perfect field of characteristic $p$

and $G$ is an abelian $p$-group. So, it will be interesting to study the two different cases when $n > p$ and $n < p$, respectively.

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