HIGHER-ORDER SUFFICIENT CONDITIONS FOR OPTIMIZATION PROBLEMS WITH GÂTEAUX DIFFERENTIABLE DATA

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Our goal is to give some higher-order sufficient efficiency conditions for a nonsmooth multiobjective optimization problem with equality constraints and an arbitrary set constraint and with sufficiently often Gâteaux differentiable data. The smooth scalar case of the problem considered receives special attention. We analyze some examples for which the Second Derivative Test for Constrained Extrema fails and some examples for which the method of Lagrange multipliers cannot be used.

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1. INTRODUCTION

This paper deals with the following nonsmooth constrained multiobjective optimization problem

$$(MPO) \qquad f(\bar{x}) = \text{ Local Extremum}\, f(x), \quad x \in G^{-1}(0) \cap S,$$

where $f = (f_1, \ldots, f_r) : U \to \mathbb{R}^r$ with one component f_s p-times Gâteaux differentiable at $\bar{x} \in G^{-1}(0) \cap S$, $G : U \to \mathbb{R}^k$ is m-times Gâteaux differentiable at $\bar{x}$, p, k, n, m, r are positive integers, $G^{-1}(0) = \{x \in U;\, G(x) = 0\}$, S is an arbitrary subset of $\mathbb{R}^n$, and U is an open subset of $\mathbb{R}^n$ that contains S.

Also we consider the problem (P) which is a nonsmooth scalar case of (MPO) and problem (P_1) which is a smooth particular case of problem (P).

$$(P) \qquad F(\bar{x}) = \text{ Local Extremum } F(x), \quad x \in G^{-1}(0) \cap S,$$

where $F : U \to \mathbb{R}$ is p-times Gâteaux differentiable at $\bar{x} \in G^{-1}(0) \cap S$, and G, U, S are as above.

$$(P_1) \qquad F(\bar{x}) = \text{ Local Extremum}\, F(x), \quad x \in G^{-1}(0),$$

where $F : U \to \mathbb{R}$ is of class C^p on an open subset U of $\mathbb{R}^n$, $G : U \to \mathbb{R}^k$ is of class C^m on U, $\bar{x} \in G^{-1}(0)$.

In this paper, we give some higher-order sufficient efficiency conditions for the nonsmooth multiobjective problem (MPO) and some higher-order sufficient optimality conditions for the nonsmooth scalar problem (P) with sufficiently often Gâteaux differentiable data (and, consequently, to its smooth particular case problem (P_1)) by means of the contingent cone to the constrained set at the extremum point.

The classical approach to solving problem (P_1) involves the Second Derivative Test for Constrained Extrema applied to nondegenerate constraint critical points found by means of the method of Lagrange multipliers.

With the aid of our results we analyze some examples for which the Second Derivative Test for Constrained Extrema fails and some examples to which the method of Lagrange multipliers cannot be applied.

In a joint paper with N.H. Pavel and I. Raykov [5], we derived second-order sufficient conditions for a strict local minimizer of a twice continuously differentiable function subject to an arbitrary set constraint via the contingent cone. In [3, 4] and [6], we obtained higher-order necessary conditions useful in excluding as nonoptimal constrained and unconstrained critical points for which the Second Derivative Test for functions of several variables fails. Our higher-order necessary conditions of [3, 6] were formulated for a smooth optimization problem with an arbitrary set constraint via the higher-order tangent cones in Pavel-Ursescu sense [24, 25]. Higher-order necessary optimality conditions via higher-order tangent cones (in Ledzewicz-Schaettler sense [17]) for a local minimizer of a sufficiently often continuously differentiable scalar objective function subject to equality constraints have also been established in [17, 18] where the problem has inequality constraints too.

Our Theorems 3.1 and 3.2 generalize the well-known second-order sufficient optimality conditions of Bertsekas, [2, Proposition 3.2.1]. Unlike the classical result of [2, Proposition 3.2.1], our Theorem 3.2 applies to a constraint function G with the derivative at the critical point not necessarily different from zero and applies to a constrained degenerate critical point as well. In our results, as in [2, Proposition 3.2.1], the constrained critical point is not required to be regular. Our Theorems 3.3 and 3.4 generalize a result concerning second-order sufficient optimality conditions for smooth scalar optimization problems, by the present author, N.H. Pavel and I. Raykov [5, Theorem 3.2].

Recently D.V. Luu, [19] introduced the notion of Gâteaux differentiability of higher-order and established in [19, Theorem 4.2] sufficient efficiency conditions for the existence of a higher-order strict local Pareto minimum for a multiobjective optimization problem involving cone-constraints and a convex set constraint with Gâteaux differentiable functions via higher-order tangential cones in Pavel-Ursescu sense. The differentiability in the classical sense implies the Gâteaux differentiability.

In this paper, we give sufficient optimality conditions for the existence of a higher-order isolated local extremum of the scalar optimization problem (P) with equality constraints and an arbitrary constraint set with Gâteaux differentiable functions via the contingent cone (Theorems 3.1 and 3.3). We then use our sufficient optimality conditions for the scalar problem (P) and a result due to B. Jimenez [14] to derive sufficient efficiency conditions for the multiobjective problem (MPO).

Thus we extend our second-order sufficient optimality conditions of [5] for a scalar optimization problem with twice continuously differentiable data and with an arbitrary set constraint to higher order sufficient optimality conditions for scalar and multiobjective optimization problems with sufficiently often Gâteaux differentiable data and with an arbitrary constraint set and equality constraints.

We present examples (Examples 3.1 and 3.3) to which [19, Theorem 4.2] is applicable but cannot recognize the origin as a higher-order isolated local minimizer for the smooth scalar equality constrained optimization problems considered in these examples because the higher-order sufficient optimality conditions of Luu's result are not verified. Also we analyze an example (Example 4.1) to which [19, Theorem 4.2] is applicable but fails to identify the origin as a strict local Pareto minimum of higher-order for the nonsmooth multiobjective (MPO) with Gâteaux differentiable data because the higher-order sufficient efficiency conditions of Luu's result are not satisfied. In Examples 3.1 and 3.3, our results (Theorem 3.2 and Corollary 3.1, respectively) help us classify the origin as an isolated local minimizer of order four, while in Example 4.1, Theorem 4.1 guarantees that the origin is a strict local Pareto minimum of order four. Also we show that the second-order sufficient optimality conditions we previously obtained in [5, Theorems 2.2, 3.2] either are not applicable or are not verified in the examples given in this paper.

The paper is organized as follows. In Section 2, we present some definitions and basic results which are used throughout the paper. In Section 3, we derive higher-order sufficient conditions for a feasible point to be a local extremum to the nonsmooth scalar optimization problem (P) and to the smooth scalar optimization problem (P_1), and we analyze some illustrative examples. In Section 4, we give higher-order sufficient efficiency conditions for the nonsmooth multiobjective problem (MPO) and an example.

2. PRELIMINARIES

We begin with some preliminary definitions and notations.

A point $\bar{x} \in D = G^{-1}(0) \cap S$ is said to be an isolated local minimizer of order p (p positive integer) of $F : \mathbb{R}^n \to \mathbb{R}$ on D if there exists a neighborhood V of $\bar{x}$ and a constant $c > 0$ such that

$$F(x) - F(\bar{x}) \geq c\|x - \bar{x}\|^p, \text{ for all } x \in D \cap V \setminus \{\bar{x}\}, \tag{1}$$

where $\|\cdot\|$ denotes the norm of the n-dimensional Euclidean space $\mathbb{R}^n$. If the sense of inequality (1) is reversed, then $\bar{x}$ is said to be an isolated local maximizer of order p of F on D.

In several articles the isolated local minimizers are investigated (for instance in [7, 8, 10, 13, 15, 16, 21, 22, 27]).

An isolated local extremum of any order is a strict local extremum.

Let us recall that the point $\bar{x} \in D$ is a local Pareto minimum (or local efficient solution) for problem (MPO) (or of f on D) if there exists a neighborhood V of $\bar{x}$ such that $(f(D \cap V) - f(\bar{x})) \cap -\mathbb{R}^r_+ = \{0\}$, where $\mathbb{R}^r_+ = \{x = (x_1, \ldots, x_r) \in \mathbb{R}^r : \ x_s \geq 0, \ s = 1, \ldots, r\}$. This means that $\bar{x} \in D$ is a local Pareto minimum if there exists a neighborhood V of $\bar{x}$ such that no $x \in V \cap D$ satisfies $f_s(x) \leq f_s(\bar{x})$ for all $s = 1, \ldots, r$ with $f_s(x) < f_s(\bar{x})$ for at least one index.

The notion of local Pareto minimum is the concept of local minimizer when $r = 1$.

A point $\bar{x} \in D$ is a strict local Pareto minimum of order p, $p \geq 1$ integer, for (MPO) (Jiménez, [15]) if there exists a constant $\alpha > 0$ and a neighborhood V of $\bar{x}$ such that

$$(f(x) + \mathbb{R}^r_+) \cap B(f(\bar{x}), \alpha\|x - \bar{x}\|^p) = \emptyset, \ \forall\, x \in D \cap V, \ x \neq \bar{x},$$

where $B(f(\bar{x}), \alpha\|x - \bar{x}\|^p)$ denotes the open ball of center $f(\bar{x})$ and radius $\alpha\|x - \bar{x}\|^p$.

This is equivalent to (1) in the scalar case ($r = 1$).

A point $\bar{x} \in D$ is a strict local Pareto minimum for (MPO) (or of f on D) (Jimenez, [15]) if there exists a neighborhood V of $\bar{x}$ such that $f(x) \nleqq f(\bar{x})$, $\forall\, x \in D \cap V$, $x \neq \bar{x}$,

or equivalently, $f(x) - f(\bar{x}) \notin -\mathbb{R}^r_+$, $\forall\, x \in D \cap V$, $x \neq \bar{x}$.

Here $z = (z_1, \ldots, z_r) \leqq w = (w_1, \ldots, w_r)$, $z, w \in \mathbb{R}^r$, means $z_s \leq w_s$, $s = 1, \ldots, r$.

The above two notions generalize the corresponding scalar notions.

The following result due to Jimenez (a particular case of [14, Proposition 3.4]) will be used to prove the theorems in Section 4.

PROPOSITION 2.1 ([14, Proposition 3.4]). *Let $f : U \to \mathbb{R}^r$, $\bar{x} \in D$, U an open subset of $\mathbb{R}^n$ that contains D, and $p \geq 1$. Then $\bar{x}$ is not a strict local Pareto minimum of order p of f on D if and only if there exist sequences $x_i \in D \setminus \{\bar{x}\}$, $b_i \in \mathbb{R}^r_+$, such that $x_i \to \bar{x}$ and*

$$\lim_{i\to\infty} \frac{f(x_i) - f(\bar{x}) + b_i}{\|x_i - \bar{x}\|^p} = 0.$$

Let us denote $[v]^k = (v, \ldots, v) \in X^k = \underbrace{X \times \ldots \times X}_{k-times}$, $k \geq 2$ integer.

Let g be a mapping from X into Y, where X and Y are real normed linear spaces. Recall that g is Gâteaux differentiable at $\bar{x}$ if there exists a continuous linear mapping Λ_1 from X into Y such that

$$g(\bar{x} + tv) = g(\bar{x}) + t\Lambda_1(v) + o(t),\ \forall\, v \in X,$$

where $\|o(t)\|/|t| \to 0$ as $t \to 0$. The mapping Λ_1 is said to be Gâteaux derivative of g at $\bar{x}$ and is denoted by $g'_G(\bar{x})$. Note that a mapping which is Gâteaux differentiable at $\bar{x}$ may not be continuous at $\bar{x}$.

The mapping $g : X \to Y$ is p-times Gâteaux differentiable at $\bar{x}$ ($p \geq 2$) if g is Gâteaux differentiable at $\bar{x}$ and there exist continuous multilinear symmetric mappings Λ_k from X^k into Y (continuous linear symmetric in k variables), $k = 2, \ldots, p$, such that

$$g(\bar{x} + tv) = g(\bar{x}) + t\Lambda_1(v) + \frac{t^2}{2!}\Lambda_2[v]^2 + \cdots + \frac{t^p}{p!}\Lambda_p[v]^p + o(t^p),\ \forall\, v \in X,$$

where $\Lambda_1 = g'_G(\bar{x})$, $\|o(t^p)\|/|t|^p \to 0$ as $t \to 0$ (see Luu [19]). Note that symmetric means it does not change under permutation of variables. For the correctness of this definition, the symmetric multilinear mapping Λ_p should be uniquely determined by the respective form $v \to \Lambda_p(v)^p$ (see, for example, [23]). The continuous multilinear symmetric mapping Λ_k is the k^{th} order Gâteaux derivative of g at $\bar{x}$ and is denoted by $g_G^{(k)}(\bar{x})$. Thus for a function g which is p-times Gâteaux differentiable at $\bar{x}$, g can be expanded as

$$g(\bar{x}+tv) = g(\bar{x})+tg'_G(\bar{x})(v)+\frac{t^2}{2!}g_G^{(2)}(\bar{x})[v]^2+\cdots+\frac{t^p}{p!}g_G^{(p)}(\bar{x})[v]^p+o(t^p), \tag{2}$$

for all $v \in X$, where $\|o(t^p)\|/|t|^p \to 0$ as $t \to 0$.

If g is p-times (Fréchet) differentiable at $\bar{x}$ (see [1, Definition 4, p. 86] and [1, Definition 1, p. 97]), we have the following Taylor expansion

$$g(\bar{x}+v) = g(\bar{x})+g'(\bar{x})(v)+\frac{1}{2!}g''(\bar{x})[v]^2+\cdots+\frac{1}{n!}g^{(p)}(\bar{x})[v]^p+\|v\|^p r(v), \tag{3}$$

for all $v \in X$, where $\|r(v)\| \to 0$, as $v \to 0$, and $g^{(k)}(\bar{x})$ is the k-th order

(Fréchet) derivative of g at $\bar{x}$, $k = 1, \ldots, p$ (see Theorem on Taylor's Formula, [1, p. 100]). In this case g is p-times Gâteaux differentiable at $\bar{x}$ and $g^{(k)}(\bar{x}) = g_G^{(k)}(\bar{x})$, $k = 1, \ldots, p$.

If the p-th order (Fréchet) derivative $g^{(p)}(x)$ of $g : X \to Y$ exists at each point x in an open set $U \subseteq X$ and $x \to g^{(p)}(x)$ is continuous in the uniform topology of the space $L(X, ..., \underbrace{L(X,Y)...)}_{p\ times}$ (generated by the norm), then g is of class $C^p(U)$ (see [1, p. 97]).

Let $g^{(p)}(\bar{x})[y]^p = \sum\limits_{1 \le i_1, \ldots, i_p \le n} g_{x_{i_1} \ldots x_{i_p}}(\bar{x}) y_{i_1} \ldots y_{i_p}$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, where $g_{x_{i_1} \ldots x_{i_p}}(\bar{x})$ is the p-order partial derivative of g at $\bar{x}$ with respect to $x_{i_1}, \ldots, x_{i_p}$, $p \ge 2$.

Our sufficient optimality conditions are formulated using the theory of contingent cones.

The contingent cone $T_x D$ to a subset D of $\mathbb{R}^n$ at x in the closure $\bar{D}$ of D is defined by

$$T_x D = \{y \in \mathbb{R}^n;\ \exists\, t_i \to 0+,\ \exists\, y_i \to y, \text{ such that } x + t_i y_i \in D,\ \forall\, i \ge 1\}.$$

It is known that $T_x D$ is a closed cone. If x is an interior point of D, then $T_x D = \mathbb{R}^n$. If $x \in \overline{A \cap B}$, then $T_x(A \cap B) \subseteq T_x A \cap T_x B$, for any A, $B \subseteq \mathbb{R}^n$.

Several equivalent definitions of the contingent cone are presented in [20].

It is well-known the method Lagrange developed for finding possible sites for extrema of a functional constrained optimization problem of type (P_1) (see, for instance, [2, Proposition 3.1.1]).

Next we remind the definitions of a constrained critical point, of a degenerate constraint critical point, and of a regular feasible point for problem (P_1). A feasible point $\bar{x} \in G^{-1}(0)$ for which there is λ such that $F'(\bar{x}) - \lambda G'(\bar{x}) = 0$ is called a constrained critical point. If $\bar{x}$ is a constrained critical point and the second order expression $[F''(\bar{x}) - \lambda G''(\bar{x})][y]^2$ is equal to zero in any direction y such that $G'(\bar{x})(y) = 0$, then $\bar{x}$ is called a degenerate constrained critical point.

A feasible point $\bar{x} \in G^{-1}(0)$ for which the constraint gradients $G_1'(\bar{x})$, $G_2'(\bar{x})$, $\ldots$, $G_k'(\bar{x})$ are linearly independent is called regular. The vectors $G_1'(\bar{x})$, $G_2'(\bar{x})$, $\ldots$, $G_k'(\bar{x})$ are linearly independent if and only if $G'(\bar{x})$ is onto.

To guarantee that a given constrained critical point is a local extremum, we need sufficient conditions for optimality. Different forms of the Second Derivative Test for regular nondegenerate constrained critical points are given in [9] and [26]. In [26], it is also established a criterion for a regular constrained critical point that may be degenerate to be a saddle point. Our higher-order necessary conditions for sufficiently often Fréchet differentiable objective and constraint functions defined on infinite dimensional linear normed spaces can be used to classify as saddle points, regular constrained critical points that are degenerate or not [6, Theorem 3]. In [5], we established second-order necessary conditions for optimization problems with C^2 data and an arbitrary constraint set [5, Theorem 2.1] via first and second-order tangent cones in Pavel-Ursescu sense, and second-order sufficient conditions via the contingent cone [5, Theorem 3.2]. Also we gave second-order sufficient conditions of extremum for regular constrained critical points for equality constrained optimization problems with C^2 data [5, Theorem 2.2].

The classical second-order sufficient condition for a constrained critical point $\bar{x}$ with $G'(\bar{x}) \neq 0$ to be a local extremum for problem (P_1) is known as the Second Derivative Test for Constrained Extrema [2, Proposition 3.2.1, p. 272].

If $\bar{x}$ is a regular degenerate constrained critical point for problem (P_1), then both the classical second-order necessary conditions for a local minimizer and for a local maximizer of Lagrange Multiplier Theorem [2, Proposition 3.1.1, p. 255], are verified at $\bar{x}$. Moreover, if $\bar{x}$ is a degenerate constrained critical point for problem (P_1), then the classical second-order sufficient conditions of Second-Derivative Test for Constrained Extrema, [2, Proposition 3.2.1] do not give any information about $\bar{x}$.

Therefore if $\bar{x}$ is a degenerate constrained critical point we must use another method to determine whether or not it is a local extremum and if it is a local minimizer or a local maximizer. In the next section, we introduce such a method.

3. HIGHER-ORDER SUFFICIENT OPTIMALITY CONDITIONS FOR SCALAR PROBLEMS

In this section, we give our main results for the nonsmooth scalar problem (P) with sufficiently often Gâteaux differentiable data.

THEOREM 3.1. *Let* $F : U \to \mathbb{R}$ *and* $G : U \to \mathbb{R}^k$ *be* p*-times Gâteaux differentiable at* $\bar{x} \in G^{-1}(0) \cap S$*, where* $G^{-1}(0) = \{x \in U;\ G(x) = 0\}$*,* $p \geq 2$*, and* S *is an arbitrary subset of* $\mathbb{R}^n$*,* $S \subseteq U$*. Suppose that there exists some* $\lambda \in \mathbb{R}^k$ *satisfying*

i) $[F_G^{(j)}(\bar{x}) - \lambda G_G^{(j)}(\bar{x})][y]^j \geq 0$*, for all* $y \in \mathbb{R}^n$*,* $1 \leq j \leq p-1$*, and*

ii) $[F_G^{(p)}(\bar{x}) - \lambda G_G^{(p)}(\bar{x})][y]^p > 0$*,* $\forall\, y \in T_{\bar{x}}(G^{-1}(0) \cap S)$*,* $y \neq 0$*.*

Then $\bar{x}$ *is an isolated local minimizer of order* p *of* F *on* $G^{-1}(0) \cap S$*.*

Proof. We can show that $F(x) - F(\bar{x}) \geq c\|x - \bar{x}\|^p$, for some $c > 0$ and for all $x \in G^{-1}(0) \cap S$ in a neighborhood of $\bar{x}$.

Assume by contradiction, that there exists a sequence $\{x_i\}_{i\geq 1}$ such that $x_i \to \bar{x}$, and for all i, $x_i \neq \bar{x}$, $G(x_i) = 0$, $x_i \in S$, and

$$F(x_i) < F(\bar{x}) + \frac{1}{i}\|x_i - \bar{x}\|^p. \tag{4}$$

Let us write $x_i - \bar{x} = \delta_i y_i$, $y_i = \dfrac{x_i - \bar{x}}{\|x_i - \bar{x}\|}$, $\delta_i = \|x_i - \bar{x}\| \to 0$ as $i \to \infty$. The sequence $\{y_i\}_{i\geq 1}$ is bounded, so it must have a subsequence converging to some $\bar{y}$ with $\|\bar{y}\| = 1$. Without loss of generality, we assume that the whole sequence $\{y_i\}_{i\geq 1}$ converges to $\bar{y}$. Since $\bar{x} + \delta_i y_i = x_i \in G^{-1}(0) \cap S$ we have that $\bar{y} \in T_{\bar{x}}(G^{-1}(0) \cap S)$.

Taking into account that F and G are p-times Gâteaux differentiable at $\bar{x}$, we can use equality (2) to get the following expansions of F and G about $\bar{x}$

$$\begin{aligned} \frac{1}{i}\|x_i - \bar{x}\|^p > F(x_i) - F(\bar{x}) = \delta_i F'_G(\bar{x})(y_i) + \frac{\delta_i^2}{2!}F''_G(\bar{x})(y_i)(y_i) + \\ + \frac{\delta_i^3}{3!}F_G^{(3)}(\bar{x})(y_i)(y_i)(y_i) + \ldots + \frac{\delta_i^p}{p!}F_G^{(p)}(\bar{x})[y_i]^p + o(\delta_i^p), \end{aligned} \tag{5}$$

$$\begin{aligned} 0 = G(x_i) - G(\bar{x}) = \delta_i G'_G(\bar{x})(y_i) + \frac{\delta_i^2}{2!}G''_G(\bar{x})(y_i)(y_i) + \\ + \frac{\delta_i^3}{3!}G_G^{(3)}(\bar{x})(y_i)(y_i)(y_i) + \ldots + \frac{\delta_i^p}{p!}G_G^{(p)}(\bar{x})[y_i]^p + o(\delta_i^p), \end{aligned} \tag{6}$$

where $o(\delta_i^p)/\delta_i^p \to 0$ as $i \to \infty$. Consider the two possible cases:

1) If $\lambda \neq 0$, we subtract (6) multiplied by λ from (5), to get

$$\begin{aligned} \frac{1}{i}\|x_i - \bar{x}\|^p > \delta_i(F'_G(\bar{x})(y_i) - \lambda G'_G(\bar{x})(y_i)) + \frac{\delta_i^2}{2!}(F''_G(\bar{x})(y_i)(y_i) - \\ - \lambda G''_G(\bar{x})(y_i)(y_i)) + \ldots + \frac{\delta_i^p}{p!}(F_G^{(p)}(\bar{x})[y_i]^p - \lambda G_G^{(p)}(\bar{x})[y_i]^p) + o(\delta_i^p), \end{aligned} \tag{7}$$

where $o(\delta_i^p)/\delta_i^p \to 0$ as $i \to \infty$.

Since $[F_G^{(j)}(\bar{x}) - \lambda G_G^{(j)}(\bar{x})][y_i]^j \geq 0$, $1 \leq j \leq p-1$, we have from (7) that

$$\frac{1}{i}\|x_i - \bar{x}\|^p > \frac{\delta_i^p}{p!}(F_G^{(p)}(\bar{x})[y_i]^p - \lambda G_G^{(p)}(\bar{x})[y_i]^p) + o(\delta_i^p), \tag{8}$$

where $o(\delta_i^p)/\delta_i^p \to 0$ as $i \to \infty$.

After dividing (8) by δ_i^p and letting i go to infinity, we obtain

$$0 \geq [F_G^{(p)}(\bar{x}) - \lambda G_G^{(p)}(\bar{x})][\bar{y}]^p.$$

On the other hand, by assumption ii), $[F_G^{(p)}(\bar{x}) - \lambda G_G^{(p)}(\bar{x})][\bar{y}]^p$ is strictly positive because $\bar{y} \in T_{\bar{x}}(G^{-1}(0) \cap S)$ and $\bar{y} \neq 0$ as $\|\bar{y}\| = 1$. This contradiction shows that $\bar{x}$ is an isolated local minimizer of order p of F on $G^{-1}(0) \cap S$.

2) If $\lambda = 0$, we divide inequality (5) by δ_i^p.

Since $F_G'(\bar{x})(y_i)$, $F_G''(\bar{x})[y_i]^2, \ldots, F_G^{(p-1)}(\bar{x})[y_i]^{p-1}$ are nonnegative, we have

$$\frac{1}{i} > \frac{1}{p!} F_G^{(p)}(\bar{x})[y_i]^p + o(\delta_i^p)/\delta_i^p, \tag{9}$$

where $o(\delta_i^p)/\delta_i^p \to 0$ as $i \to \infty$.

Letting $i \to \infty$, we obtain from (9) that $0 \geq F_G^{(p)}(\bar{x})[\bar{y}]^p$.

We arrive at a contradiction because, by assumption ii) with $\lambda = 0$,

$$F_G^{(p)}(\bar{x})[y]^p > 0, \text{ for any } y \in T_{\bar{x}}(G^{-1}(0) \cap S),\ y \neq 0.$$

Thus, $\bar{x}$ is an isolated local minimizer of order p of F on $G^{-1}(0) \cap S$ in this case as well. $\square$

Note that the condition $[F_G'(\bar{x}) - \lambda G_G'(\bar{x})](y) \geq 0$, for all $y \in \mathbb{R}^n$ is equivalent to $F_G'(\bar{x}) - \lambda G_G'(\bar{x}) = 0$ because the mapping $F_G'(\bar{x}) - \lambda G_G'(\bar{x}) : \mathbb{R}^n \to \mathbb{R}$ is linear.

By taking $S = \mathbb{R}^n$ in our Theorem 3.1, we obtain the theorem below that gives higher-order sufficient conditions of extremum for the smooth equality constrained optimization problem (P_1).

THEOREM 3.2. *Let $F : U \to \mathbb{R}$ and $G : U \to \mathbb{R}^k$ be of class C^p on an open subset U of $\mathbb{R}^n$, $p \geq 2$, and $\bar{x} \in G^{-1}(0) = \{x \in U;\ G(x) = 0\}$.*

Suppose that there exists some $\lambda \in \mathbb{R}^k$ such that

i) $[F^{(j)}(\bar{x}) - \lambda G^{(j)}(\bar{x})][y]^j \geq 0$, $\forall\, y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, and

ii) $[F^{(p)}(\bar{x}) - \lambda G^{(p)}(\bar{x})][y]^p > 0$, $\forall\, y \in T_{\bar{x}}G^{-1}(0)$, $y \neq 0$.

Then $\bar{x}$ is an isolated local minimizer of order p of F on $G^{-1}(0)$.

LEMMA 3.1. i) *If $G : U \to \mathbb{R}^k$ is m-times Gâteaux differentiable at $\bar{x} \in G^{-1}(0) = \{x \in U;\ G(x) = 0\}$ and $G_G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, while $G_G^{(m)}(\bar{x})$ is not identically zero, then $T_{\bar{x}}G^{-1}(0) \subseteq \{y \in \mathbb{R}^n;\ G_G^{(m)}(\bar{x})[y]^m = 0\}$.*

ii) *If $G : U \to \mathbb{R}^k$ is m-times (Fréchet) differentiable at $\bar{x} \in G^{-1}(0)$ and $G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, while $G^{(m)}(\bar{x})$ is not identically zero, then $T_{\bar{x}}G^{-1}(0) \subseteq \{y \in \mathbb{R}^n;\ G^{(m)}(\bar{x})[y]^m = 0\}$.*

Here $G_G^{(0)}(\bar{x}) = G(\bar{x})$.

Proof. i) Let $y \in T_{\bar{x}}G^{-1}(0)$. We will show that $G_G^{(m)}(\bar{x})[y]^m = 0$. Since there are $t_i \to 0+$ and $y_i \to y$, $i \geq 1$ such that $x_i = \bar{x} + t_i y_i \in G^{-1}(0)$, using the fact that G is m-times Gâteaux differentiable at $\bar{x}$, we get

$$0 = G(\bar{x} + t_i y_i) - G(\bar{x}) = t_i G_G^{'}(\bar{x})(y_i) + \frac{t_i^2}{2!} G_G^{''}(\bar{x})(y_i)(y_i) + \\ + \frac{t_i^3}{3!} G_G^{(3)}(\bar{x})(y_i)(y_i)(y_i) + \ldots + \frac{t_i^m}{m!} G_G^{(m)}(\bar{x})[y_i]^m + o(t_i^m),$$

where $o(t_i^m)/t_i^m \to 0$ as $i \to \infty$.

By taking the limit as $t_i \to 0$ in the relation

$$0 = \frac{G(x_i) - G(\bar{x})}{t_i^m} = G_G^{(m)}(\bar{x})[y_i]^m + \frac{o(t_i^m)}{t_i^m},$$

we obtain $G_G^{(m)}(\bar{x})[y]^m = 0$.

ii) The proof follows from part i) in view of Remark 2.1. $\square$

Example 3.1. Let us consider the objective function $F(x_1, x_2, x_3) = 3x_3^4 + 5x_1 + 3x_2 + 5x_3$, subject to the equality constraints $G_1(x_1, x_2, x_3) = x_1^5 - x_2^5 + x_1 + x_2 + x_3 = 0$, and $G_2(x_1, x_2, x_3) = 2x_1^4 + x_2^4 + x_1 - x_2 + x_3 + x_1 x_3^3 = 0$, $F, G_1, G_2 : \mathbb{R}^3 \to \mathbb{R}$.

We have that $\bar{x} = (0, 0, 0)$ is a constrained critical point because $G(0, 0, 0) = 0$, where $G = (G_1, G_2)$, the vectors $G_1'(0, 0, 0) = (1, 1, 1)$ and $G_2'(0, 0, 0) = (1, -1, 1)$ are linearly independent (so $G'(0, 0, 0)$ is onto), and the pair $(\lambda, \bar{x}) = ((\lambda_1, \lambda_2); (\bar{x}_1, \bar{x}_2, \bar{x}_3)) = ((4, 1); (0, 0, 0))$ is a solution of the equation $F^{'}(x_1, x_2, x_3) = \lambda G'(x_1, x_2, x_3) = \lambda_1 G_1^{'}(x_1, x_2, x_3) + \lambda_2 G_2^{'}(x_1, x_2, x_3)$.

We have $F^{''}(0, 0, 0) = G^{''}(0, 0, 0) = F^{^{(3)}}(0, 0, 0) = G^{^{(3)}}(0, 0, 0) = 0$.

Since $F''(0, 0, 0) - \lambda G^{''}(0, 0, 0) = 0$, the classical second-order necessary conditions [2, Proposition 3.1.1] are verified at $(0, 0, 0)$ but the Second Derivative Test for Constrained Extrema [2, Proposition 3.2.1] does not give any information.

We apply our sufficient conditions of Theorem 3.2 with $p = 4$, $G = (G_1, G_2)$, and $\lambda = (4, 1)$.

Lyusternik's Theorem [12] implies that $T_{(0,0,0)} G^{-1}(0) = \{y \in \mathbb{R}^3;\ G_1'(0, 0, 0)(y) = 0,\ G_2'(0, 0, 0)(y) = 0\}$.

We notice that $F'(0, 0, 0) - \lambda G^{'}(0, 0, 0) = 0$, $F''(0, 0, 0) - \lambda G^{''}(0, 0, 0) = 0$, and $F^{(3)}(0, 0, 0) - \lambda G^{^{(3)}}(0, 0, 0) = 0$.

We have that $[F^{(4)}(0, 0, 0) - \lambda G^{^{(4)}}(0, 0, 0)][y]^4 = F_{x_3 x_3 x_3 x_3}(0, 0, 0) y_3^4 - \lambda_2 [(G_2)_{x_1 x_1 x_1 x_1}(0, 0, 0) y_1^4 + (G_2)_{x_2 x_2 x_2 x_2}(0, 0, 0) y_2^4 + 4(G_2)_{x_1 x_3 x_3 x_3}(0, 0, 0) y_1 y_3^3]$ $= 3(4! y_3^4) - [2(4! y_1^4) + 4! y_2^4 + 4(3!) y_1 y_3^3] = 48 y_1^4 > 0$, for all $y \in T_{(0,0,0)} G^{-1}(0)$, $y \neq 0$, *i.e.*, for all $y \in \{y \in \mathbb{R}^3; y_1 + y_2 + y_3 = 0,\ y_1 - y_2 + y_3 = 0\} = \{y \in \mathbb{R}^3; y_1 + y_3 = 0,\ y_2 = 0\}$, as $y_1 = 0$ yields $y_3 = 0$, and thus, $y = 0$.

In conclusion, according to Theorem 3.2, $(0, 0, 0)$ is a strict local minimizer of F on $G^{-1}(0)$, it is an isolated local minimizer of order four of F on $G^{-1}(0)$.

Theorem 4.2, [19] is applicable to Example 3.1 with $f_s = F$, $h = G$, $g(x) = 0$, $\forall x \in \mathbb{R}^3$, and $C = \mathbb{R}^3$. The higher-order sufficient optimality conditions of D.V. Luu's result are not verified at $(0,0,0)$ as $F^{'}(0,0,0)(y) = 5y_1 + 3y_2 + 5y_3$ is not necessarily nonnegative for any direction $y = (y_1, y_2, y_3)$ with $\|y\| = 1$, for example $F'(0,0,0)(-1,0,0) = -5 \not\geq 0$. Thus the origin cannot be recognized as a higher-order isolated local minimizer of F on $G^{-1}(0)$ by means of [19, Theorem 4.2].

Remark 3.1. Taking $p = 2i$ and $\lambda = (4,1)$ in Theorem 3.2, it can be shown that $(0,0,0)$ is a strict local minimizer of $F(x_1,x_2,x_3) = 3x_3^{2i} + 5x_1 + 3x_2 + 5x_3$, $(x_1,x_2,x_3) \in \mathbb{R}^3$, i positive integer, $i \geq 3$, subject to the constraints $G_1(x_1,x_2,x_3) = x_1^{2i+1} - x_2^{2i+1} + x_1 + x_2 + x_3 = 0$, and $G_2(x_1,x_2,x_3) = 2x_1^{2i} + x_2^{2i} + x_1 - x_2 + x_3 + x_1 x_3^{2i-1} = 0$. The origin is an isolated local minimizer of order $2i$ of F on $G^{-1}(0)$.

Theorem 3.2 as well as Corollary 3.1 deal with the situation where G has the gradient equal to zero and therefore the method of Lagrange multipliers cannot be applied.

Example 3.2. Let us consider the objective function $F(x_1,x_2) = x_1^4 - 4x_2^2$, subject to $G(x_1,x_2) = x_1^5 - x_2^4 + x_2^2 - x_1^3 x_2 = 0$, $F, G : \mathbb{R}^2 \to \mathbb{R}$.

We notice that $\bar{x} = (0,0)$ belongs to the constraint set $G^{-1}(0)$. Obviously, $\bar{x}$ verifies the well-known first-order necessary optimality conditions $F'(\bar{x})(v) \geq 0$, for all $v \in T_{\bar{x}}G^{-1}(0)$, as $F'(0,0) = 0$. Also it can be seen that $\bar{x}$ satisfies our second-order necessary optimality conditions given in [5, Theorem 2.1]: $F''(\bar{x})(v)(v) + F'(\bar{x})(w) = F''(\bar{x})(v)(v) = -8v_2^2 \geq 0$, for all $v = (v_1, v_2) \in S_w \subset T_{\bar{x}}G^{-1}(0) \subseteq \{v \in \mathbb{R}^2; G''(0,0)[v]^2 = 0\} = \{v \in \mathbb{R}^2; v_2 = 0\}$ (see [5] for the definition of the set S_w).

The gradient of G at $\bar{x}$ is equal to zero, and thus the Method of Lagrange Multipliers and the classical second-order optimality conditions [2, Propositions 3.1.1 and 3.2.1] cannot be used for the point $\bar{x}$.

The conditions of Theorem 3.2 are fulfilled with $p = 4$, and $\lambda = -4$. Indeed, first of all we notice that $F^{'}(\bar{x}) - (-4)G^{'}(\bar{x}) = 0$, $F^{^{(3)}}(\bar{x}) - (-4)G^{^{(3)}}(\bar{x}) = 0$, as all the first and third-order partial derivatives of F and G at $\bar{x}$ are equal to zero. Then we get that $[F^{''}(\bar{x}) - (-4)G^{''}(\bar{x})][y]^2 = -4(2y_2^2) - (-4)(2y_2^2) = 0$, $\forall\, y \in \mathbb{R}^2$, and $[F^{^{(4)}}(\bar{x}) - (-4)G^{^{(4)}}(\bar{x})][y]^4 = 24y_1^4 + 4(-24y_2^4 - 24y_1^3 y_2) = 24y_1^4 > 0$, $\forall\, y \neq 0$, $y \in T_{\bar{x}}G^{-1}(0) \subseteq \{y \in \mathbb{R}^2; y_2 = 0\}$, as if $y_1 = 0$ then $y = 0$ as well.

Thus $\bar{x} = (0,0)$ is a strict local minimizer of F on $G^{-1}(0)$, it is an isolated local minimizer of order four of F on $G^{-1}(0)$.

Here we used the fact that the explicit forms of the involved derivatives of F are given by $F^{''}(\bar{x})[y]^2 = F_{x_1x_1}(\bar{x})y_1^2 + F_{x_2x_2}(\bar{x})y_2^2 + 2F_{x_1x_2}(\bar{x})y_1y_2$,

$F^{(3)}(\bar{x})[y]^3 = F_{x_1x_1x_1}(\bar{x})y_1^3 + 3F_{x_1x_1x_2}(\bar{x})y_1^2y_2 + 3F_{x_1x_2x_2}(\bar{x})y_1y_2^2 + F_{x_2x_2x_2}(\bar{x})y_2^3$, and $F^{(4)}(\bar{x})[y]^4 = F_{x_1x_1x_1x_1}(\bar{x})y_1^4 + 4F_{x_1x_1x_1x_2}(\bar{x})y_1^3y_2 + 6F_{x_1x_1x_2x_2}(\bar{x})y_1^2y_2^2 + 4F_{x_1x_2x_2x_2}(\bar{x})y_1y_2^3 + F_{x_2x_2x_2x_2}(\bar{x})y_2^4$.

Remark 3.2. Taking $p = 2i$ and $\lambda = -4$ in Theorem 3.2, it can be seen that $(0,0)$ is a strict local minimizer of $F(x_1, x_2) = x_1^{2i} - 4x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$, i positive integer, $i \geq 3$, subject to $G(x_1, x_2) = x_1^{2i+1} - x_2^{2i} + x_2^2 - x_1^{2i-1}x_2 = 0$.

THEOREM 3.3. *Suppose that*

a) $F : U \to \mathbb{R}$ *is p-times Gâteaux differentiable at* $\bar{x} \in G^{-1}(0) \cap S$, *where* $G^{-1}(0) = \{x \in U,\, G(x) = 0\}$, $G : U \to \mathbb{R}^k$, $p \geq 2$, *and S is an arbitrary subset of* $\mathbb{R}^n$, $S \subseteq U$.

b) *There is a positive integer m such that G is m-times Gâteaux differentiable at* $\bar{x}$, $G_G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, *and* $G_G^{(m)}(\bar{x})$ *is not identically zero.*

c) $F_G^{(j)}(\bar{x})[y]^j \geq 0$, $\forall\, y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, *and* $F_G^{(p)}(\bar{x})[y]^p > 0$, $\forall\, y \neq 0$ *with* $G_G^{(m)}(\bar{x})[y]^m = 0$ *and* $y \in T_{\bar{x}}S$.

Then $\bar{x}$ is an isolated local minimizer of order p of F on $G^{-1}(0) \cap S$.

Proof. We assume by contradiction that $\bar{x}$ is not an isolated local minimizer of order p of F on $G^{-1}(0) \cap S$, *i.e.*, that (4) holds for some sequence $x_i \to \bar{x}$, $G(x_i) = 0$, $x_i \in S$, $x_i \neq \bar{x}$ for all i. Then we obtain (5) for $\delta_i = \|x_i - \bar{x}\| \to 0$, $y_i \to \bar{y} \in T_{\bar{x}}(G^{-1}(0) \cap S)$, $\|\bar{y}\| = 1$ as in the proof of Theorem 3.1. Since $T_{\bar{x}}(G^{-1}(0) \cap S) \subseteq T_{\bar{x}}G^{-1}(0) \cap T_{\bar{x}}S$, by Lemma 3.1 we get $G_G^{(m)}(\bar{x})[\bar{y}]^m = 0$.

Since $F_G^{(j)}(\bar{x})[y_i]^j \geq 0$, $1 \leq j \leq p-1$, it follows from (5) that

$$\frac{1}{i}\|x_i - \bar{x}\|^p > \frac{\delta_i^p}{p!}F_G^{(p)}(\bar{x})[y_i]^p + o(\delta_i^p), \tag{10}$$

where $o(\delta_i^p)/\delta_i^p \to 0$ as $i \to \infty$.

After dividing (10) by δ_i^p and letting i go to infinity, we obtain $F_G^{(p)}(\bar{x})[\bar{y}]^p \leq 0$. On the other hand, by assumption c), $F_G^{(p)}(\bar{x})[\bar{y}]^p > 0$ because $\bar{y} \in T_{\bar{x}}S$, $G_G^{(m)}(\bar{x})[\bar{y}]^m = 0$, and $\bar{y} \neq 0$ as $\|\bar{y}\| = 1$. This contradiction implies that $\bar{x}$ is an isolated local minimizer of order p of F on $G^{-1}(0) \cap S$. $\square$

By arguments similar to those used in the proof of Theorem 3.3, we can get the following theorem for optimization problems with an arbitrary constraint set only.

THEOREM 3.4. *Suppose that*

a) $F : U \to \mathbb{R}$ *is p-times Gâteaux differentiable at* $\bar{x} \in S$, *where* $p \geq 2$, *and S is an arbitrary subset of* $\mathbb{R}^n$, $S \subseteq U$.

b) $F_G^{(j)}(\bar{x})[y]^j \geq 0$, $\forall\, y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, *and* $F_G^{(p)}(\bar{x})[y]^p > 0$, $\forall\, y \in T_{\bar{x}}S$, $y \neq 0$.

Then $\bar{x}$ *is an isolated local minimizer of order* p *of* F *on* S.

Remark 3.3. Theorem 3.4 extends to higher-order sufficient conditions of extremum for nonsmooth optimization problems with sufficiently often Gâteaux differentiable data the second-order sufficient conditions of extremum obtained in [5, Theorem 3.2] for smooth optimization problems with C^2 data. Indeed, if the objective function is of class C^2 on X, Theorem 3.2 [5] follows from Theorem 3.4 with $p = 2$ because the condition $F'(\bar{x})(y) \geq 0$, $\forall\, y \in \mathbb{R}^n$ of Theorem 3.4 is equivalent to the condition $F'(\bar{x}) = 0$ of [5, Theorem 3.2] as $F'(\bar{x})$ is linear. Also the cone $T_{\bar{x}}^l S$ used in [5] is the contingent cone (see [11,20] for various equivalent definitions of the contingent cone).

Theorem 3.3 concerning nonsmooth optimization problems with equality constraints and an arbitrary constraint set generalizes [5, Theorem 3.2] concerning smooth optimization problems with an arbitrary constraint set only.

Theorem 3.3 implies the result below if we take $S = \mathbb{R}^n$. Corollary 3.1 gives higher-order sufficient conditions of extremum for the smooth equality constrained optimization problem (P_1).

COROLLARY 3.1. *Suppose that*

a) $F : U \to \mathbb{R}$ *is of class* C^p *on an open subset* U *of* $\mathbb{R}^n$, $p \geq 2$, *and* $\bar{x} \in G^{-1}(0) = \{x \in U,\, G(x) = 0\}$, $G : U \to \mathbb{R}^k$.

b) *There is a positive integer* m *such that* G *is of class* C^m *on* U, $G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, *and* $G^{(m)}(\bar{x})$ *is not identically zero.*

c) $F^{(j)}(\bar{x})[y]^j \geq 0$, $\forall\, y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, *and* $F^{(p)}(\bar{x})[y]^p > 0$, $\forall\, y \neq 0$ *with* $G^{(m)}(\bar{x})[y]^m = 0$.

Then $\bar{x}$ *is an isolated local minimizer of order* p *of* F *on* $G^{-1}(0)$.

Remark 3.4. Theorems 2.2 and 3.2 and Corollary 3.1 do not require the constrained critical point $\bar{x}$ to be regular. Moreover, the case $G'(\bar{x}) = 0$ is covered by our results.

Example 3.3. Let us consider the objective function $F(x_1, x_2) = x_1^4 + 3x_2^5$, subject to $G(x_1, x_2) = 2x_1^{7/3} + x_1x_2 + x_2^2 = 0$, $F, G : \mathbb{R}^2 \to \mathbb{R}$.

The method of Lagrange multipliers and the classical second-order optimality conditions [2, Propositions 3.1.1 and 3.2.1] cannot be applied to this example because $G'(x) = 0$, for all $x \in \mathbb{R}^2$.

We notice that $\bar{x} = (0, 0)$ belongs to the constraint set $G^{-1}(0)$. Obviously, $\bar{x}$ verifies the well-known first-order necessary optimality conditions $F'(\bar{x})(v) \geq 0$, for all $v \in T_{\bar{x}}G^{-1}(0)$ as $F'(0, 0) = 0$. Moreover, $\bar{x}$ satisfies our second-order necessary optimality conditions of [5, Theorem 2.1] as $F'(\bar{x}) = F''(\bar{x}) = 0$.

The conditions of Corollary 3.1 are fulfilled with $m = 2, p = 4$ since $F'(\bar{x}) = F''(\bar{x}) = F^{(3)}(\bar{x}) = 0$, and $F^{(4)}(\bar{x})[y]^4 = 24y_1^4 > 0$, $\forall\, y = (y_1, y_2)$, $y \neq 0$ such that $G''(\bar{x})[y]^2 = 0$, *i.e.*, for all $y \neq 0$ with $y_1y_2 + y_2^2 = 0$, as $y_1 = 0$ implies $y_2 = 0$ too, which contradicts $y \neq 0$.

By Corollary 3.1, we conclude that $\bar{x} = (0, 0)$ is a strict local minimizer of F on $G^{-1}(0)$, it is an isolated local minimizer of order four of F on $G^{-1}(0)$.

Theorem 4.2 [19] is applicable to Example 3.3 with $f_s = F$, $h = G$, $g(x) = 0$, $\forall\, x \in \mathbb{R}^2$, and $C = \mathbb{R}^2$. The higher-order sufficient optimality conditions of D.V. Luu's result are not verified at $(0, 0)$ as $F^{(4)}(0, 0)[y]^4$ is not necessarily strictly positive for any direction y with $\|y\| = 1$, for example $F^{(4)}(0, 0)[(0, 1)]^4 = 0$. Thus the origin cannot be recognized as a higher-order isolated local minimizer of F on $G^{-1}(0)$ by means of [19, Theorem 4.2].

Remark 3.5. Taking $p = 2i$, $m = 2l$ in Corollary 3.1, it can be seen that $(0, 0)$ is a strict local minimizer of $F(x_1, x_2) = x_1^{2i} + 3x_2^{2i+1}$, $(x_1, x_2) \in \mathbb{R}^2$, subject to $G(x_1, x_2) = 2x_1^{(6l+1)/3} + x_1^l x_2^l + x_2^{2l} = 0$, i and l positive integers, $i \geq 3$, $l \geq 1$. The origin is an isolated local minimizer of order $2i$ of F on $G^{-1}(0)$.

Remark 3.6. In Examples 3.1–3.3, Theorem 3.2 [5] is applicable with $D = G^{-1}(0)$ as $F'(\bar{x}) = 0$. In Examples 3.1 and 3.3, the second-order sufficient optimality conditions of [5, Theorem 3.2] are not verified because $F''(\bar{x}) = 0$ and therefore $F''(\bar{x})(y)(y)$ is not strictly positive in any nonzero direction $y \in T_{\bar{x}}G^{-1}(0)$. In Example 3.2, the second-order sufficient optimality conditions of [5, Theorem 3.2] are not satisfied as $F''(\bar{x})(y)(y) = -8y_2^2 = 0 \not> 0$ for any nonzero $y \in T_{\bar{x}}G^{-1}(0) \subseteq \{y \in \mathbb{R}^2;\ y_2 = 0\}$. Thus, in all the examples in this paper, the origin cannot be identified as a strict local minimizer of F on $G^{-1}(0)$ with the aid of [5, Theorem 3.2].

In Examples 3.2 and 3.3, [5, Theorem 2.2] is not applicable as $G'(\bar{x})$ is not onto. In Example 3.1, the second-order sufficient optimality conditions of [5, Theorem 2.2] are not satisfied at any $x \in G^{-1}(0)$. Indeed, since $F''(x) = G''(x) = 0$, for all $x \in \mathbb{R}^3$ and there exists $\lambda = (4, 1)$ such that $F'(x) = \lambda G'(x)$, for all $x \in \mathbb{R}^3$, we get $F''(x)(v)(v) + F'(x)(w) = F'(x)(w) = -\lambda G'(x)(w) = -\lambda G''(x)(v)(v) = 0 \not> 0$, for any $(v, w) \neq 0$ such that $F'(x)(v) = 0$, $G'(x)(v) = 0$, $G''(x)(v)(v) + G'(x)(w) = 0$. Thus, in all the examples in this paper, the origin cannot be recognized as a local minimizer of F on $G^{-1}(0)$ by means of [5, Theorem 2.2].

Remark 3.7. If we reverse the sense of all the inequalities in our results of this section, then we obtain sufficient conditions for $\bar{x}$ to be an isolated local maximizer of order p.

4. HIGHER-ORDER SUFFICIENT EFFICIENCY CONDITIONS FOR MULTIOBJECTIVE PROBLEMS

In this section, we give our main results for the nonsmooth multiobjective problem (MPO) with sufficiently often Gâteaux differentiable data.

THEOREM 4.1. *Let $f = (f_1, \dots, f_r) : U \to \mathbb{R}^r$ be defined on an open subset U of $\mathbb{R}^n$. Suppose that $G : U \to \mathbb{R}^k$ and $f_s : U \to \mathbb{R}$ for some $s \in \{1, \dots, r\}$ are p-times Gâteaux differentiable at $\bar{x} \in G^{-1}(0) \cap S$, where $G^{-1}(0) = \{x \in U;\ G(x) = 0\}$, $p \geq 2$, S is an arbitrary subset of $\mathbb{R}^n$, $S \subseteq U$. Suppose that there exists some $\lambda \in \mathbb{R}^k$ such that*

i) $[(f_s)_G^{(j)}(\bar{x}) - \lambda G_G^{(j)}(\bar{x})][y]^j \geq 0$, for all $y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, and

ii) $[(f_s)_G^{(p)}(\bar{x}) - \lambda G_G^{(p)}(\bar{x})][y]^p > 0$, for all $y \in T_{\bar{x}}(G^{-1}(0) \cap S)$, $y \neq 0$.

Then $\bar{x}$ is a strict local Pareto minimum of order p of f on $G^{-1}(0) \cap S$.

Proof. By Theorem 3.1, $\bar{x}$ is an isolated local minimizer of order p of f_s on $G^{-1}(0) \cap S$.

Assume by contradiction that $\bar{x}$ is not a strict local Pareto minimum of order p of f on $G^{-1}(0) \cap S$. Then by Proposition 2.1 [14, Proposition 3.4], there would exist $x_i \in G^{-1}(0) \cap S$, $x_i \neq \bar{x}$ for all i, $x_i \to \bar{x}$, and $b_i = (b_{i,1}, \dots, b_{i,r}) \in \mathbb{R}^r_+$ such that

$$\lim_{i\to\infty} \frac{f(x_i) - f(\bar{x}) + b_i}{\|x_i - \bar{x}\|^p} = 0,$$

which implies that

$$\lim_{i\to\infty} \frac{f_s(x_i) - f_s(\bar{x}) + b_{i,s}}{\|x_i - \bar{x}\|^p} = 0.$$

Then Proposition 2.1 guarantees that $\bar{x}$ is not an isolated local minimizer of order p of f_s on $G^{-1}(0) \cap S$, and we arrive at a contradiction. $\square$

Example 4.1. Let us consider the function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ subject to $G(x_1, x_2) = 0$, where $f_1 = F$ and G are the functions from Example 3.2, and

$$f_2(x_1, x_2) = \begin{cases} -x_2, \text{ if } x_1 = x_2^2 \\ 0, \text{ if otherwise.} \end{cases}$$

The functions $f_1(x_1, x_2) = x_1^4 - 4x_2^2$ and $G(x_1, x_2) = x_1^5 - x_2^4 + x_2^2 - x_1^3 x_2$ are polynomials so they are (Fréchet) differentiable at $(0, 0)$ of any order and therefore Gâteaux differentiable of any order. The function f_2 is Gâteaux differentiable at $(0, 0)$ with $f'_{2,G}(0, 0) = 0$, but f_2 is not (Fréchet) differentiable at $(0, 0)$. Indeed, $\lim\limits_{\|h\|\to 0} \dfrac{|f_2((0, 0) + h) - f_2(0, 0)|}{\|h\|} = 1 \neq 0$, for $h(t) = (t^2, t)$ with $t \to 0+$, so the (Fréchet) derivative of f_2 does not exist at $(0, 0)$.

In view of Example 3.2, the hypotheses of Theorem 4.1 are verified with $f_s = f_1 = F$, $S = \mathbb{R}^2$, $p = 4$, $m = 2$, and therefore, the origin is a strict local Pareto minimum of order four of f subject to $G(x_1, x_2) = 0$.

Theorem 4.2 [19] is applicable with $f_s = f_1$, $h = G$, $g(x) = 0$, for all $x \in \mathbb{R}^2$, $C = \mathbb{R}^2$, but the hypotheses of Luu's result are not satisfied as $f_1''(0,0)[y]^2 = F''(0,0)[y]^2 = -8y_2^2 \not\geq 0$, for all y with $\|y\| = 1$. Thus, the origin cannot be recognized as a strict local Pareto minimum of order four by means of [19, Theorem 4.2].

Theorem 4.2. *Suppose that*

a) $f = (f_1, \ldots, f_r) : U \to \mathbb{R}^r$ *is defined on an open subset* U *of* $\mathbb{R}^n$, $f_s : U \to \mathbb{R}$ *for some* $s \in \{1, \ldots, r\}$ *is* p*-times Gâteaux differentiable at* $\bar{x} \in G^{-1}(0) \cap S$, *where* $G^{-1}(0) = \{x \in U;\ G(x) = 0\}$, $p \geq 2$, S *is an arbitrary subset of* $\mathbb{R}^n$, $S \subseteq U$.

b) *There is a positive integer* m *such that* G *is* m*-times Gâteaux differentiable at* $\bar{x}$, $G_G^{(j)}(\bar{x}) = 0$, $0 \leq j \leq m-1$, *and* $G_G^{(m)}(\bar{x})$ *is not identically zero.*

c) $(f_s)_G^{(j)}(\bar{x})[y]^j \geq 0$, $\forall\, y \in \mathbb{R}^n$, $1 \leq j \leq p-1$, *and* $(f_s)_G^{(p)}(\bar{x})[y]^p > 0$, $\forall\, y \neq 0$ *with* $G_G^{(m)}(\bar{x})[y]^m = 0$ *and* $y \in T_{\bar{x}}S$.

Then $\bar{x}$ *is a strict local Pareto minimum of order* p *of* f *on* $G^{-1}(0) \cap S$.

Proof. The proof is similar to the proof of Theorem 4.1, only that it makes use of Theorem 3.3 instead of Theorem 3.1.

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