GENERALIZED q-GAUSSIAN VON NEUMANN ALGEBRAS WITH COEFFICIENTS III. UNIQUE PRIME FACTORIZATION RESULTS

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We prove some unique prime factorization results for tensor products of type II_1 factors of the form $\Gamma_q(\mathbb{C}, S \otimes H)$ arising from symmetric independent copies with sub-exponential dimensions of the spaces $D_k(S)$ and $\dim(H)$ finite and greater than a constant depending on q.

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1. INTRODUCTION

This article is a continuation of the program initiated in [2]. In [2], we introduced the generalized q-gaussian von Neumann algebras $\Gamma_q(B, S \otimes H)$ with coefficients in B and proved their strong solidity relative to B under the assumptions of $dim_B(D_k(S))$ sub-exponential and $dim(H) < \infty$ (see [2], Definition 3.18 and Corollary 7.4). In a subsequent paper [3], we investigated the presence of non-trivial central sequences and we showed they do not exist when B is a finite dimensional factor, the dimensions over B of the modules $D_k(S)$ are sub-exponential and the dimension of H is finite and greater than a constant depending on q. In the present work, we prove some unique prime factorization results for tensor products of von Neumann algebras of the form $\Gamma_q(\mathbb{C}, S \otimes H)$ arising from a sequence of symmetric independent copies over $\mathbb{C}$ and having sub-exponential dimensions (over $\mathbb{C}$) of the spaces $D_k(S)$ introduced in [2], Definition 3.18. The first results of this kind for type II_1 factors – arising from either (discrete) ICC non-amenable hyperbolic groups or (discrete) subgroups of connected simple Lie groups of rank one – have been obtained by Ozawa and Popa in [7], through a combination of Ozawa's C^*-algebraic techniques previously used in [6] and the powerful intertwining and unitary conjugacy techniques of Popa (see *e.g.* [8], Appendix and [9], Theorem 2.1). Let's recall that for $(\mathcal{M}, \tau)$ a type II_1 factor and $t > 0$, the amplification of $\mathcal{M}$ by t is defined as $\mathcal{M}^t = p(M_n(\mathbb{C}) \otimes \mathcal{M})p$, where $n > t$ and $p \in M_n(\mathbb{C}) \otimes \mathcal{M}$ is a projection with $\tau(p) = t/n$. Our main result is (see also Theorem 1 in [7]):

THEOREM 1.1. *Let* $M_k = \Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$ *coming from a sequence of symmetric independent copies* $(\pi_j^k, \mathbb{C}, A_k, D_k)$ *for* $-1 < q_k < 1$ *and* $1 \le k \le n$. *Assume that for all* $1 \le k \le n$, H_k *is finite dimensional and* $dim_{\mathbb{C}}((D_k)_i(S_k)) < Cd^i$ *for all* i *and some constants* $C, d > 0$. *Suppose that* $M = \overline{\bigotimes}_{k=1}^n M_k = N_1 \bar{\otimes} N_2$ *for some type* II_1 *factors* N_1 *and* N_2. *Then there exists* $t > 0$ *and a partition* $I_1 \sqcup I_2 = \{1, \ldots, n\}$ *such that, modulo conjugacy by an unitary in* M, *we have* $N_1^t = \overline{\bigotimes}_{k \in I_1} M_k$ *and* $N_2^{1/t} = \overline{\bigotimes}_{k \in I_2} M_k$.

To prove Theorem 1.1, instead of relying on C^*-algebraic techniques and the property (AO) as in [6, 7], we use our relative strong solidity result in [2], Corollary 7.4. We should note that the von Neumann algebras $\Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$ are automatically factors if $dim(H_k) \ge d(q_k)$ (Proposition 3.23 in [2]). By repeatedly applying Theorem 1.1 one obtains

COROLLARY 1.2. *Let* $M_k = \Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$ *with the dimensions (over* $\mathbb{C}$*) of the spaces* $(D_k)_i(S_k)$ *sub-exponential and* $\infty > dim(H_k) \ge d(q_k)$ *for all* $1 \le k \le n$. *Assume that*

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n = N_1 \bar{\otimes} \cdots \bar{\otimes} N_m,$$

for $m \ge n$ *and some type* II_1 *factors* $N_1, \ldots, N_m$. *Then* $m = n$ *and there exist* $t_1, \ldots, t_n > 0$ *with* $t_1 t_2 \cdots t_n = 1$ *such that, after permutation of indices and unitary conjugacy, we have* $N_k^{t_k} = M_k$.

When the factors N_j are assumed to be prime the assumption $m \ge n$ becomes unnecessary and hence we obtain

COROLLARY 1.3. *Let* $M_1, \ldots, M_n$ *be generalized* q*-gaussians as above. Suppose that for some* $m \in \mathbb{N}$ *and prime type* II_1 *factors* $N_1, \ldots, N_m$ *we have*

$$M_1 \bar{\otimes} \cdots \bar{\otimes} M_n = N_1 \bar{\otimes} \cdots \bar{\otimes} N_m.$$

Then $m = n$ *and there exist* $t_1, \ldots, t_n > 0$ *with* $t_1 t_2 \cdots t_n = 1$ *such that, after permutation of indices and unitary conjugacy, we have* $N_k^{t_k} = M_k$. *In particular this holds if each* $N_j = \Gamma_{q_j}(\mathbb{C}, T_j \otimes K_j)$ *is a generalized* q_j*-gaussian with scalar coefficients, sub-exponential dimensions of* $(D_j)_i(T_j)$ *and* $dim(K_j) < \infty$.

In particular, M_k and / or N_j could be any of the examples in 4.4.1, 4.4.2, 4.4.3 in [2]. Thus, if $M_i, 1 \le i \le n$, and $N_j, 1 \le j \le m$, are generalized q-gaussian von Neumann algebras as above and $m \ne n$, then $M_1 \bar{\otimes} \cdots \bar{\otimes} M_n \not\cong N_1 \bar{\otimes} \cdots \bar{\otimes} N_m$.

2. PROOF OF THE MAIN THEOREM

Throughout this section, we freely use notations and results from Section 3 of [2]. We start by stating some preliminary technical results. The first one is Proposition 2.7 in [12]. If (M,τ) is a tracial von Neumann algebra and $P,Q \subset M$ are von Neumann subalgebras, we say that P is amenable relative to Q (inside M) if there exists a P-central state Ω on $B(L^2(M)) \cap (Q^{op})'$ such that $\Omega|_M = \tau$ (see *e.g.* Definition 2.2 in [12]).

PROPOSITION 2.1. *Let (M,τ) be a tracial von Neumann algebra and let $Q_1, Q_2 \subset M$ be von Neumann subalgebras. Assume that Q_1, Q_2 form a commuting square, which means $E_{Q_1} \circ E_{Q_2} = E_{Q_2} \circ E_{Q_1}$, where E_{Q_1}, E_{Q_2} are the conditional expectations of M onto Q_1, Q_2 respectively, and that Q_1 is regular in M. Let $P \subset M$ be a von Neumann subalgebra which is amenable relative to both Q_1 and Q_2. Then P is amenable relative to $Q_1 \cap Q_2$.*

The next result is Proposition 12 in [7].

PROPOSITION 2.2. *Let $M = M_1 \bar{\otimes} M_2$ and $N \subset M$ be type II_1 factors. Assume that $N \prec_M M_1$ and $N' \cap M$ is a factor. Then there exists a decomposition $M = M_1^t \bar{\otimes} M_2^{1/t}$ for some $t > 0$ and a unitary $u \in \mathcal{U}(M)$ such that $uNu^* \subset M_1^t$.*

The next result will be needed in the proof of Theorem 1.1. It is an analogue of Proposition 15 in [7]. For convenience, if $M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_n$ and $1 \leq k \leq n$, let's denote by

$$\widehat{M_k} = M_1 \bar{\otimes} \cdots M_{k-1} \bar{\otimes} 1 \bar{\otimes} M_{k+1} \cdots \bar{\otimes} M_n \subset M.$$

More generally, for every subset $I \subset \{1, \ldots, n\}$, we will denote by $\widehat{M_I}$ the von Neumann algebra

$$\widehat{M_I} = \overline{\bigotimes}_{i \notin I} M_i \subset M.$$

PROPOSITION 2.3. *Let $M_i = \Gamma_{q_i}(\mathbb{C}, S_i \otimes H_i)$ be generalized q-gaussian von Neumann algebras with scalar coefficients coming from symmetric independent copies and having sub-exponential dimensions over $\mathbb{C}$ of the spaces $(D_i)_k(S_i)$, for all $1 \leq i \leq n$. Let $M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_n$ and assume that $N \subset M$ is a type II_1 factor such that $N' \cap M$ is a non-amenable factor. Then there exists $t > 0$, $1 \leq k \leq n$ and a unitary $u \in \mathcal{U}(M)$ such that $uNu^* \subset (\widehat{M_k})^t$.*

Proof. Let's first note that there exists a $1 \leq k \leq n$ such that $N' \cap M$ is not amenable relative to $\widehat{M_k}$. Indeed, if this were not the case, since the subalgebras $\widehat{M_I}, \widehat{M_J}$ form a commuting square for all subsets $I, J \subset \{1, \ldots, n\}$ and all of them are regular in M, by repeatedly applying Proposition 2.1, we would obtain that $N' \cap M$ is amenable relative to $\bigcap_{k=1}^n \widehat{M_k} = \mathbb{C}$, *i.e.* $N' \cap M$ is amenable, a contradiction. Fix a k such that $N' \cap M$ is not amenable relative to $\widehat{M_k}$. Suppose that $N \not\prec_M \widehat{M_k}$. By Corollary F.14 in [1] there exists an abelian von Neumann subalgebra $\mathcal{A} \subset N$ such that $\mathcal{A} \not\prec_M \widehat{M_k}$. Let's make the following general remark. Suppose $\Gamma_q(B, S \otimes H)$ is associated to a sequence of symmetric independent copies (π_j, B, A, D) and let $\mathcal{M}$ be any tracial von Neumann algebra. Then the von Neumann algebra

$$\mathcal{M}\bar{\otimes}\Gamma_q(B, S \otimes H) = \Gamma_q(B\bar{\otimes}\mathcal{M}, S \otimes H)$$

is associated to a new sequence of symmetric independent copies $(\tilde{\pi}_j, \tilde{B}, \tilde{A}, \tilde{D})$, defined by $\tilde{B} = B\bar{\otimes}\mathcal{M}$, $\tilde{A} = A\bar{\otimes}\mathcal{M}$, $\tilde{D} = D\bar{\otimes}\mathcal{M}$ and $\tilde{\pi}_j : \tilde{A} \to \tilde{D}$ are given by $\tilde{\pi}_j(a \otimes x) = \pi_j(a) \otimes x$, for $a \in A, x \in \mathcal{M}$. Now note that

$$\mathcal{A} \subset M = \widehat{M_k}\bar{\otimes}M_k = \widehat{M_k}\bar{\otimes}\Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k) = \Gamma_{q_k}(\widehat{M_k}, S_k \otimes H_k).$$

It's trivial to check that $\dim_{\widehat{M_k}}((D_k)_i(S_k))$ $=\dim_{\mathbb{C}}(D_k)_i(S_k)$ are sub-exponential. Since $\mathcal{A}$ is amenable relative to $\widehat{M_k}$, by applying Corollary 7.4 in [2], we must have that either $\mathcal{A} \prec_M \widehat{M_k}$ or $\mathcal{N}_M(\mathcal{A})''$ is amenable relative to $\widehat{M_k}$. The first half of the alternative is precluded by the choice of $\mathcal{A}$, and the second would imply that $N' \cap M \subset \mathcal{N}_M(\mathcal{A})''$ is also amenable relative to $\widehat{M_k}$, which is a contradiction. Thus $N \prec_M \widehat{M_k}$, and by Proposition 2.2 we see that there exists a unitary $u \in \mathcal{U}(M)$ and a $t > 0$ such that $uNu^* \subset (\widehat{M_k})^t$. □

Now we can prove Theorem 1.1. The proof proceeds verbatim as in [7]. We nevertheless give details for completeness.

Proof of Theorem 1.1. We use induction over n. The case $n = 0$ is trivial. Let $M = \overline{\bigotimes}_{k=1}^n M_k = N_1\bar{\otimes}N_2$. Since M is non-amenable, we can assume that N_2 is non-amenable. By Proposition 2.3 there exist $t > 0$, $1 \leq k \leq n$ and $u \in \mathcal{U}(M)$ such that $uN_1u^* \subset (\widehat{M_k})^t$. Set $\mathcal{M}_1 = \widehat{M_k}$, $\mathcal{M}_2 = M_k$ and $N_{2,1} = N_1' \cap u\mathcal{M}_1^t u^*$. Then we see that

$$N_2 = N_1' \cap M = u^*(N_{2,1}\bar{\otimes}\mathcal{M}_2^{1/t})u \subset u^*(\mathcal{M}_1^t\bar{\otimes}\mathcal{M}_2^{1/t})u = M,$$

and $u\mathcal{M}_1^t u^*$ is generated by N_1 and $N_{2,1}$. Using the induction hypothesis, we can find an $s > 0$, a partition $I_1, I_{2,1}$ of $\{1, \ldots, n\} \setminus \{k\}$ such that $N_1 = (\overline{\bigotimes}_{j \in I_1} M_j)^s$ and $N_{2,1} = (\overline{\bigotimes}_{j \in I_{2,1}} M_j)^{t/s}$ after conjugating with a unitary element in $(\widehat{M_k})^t$. If we now set $I_2 = I_{2,1} \cup \{k\}$, the proof is complete □

Remark 2.4. The statement of the results and the proofs remain verbatim the same if one assumes that B_k is a finite dimensional factor for every $1 \leq k \leq n$.

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