BAND FUNCTIONS OF IWATSUKA MODELS: POWER-LIKE AND FLAT MAGNETIC FIELDS

PABLO MIRANDA and NICOLAS POPOFF

In this note, we consider the Iwatsuka model with a positive increasing magnetic field having finite limits. The associated magnetic Laplacian is fibred through partial Fourier transform, and, for large frequencies, the band functions tend to the Landau levels, which are thresholds in the spectrum. The asymptotics of the band functions is already known when the magnetic field converge polynomially to its limits. We complete this analysis by giving the asymptotics for a regular magnetic field which is constant at infinity, showing that the band functions converge now exponentially fast toward the thresholds. As an application, we give an estimate on the current of quantum states localized in energy near a threshold.

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1. THE IWATSUKA MODEL

In this article, we review and complete some results about the band function of the Iwatsuka model with an increasing positive magnetic field having finite limits. Assume that the magnetic field $b : \mathbb{R}^2 \to (0, +\infty)$ depends only on one variable in the sense that $b(x, y) = b(x)$. We assume moreover that

(1)
$$b \text{ is } C^0, \text{ increasing, and has finite limits } b^{\pm} \text{ as } x \to \pm\infty, \text{with } 0 < b_- < b_+.$$

The model is gauge invariant, and we choose the magnetic potential

$$A(x, y) := (0, a(x)); \quad \text{with} \quad a(x) := \int_0^x b(t)\mathrm{d}t.$$

The magnetic Laplacian is then defined by

$$H_0 := (-i\nabla - A)^2 = -\partial_x^2 + (-i\partial_y - a(x))^2$$

acting in $L^2(\mathbb{R}^2)$.

Historically, this operator was introduced in order to provide an example of a magnetic Laplacian with purely absolutely continuous spectrum, see [12]. After, this has been proved under various conditions on b, see [6,12,13,20], but the fact that this is true as long as b is non-constant is still open, since [3].

Over the years, this model has been widely studied, as a source of interesting questions linked to transport phenomena in the translation invariant direction. In this note, we describe and complete results on the asymptotics of the band functions, which is an important step when describing spectral properties at energies near the thresholds.

A key tool for studying operators having a translation invariance is fibration through partial Fourier transform. In our case, denote by $\mathcal{F}_y$ the partial Fourier transform in the y variable. Then there holds

$$\mathcal{F}_y H_0 \mathcal{F}_y^\star = \int^{\oplus} \mathfrak{h}(k) \mathrm{d}k$$

where $\mathfrak{h}(k)$ is the unidimensional Sturm-Liouville operator defined by

$$\mathfrak{h}(k) := -\partial_x^2 + (a(x) - k)^2 \tag{2}$$

acting on $L^2(\mathbb{R})$. It is positive, self-adjoint with compact resolvent. We denote by $\{E_n(k), n \geq 1\}$ the increasing sequence of its eigenvalues. They are simple, see [12, Lemma 2.3], therefore the functions $E_n(\cdot)$ are analytic with respect to k on $\mathbb{R}$. They are called the band functions (or dispersion curves) of H_0.

With the hypotheses made on b (see (1)), the band functions $k \mapsto E_n(k)$ are increasing and converge to $\Lambda_n b^\pm$ as $k \to \pm\infty$, where $\Lambda_n := 2n - 1$. In this case, the spectrum of H_0 is obviously purely absolutely continuous.

The values $\Lambda_n b^\pm$ are thresholds in the spectrum of H_0. The nature of these thresholds, and more refined properties of the operator (and its perturbations), are deeply linked to the behavior of the band functions at these limits.

The trajectory of a classical particle submitted to this kind of magnetic fields is quite easy to picture. Generically, the particles exhibit a drift in the invariance direction y. Because the magnetic field varies slowly when $|x|$ is large, if the particle is located initially in such a zone, the drift will be weak, and the trajectory of the particle will be close to a circle. For a spinless quantum particle of a given energy, the evolution is linked to the band functions crossing this energy, the velocity in the y direction being related to the derivative of the band functions. If the energy is far from thresholds, the particle is usually called an *edge state*, because it will show some propagation, as it is the case for models involved in Quantum Hall Effect, where edges induce transport. We refer to [11, 13, 17] for a study of this case. On the other hand, if the energy is closed to a threshold, the particle will bear a *bulk component*, that is a component whose velocity is small. Quantitative estimates on the velocity requires asymptotics of the band functions near thresholds. Information on the eigenfunctions of the fiber operator allow to describe the zone where the particle is localized. This is done by an analysis in phase space (see [10, 15], and also [4] for a rough analysis on a similar model).

2. ASYMPTOTICS OF BAND FUNCTIONS

In [15], it is assumed that the magnetic field converges to its limit like a negative power of x, the model case being

$$\exists x_0 \in \mathbb{R}, \forall x \geq x_0, \quad b(x) = b_+ - \langle x \rangle^{\alpha}, \quad \alpha > 0.$$

Our hypotheses bear on the behavior of the magnetic field at $+\infty$ and the asymptotics of the band function when $k \to +\infty$. The same hypotheses in the direction $x \to -\infty$ would provide similar results as $k \to -\infty$.

Under this condition, the behavior of the band functions is provided in [15, Theorem 2.2 and Corollary 2.4]:

$$E_n(k) = b_+ \Lambda_n - \frac{\Lambda_n b_+^M}{k^M} + O\Big(\frac{1}{k^{M+2}}\Big). \tag{3}$$

Here we will consider another physically relevant class of magnetic fields, those which are *equal* to their limit for large x:

$$\exists x_\infty \in \mathbb{R}, \forall x \geq x_\infty, \quad b(x) = b_+. \tag{4}$$

Up to picking the smallest real satisfying the above relation, we may assume that $b(x) < b_+$ for $x < x_\infty$. The case of a piecewise constant magnetic field was treated in [9, 17], but in that case, the use of special functions allows precise computations of the asymptotics that are not available in a general context. Moreover, in this article we are interested in more regular magnetic fields.

We say that the contact at x_∞ is of order $p \in \mathbb{N} \cup \{+\infty\}$ when b is $C^p((-\infty, x_\infty)$,

$$\lim_{\substack{x \to x_\infty \\ x < x_\infty}} b^{(p)}(x) := b^{(p)}(x_\infty^-) \neq 0$$

and $b^{(j)}(x_\infty) = 0$ for all $j = 1, \ldots, p-1$.

The potential in (2) vanishes at a unique point $a^{-1}(k) =: x_k$, a^{-1} being the inverse function of a. The proof of the asymptotics relies on the construction of quasi-modes in the spirit of the harmonic approximation [5]. Indeed, after the change of variable $x = b_+^{-1/2} t + x_k$, the operator $\mathfrak{h}(k)$ is transformed in $b_+ \widetilde{\mathfrak{h}}(k)$, with

$$\widetilde{\mathfrak{h}}(k) = -\partial_t^2 + W(t, k). \tag{5}$$

In both cases, (3) and (4), W has a unique minimum at 0 which is non degenerate, in the sense that $\partial_t^2 W(0,k) = 2$.

Writing $W(t,k) = t^2 + d_k(t)$, then, in case (3), d_k is not zero near $t = 0$, and $d_k(t) = \alpha(k)O(t^3)$, where $\alpha(k) \to 0$ as $k \to +\infty$. Therefore, perturbation theory provides the asymptotics of the eigenvalues of $\mathfrak{h}(k)$ as $k \to +\infty$. But in case (4), $W(t,k) = t^2$ in a neighborhood of 0 and it is not clear on which quantity depends the asymptotics of the band function, and which is the convergence rate. We define $a_\infty := a(x_\infty)$, and

$$t_k := \sqrt{b_+}(x_\infty - x_k) = \frac{a_\infty - k}{b_+^{1/2}}.$$

Notice that for k large enough, $t_k < 0$, and $W(t,k) = t^2$ for $t \in (t_k, +\infty)$. The asymptotics of the band functions is now very different from (3): as one expects, in case (4) the band functions converge faster to their limits, more precisely the quantity $E_n(k) - b_+\Lambda_n$ is now exponentially small as $k \to +\infty$, and the first order term depends only on the contact point, as follows:

THEOREM 2.1. *Assume that the contact at x_∞ is of order $p \geq 1$. Then, as $k \to +\infty$:*

$$E_n(k) = \Lambda_n + C(n,p,b_+) b^{(p)}(x_\infty^-) k^{2n-p-3} e^{-t_k^2} + o(k^{2n-p-3} e^{-t_k^2}), \tag{6}$$

where $C(n,p,b_+) = (-1)^p \frac{2^{n-p-2}}{\sqrt{\pi}(n-1)! b_+^{n-\frac{3}{2}}}$.

If the contact is of order ∞, then

$$e^{t_k^2}(E_n(k) - \Lambda_n) = o(k^{-\infty}). \tag{7}$$

Proof. Assume (4) and note that

$$x_k = x_\infty + \frac{k - a_\infty}{b_+}, \tag{8}$$

moreover, b is constant on (x_∞, x_k). Recall that $d_k(t) = W(k,t) - t^2$, with

$$W(t,k) = \frac{1}{b_+}(a(b_+^{-1/2}t + x_k) - a(x_k))^2 = \frac{1}{b_+}\left(\int_{x_k + \frac{t}{\sqrt{b_+}}}^{x_k} b(s)\mathrm{d}s\right)^2.$$

Writing

$$t^2 = \frac{1}{b_+}\left(\int_{x_k + \frac{t}{\sqrt{b_+}}}^{x_k} b_+\mathrm{d}s\right)^2,$$

we get for all $t \in \mathbb{R}$ that:

$$d_k(t) = \frac{1}{b_+} \int_{x_k + \frac{t}{\sqrt{b_+}}}^{x_k} (b(s) + b_+)\mathrm{d}s \int_{x_k + \frac{t}{\sqrt{b_+}}}^{x_k} (b(s) - b_+)\mathrm{d}s. \tag{9}$$

Note that $|d_k(t)| \leq Ct^2$ and that $d_k \to 0$ point-wise as $k \to +\infty$. Therefore, we will approximate $\widetilde{\mathfrak{h}}(k)$ by the harmonic oscillator $\mathfrak{h}_0 := -\partial_t^2 + t^2$ as $k \to +\infty$.

The same computations as in [15, Section 2.1], based on construction of quasi-modes for $\widetilde{\mathfrak{h}}(k)$ from Hermite's functions, and estimations of remainders in the spirit of the theory of perturbations, apply. We describe here the first step of the procedure, which is sufficient for our purposes. Denote by $\Psi_n(t) = P_n(t)e^{-\frac{t^2}{2}}$ the n-th normalized Hermite's function, starting from $n = 1$, where P_n, the associated n-th Hermite's polynomial, is of degree $n-1$ with leading coefficient $\gamma_n := (2^{n-1}/((n-1)!\pi^{1/2}))^{1/2}$. It satisfies

$$\mathfrak{h}_0 \Psi_n = \Lambda_n \Psi_n. \tag{10}$$

We look for a quasi-mode for $\widetilde{\mathfrak{h}}(k)$ under the form $\Psi_n + \Gamma_n$, where $\Gamma_n = o(1)$ as $k \to +\infty$, associated with an approximated eigenvalue $\Lambda_n + \mu_n$. This will be satisfied if we can write formally

$$\widetilde{\mathfrak{h}}(k)(\Psi_n + \Gamma_n + \ldots) = (\Lambda_n + \mu_n + \ldots)(\Psi_n + \Gamma_n + \ldots),$$

Since $\widetilde{\mathfrak{h}}(k) = \mathfrak{h}_0 + d_k$, using (10), this involves

$$(\mathfrak{h}_0 - \Lambda_n)\Gamma_n = (\mu_n - d_k)\Psi_n. \tag{11}$$

We can now construct our quasi-modes, based on the above anzatz. We make the choice

$$\mu_n(k) = \langle d_k \Psi_n, \Psi_n \rangle = \int_{\mathbb{R}} \Psi_n(t)^2 d_k(t)\mathrm{d}t = \int_{-\infty}^{t_k} P_n^2(t) e^{-t^2} d_k(t)\mathrm{d}t, \tag{12}$$

so that, by the Fredholm alternative, we can find a unique solution Γ_n for equation (11). Now, in order to evaluate the quasi-mode, we compute

$$D_{n,k} := \widetilde{\mathfrak{h}}(k)(\Psi_n + \Gamma_n) - (\Lambda_n + \mu_n)(\Psi_n + \Gamma_n) = (d_k - \mu_n)\Gamma_n.$$

Then we easily prove that $\|D_{n,k}\| = o(\mu_n)$ and that $\|\Psi_n + \Gamma_n\| = 1 + o(1)$, as $k \to +\infty$. Denote by $\widetilde{E_n}(k)$ the n-th eigenvalue of $\widetilde{\mathfrak{h}}(k)$, for k large enough, we have a spectral gap near $\widetilde{E_n}(k)$, in the sense that

$$\exists k_n \in \mathbb{R}, \exists c > 0 \forall k \geq k_n, \quad \mathrm{dist}(\widetilde{E_n}(k), \sigma(\widetilde{\mathfrak{h}}(k)) \setminus \{\widetilde{E_n}(k)\}) > c.$$

Therefore, the spectral Theorem shows that, as $k \to +\infty$,

$$\widetilde{E_n}(k) = \Lambda_n + \mu_n(k) + o(\mu_n(k)), \tag{13}$$

Using the form of Ψ_n, we get

$$\mu_n(k) \underset{k\to+\infty}{\sim} \gamma_n^2 \int_{-\infty}^{t_k} t^{2n-2} e^{-t^2} d_k(t)\mathrm{d}t. \tag{14}$$

Relation (9) and the definition of t_k provides $d_k^{(j)}(t_k) = 0$ for all $j = 1, \ldots, p$, moreover $|d_k^{(p+2)}(t)| \leq C_p|t|$ with $C_p > 0$. Therefore, using $(p+1)$ integrations by parts, we get:

$$\int_{-\infty}^{t_k} t^{2n-2} e^{-t^2} d_k(t)\mathrm{d}t \underset{k\to+\infty}{\sim} -t_k^{2n-2} \frac{e^{-t_k^2}}{(2t_k)^{p+2}} d^{(p+1)}(t_k).$$

Now, from (8):

$$d_k^{(p+1)}(t_k) = -2(k - a_\infty) b_+^{-\frac{p+3}{2}} b^{(p)}(x_\infty^-). \tag{15}$$

Therefore, by the definition of t_k:

$$\mu_n(k) \underset{k\to+\infty}{\sim} \frac{2^{n-p-2}}{\sqrt{\pi}(n-1)! b_+^{n-\frac{1}{2}}} b^{(p)}(x_\infty^-) k^{2n-p-3} e^{-t_k^2}.$$

Using (13) and $E_n(k) = b_+ \widetilde{E_n}(k)$, we get (6).

When the contact is of infinite order, we have $d^{(j)}(t_k) = 0$ for all $j \in \mathbb{N}$. Then, (7) follows easily from (12). □

Remark 2.2. In case of an infinite contact point, the proof shows that the asymptotics of the band function is still given by $E_n(k) = b_+\Lambda_n + b_+\mu_n(k) + o(\mu_n(k))$, where μ_n is given in (12). For k large enough, $\mu_n(k) < 0$, but there is no natural expansion for this quantity without additional hypotheses.

3. CONSEQUENCES NEAR THE THRESHOLDS

3.1. Bulk states

The current operator is defined as the commutator $J_y := -i[H_0, y]$, on $\mathrm{Dom}(H_0)$. The evolution through the unitary group defined by H_0 of this self-adjoint operator is the velocity in the y direction, indeed, defining $y(t) := e^{-itH_0} y e^{itH_0}$, the evolution of the position along the y direction, there holds

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = e^{-iH_0} J_y e^{itH_0}.$$

In [8], the authors use this commutator to establish a Mourre estimate for general analytically fibered Hamiltonian for any energies except for a discrete set, called thresholds. Their technique relies on the stratification of the projection from the Bloch Variety into the spectrum, according to the algebraic multiplicity of values in the spectrum, the thresholds being the set of dimension 0 in the spectrum associated with this stratification, see [8, Definition 3.2]. In some sense, they provide a general estimate from below from the current: it is bounded from below at energies away from thresholds.

Iwatsuka models, among others magnetic models, have the property that their fibers operators are 1d, and the algebraic machinery described above can be avoided by controlling the commutator in a more direct way. The key tool to do this is the Feymann-Hellman formula. Given a function $\varphi \in \mathrm{Dom}(H_0)$ (see [15, Section 3] for a precise definition), there holds:

$$\langle J_y \pi_n \varphi, \pi_n \varphi \rangle = \int_{k\in\mathbb{R}} |\pi_n \varphi(k)|^2 \Lambda_n'(k) \mathrm{d}k,$$

where π_n is the projection along the n-th harmonic, see [13, Section 5] and [11].

Therefore, estimates on the derivative on the band function turn into control on the current operator, more precisely, if we assume in addition that φ is localized in energy in an interval I, then

$$\inf_{E_n^{-1}(I)} E_n' \leq \frac{\langle J_y \pi_n \varphi, \pi_n \varphi \rangle}{\|\pi_n \varphi\|^2} \leq \sup_{E_n^{-1}(I)} E_n'. \tag{16}$$

If thresholds are defined as set of energies for which the derivative of the band function can be small, then it becomes obvious that the current is bounded from below, away from thresholds. In case where the band functions are proper, such energies correspond to critical point of band functions. Note that being proper is also an hypothesis from [8]. But in our case, the band functions tend to finite limit, giving rise to a different kind of thresholds $\mathcal{T} := \{b_\pm \Lambda_n, n \geq 1\}$. For a particle whose energy interval contains a threshold, no bound from below is available for the current. A more quantitative approach is given in [10]: we consider an energy interval I at a small distance from the set of threshold $\mathcal{T}$. Then (16) shows that a precise asymptotics of E_n and its derivative provides a good control on the current of states localized in energy in I. Following this strategy, it is proved in [15] that in case (3), when $I = (\Lambda_n - \delta_2, \Lambda_n - \delta_1)$:

$$\delta_1^{1+\frac{1}{M}} \lesssim \frac{\langle J_y \pi_n \varphi, \pi_n \varphi \rangle}{\|\pi_n \varphi\|^2} \lesssim \delta_2^{1+\frac{1}{M}}, \quad \delta_i \to 0.$$

In the case where the magnetic field satisfies (4), then the strategy is similar. First, note that for a fixed $\delta > 0$ small enough, the equation $\Lambda_n - \delta = E_n(k)$ has a unique solution $k(\delta)$, which satisfies, $k(\delta) = \sqrt{|b_+ \log \delta|} + o(1)$ as $\delta \to 0$.

Next, one needs to show that the asymptotics of $E_n'(k)$ can be derived from (6). Since the magnetic field does not satisfy any analyticity hypothesis, no method based on special functions can be used (as it was the case in [7, 9]).

A direct method is to use the following *Feynman-Hellmann* integral formula for $E_n'(k)$:

$$E_n'(k) = -2\int_{\mathbb{R}} (a(x)-k)u_n(x,k)^2 \mathrm{d}x,$$

where $u_n(\cdot,k)$ is a normalized eigenfunction associated with $E_n(k)$. Then, it is sufficient to prove that the quasi-mode $(\Psi_n + \Gamma_n)$ constructed in the previous section is an accurate approximation of the eigenfunction, and to use this quasi-mode in the evaluation of the integral. This procedure, based on a refinement of the spectral Theorem, is done in details in [15, Section 2.3]. As a consequence, we get

$$E_n'(k(\delta)) \underset{\delta\to 0}{\sim} \delta\sqrt{|\log\delta|}$$

and therefore the estimates on the current follow:

$$\delta_1\sqrt{|\log\delta_1|} \lesssim \frac{\langle J_y\pi_n\varphi, \pi_n\varphi\rangle}{\|\pi_n\varphi\|^2} \lesssim \delta_2\sqrt{|\log\delta_2|},$$

showing the difference with case (3). Estimations on the localizations on states localized in this interval are possible, as done in [15, Section 3.2].

3.2. **Perturbations**

Here we describe other possible applications of the asymptotics of band functions, without entering into the details. Giving a measurable sign-definite potential $V:\mathbb{R}^2\to\mathbb{R}$ going to 0 at infinity, a physically relevant question concerns the effect of this perturbation on the system. Some of the classical trajectories may become bounded, corresponding to *trapped modes*. These ones correspond in the quantum system to eigenvalues of the operator H_0+V. If there are gaps in the essential spectrum of H_0, then under these assumptions on V, these gaps remain the same for the essential spectrum of H_0+V. But discrete eigenvalues can appear inside these gaps. Finiteness (or asymptotics) of these eigenvalues is an important topic that has received a lot of attention. For example, the general case of thresholds corresponding to critical points has been described in [16], but in our case, these techniques do not apply since the thresholds are limit of band functions. These cases are treated under various hypotheses on the potential in [14, 18, 19]. A delicate extension of these questions concerns the behavior of the spectral shift function for the pair (H_0+V, H_0). This function can be used to describe the counting function of eigenvalues outside the essential spectrum, and its singularity at thresholds is a natural question. This problematic has been treated in [15, Section 4] for the Iwatsuka model, under condition (3).

In case (4), the exponential convergence of the band functions toward their limits is closer to half-planes model with constant magnetic field. Therefore the precise behavior of the eigenvalues counting function, and of the SSF, can be obtained by adapting the methods from [1,2], using the asymptotics (6) for E_n (and its derivatives).

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Universidad de Santiago de Chile,
Departamento de Matemática y Ciencia de la Computación,
Las Sophoras 173, Santiago, Chile
pablo.miranda.r@usach.cl

Université de Bordeaux,
IMB, UMR 5251, 33405 TALENCE Cedex, France
Nicolas.Popoff@math.u-bordeaux1.fr