ESSENTIAL SPECTRUM, QUASI-ORBITS AND COMPACTIFICATIONS: APPLICATION TO THE HEISENBERG GROUP

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Let H be the Heisenberg group and $\overline{H} = H \cup \Sigma_H$ be its radial compactification. Our main result is a decomposition of the essential spectrum of the Schrödinger-type operator $T = -\Delta + V$, where V is a continuous real function on $\overline{H}$. Our result extends classical results of HVZ-type from Euclidean spaces to the Heisenberg group. While many features are preserved in the Heisenberg group case, there are also some notable differences. First, the action of H on itself extends to an action of H on $\overline{H}$ and we compute the quasi-orbits (the closure of the orbits) of the action, whose structure is more complicated in the Heisenberg case. Following [16], we show that the essential spectrum of any operator T contained in (or affiliated to) $\mathcal{C}(\overline{H}) \rtimes H$ is the union of the spectra of a family $(T_\alpha)_{\alpha \in \mathcal{F}}$ of simpler operators indexed by a family of quasi-orbits that cover $\overline{H} \smallsetminus H$, that is, $\sigma_{ess}(T) = \bigcup\limits_{\alpha \in \mathcal{F}} \sigma(T_\alpha)$. We obtain similar results also for an other H-equivariant compactification of H.

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INTRODUCTION

We study the essential spectrum and Fredholm conditions for Schrödinger-type operators with homogeneous potentials at the infinity acting on $L^2(H)$, where H is the Heisenberg group. The approach of this paper is based on the articles [9, 10, 13, 16]. The techniques presented here also work for more general locally compact groups.

We consider $\overline{H}$ a compactification of H such that the algebra of continuous function on the compactification $\mathcal{C}(\overline{H})$ is separable. The compactification induces a natural family of translations at the "infinity" R_α. We also show that the action of H on itself extends to the compactification and we compute the quasi-orbits. Recall that a quasi-orbit is the closure of an orbit. We show that the family R_α is, in fact, an *exhaustive family* of morphisms for a suitable operator algebra contained in the Calkin algebra. In particular, for the operator $T = -\Delta + V$ with potential V, a continuous function on the compactification $\overline{H}$, we obtain:

$$\sigma_{ess}(T) = \bigcup_{\alpha\in\mathcal{F}} \sigma(T_\alpha).$$

Here $\mathcal{F}$ is a covering by quasi-orbits of the part at the infinity of the compactification and $T_\alpha = -\Delta + V_\alpha$, with V_α a simpler potential associated to a quasi-orbit corresponding of α. The operator algebra that we consider contains the resolvent of the Schrödinger-type operator and is generated by some integral operators. For more details on the concept of exhaustive family of morphisms, see [17] and [21]. The idea of considering the algebra of the resolvent operators is not new and has been well developed by Georgescu and others in [1, 9]. The method of spectral decomposition used here involved crossed product of C^*-algebras and has been applied to magnetic fields [14] and to the N-body problem [10, 12]. However, every crossed product of a commutative C^*-algebra can be viewed as a C^*-algebra of a groupoid [20]. Examples of C^*-algebras associated to groupoids, and more generally algebras of pseudo-differential operators on groupoids, can be found in [4–6] and the references therein. The advantage of the crossed product C^*-algebras is that most of them have the *quasi-regularity* property. The quasi-regularity allows us to express the spectrum of the crossed product C^*-algebra in term of quasi-orbits of the action of the group on the spectrum of the initial C^*-algebra. More details on quasi-regularity will be given in Section 3 and can be found in [22]. For results on Fredholm conditions and decompositions of the essential spectrum, see [3], published also in this Special Issue.

Contents of the paper

We briefly describe the contents of the paper. Section 1 is dedicated to the study of the operator algebra that contains the resolvents of T. We also recall some earlier results from [9,12,16,17]. We finish the first section with elementary facts on the Heisenberg group. The second section focuses on the definition and a convenient description of the Laplacian on a Lie group. In Section 3, we discuss the quasi-regularity property. Thanks to Williams [22], we have convenient conditions that imply the quasi-regularity. In Section 4, we compute explicitly the quasi-orbits for two differents (but similar) compactification of H. Section 5 is devoted to the study of translations at the infinity and to the proof of the main theorem. The last section describes other kinds of algebras of potentials associated to repeated compactifications. We also discuss the relation between our results and a result of Power in [19] characterizing the spectrum of mixed algebras.

1. BACKGROUNDS

By $\mathcal{H}$, we will always denote a Hilbert space and by $\mathcal{B}(\mathcal{H})$ the bounded operator acting on $\mathcal{H}$.

1.1. C^*-algebras

Recall that a C^*-algebra A is a complex algebra with a norm $\|.\|$, an involution "$*$" such that $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$, $(ab)^* = b^* a^*$, $\|ab\| \leq \|a\|\|b\|$, and $\|aa^*\| = \|a\|^2$ for $\lambda, \mu \in \mathbb{C}$ and $a, b \in A$. In addiction, A is complete for the topology induces by the norm $\|.\|$. Every commutative C^*-algebra A is isomorphic to $\mathcal{C}_0(\Omega)$, the continuous function that vanishes at infinity on a locally compact space Ω. When $1 \in A$, the space Ω is compact. See [7] for more basic material on C^*-algebra.

1.2. Affiliated operators

Let P be a self-adjoint, not necessarily bounded, operator acting on $\mathcal{H}$. We consider a self-adjoint, norm closed subalgebra $A \subset \mathcal{B}(\mathcal{H})$. Then A is a C^*-algebra. We say that P is affiliated to A, writing $P \in' A$, if for all $\varphi \in \mathcal{C}_0(\mathbb{R})$ the operator $\varphi(P) \in A$. It is known (see [17] for example), that in order to check that $P \in' A$, it is enough to show that $(P - z)^{-1} \in A$ for some z in the resolvent set of P. Of course every self-adjoint operator is affiliated to $\mathcal{B}(\mathcal{H})$, however, we are interested in smaller C^*-algebras. If $P \in' A$ and $\phi : A \to B$ is a morphism of C^*-algebra, then $\phi(P)$ is defined. More details on the notion of affiliated operator can be found in [1]. In [8], we can find a generalization of affiliated operators for densely defined and non self-adjoints operators. A related (but different) notion of "affiliated operators" has been studied in [2] and [23]. See [17] for a comparison of these notions.

1.3. Spectrum of a C^*-algebra

A *character* of a commutative C^*-algebra A is a nonzero morphism $\chi : A \to \mathbb{C}$. The set of all characters endowed with the weak topology is the *spectrum* of A, denoted $\widehat{A}$. We recall that $\widehat{A}$ is compact, if and only if, A has a unit. In this case, there is an isomorphism $A \simeq \mathcal{C}(\widehat{A})$.

For a non commutative C^*-algebra, the *spectrum* $\widehat{A}$ is formed of classes of equivalence of irreducible representations of A. More precisely, a representation of A is a $*$-morphism $\pi : A \to \mathcal{B}(\mathcal{H})$. A representation π is said to be *irreducible* if the only invariant closed subspaces of $\mathcal{H}$ under the action of $\pi(A)$ are $\{0\}$ and $\mathcal{H}$. In general, the space $\widehat{A}$ is too big, so we prefer studying the *primitive spectrum* of A denoted by $Prim(A)$, with $Prim(A) := \{\ker(\pi), \pi \in \widehat{A}\}$.

1.4. C^*-dynamical systems

We recall now some basic facts concerning C^*-dynamical systems and their associated crossed products, see [18], [22] for more details.

A *C^*-dynamical system* consists of a C^*-algebra A and a locally compact group G with a strongly continuous action $\theta : G \to Aut(A)$. Let $\mathcal{LUC}(G)$ be the C^*-algebra of left uniformly continuous functions on G. We focus our study on the case when A is an unital C^*-subalgebra of $\mathcal{LUC}(G)$ and $\theta_y(f)(x) = f(y^{-1}x)$ for $x, y \in G$. Note that, with this choice of A and θ, the C^*-algebra A is commutative and we have to assume that A is invariant by translation. We consider $L^1(G, A)$, the Bochner space of integrable functions. The norm on $L^1(G, A)$ is defined using the norm of A. We endow $L^1(G, A)$ with a structure of $*$-algebra with the product and the involution defined by:

$$\phi * \psi(x) = \int_G \phi(y)\theta_y[\psi(y^{-1}x)]\mathrm{d}\mu(y), \quad \phi^*(x) = m(x)^{-1}\theta_x[\phi(x^{-1})^*],$$

where $\phi, \psi \in L^2(G, A), x \in G$ and μ is the Haar measure on G, and m is the modular function of G. The crossed product, $A \rtimes G$ is defined as the completion of $L^1(G, A)$ for the norm $||\phi|| := sup_\Pi ||\Pi(\phi)||$, where the supremum is taken over all non-denegerate $*$-representation $\Pi : L^1(G, A) \to \mathcal{B}(\mathcal{H})$.

To characterize the representations of $A \rtimes G$, we need the notion of covariant pair.

Definition 1.1. For a C^*-dynamical system (A, θ, G), a triplet $(\pi, U, \mathcal{H})$ is a *covariant pair* of (A, θ, G) if:

- $\mathcal{H}$ is a Hilbert space,
- $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*-$representation of A,
- $U : G \to \mathcal{B}(\mathcal{H})$ is a strongly continuous unitary morphisms,
- for all $a \in A, g \in G$, we have $U(g)\pi(a)U(g^{-1}) = \pi(\theta_g(a))$.

If there is no ambiguity, we will drop the Hilbert space $\mathcal{H}$ and simply write (π, U) for a covariant pair.

Definition 1.2. Let (A, θ, G) be a C^*-dynamical system and (π, U) be a covariant pair. To (A, θ, G), we associate a representation $\pi \rtimes U : A \rtimes G \to \mathcal{B}(\mathcal{H})$

called the *integrated form* of (π, U). This integrated form is defined by:

$$\pi \rtimes U(f) = \int_G \pi(f(x))U_x \mathrm{d}\mu(x),$$

where, $f \in L^2(G, A)$.

The following proposition corresponds to Proposition 2.40 in [22].

PROPOSITION 1.3 (Williams). *Let (A, θ, G) be a dynamical system with A not necessarily commutative. Every non degenerate representation Π of $A \rtimes G$ comes from a unique integrated form of a covariant pair of the dynamical system (A, θ, G). That is $\Pi = \pi \rtimes U$, where the pair (π, U) is a covariant representation of the (A, θ, G) and the pair (π, U) is unique (up to equivalence).*

Remark 1.4. There exists another definition of the crossed product, called the *reduced* crossed product, which is the completion by another norm of $L^1(G, A)$ induces by a particular covariant pair. However, for amenable groups like H the Heisenberg group, the two definitions of the crossed product coincide.

If we assume as before that A is a C^*-subalgebra of $\mathcal{LUC}(G)$, any function $\phi : G \to A$ is identified with a function $G^2 \to \mathbb{C}$. Let $x, y \in G$, we will use the notation $\phi(x; y) = [\phi(x)](y)$ to keep in mind the dependence on the two variables. The following proposition gives a convenient representation of the crossed product $A \rtimes G$.

PROPOSITION 1.5. *Let $\phi \in L^1(G, A)$, we define $Sch(\phi)$, an operator on $L^2(G)$, by*

$$|Sch(\phi)f](x) = \int_G \phi(x; y)f(y^{-1}x)\mathrm{d}\mu(y).$$

where $f \in L^2(G)$ and $x \in G$. The application Sch can be extended from $L^1(G, A)$ to $A \rtimes G$. The extension of Sch is a faithful representation of $A \rtimes G$ on $\mathcal{B}(L^2(G))$.

The preceding proposition corresponds to the Proposition 7.9 in [15].

Remark 1.6. The category of commutative C^*-algebras and the category of locally compact spaces are equivalent, via the isomorphism $A \simeq \mathcal{C}_0(\widehat{A})$. A C^*-dynamical system (A, θ, G), where A is a commutative C^*-algebra, can be defined by a triplet (Ω, θ, G), where θ is a continuous action of the group G on the locally compact space $\Omega = \widehat{A}$. When A is a commutative and unitary C^*-algebra, the corresponding space Ω is compact and we will call the triplet (Ω, θ, G) a compact dynamical system. We will often navigate between the C^*-algebra formalism and the locally compact space formalism for C^*-dynamical system.

For a compact dynamical system (Ω, θ, G), we will denote by O^ω, the orbit of ω. The closure of O^ω is called the *quasi-orbit* of ω and we will denote it by Q^ω.

Remark 1.7. There exists a natural identification between $L^1(G, A)$ and the projective tensor product $A\overline{\otimes}L^1(G)$. The assumption $1 \in A$ allows us to consider $Sch(1 \otimes \psi)$ for $\psi \in L^1(G)$. In this case, $Sch(1 \otimes \psi)$ is an operator of convolution acting on $L^2(G)$.

The discussion in Section 4 of [9] give details on the crossed product in the particular case $G = \mathbb{R}^n$ as additive group.

1.5. The case $G = \mathbb{R}^n$

Assume $G = \mathbb{R}^n$, regarded as additive group. We denote by $\mathcal{B}(L^2(G))$ the algebra of bounded linear operator on $L^2(G)$ and by $\mathcal{K}(G)$ the subalgebra of compact operators of $\mathcal{B}(L^2(G))$.

Let $\mathcal{C}_b(G)$ be the algebra of bounded continuous functions on G with complex values, $\mathcal{C}_c(G)$ the subalgebra of functions with compact support and $\mathcal{C}_0(G)$ the subalgebra of functions that go to zero at infinity.

Any measurable function $\phi : G \to \mathbb{C}$ induces an operator of multiplication by ϕ acting on $L^2(G)$. To avoid the ambiguity, the operator of multiplication will be denoted by $\phi(q)$. We also introduce the operator $\phi(p)$ defined using the Fourier transform: $\phi(p) := \mathcal{F}^{-1}\phi(q)\mathcal{F}$. When $\phi \in \mathcal{C}_0(G)$, the operator $\phi(p)$ is an element of $\mathcal{B}(L^2(G))$. There is a natural action by translations of G on $\mathcal{C}_b(G)$, we denote by τ_x this translation for each $x \in G$. With this notation, we can describe the crossed product $\mathcal{A} = A \rtimes G$, where A is a C^*-algebra of functions on G such that $1 \in A$ and $\tau_x(A) \subset A$ for every $x \in X$. Moreover, we suppose that $\mathcal{C}_0(G) \subset A \subset \mathcal{C}_b(G)$. With this assumption, $A \rtimes G$ can be represented into $\mathcal{B}(L^2(G))$ as the norm closure and stable by involution of the linear subspace generated by $\phi(q)\psi(p)$ for $\phi \in A, \psi \in \mathcal{C}_0(G)$. See Remark 1.7 to relate it with the representation $\phi(q)\psi(p)$ and Proposition 1.5.

We recall that $\widehat{A}$ is compact if and only if A has a unit and there is an isomorphism between A and $\mathcal{C}(\widehat{A})$. Each element $x \in G$ can be identified with a character χ_x of A defined by $\chi_x(\phi) = \phi(x)$. We can show that $\widehat{A}$ is actually a compactification of G and let $\mathcal{A}^\dagger$ be the part at infinity of this compactification space, that is

$$A^\dagger = \widehat{A} \setminus G = \{\varkappa \in \widehat{A} | \varkappa(\phi) = 0, \forall \phi \in \mathcal{C}_0(G)\}.$$

Let $\varkappa \in A^\dagger$ and let $\phi \in A$, then there exists a net $(x_i)_{i\in I}$ of elements of G such that $\lim_{i\in I} x_i = \varkappa$. Using this net, we can define $\tau_\varkappa$, the translation at the infinity by $\varkappa$, that is:

$$\tau_\varkappa[\phi](x) = \lim_{i\in I} \tau_{x_i}[\phi](x) = \lim_{i\in I} \phi(x+x_i) = \phi(x+\varkappa).$$

We can extend the translations τ_x and $\tau_\varkappa$ from A to $A\rtimes G$ such these extensions leave invariant the operator $\psi(p)$ and acts on $\phi(q)$ as on ϕ viewed as a element of A. We keep the same notation for the extension. For $P\in A\rtimes G$, we have $\tau_x(P) = e^{ixp}Pe^{-ixp}$, where e^{ixp} is the translation operator by x and

$$\tau_\varkappa(P) = \lim_{j\in I} e^{ix_jp}Pe^{-ix_jp} := P_\varkappa.$$

The convergence holds for the topology induces by the family of semi-norms $||P||_\theta = ||P\theta(q)||$, with $\theta\in\mathcal{C}_0(G)$ on $\mathcal{B}(L^2(G))$. We have the following theorem in [8].

Theorem 1.8 (Georgescu). *For any operator $P\in A\rtimes G$, we have*

$$\sigma_{ess}(P) = \bigcup_{\varkappa\in A^\dagger} \sigma(P_\varkappa).$$

1.6. **Generalities on the Heisenberg group**

Definition 1.9. We recall that the Heisenberg group H is the sub-group of $GL(3,\mathbb{R})$ of upper triangular matrices and that we have a natural bijection between H and $\mathbb{R}^3$:

$$H \ni \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \leftrightarrow (a,b,c)\in\mathbb{R}^3 \tag{1}$$

Let $\mathfrak{h}$ be the Lie algebra of H, it's well known that $\mathfrak{h}$ is the space of strictly upper triangular matrix. Like for H and $\mathbb{R}^3$, there is a natural bijection between $\mathfrak{h}$ and $\mathbb{R}^3$:

$$\mathfrak{h} \ni \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \leftrightarrow (a,b,c)_0\in\mathbb{R}^3 \tag{2}$$

To distinguish the identification between H and $\mathbb{R}^3$ and between $\mathfrak{h}$ and $\mathbb{R}^3$, we will denote by $(a,b,c)_0$ with the label "0" when we consider an element of the Lie algebra. The Heisenberg group is nilpotent, hence the exponential map is a diffeomorphism from the Lie algebra $\mathfrak{h}$ to H. Recall that he exponential map is given by $\exp((a,b,c)_0) = (a,b,c+\frac{ab}{2})$ with inverse $\exp^{-1}((a,b,c)) = (a,b,c-\frac{ab}{2})_0$. For each $X\in H$, let L_X (resp R_X): $H\to H$, the left (resp. right) multiplication by X. In the next sections, we will consider potentials equivariant for various compactification.

2. LAPLACIAN OF A LIE GROUP

This section is dedicated to a convenient presentation of the Laplacian for a Lie group and to make connection with the notion of affiliated operators to a C^*-algebra. Let G be a Lie group. We denote by $Lie(G)$ the Lie algebra of G. It consists of first-order differential operators without constant term acting on $\mathcal{C}_c^\infty(G)$ and commuting with the right translation. We take $X_1, \ldots, X_n$ an orthonormal basis of $Lie(G)$ with respect to a right invariant metric and we consider

$$L = \sum_{i=1}^{n} X_i^2.$$

The operator L is an unbounded operator acting on $L^2(G)$ with core $\mathcal{C}_c^\infty(G)$, which is dense in $L^2(G)$. The Laplacian is defined as

$$\Delta = L^*.$$

The following proposition is from [11].

PROPOSITION 2.1. *For $\lambda > 0$, the resolvent $\rho_\lambda = (\lambda - \Delta)^{-1}$ is a bounded operator acting on $L^2(G)$. Moreover there exists ν_λ, an absolutely continuous measure with respect to the Haar measure of G, such that, for every $f \in L^2(G)$, we have:*

$$\rho_\lambda f = \nu_\lambda * f.$$

The Radon-Nikodym derivative of ν_λ, denoted by k_λ, is an element of $L^1(G)$ and is given by

$$k_\lambda = \int_0^\infty e^{-\lambda t} p_t dt,$$

where, for every $t > 0$, p_t is a non-negative function that fulfills the following conditions:

$$p_t \in L^1(G) \cap L^2(G), \quad \|p_t\|_1 = 1.$$

COROLLARY 2.2. *We have the following consequence of Remark 1.7 and 2.1, the crossed product contains the operator $Sch(1 \otimes \psi) = \psi(p)$ for $\psi \in L^1(G)$. In particular, ρ_λ is an operator of convolution by a function in $L^1(G)$. That means that Δ is affiliated to $\mathcal{C}(G^+) \rtimes G$, where G^+ is the one-point compactification of G.*

3. EXHAUSTIVE FAMILIES AND QUASI-REGULAR DYNAMICAL SYSTEM

3.1. Exhaustive family

We shall need the notion of an exhaustive family of morphisms, which is due to Nistor and Prudhon in [17].

Definition 3.1. Let A be a C^*-algebra and $\mathcal{F}$ be a family of *-morphisms $\phi : A \to B_\phi$. We say the family $\mathcal{F}$ is *exhaustive* if

$$\forall J \in Prim(A), \exists \phi \in \mathcal{F}, \ker(\phi) \subset J.$$

The main reason why we are interested in exhaustive families of morphisms is because of the following result from [17].

PROPOSITION 3.2. *Let A be C^*-algebra and $\mathcal{F}$ be an exhaustive family of morphisms. Then for every operator P affiliated to A, we have:*

$$\sigma(P) = \bigcup_{\phi \in \mathcal{F}} \sigma(\phi(P)).$$

The property described in Proposition 3.2 is the definition of a *spectral family* of morphisms. However, we introduced the notion of exhaustive family of morphisms because it is an easier condition to show than the condition that a family of morphisms is a spectral family.

3.2. **Quasi-regular dynamical systems**

Let (A, θ, G) be a C^*-dynamical system, where $A = \mathcal{C}(\Omega)$ is a commutative and a unital C^*-algebra. The quasi-regularity is a property of the C^*-dynamical system that links the quasi-orbits of the action G on Ω and the primitive spectrum of $A \rtimes G$. Topological conditions on the system leads to quasi-regularity and then to a convenient algebraic decomposition of the primitive spectrum of the crossed product.

Recall that for a compact dynamical system (Ω, θ, G), we denote by O^ω, the orbit of $\omega \in \Omega$. The quasi-orbit of ω, which is the closure of O^ω, will be denoted by Q^ω.

Definition 3.3. Let (Ω, θ, G) be a compact dynamical system, this dynamical system is said to be *quasi-regular* if each irreducible representation of $C(\Omega) \rtimes G$ lives on a quasi-orbit. More precisely, let Π be an irreducible representation of $C(\Omega) \rtimes G$ and (π, U) the covariant pair that realizes Π (see Proposition 1.3). The representation Π lives on a quasi-orbit if there exists $\omega \in \Omega$ such that $Res(\ker(\Pi)) := \ker(\pi) = C^{\Omega^\omega}(\Omega) = \{f \in \mathcal{C}(\Omega), f_{|Q^\omega} = 0\}$.

The following proposition gives a topological condition that implies the quasi-regularity.

PROPOSITION 3.4 (Gootman-Rosenberg-Sauvageot). *Let (A, θ, G) be a C^*-dynamical system. We suppose that A is separable and G is second countable. In this case, the C^*-dynamical system is quasi-regular.*

Proposition 3.4 is a consequence of Theorem 8.21 in [22]. It states that with the same assumptions, the C^*-dynamical system is *EH-regular*, which is a stronger condition than quasi-regularity. However, we only need quasi-regularity is this paper.

Definition 3.5. A set $\{\omega_i, | i \in I\}$ of points of Ω is a called a *sufficient family* if the associated quasi-orbit $\{Q^{\omega_i} | i \in I\}$ form a covering of the space Ω, in other words, $\bigcup_{i \in I} Q^{\omega_i} = \Omega$.

An important propriety is given by the following proposition, which is Proposition 6.3 in [16].

PROPOSITION 3.6. *Let (Ω, θ, G) be a compact quasi-regular dynamical system, and $\{\omega_i, i \in I\}$ a sufficient family of points of Ω. We consider for every $i \in I$, the restriction morphism:*

$$p_i : \mathcal{C}(\Omega) \to C(Q^{\omega_i}), \quad p_i(f) = f|_{Q^{\omega_i}}$$

and then extend it to the crossed product

$$P_i := p_i^{\rtimes} : \mathcal{C}(\Omega) \rtimes G \to \mathcal{C}(Q^{\omega_i}) \rtimes G.$$

The family of morphims $\mathcal{P} := \{P_i, i \in I\}$ is then an exhaustive family of morphisms of the C^*-algebra $\mathcal{C}(\Omega) \rtimes G$.

4. COMPACTIFICATION OF *H*

In this subsection, we will consider two compactifications of the Heisenberg group and the induced action of H on each of these compactifications. We will denote by $\|.\|$ the usual euclidean norm.

4.1. The spherical compactification

We consider the spherical compactification of H induced by the canonical identification (1). We will denote by $\overline{H}$ this compactification, hence, we have $\overline{H} = H \cup \Sigma_H$ with $\Sigma_H \simeq (\mathbb{R}^3)^*/(\mathbb{R}_+^*) \simeq \mathbb{S}^2$. We shall need an explicit identification between $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$ and $\mathbb{S}^2$. Let $\alpha \in (\mathbb{R}^3)^*/(\mathbb{R}_+^*)$, then by definition, there exists $V \in H \setminus \{0\}$ such that the equivalence class α is the half-line $\mathbb{R}_+^* V$. We choose the point $v = (a, b, c) \in \mathbb{R}_+^* V$ such that $\|v\| = 1$, then the point $v \in \mathbb{S}^2$ characterizes α. Conversely to each point of the sphere, $z \in \mathbb{S}^2$, we can associate the half-line $\mathbb{R}_+^* z$ to obtain an element of $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$. A sequence $U_n = (a_n, b_n, c_n) \in H$ converges to the point $(a, b, c) \in \Sigma_H$ if:

$$\lim_{n \to +\infty} \|U_n\| = +\infty, \quad \lim_{n \to +\infty} \frac{U_n}{\|U_n\|} = (a, b, c). \tag{3}$$

4.1.1. TRANSLATION AT INFINITY

Let l_X be the left translation by the vector $X \in H$. The left translation l_X can be extended from $H \to H$ to $H \to \overline{H}$ that is, we want to define $l_X(\alpha)$ for $\alpha \in \Sigma_H$. It is convenient to have an explicit formula of $l_X(\alpha)$, with $X = (x, y, z) \in H$ and $\alpha = (a, b, c) \in \Sigma_H$. Let U_n be as in equation (3), we have

$$l_X(U_n) = (x + a_n, y + b_n, z + c_n + xb_n)$$

Then

$$\begin{aligned}\|l_X(U_n)\|^2 &= x^2 + y^2 + z^2 + a_n^2 + b_n^2 + c_n^2 + x^2 b_n^2 \\ &\quad +2(xa_n + yb_n + zc_n + xb_n c_n + xzb_n) \\ &= \|U_n\|^2 \left(\frac{a_n^2 + b_n^2 + (c_n + xb_n)^2}{\|U_n\|^2} + \epsilon_n \right)\end{aligned}$$

with $\epsilon_n = \dfrac{\|X\|^2 + 2\langle X, U_n\rangle + 2xzb_n}{\|U_n\|^2} \to 0$. Moreover, $a^2 + b^2 + c^2 = 1$, hence for every $x \in \mathbb{R}$, the quantity $C := a^2 + b^2 + (c + xb)^2$ is always positive. Indeed, if $b \neq 0$ then $C \neq 0$, and if $b = 0$ then $C = a^2 + c^2 = 1$. This leads to:

$$\|l_X(U_n)\|^2 \sim ||U_n||^2 \left(a^2 + b^2 + (c + xb)^2\right) = +\infty.$$

We obtain:

$$\frac{l_X(U_n)}{\|l_X(U_n)\|} \to \frac{1}{\sqrt{a^2 + b^2 + (c + xb)^2}}(a, b, c + xb).$$

The two points $(a, b, c + xb)$ and $\dfrac{1}{\sqrt{a^2 + b^2 + (c + xb)^2}}(a, b, c + xb)$ have the same equivalence class in $(\mathbb{R}^3)^*/(\mathbb{R}_+^*)$, hence we can drop the constant and only consider $(a, b, c + bx)$ for the limit of $l_X(U_n)$.

4.1.2. CHARACTERIZATION OF THE QUASI-ORBITS

We can sum up the preceding subsection with this equality:

$$l_X((a, b, c)) = \lambda(a, b, c + bx) \tag{4}$$

where $X = (x, y, z) \in H, \alpha = (a, b, c) \in \Sigma_H$ and $\lambda \in \mathbb{R}_+^*$. Using this relation, we can characterize the set of fixed points: $O_{fix} := \{(a, 0, c), a^2 + c^2 = 1\}$. For $\alpha \in \Sigma_H \setminus O_{fix}$, that is $\alpha = (a, b, c)$ with $b \neq 0$. The orbit of α is:

$$O^\alpha = \{\mathbb{R}_+^*(a, b, x), x \in \mathbb{R}\}.$$

The set $Z_\alpha = \{(a, b, x), x \in \mathbb{R}\}$ is a line in $\mathbb{R}^3$ not containing the origin. The orbit O^α is the set of the half-lines starting at 0 that intersect Z_α. The pre-image in the projective space of O^α is a half-space. Then the intersection of this half-space and $\mathbb{S}^2$ is the open great half-circle of $\mathbb{S}^2$ with end points $(0,0,1),(0,0,-1)$ and passing through the point $\frac{1}{\sqrt{a^2+b^2}}(a,b,0)$. The quasi-orbit is then the closed great half-circle. The difference between the orbit and the quasi-orbit is:

$$Q^\alpha = O^\alpha \sqcup \{(0,0,1),(0,0,-1)\}.$$

Let $\mathcal{F}$ be a family of points of Σ_H such that $\bigcup_{\alpha\in\mathcal{F}} Q^\alpha = \Sigma_H$. We also assume that $\mathcal{F}$ is minimal in the sense that each points of $\mathcal{F}$ are necessary to make the cover of Σ_H. With these choices, $\mathcal{F}$ is a sufficient family of the dynamical system (Σ_H, l, H), that is $\Sigma_H = \bigcup_{\alpha\in\mathcal{F}} Q^\alpha$. An example of a minimal sufficient family is

$$\mathcal{F} = \{(a,b,0), a^2+b^2=1\} \cup \{(a,0,c), a^2+c^2=1, a\neq 0\}.$$

4.2. **Spherical compactification via the exponential map**

We consider another compactification of H. Let $\overline{\mathfrak{h}}$ be the spherical compactification of $\mathfrak{h}$ via the identification (2). We define $\widetilde{H}$ as the image of the exponential map of $\overline{\mathfrak{h}}$, formally $\widetilde{H} = \exp(\overline{\mathfrak{h}})$. Each point $s \in \widetilde{H}\setminus H$ is the limit of a sequence $\exp(U_n)$, where $U_n = (a_n, b_n, c_n)_0 \in \mathfrak{h}$ and

$$\lim_{n\to+\infty} ||U_n|| = +\infty, \qquad \lim_{n\to+\infty} \frac{U_n}{||U_n||} = (a,b,c)_0$$

with $||(a,b,c)_0|| = 1$.

4.2.1. TRANSLATION AT INFINITY AND THE EXPONENTIAL MAP

As before, we want to extend the operator l_X to $\widetilde{H}$. That is, we want to find the point $(a',b',c') \in \widetilde{H}\setminus H$ such that

$$\lim_{n\to+\infty} \frac{V_n}{||V_n||} = (a',b',c')_0,$$

where $V_n \in \mathfrak{h}$ and checks $\exp(V_n) = l_X(\exp(U_n))$. We have:

$$X.\exp(U_n) = (a_n + x, b_n + y, c_n + z + xb_n + \frac{a_n b_n}{2}).$$

If we take $V_n = (a_n + x, b_n + y, c_n + z + \frac{1}{2}(xb_n - ya_n - xy))_0$, we obtain:

$$\|V_n\|^2 = \|U_n\|^2 \left(\frac{a_n^2 + b_n^2 + (c_n + \frac{xb_n}{2} - \frac{ya_n}{2})^2}{\|U_n\|^2} + \epsilon_n \right),$$

with

$$\epsilon_n = \frac{1}{||U_n||^2}\left(\|X\|^2 + 2\langle X, U_n\rangle + \frac{x^2y^2}{4} - xyc_n + xzb_n - xyz - \frac{xy}{2}(xb_n - ya_n)\right).$$

Note that $\lim_{n\to+\infty} \epsilon_n = 0$. Moreover, $a^2+b^2+c^2 = 1$, hence for every $X \in H$, the quantity $C' := a^2 + b^2 + (c + \frac{xb}{2} - \frac{ya}{2})^2$ is always positive. To see this, suppose that $a = b = 0$ then $C' = c^2 = 1$. Otherwise, $0 < a^2 + b^2 \leq C'$. This leads to:

$$||V_n||^2 \sim ||U_n||^2 \left(a^2 + b^2 + (c + \frac{xb}{2} - \frac{ay}{2})^2\right) \to +\infty.$$

4.2.2. CHARACTERIZATION OF THE ORBIT OF EXPONENTIAL-RADIAL COMPACTIFICATION

We can sum up the preceding subsection with this equality:

$$l_X(\exp((a, b, c)_0)) = \exp(\lambda(a, b, c + \frac{1}{2}(xb - ay))_0) \quad (5)$$

where $X = (x, y, z) \in H, \alpha = (a, b, c)_0 \in \widetilde{H}\backslash H$ and $\lambda \in \mathbb{R}_+^*$. Using this relation, we note that $(0, 0, 1)$ and $(0, 0, -1)$ are the only fixed points. If $\alpha = (a, b, c)$ is not a fixed point then

$$O^{\exp(\alpha)} = \{(\mathbb{R}_+^*(a, b, x), x \in \mathbb{R}\}.$$

An example of a minimal sufficient family for $\exp(\overline{\mathfrak{h}})$ is

$$\mathcal{F} = \{(a, b, 0), a^2 + b^2 = 1\}.$$

4.2.3. COMPARISON OF THE (QUASI)-ORBITS

We stress the differences between the two structure of orbits and quasi-orbits of the two compactifications $\overline{H}$ and $\widetilde{H} = \exp(\overline{\mathfrak{h}})$. In the two cases, we can identify the part at infinity with $\mathbb{S}^2$. For $\alpha \in \mathbb{S}^2$, we recall that the notation O^α is the orbit for the action of H on $\overline{H}$ and $O^{\exp(\alpha)}$ for the action of H on $\exp(\overline{\mathfrak{h}})$. By the equations (4) and (5), we obtain for $\alpha = (a, b, c) \in \mathbb{S}^2$:

- The two points $(0, 0, 1)$ and $(0, 0, -1)$ are fixed points in both cases.
- If $b \neq 0$ then $O^\alpha = O^{\exp(\alpha)}$.
- If $b = 0$ and $a \neq 0$, then $O^\alpha = \{\alpha\}$ and $O^{\exp(\alpha)} = \{(a, 0, x), x \in \mathbb{R}|a^2 + x^2 = 1\}$.

In other words, the main difference is that the great circle passing through the points $(0, 0, -1), (0, 0 - 1)$ and $(1, 0, 0)$ is the set of fixed points for $\overline{H}$. For $\exp(\overline{\mathfrak{h}})$, this great circle splits into two open half great circles: $O^{\exp(-1,0,0)}$ and $O^{\exp(1,0,0)}$ and two fixed point: $(0, 0, -1)$ and $(0, 0, 1)$.

5. AN EXHAUSTIVE FAMILY FOR $\mathcal{C}(\Sigma_H) \rtimes H$

For each function $f : H \to \mathbb{C}$, as in Subsection 1.5, we will denote by $f(q)$ the operator of multiplication acting on $L^2(X)$. This kind of operator is well defined for instance when $f \in \mathcal{C}_b(H)$. Let $\alpha \in \Sigma_H$ and U_n a sequence of elements of H that converges to α. We want to characterize the function f_α such that $f_\alpha(q)$ is "invariant by right translations at the infinity." Thus, we consider f_α, which is the limit of the operator $f_n(q) := R_{U_n} f(q) R^*_{U_n}$, with $f \in \mathcal{C}(\widetilde{H})$. For any element $X \in H$ and $\phi \in L^2(H)$, we write

$$(f_n(q)\phi)(X) = R_{U_n} f(q) R^*_{U_n}(\phi)(X) = f(R_{U_n}(X))\phi(X) = f(XU_n)\phi(X).$$

When n goes to infinity $XU_n \to X\alpha$ and the value of $L_X(\alpha) = X\alpha$ hence it is an element of the quasi-orbits of Q^α. Hence, the function $f_\alpha := \lim f_n$ associated to the limit operator of $f_n(q)$ can be view as an element of $\mathcal{C}(Q^\alpha)$. At the level of functions, the link between f and f_α is $R_\alpha(f) = f_\alpha$, where R_α is the pointwise limit of the right translation R_{U_n}. Moreover, using the translation R_α, we can build an exhaustive family.

THEOREM 5.1. *Let $\widetilde{H} = H \cup \Sigma_H$ be one of the two compactifications of H considered in Section 4. We consider a sufficient family $\mathcal{F}$ of points of Σ_H with the extended action of H. For each $\alpha \in \mathcal{F}$, we extend the translation R_α to the crossed product: $\mathcal{C}(\widetilde{H}) \rtimes H \to \mathcal{C}(Q^\alpha) \rtimes H$. With this assumption, the family $\{R_\alpha\}_{\alpha \in \mathcal{F}}$ is an exhaustive family of morphisms of $\mathcal{C}(\Sigma_H) \rtimes H$.*

The two main tools of the proof are the following lemma and the well behavior of the product constructions.

LEMMA 5.2. *Let R_α be the right translation at the infinity defined as before. We have:*

$$\ker(R_\alpha) \subset \mathcal{C}^{\Omega^\alpha}(\widetilde{H}).$$

Where $C^{Q^\alpha}(\widetilde{H}) = \{f \in \mathcal{C}(\widetilde{H}), f_{|Q^\alpha} = 0\}$.

Proof. Let $f \in \mathcal{C}(\widetilde{H})$ such that $f(\beta) \neq 0$, for $\beta \in Q^\alpha$. We want to show that $R_\alpha(f) \neq 0$. By definition of β, there exists a sequence U_n of elements of H such that $l_{U_n}(\alpha) = U_n.\alpha \to \beta$. The continuity of f implies $R_\alpha(f)(U_n) = f(U_n.\alpha) \to f(\beta) \neq 0$. We conclude that $R_\alpha(f) \neq 0$ and finish the proof. □

Proof of Proposition 5.1. For each $\alpha \in \Sigma_H$, we have $R_\alpha(\mathcal{C}_0(H)) = 0$ hence the function R_α can be defined on $\mathcal{C}(\widetilde{H})/\mathcal{C}_0(H) \simeq \mathcal{C}(\Sigma_H)$. We extend R_α to the crossed product $\mathcal{C}(\Sigma_H) \rtimes H$. Now, we show that the family $(R_\alpha)_{\alpha \in \mathcal{F}}$ is exhaustive. Let J be a primitive ideal of $\mathcal{C}(\Sigma_H) \rtimes H$ and Π be an irreducible representation of $\mathcal{C}(\Sigma_H) \rtimes H$ with kernel J. In view of Proposition 3.4, the separability of the $\mathcal{C}(\Sigma_H)$ implies the quasi-regularity of $(\mathcal{C}(\Sigma_H), l, H)$. Quasi-regularity implies the existence of a covariant pair (π, U) and $\beta \in \Sigma_H$ such that

$\Pi = \pi \rtimes U$ and $\ker \pi = \mathcal{C}^{Q^\beta}(\widetilde{H})$. Moreover, the family $\mathcal{F}$ is a sufficient family hence, there exists $\alpha \in \mathcal{F}$ such that $\beta \in Q^\alpha$ then $Q^\beta \subset Q^\alpha$ or equivalently $\mathcal{C}^{Q^\alpha}(\Sigma_H) \subset \mathcal{C}^{Q^\beta}(\Sigma_H)$. In view of Lemma 5.2, we have

$$\ker(R_\alpha) \subset \mathcal{C}^{Q^\alpha}(\Sigma_H) \subset \mathcal{C}^{Q^\beta}(\Sigma_H).$$

Then the equality holds when we pass to the crossed product and $\ker(R_\alpha) \subset J$. □

PROPOSITION 5.3. *Let T be an operator affiliated to $\mathcal{C}(\widetilde{H}) \rtimes H$. We have the following spectral decomposition:*

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(T_\alpha), \tag{6}$$

where T_α is the image of T through R_α.

COROLLARY 5.4. *In particular, when $T \in \mathcal{C}(\widetilde{H}) \rtimes H$ and the operator T is associated to $\phi \in L^1(H, \mathcal{C}(\widetilde{H}))$ via the formula $T = Sch(\phi)$, we obtain*

$$\sigma_{ess}(T) = \bigcup_{\alpha \in \mathcal{F}} \sigma(R_\alpha(Sch(\phi))).$$

COROLLARY 5.5. *In particular, for $T = -\Delta + V$, an operator of Schrödinger-type with a potential $V \in \mathcal{C}(\widetilde{H})$ such that V is a real potential, the decomposition* (6) *holds.*

Proof. By Corollary 2.2, the Laplacian is affiliated to $\mathcal{C}(H^+) \rtimes H \subset \mathcal{C}(\widetilde{H}) \rtimes H$. The identity

$$(T + i) = -\Delta + V + i = (-\Delta + i)[1 + (-\Delta + i)^{-1} V]$$

leads to $T \in' \mathcal{C}(\widetilde{H}) \rtimes H$ hence, we can apply Proposition 5.3 on T. □

Proof of Proposition 5.3. The isomorphism $\mathcal{C}(\widetilde{H})/\mathcal{C}_0(H) \simeq \mathcal{C}(\Sigma_H)$ can be extended to an isomorphism of $(\mathcal{C}(\widetilde{H}) \rtimes H)/(\mathcal{C}_0(H) \rtimes H) \simeq \mathcal{C}(\Sigma_H) \rtimes H$, since H is amenable. Moreover $\mathcal{C}_0(H) \rtimes H \simeq \mathcal{K}(L^2(H))$ and it is well-known that the essential spectrum of an element coincides with the usual spectrum of its image in the quotient by the compact operators. In other words, for T an operator affiliated to $\mathcal{C}(\widetilde{H}) \rtimes H$, we have $\sigma_{ess}(T) = \sigma(\pi(T))$, where $\pi : \mathcal{C}(\widetilde{H}) \rtimes H \to \mathcal{C}(\Sigma_H) \rtimes H$. By Proposition 5.1, the family $(R_\alpha)_{\alpha \in \mathcal{F}}$ is an exhaustive family of morphisms, which gives the following spectral decomposition

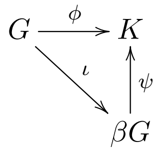
$$\sigma_{ess}(T) = \sigma(\pi(T)) = \bigcup_{\alpha \in \mathcal{F}} \sigma(R_\alpha(\pi(T))),$$

and hence the decomposition (6). □

6. ALGEBRAS GENERATED BY A FAMILY OF COMPACTIFICATIONS

A similar approach could be used to study more complicated compactification of H (or more generally locally compact group). They often arise from continuous map $\phi : H \to K$ with, $\phi(H)$ is dense in a compact set K. The problem is to understand these compactifications. We describe a convenient approach in this section.

Let G be a locally compact group and K be a compact space with a continuous map $\phi : X \to K$. We also assume that $\phi(G)$ is dense in K. Let βG the Stone-Čech compactification of G and $\iota : G \to \beta G$ the embedding of G in βG. By the universal propriety of the Stone-Čech compactification, there exists a unique continuous map $\psi : \beta G \to K$ that makes the following diagram commutative:



Let $\psi_* : \mathcal{C}(K) \to \mathcal{C}(\beta G)$ be the pull-back induced by ψ. The commutativity of the preceding diagram and the density of $\phi(G)$ in K implies the injectivity of ψ_*. We can view $\mathcal{C}(K)$ as unital C^*-subalgebra of $\mathcal{C}(\beta G)$. The space βG is the spectrum of the C^*-algebra $\mathcal{C}_b(G)$, the continuous bounded functions on G equipped with the supremum norm. This leads to the isomorphism $\mathcal{C}(\beta G) \simeq \mathcal{C}_b(G)$. With all this identification, we can view $\mathcal{C}(K)$ as C^*-subalgebra of $\mathcal{C}_b(G)$.

We consider a family $\mathcal{K}$ of compact spaces and a continuous map $\phi_K : G \to K$ for each $K \in \mathcal{K}$ such that $\phi_K(X)$ is dense in K. We define the C^*-algebra generated by:

$$\mathcal{E}_{\mathcal{K}}(X) := \langle \mathcal{C}(K), \mathcal{C}_0(G), K \in \mathcal{K} \rangle.$$

The algebra $\mathcal{E}_{\mathcal{K}}(X)$ remains a C^*-algebra of $\mathcal{C}_b(G)$ because each generator is contained in $\mathcal{C}_b(G)$. Following Theorem 4.4 in [12], we will give a characterization of the spectrum of $\mathcal{E}_{\mathcal{K}}(G)$. We combine all the functions ϕ_K with the identity map on X to define Φ:

$$\Phi : id_G \times \prod_{K\in\mathcal{K}} \phi_k : G \to G \times \prod_{K\in\mathcal{K}} K, \quad \Phi(x) = (x, (\phi_K(x))_{K\in\mathcal{K}}).$$

For each $K \in \mathcal{K}$, we consider γ_K, the restriction map defined by:

$$\gamma_K : \widehat{\mathcal{E}_{\mathcal{K}}(X)} \to \widehat{\mathcal{C}(K)} \simeq K, \quad \chi \mapsto \chi_{|\mathcal{C}(K)} = x_K.$$

As before, we combine all the map γ_K and add the restriction to $\mathcal{C}_0(X)$ to define $\Gamma : \widehat{\mathcal{E}_{\mathcal{K}}(X)} \to G \times \prod_{K \in \mathcal{K}} K$ with:

$$\chi \mapsto (x, x_K), \text{where } \chi(f) = f(x_K), f \in \mathcal{C}(K), K \in \mathcal{K}.$$

With this assumption, we can generalize the lemma 4.3 of [12].

LEMMA 6.1. *The map* Γ *is continuous and a homeomorphism onto its image.*

Proof. We recall the argument in [12]. The continuity comes from the fact that the restriction of a character is continuous. The injectivity is a consequence of the construction of $\mathcal{E}_{\mathcal{K}}(G)$: the values of the character on the generators determines the character everywhere. We have a continuous map between two compact spaces hence, a homeomorphism on its image. $\square$

Let $j : G \to \widehat{\mathcal{E}_{\mathcal{K}}(G)}$ be the extension of a character from the ideal $\mathcal{C}_0(G)$ to the algebra $\mathcal{E}_{\mathcal{K}}(G)$. We have all the notation to generalize Theorem 4.4 of [12].

THEOREM 6.2. *The following diagram is commutative:*

$$\begin{array}{ccc} \widehat{\mathcal{E}_{\mathcal{K}}(G)} & \xrightarrow{\Gamma} & G \times \prod_{K \in \mathcal{K}} K \\ & {}_{j}\nwarrow \quad \nearrow_{\Phi} & \\ & G & \end{array}$$

Moreover, the diagram induces a homeomorphism between $\widehat{\mathcal{E}_{\mathcal{K}}(G)}$ *and* $\overline{\Phi(G)}$*, the closure of the image of* $\Phi(G)$*.*

Proof. For each $K \in \mathcal{K}$, the image of $\phi_K \circ j$ is given by the extension a character χ_x of $\mathcal{C}_0(G)$ to the algebra $\mathcal{E}_{\mathcal{K}}(G)$ and the restrict to $\mathcal{C}(K)$. This extension is unique and corresponds to the character $\chi_{\phi_K(x)}$, that is, to the evaluation map at $\phi_K(x)$. Recall that $\mathcal{C}_0(G)$ is an essential ideal of $\mathcal{E}_{\mathcal{K}}(G)$ and then G is dense in the spectrum of $\mathcal{E}_{\mathcal{K}}(G)$. The continuity of j, Γ and Φ implies

$$\overline{j(G)} = \widehat{\mathcal{E}_{\mathcal{K}}(G)}, \quad \Gamma(\widehat{\mathcal{E}_{\mathcal{K}}(G)}) = \Gamma(\overline{j(G)}) = \overline{\Gamma(j(G))} = \overline{\Phi(G)}. \quad \square$$

Example 6.3. Let G be a locally compact group and $\mathcal{F}$ a family of subgroups of G. For each $H \in \mathcal{F}$, we suppose that there exists a compactification of the quotient G/H such that the action of G on G/H extend to $\widetilde{G/H}$. The continuous map with dense image is given by the canonical map $\pi_H : G \to \widetilde{G/H}$.

Similar mixed algebras have been studied in [13] and by Power [19]. In particular, Power introduced the notion of *permanent point* to characterize the spectrum of the mixed (repeated compactification) algebras.

Definition 6.4. Let A be a unitary C^*-algebra and $(A_i)_{i\in I}$ a family of unitary C^*-subalgebra of A. A point $\chi = (\chi_i)_{i\in I} \in \prod_{i\in I} \widehat{A_i}$ is called *permanent point* if for every Γ, a finite subset of I, the character χ verifies the following propriety:

If, for every index $\gamma \in \Gamma$ and every contraction $a_\gamma \in A_\gamma$, $0 \leq a_\gamma \leq 1$, the equality $\chi_\gamma(a_\gamma) = 1$ is fulfilled, then $\| \prod_{\gamma\in\Gamma} a_\gamma \| = 1$.

The notion of "permanent point" is convenient to describe the spectrum of $\mathcal{A} := \langle A_i, i \in I\rangle$.

Proposition 6.5 (Power). *The spectrum $\widehat{\mathcal{A}}$ can be embedded in $\prod_{i\in I} \widehat{A_i}$ via the restriction map. As subset of $\prod_{i\in I} \widehat{A_i}$, the spectrum $\widehat{\mathcal{A}}$ is exactly the set of permanent points.*

This provides a new way of looking at Theorem 6.2.

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