# QUASI-UNITARY EQUIVALENCE AND GENERALISED NORM RESOLVENT CONVERGENCE

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The purpose of this article is to give a short introduction to the concept of quasi-unitary equivalence of quadratic forms and its consequences. In particular, we improve an estimate concerning the transitivity of quasi-unitary equivalence for forms. We illustrate the abstract setting by two classes of examples.

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## 1. GENERALISED NORM-RESOLVENT CONVERGENCE

In this article, we give an overview of the concept of *quasi-unitary equivalence* for non-negative and self-adjoint operators and quadratic forms acting in *different* Hilbert spaces. Quasi-unitary equivalence provides a sort of "distance" between two operators resp. two quadratic forms. The concept was introduced by the first author in [9] and later explained in great detail in the monograph [10, Ch. 4]. In this article, we also improve an estimate concerning the transitivity of the notion of quasi-unitary equivalence of quadratic forms in Proposition 1.6 (cf. [10, Proposition 4.4.16]). Moreover, we also specify the convergence rate of functions of the operators in Theorem 1.8 which also follows from quasi-unitary equivalence and thus simplifying some earlier results (cf. [10, Theorem 4.2.9 and following pages]). In particular, we make both estimates more explicit.

To illustrate the strength of the concept, we give several examples. We start with the approximation of Laplacians on post-critically finite fractals such as the Sierpiński gasket by their finite-dimensional analogues, see Section 2; more details can be found in [11]. In [13, 14] we extend the results to *magnetic* Laplacians and more general spaces such as finitely ramified fractals. In [12] we use a similar strategy to compare Laplacians on discrete graphs with metric spaces such as metric graphs and graph-like manifolds. In particular, using the above-mentioned transitivity, we could show that Laplacians on a post-critically finite fractal can be approximated by a sequence of Laplacians on manifolds.

The concept of quasi-unitary equivalence is not restricted to such discretisations: it can also be applied in other situations where the underlying spaces change: it was originally developed for a family of thin manifolds shrinking to a so-called metric (or quantum) graph (see [9]), but can also be applied to other (drastic) changes of the space as in [1]. We give a flavour of such arguments in Section 3 where we apply the concept to the case of a manifold with (small) obstacles taken out: we show that the Neumann Laplacian on the remaining set is close to the original (unperturbed) Laplacian.

Other applications are possible, *e.g.* in [7] we could apply it to homogenisation problems and show a generalised *norm* resolvent convergence. Typically, results in homogenisation theory only include *strong* resolvent convergence. Due to the lack of spectral exactness (the limit spectrum can suddenly shrink: there may be approximating sequences that do not correspond to spectral values in the limit, so-called *spectral pollution*). Norm resolvent convergence (and also our generalised version of it, see Definition 1.2) implies that the spectra converge, see Subsection 1.3 for details. Our concept is also closely related to generalised norm resolvent convergence in the sense of Weidmann, see [18, Sec. 9.3]. We also refer to the recent work [2] and references therein; we deal with these aspects in a forthcoming publication. Our results also have a link to numerical analysis where elliptic problems are typically approximated by finite dimensional problems. We will treat such questions also in a forthcoming publication.

We restrict our analysis to non-negative operators and forms mostly for simplicity only. The concept of quasi-unitary equivalence for operators in Subsection 1.1 can be extended to any self-adjoint pair of operators using then the resolvents in $\pm\mathrm{i}$, i.e., $R_\pm := (\Delta \mp \mathrm{i})^{-1}$. Moreover, $\mathscr{H}^2$ is then the domain of $\Delta$ together with its graph norm $\|f\|_2^2 := \|\Delta f\|^2 + \|f\|^2$, and similarly for $\widetilde{\Delta}$, see also [4, Sec. 3]; for non-self-adjoint operators, see [10, Sec. 4.5–4.6]. The concept of quasi-unitary equivalence for energy forms can be generalised to more general forms once there is a good theory of associated operators, *e.g.* for *sectorial* operators, see [8] or [10, Sec. 4.7] for details.

### 1.1. **Quasi-unitary equivalence for operators**

We first start defining a “distance” between two non-negative and self-adjoint operators $\Delta$ and $\widetilde{\Delta}$ acting in *different* Hilbert spaces $\mathscr{H}$ and $\widetilde{\mathscr{H}}$. The distance is expressed in terms of a parameter $\delta \geq 0$, and appears in the concept of *$\delta$-quasi-unitary equivalence*, which we will explain now.

Associated with such a $\Delta$ we define a so-called *scale of Hilbert spaces* $\mathscr{H}^k := \operatorname{dom} \Delta^{k/2}$ with norm $\|f\|_k := \|(\Delta+1)^{k/2} f\|_{\mathscr{H}}$ for $k \geq 0$. For negative powers, we let $\mathscr{H}^{-k}$ be the completion of $\mathscr{H}$ under the norm $\|f\|_{-k} := \|(\Delta+1)^{-k/2} f\|_{\mathscr{H}}$; moreover the inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ extends continuously onto the dual pairing $\mathscr{H}^{-k} \times \mathscr{H}^k$. Similarly, we have a scale of Hilbert spaces $\widetilde{\mathscr{H}}^k$ associated with $\widetilde{\Delta}$.

*Definition* 1.1. Let $\delta \geq 0$.

(i) We say that a linear operator $J \colon \mathscr{H} \longrightarrow \widetilde{\mathscr{H}}$ is *$\delta$-quasi-unitary* with *$\delta$-quasi-adjoint* $J' \colon \widetilde{\mathscr{H}} \longrightarrow \mathscr{H}$ (for the operators $\Delta$ and $\widetilde{\Delta}$) if

$$\|Jf\| \leq (1+\delta)\|f\|, \quad \big|\langle Jf, u\rangle - \langle f, J'u\rangle\big| \leq \delta \|f\| \|u\| \quad (f \in \mathscr{H}, u \in \widetilde{\mathscr{H}}), \tag{1.1a}$$

$$\|f - J'Jf\| \leq \delta \|f\|_2, \quad \|u - JJ'u\| \leq \delta \|u\|_2 \qquad (f \in \mathscr{H}^2, u \in \widetilde{\mathscr{H}}^2). \tag{1.1b}$$

We call $J$ and $J'$ *identification operators.*

(ii) We say that the operators $\Delta$ and $\widetilde{\Delta}$ are *$\delta$-close* if

$$\big|\langle Jf, \widetilde{\Delta} u\rangle_{\widetilde{\mathscr{H}}} - \langle J\Delta f, u\rangle_{\mathscr{H}}\big| \leq \delta \|f\|_2 \|u\|_2 \qquad (f \in \mathscr{H}^2, u \in \widetilde{\mathscr{H}}^2). \tag{1.1c}$$

(iii) We say that $\Delta$ and $\widetilde{\Delta}$ are *$\delta$-(operator-)quasi-unitarily equivalent*, if (1.1a)–(1.1c) are fulfilled, *i.e.*, we have the following equivalent operator norm estimates

$$\|J\| \leq 1+\delta, \qquad \|J^* - J'\| \leq \delta \tag{1.1a'}$$

$$\|(\mathrm{id}_{\mathscr{H}} - J'J)R\| \leq \delta, \qquad \|(\mathrm{id}_{\widetilde{\mathscr{H}}} - JJ')\widetilde{R}\| \leq \delta, \tag{1.1b'}$$

$$\|\widetilde{R}J - JR\| \leq \delta, \tag{1.1c'}$$

where $R := (\Delta+1)^{-1}$ and $\widetilde{R} := (\widetilde{\Delta}+1)^{-1}$.

Note that we also have

$$\|J'\| \leq \|J' - J^*\| + \|J^*\| \leq 1 + 2\delta, \tag{1.2}$$

using $\|J^*\| = \|J\|$ and (1.1a').

If $\delta = 0$ in the above definition then $J$ is unitary with inverse $J^* = J'$ by (1.1a') and (1.1b'). Thus the corresponding operators $\Delta$ and $\widetilde{\Delta}$ are *unitarily equivalent* by (1.1c'). Hence, quasi-unitary equivalence *generalises unitary equivalence.*

Note that quasi-unitary equivalence allows to define a sort of "distance" between two operators $\Delta$ and $\widetilde{\Delta}$ as the infimum of all $\delta \geq 0$ such that (1.1a')–(1.1c') are fulfilled. Then the distance is 0 if and only if $\Delta$ and $\widetilde{\Delta}$ are approximately unitarily equivalent. For more details, we refer to [15].

*Definition* 1.2. Let $\Delta_m$ be a self-adjoint and non-negative operator acting in $\mathscr{H}_m$ for $m \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. We say that the sequence $\{\Delta_m\}_{m\in\mathbb{N}}$ *converges in generalised norm resolvent sense* (with error estimate $\{\delta_m\}_{m\in\mathbb{N}}$) to $\Delta_\infty$, if $\Delta_m$ and $\Delta_\infty$ are $\delta_m$-quasi-unitarily equivalent with $\delta_m \to 0$.

The notion also *generalises* the concept of *norm resolvent convergence*: Assume that the operators all act in the same Hilbert space, *i.e.*, that $\mathscr{H} := \mathscr{H}_m = \mathscr{H}_\infty$ for all $m \in \mathbb{N}$. Then the sequence $\{\Delta_m\}_{m\in\mathbb{N}}$ converges in *norm resolvent sense* to $\Delta_\infty$ if and only if

$$\big\|(\Delta_m + 1)^{-1} - (\Delta_\infty + 1)^{-1}\big\| \to 0 \qquad \text{as} \quad m \to \infty. \tag{1.3}$$

If we choose $J$ and $J'$ to be the identity operator on $\mathscr{H}$, then (1.1a') and (1.1b') are fulfilled with $\delta_m = 0$, and (1.1c') with $\delta_m \to 0$ is equivalent with (1.3). A sequence of operators also converges in *generalised* norm resolvent sense if there is a sequence of unitary operators $J_m \colon \mathscr{H}_m \longrightarrow \mathscr{H}_\infty$ such that

$$\|J_m(\Delta_m + 1)^{-1} J_m^* - (\Delta_\infty + 1)^{-1}\| \to 0.$$

The notion of operator-quasi-unitary equivalence is transitive in the following sense (the proof is similar to the one of Proposition 1.6, and we slightly improved the error term $\hat{\delta}$ compared to the one given in [10, Proposition 4.2.5]):

Proposition 1.3. *Assume that $\delta, \widetilde{\delta} \in [0,1]$. Assume in addition that $\Delta$ and $\widetilde{\Delta}$ are $\delta$-quasi-unitarily equivalent with identification operators $J$ and $J'$, and that $\widetilde{\Delta}$ and $\widehat{\Delta}$ are $\widetilde{\delta}$-quasi-unitarily equivalent with identification operators $\widetilde{J}$ and $\widetilde{J}'$. Then $\Delta$ and $\widehat{\Delta}$ are $\hat{\delta}$-quasi-unitarily equivalent with identification operators $\hat{J} = \widetilde{J}J$ and $\hat{J}' = J'\widetilde{J}'$, where $\hat{\delta} = 5\delta + 5\widetilde{\delta}$.*

## 1.2. Quasi-unitary equivalence for energy forms

It is actually more convenient to start with the quadratic forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ associated with the non-negative operators $\Delta$ and $\widetilde{\Delta}$, and develop a slightly more elaborated version of quasi-unitary equivalence. This approach avoids dealing with the sometimes complicated operator domains and graph norms. Nevertheless, in applications, the more elaborated conditions are easily verified.

Let $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ be two separable (complex) Hilbert spaces. We say that $\mathcal{E}$ is an *energy form in $\mathscr{H}$* if $\mathcal{E}$ is a closed, non-negative quadratic form in $\mathscr{H}$, *i.e.*, if $\mathcal{E}(f) := \mathcal{E}(f,f)$ for some sesquilinear form $\mathcal{E} \colon \mathscr{H}^1 \times \mathscr{H}^1 \longrightarrow \mathbb{C}$, denoted by the same symbol, if $\mathcal{E}(f) \geq 0$ and if $\mathscr{H}^1 := \operatorname{dom} \mathcal{E}$, endowed with the norm defined by

$$\|f\|_{\mathcal{E}}^2 := \|f\|_{\mathscr{H}}^2 + \mathcal{E}(f), \tag{1.4}$$

is itself a Hilbert space and dense (as a set) in $\mathscr{H}$. We call the corresponding non-negative, self adjoint operator $\Delta$ (see *e.g.* [5, Sec. VI.2]) the *Laplacian* associated with $\mathcal{E}$. Similarly, let $\widetilde{\mathcal{E}}$ be an energy form in $\widetilde{\mathscr{H}}$ with Laplacian $\widetilde{\Delta}$. Note that $\|f\|_1 = \|f\|_{\mathcal{E}}$ and that $\|f\|_{\mathcal{E}} \le \|f\|_2$ in the terminology of Subsection 1.1.

We now also need identification operators $J^1$ and $J'^1$ acting on the form domains.

*Definition* 1.4. Let $\delta \ge 0$, $J\colon \mathscr{H} \longrightarrow \widetilde{\mathscr{H}}$ and $J'\colon \widetilde{\mathscr{H}} \longrightarrow \mathscr{H}$, resp. $J^1\colon \mathscr{H}^1 \longrightarrow \widetilde{\mathscr{H}}^1$ and $J'^1\colon \widetilde{\mathscr{H}}^1 \longrightarrow \mathscr{H}^1$ be linear operators on the Hilbert spaces and energy form domains.

(i) We say that $J$ is *$\delta$-quasi-unitary* with *$\delta$-quasi-adjoint* $J'$ (for the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$) if

$$\|Jf\| \le (1+\delta)\|f\|, \ \big|\langle Jf, u\rangle - \langle f, J'u\rangle\big| \le \delta\|f\|\|u\| \ (f \in \mathscr{H}, u \in \widetilde{\mathscr{H}}), \tag{1.5a}$$

$$\|f - J'Jf\| \le \delta\|f\|_{\mathcal{E}}, \quad \|u - JJ'u\| \le \delta\|u\|_{\widetilde{\mathcal{E}}} \quad (f \in \mathscr{H}^1, u \in \widetilde{\mathscr{H}}^1). \tag{1.5b}$$

(ii) We say that $J^1$ and $J'^1$ are *$\delta$-compatible* (with identification operators $J$ and $J'$) if

$$\|J^1 f - Jf\| \le \delta\|f\|_{\mathcal{E}}, \quad \|J'^1 u - J'u\| \le \delta\|u\|_{\widetilde{\mathcal{E}}} \quad (f \in \mathscr{H}^1, u \in \widetilde{\mathscr{H}}^1). \tag{1.5c}$$

(iii) We say that the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are *$\delta$-close* if

$$\big|\widetilde{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| \le \delta\|f\|_{\mathcal{E}}\|u\|_{\widetilde{\mathcal{E}}} \qquad (f \in \mathscr{H}^1, u \in \widetilde{\mathscr{H}}^1). \tag{1.5d}$$

(iv) We say that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are *$\delta$-quasi-unitarily equivalent*, if (1.5a)–(1.5d) are fulfilled.

We have the following relation between quasi-unitary equivalence for quadratic forms and operators; the last conclusion has already been shown in [10, Proposition 4.4.15]:

PROPOSITION 1.5. *If the forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent then we have*

$$\|\widetilde{R}(z)J - JR(z)\| \le C(z)\delta, \tag{1.6}$$

*where $R(z) := (\Delta - z)^{-1}$ and $\widetilde{R}(z) := (\widetilde{\Delta} - z)^{-1}$ for $z \in \mathbb{C} \setminus (\sigma(\Delta) \cup \sigma(\widetilde{\Delta}))$ and*

$$C(z) := 4\Big(1 + \frac{|z+1|}{d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta}))}\Big)^2.$$

*In particular, the associated operators $\Delta$ and $\widetilde{\Delta}$ are $4\delta$-quasi-unitarily equivalent.*

*Proof.* For $g \in \mathscr{H}$ and $v \in \widetilde{\mathscr{H}}$, we have

$$\begin{aligned}
&\bigl|\langle(\widetilde{R}(z)J - JR(z))g, v\rangle\bigr| = \bigl|\langle g, J^*\widetilde{R}(\overline{z})v\rangle - \langle JR(z)g, v\rangle\bigr| \\
&= \bigl|\langle \Delta f, J^* u\rangle - \langle Jf, \widetilde{\Delta} u\rangle\bigr| \\
&\le \bigl|\bigl\langle \Delta f, \bigl((J^* - J') + (J' - J'^1)\bigr)u\bigr\rangle\bigr| + \bigl|\mathcal{E}(f, J'^1 u) - \widetilde{\mathcal{E}}(J^1 f, u)\bigr| \\
&+ \bigl|\langle (J^1 - J)f, \widetilde{\Delta} u\rangle\bigr| \\
&\le 2\delta\|f\|_2\|u\| + \delta\|f\|_1\|u\|_1 + \delta\|f\|\,\|u\|_2 \le 4\delta\|f\|_2\|u\|_2,
\end{aligned}$$

where $f = R(z)g$ and $u = \widetilde{R}(\overline{z})v$. We have $\|f\|_2 = \|(\Delta+1)R(z)g\| \le \|(\Delta+1)R(z)\|\,\|g\|$ and

$$\|(\Delta+1)R(z)\| = \sup_{\lambda\in\sigma(\Delta)} \frac{\lambda+1}{|\lambda - z|} \le 1 + \sup_{\lambda\in\sigma(\Delta)} \frac{|z+1|}{|\lambda - z|} = 1 + \frac{|z+1|}{d(z, \sigma(\Delta))}$$

using the spectral theorem. A similar estimate holds for $\|u\|_2$. In particular, the resolvent estimate follows. For the second statement, note that for $z = -1$ we have $C(-1) = 1$, hence (1.1c) holds with $4\delta$. The remaining estimates (1.1a)–(1.1b) follow from the quasi-unitary equivalence of the forms and the fact that $\|f\|_1 \le \|f\|_2$ and similarly for $u$. $\square$

In particular, if we choose the rough estimate $\sigma(\Delta)\cup\sigma(\widetilde{\Delta}) \subset [0,\infty)$, then $C(z) \le 1 + |z+1|/d(z,[0,\infty))$. For $\operatorname{Re} z \ge 0$, the latter equals $1 + |z+1|/|\operatorname{Im} z|$ and for $\operatorname{Re} z < 0$ the latter equals $1 + |z+1|/|z|$. Hence, we have

$$C(z) \le 1 + \frac{|z+1|}{|\operatorname{Im} z|} \quad \text{resp.} \quad C(z) \le 1 + \frac{|z+1|}{|z|}$$

for $\operatorname{Re} z \ge 0$ resp. $\operatorname{Re} z < 0$.

Let us mention a special case here, namely $\delta = 0$ in (1.5a)–(1.5c). In this situation, $J$ is a unitary operator with $J' = J^*$, $J^1 = J\restriction_{\mathscr{H}^1}$ and $J'^1 = J^*\restriction_{\widetilde{\mathscr{H}}^1}$; hence without loss of generality we can assume $\mathscr{H} = \widetilde{\mathscr{H}}$, $J = J' = \mathrm{id}_{\mathscr{H}}$ and $\operatorname{dom}\mathcal{E} = \operatorname{dom}\widetilde{\mathcal{E}}$. In particular, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are $\delta$-quasi-unitarily-equivalent if and only if

$$|\widetilde{\mathcal{E}}(f,u) - \mathcal{E}(f,u)| \le \delta\|f\|_{\mathcal{E}}\|u\|_{\widetilde{\mathcal{E}}} \tag{1.7a}$$

for all $f, u \in \mathscr{H}^1 := \operatorname{dom}\mathcal{E} = \operatorname{dom}\widetilde{\mathcal{E}}$. Using the fact that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are symmetric, it is sufficient if (1.7a) only holds for $f = u$, *i.e.*, (1.7a) is equivalent with

$$|\widetilde{\mathcal{E}}(f) - \mathcal{E}(f)| \le \hat{\delta}\|f\|_{\mathcal{E}}^2 \tag{1.7b}$$

for all $f \in \mathscr{H}^1$. For the implication (1.7a) $\Rightarrow$ (1.7b) one can use $\hat{\delta} = \delta\sqrt{(2+\delta)/(2-\delta)}$ (provided $\delta < 2$) and for (1.7b)$\Rightarrow$(1.7a) one can use $\delta = \hat{\delta}/\sqrt{1-\hat{\delta}}$ (provided $\hat{\delta} < 1$). This situation has also been studied in [3][1]; basi-

[1] In memoriam Johannes Brasche, who suddenly passed away in December 2018.

cally, their Theorem 2 is the implication (1.7b)$\Rightarrow$(1.7a) together with Proposition 1.5 (with $z=-1$ and $4\delta$ replaced by $\delta$, as $\delta=0$ in (1.5a)–(1.5c)).

In particular, if $\{\mathcal{E}_m\}_{m\in\mathbb{N}}$ is a sequence of energy forms acting in the *same* Hilbert space as $\mathcal{E}_\infty$, *i.e.*, $\mathscr{H}_m=\mathscr{H}_\infty$ with the *same* domain $\operatorname{dom}\mathcal{E}_m=\operatorname{dom}\mathcal{E}_\infty$ for all $m\in\mathbb{N}$, then (with all identification operators being the corresponding identity operators) (1.5a)–(1.5c) are trivially fulfilled with $\delta=0$. Moreover, (1.5d) with $\delta=\delta_m\to 0$ is equivalent with

$$\big|\mathcal{E}_\infty(f)-\mathcal{E}_m(f)\big|\le\hat{\delta}_m\|f\|^2_{\mathcal{E}_\infty}$$

for all $f\in\operatorname{dom}\mathcal{E}_\infty$ with $\delta_m$ and $\hat{\delta}_m$ related as above. This is the classical situation of Kato [5, Theorem VI.3.6] or [16, Theorem VIII.25(c)], and we conclude (using Proposition 1.5) that the operators $\Delta_m$ associated with $\mathcal{E}_m$ *converge* to $\Delta_\infty$ in *norm resolvent sense*, see (1.3). Note that both classical results do not state the convergence speed of the norm of the resolvent difference.

Another useful implication is the transitivity of quasi-unitary equivalence for energy forms; it was originally proved in [10, Proposition 4.4.16]; we give here a simpler proof.

Proposition 1.6. *Let $\delta,\widetilde{\delta}\in[0,1]$. Assume that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent with identification operators $J$, $J^1$, $J'$ and $J'^1$. Moreover, assume that $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ are $\widetilde{\delta}$-quasi-unitarily equivalent with identification operators $\widetilde{J}$, $\widetilde{J}^1$, $\widetilde{J}'$ and $\widetilde{J}'^1$. Assume in addition that, for all $f\in\mathscr{H}^1$ and $w\in\widehat{\mathscr{H}}^1$,*

$$\|J^1f\|_{\widetilde{\mathcal{E}}}\le(1+\delta)\|f\|_{\mathcal{E}}\quad\text{and}\quad\|\widetilde{J}'^1w\|_{\widetilde{\mathcal{E}}}\le(1+\widetilde{\delta})\|w\|_{\widehat{\mathcal{E}}}.$$

*Then $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are $\hat{\delta}$-quasi-unitarily equivalent with $\hat{\delta}=14(\delta+\widetilde{\delta})$.*

*Proof.* We define the identification operators by $\widehat{J}:=\widetilde{J}J$, $\widehat{J}^1:=\widetilde{J}^1J^1$, $\widehat{J}':=J'\widetilde{J}'$ and $\widehat{J}'^1:=J'^1\widetilde{J}'^1$ and we set $R:=(\Delta+1)^{-1}$, $\widetilde{R}:=(\widetilde{\Delta}+1)^{-1}$ and $\widehat{R}:=(\widehat{\Delta}+1)^{-1}$. Then $\widehat{J}$ is bounded, because

$$\|\widehat{J}\|=\|\widetilde{J}J\|\le(1+\delta)(1+\widetilde{\delta})\le 1+\frac{3}{2}(\delta+\widetilde{\delta}).$$

The second inequality in (1.5a) follows from

$$\|\widehat{J}^*-\widehat{J}'\|\le\|J^*(\widetilde{J}^*-\widetilde{J}')\|+\|(J^*-J')\widetilde{J}'\|\le(1+\delta)\widetilde{\delta}+\delta(1+2\widetilde{\delta})\le\frac{5}{2}\delta+\frac{5}{2}\widetilde{\delta}$$

as $\|\widetilde{J}'\|\le 1+2\delta$ by (1.2). The first inequality in (1.5b) is also satisfied because

$$\begin{aligned}\|f-\widehat{J}'\widehat{J}f\|&\le\|f-J'Jf\|+\|J'(J-J^1)f\|+\|J'(\mathrm{id}_{\widetilde{\mathscr{H}}}-\widetilde{J}'\widetilde{J})J^1\|\\&+\|J'\widetilde{J}'\widetilde{J}(J^1-J)f\|\le\Big(\delta+(1+2\delta)\big(\delta+\widetilde{\delta}(1+\delta)+(1+2\widetilde{\delta})(1+\widetilde{\delta})\delta\big)\Big)\|f\|_{\mathcal{E}}\\&\le 14(\delta+\widetilde{\delta})\|f\|_{\mathcal{E}}\end{aligned}$$

and the second one follows by similar arguments. Next we prove that the two inequalities in (1.5c) also hold. We estimate

$$\begin{aligned}\|(\widehat{J}^1 - \widehat{J})f\| &\le \|(\widetilde{J}^1 - \widetilde{J})J^1 f\| + \|\widetilde{J}(J^1 - J)f\| \\ &\le \bigl(\widetilde{\delta}(1+\delta) + (1+\widetilde{\delta})\delta\bigr)\|f\|_{\mathcal{E}} \le 2(\delta+\widetilde{\delta})\|f\|_{\mathcal{E}}\end{aligned}$$

and

$$\begin{aligned}\|(\widehat{J}'^1 - \widehat{J}')w\| &\le \|(J'^1 - J')\widetilde{J}'^1 w\| + \|J'(\widetilde{J}'^1 - \widetilde{J}')w\| \\ &\le \bigl(\delta(1+\widetilde{\delta}) + (1+2\delta)\widetilde{\delta}\bigr)\|w\|_{\widehat{\mathcal{E}}} \le \frac{5}{2}(\delta+\widetilde{\delta})\|w\|_{\widehat{\mathcal{E}}}.\end{aligned}$$

For inequality (1.5d) we estimate

$$\begin{aligned}&\bigl|\widehat{\mathcal{E}}(\widehat{J}^1 f, w) - \mathcal{E}(f, \widehat{J}'^1 w)\bigr| \\ &\qquad \le \bigl|\widehat{\mathcal{E}}(\widetilde{J}^1 J^1 f, w) - \widetilde{\mathcal{E}}(J^1 f, \widetilde{J}'^1 w)\bigr| + \bigl|\widetilde{\mathcal{E}}(\widetilde{J}^1 f, \widetilde{J}'^1 w) - \mathcal{E}(f, J'^1 \widetilde{J}'^1 w)\bigr| \\ &\qquad \le \widetilde{\delta}\|J^1 f\|_{\widetilde{\mathcal{E}}}\|w\|_{\widehat{\mathcal{E}}} + \delta\|f\|_{\mathcal{E}}\|\widetilde{J}'^1 w\|_{\widetilde{\mathcal{E}}} \\ &\qquad \le \bigl(\widetilde{\delta}(1+\delta) + \delta(1+\widetilde{\delta})\bigr)\|f\|_{\mathcal{E}}\|w\|_{\widehat{\mathcal{E}}} \le 2(\delta+\widetilde{\delta})\|f\|_{\mathcal{E}}\|w\|_{\widehat{\mathcal{E}}}. \quad \square\end{aligned}$$

It is a useful feature of Definition 1.4 that it provides us with some flexibility in terms of the inequalities. The next lemma is one example. In [11] it was applied to avoid a Poincaré-type estimate, *i.e.*, to bypass an estimate of the first non-zero eigenvalue.

LEMMA 1.7 ([11, Lem. 2.4]). *Assume that* (1.5a) *is fulfilled with* $\delta_{\mathrm{a}} > 0$ *and* (1.5c) *with* $\delta_{\mathrm{c}} > 0$. *If*

(1.5b') $$\|u - JJ'^1 u\| \le \delta' \|u\|_{\widetilde{\mathcal{E}}} \qquad (u \in \widetilde{\mathscr{H}^1})$$

*holds, then the second inequality in* (1.5b) *is fulfilled with* $\delta = \delta' + (1+\delta_{\mathrm{a}})\delta_{\mathrm{c}}$.

*In particular, if all conditions* (1.5) *are fulfilled for some* $\delta > 0$, *except for the second one in* (1.5b) *which is replaced by* (1.5b'), *then* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}$ *are* $\widetilde{\delta}$*-quasi-unitarily equivalent with* $\widetilde{\delta} = \delta' + (1+\delta)\delta$.

## 1.3. **Consequences of quasi-unitary equivalence**

Let $\Delta$ be non-negative and self-adjoint and $R(z) := (\Delta - z)^{-1}$ be its resolvent. Let $U$ be an open neighbourhood of $\sigma(\Delta) \subset \mathbb{C}$ such that $\partial U$ is locally the graph of a Lipschitz continuous function and such that $\partial U \cap \sigma(\Delta) = \emptyset$. Moreover, let $\eta\colon U \longrightarrow \mathbb{C}$ be a holomorphic function. Then the integral

(1.9) $$\eta(\Delta) := -\frac{1}{2\pi \mathrm{i}} \int_{\partial U} \eta(z) R(z)\, \mathrm{d}z$$

is defined in the operator norm topology provided

$$C_{\eta,\sigma} := \frac{1}{2\pi} \int_{\partial U} \frac{|\eta(z)|}{d(z,\sigma)} \, \mathrm{d}|z| < \infty$$

for $\sigma := \sigma(\Delta)$. For example, if $U$ encloses a compact subset $K$ of $\sigma(\Delta)$, then $\mathbb{1}_U(\Delta)$ (defined with $\eta = \mathbb{1}_U$ in (1.9)) is the spectral projection onto $K$.

Theorem 1.8. *Assume that the forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ corresponding to the operators $\Delta$ and $\widetilde{\Delta}$ are $\delta$-quasi-unitarily equivalent (or that* (1.6) *holds), and that $U$ is an open subset such that $\partial U$ is locally Lipschitz and such that $\partial U \cap (\sigma(\Delta) \cup \sigma(\widetilde{\Delta})) = \emptyset$ then*

$$\|\eta(\widetilde{\Delta})J - J\eta(\Delta)\| \le C_\eta \delta, \tag{1.10}$$

*where $C_\eta$ is defined in* (1.11).

*Proof.* Since the integrals for $\eta(\Delta)$ and $\eta(\widetilde{\Delta})$ exist in operator norm, we have

$$\eta(\widetilde{\Delta})J - J\eta(\Delta) = -\frac{1}{2\pi\mathrm{i}} \int_{\partial U} \eta(z)\big(\widetilde{R}(z)J - JR(z)\big) \, \mathrm{d}z.$$

Taking the operator norm on both sides and using (1.6), we obtain

$$\|\eta(\widetilde{\Delta})J - J\eta(\Delta)\| \le \underbrace{\frac{1}{2\pi} \int_{\partial U} |\eta(z)| 4\Big(1 + \frac{|z+1|}{d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta}))}\Big)^2 \mathrm{d}|z|}_{=:C_\eta} \cdot \delta. \tag{1.11}$$ □

Note that $C_\eta < \infty$ implies that $C_{\eta,\sigma(\Delta)} < \infty$ and $C_{\eta,\sigma(\widetilde{\Delta})} < \infty$.

*Remark* 1.9. Note that we have also a functional calculus for measurable functions continuous in a neighbourhood of $\sigma(\Delta)$. Then the error $C_\eta \delta$ has to be replaced by a function $\Phi_\eta(\delta)$ with the property that $\Phi_\eta(\delta) \to 0$ as $\delta \to 0$; in particular, we lose the information about the convergence speed (see [9, Theorem A.8] for details). Nevertheless, the following result remains true (with a modified error term, see [9, Theorem A.10].

Proposition 1.10 ([10, Lem. 4.2.13]). *Assume that* (1.10) *holds. Then*

$$\|\eta(\widetilde{\Delta}) - J\eta(\Delta)J'\| \le C'_\eta \delta \quad \text{and} \quad \|\eta(\Delta) - J'\eta(\widetilde{\Delta})J\| \le C'_\eta \delta$$

*with*

$$C'_\eta := 5 \sup_{\lambda \in [0,\infty) \cap U} |\eta(\lambda)(\lambda+1)^{1/2}| + 3C_\eta$$

*for all energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ (with corresponding operators $\Delta$ and $\widetilde{\Delta}$, respectively) being $\delta$-quasi-unitarily equivalent with identification operators $J$ and $J'$ and $\delta \in [0,1]$.*

Let us calculate explicitly the constants $C_\eta$ and $C'_\eta$ for two examples of the function $\eta$:

*Example* 1.11.

(i) *Spectral projections.* Let $I := (a,b)$ such that $-1 < a < b$ and $a, b \notin \sigma(\Delta) \cup \sigma(\widetilde{\Delta}) =: S$ with $d(\{a,b\}, S) \geq \varepsilon$ for some $\varepsilon > 0$.[2] We want to compare the spectral projections $\mathbb{1}_I(\Delta)$ and $\mathbb{1}_I(\widetilde{\Delta})$, defined via the functional calculus for self-adjoint operators. Let $U := I \times \mathrm{i}(-\varepsilon, \varepsilon) \subset \mathbb{C}$ be a rectangle enclosing $I$. Note that we have $\mathbb{1}_I(\Delta) = \eta(\Delta)$ with $\eta = \mathbb{1}_U$ where the latter operator function is defined via the holomorphic functional calculus (1.9); a similar statement holds for $\widetilde{\Delta}$. A straightforward estimate shows that

$$C_\eta = \frac{4}{\pi}(b - a + \varepsilon)\Big(1 + \sqrt{1 + \Big(\frac{b+1}{\varepsilon}\Big)^2}\Big)^2 = \mathrm{O}(b).$$

Moreover, $C'_\eta = 5\sqrt{b+1} + 3C_\eta = \mathrm{O}(b)$.

(ii) *Heat operator.* For the heat operator, we have $\eta_t(\lambda) = \mathrm{e}^{-t\lambda}$ for $t \geq 0$. As open neighbourhood of the spectrum we let $U$ be the open sector with half-angle $\theta \in (0, \pi/2)$ and vertex at $-1$, and symmetric with respect to the real axis. Then we have

$$C_{\eta_t} \leq \frac{4}{\pi} \int_0^\infty \mathrm{e}^{-tr\cos\theta}\Big(1 + \frac{1}{\sin\theta}\Big)^2 \,\mathrm{d}r = \frac{4}{\pi\cos\theta}\Big(1 + \frac{1}{\sin\theta}\Big)^2 \cdot \frac{1}{t},$$

since $d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta})) \geq |z+1| \sin\theta$. Now, the minimum of the right hand side over $\theta \in (0, \pi/2)$ is achieved when $\theta = \pi/4$, and hence is $4(4 + 3\sqrt{2})/\pi \cdot 1/t \leq 11/\pi$. Moreover, as

$$\sup_{\lambda \in [0,\infty)} |\mathrm{e}^{-t\lambda}(\lambda+1)^{1/2}| \leq \begin{cases} 1, & t \geq 1/2, \\ 1/(2t)^{1/2}, & t \in [0, 1/2], \end{cases}$$

we conclude that a rough estimate is

$$C'_{\eta_t} = \frac{39}{t} + 5. \tag{1.12}$$

In particular, we conclude the following convergence result for the solution of the heat equation:

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[2] If we aim in operator convergence of spectral projections, it is a standard assumption that $\partial I$ is in the resolvent set of at least one of the operators; if $\delta$ is small enough, it can then be shown that $\partial I$ is also in the resolvent set of the other operator, see *e.g.* [16, Theorem VIII.23 (b)]. For *strong* convergence, the assumption can be weakened to exclude that $\partial I$ are eigenvalues.

COROLLARY 1.12. *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be two $\delta$-quasi-unitarily equivalent energy forms with associated operators $\Delta$ and $\widetilde{\Delta}$. Assume that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are $\delta$-quasi-unitarily equivalent. Let $f_t$ resp. $u_t$ be the solution of the heat equations*

$$\partial_t f_t + \Delta f_t = 0 \quad \textit{resp.} \quad \partial_t u_t + \widetilde{\Delta} u_t = 0$$

*for $t > 0$. If $f_0 = J' u_0$, then for any $T > 0$ we have*

$$\|u_t - J f_t\|_{\widetilde{\mathscr{H}}} \le C_{\eta_T} \delta \|u_0\|_{\widetilde{\mathscr{H}}}$$

*for all $t \in [T, \infty)$ with $C'_{\eta_T} = \mathrm{O}(1/T)$ $(T \to 0)$ given in* (1.12).

*Proof.* We have $u_t = \mathrm{e}^{-t\widetilde{\Delta}} u_0$ and $f_t = \mathrm{e}^{-t\Delta} J' u_0$. Then

$$\|u_t - J f_t\|_{\widetilde{\mathscr{H}}} = \|(\mathrm{e}^{-t\widetilde{\Delta}} - J\mathrm{e}^{-t\Delta} J') u_0\|_{\widetilde{\mathscr{H}}} \le C'_{\eta_t} \delta \|u_0\|_{\widetilde{\mathscr{H}}}$$

We apply Proposition 1.10 and the concrete estimate for $C'_{\eta_t}$ to conclude the desired estimate. □

As in the case of usual norm convergence the operator norm convergence of spectral projections implies the *convergence of spectra* (also called *spectral exactness*):

COROLLARY 1.13 ([10, Theorem 4.3.3]). *If $\Delta_m$ converges in generalised norm resolvent sense to $\Delta_\infty$, then*

$$\bar{d}(\sigma(\Delta_m), \sigma(\Delta_\infty)) \to 0$$

*as $m \to \infty$, where*

$$\bar{d}(A, B) := \max\{\sup_{a \in A} \inf_{b \in B} d(a, b), \sup_{b \in B} \inf_{a \in A} d(a, b)\}$$

*defines a weighted Hausdorff metric between two closed sets $A, B \subset [0, \infty)$. Here, $d(a, b) := |(a+1)^{-1} - (b+1)^{-1}|$ is a weighted metric on $[0, \infty)$.*

If the operators have purely discrete spectrum, we can specify the error estimate:

COROLLARY 1.14. *Let $\lambda_k(\Delta_m)$ resp. $\lambda_k(\Delta_\infty)$ denote the $k^{th}$ eigenvalue of $\Delta_m$ resp. $\Delta_\infty$ (in increasing order and repeated according to their multiplicity). Then*

$$\big|\lambda_k(\Delta_m) - \lambda_k(\Delta_\infty)\big| \le C_k \delta_m$$

*for all $m \in \mathbb{N}$ such that $\dim \mathscr{H}_m \ge k$, where $C_k$ depends only on $\lambda_k(\Delta_\infty)$.*

In the case of purely discrete spectrum (or isolated eigenvalues) we can approximate an eigenfunction also in energy norm:

PROPOSITION 1.15 ([11, Proposition 2.6]). *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be two $\delta$-quasi-unitarily equivalent energy forms with associated operators $\Delta$ and $\widetilde{\Delta}$. Assume* *that* $\widetilde{\Phi}$ *is an eigenvector of* $\widetilde{\Delta}$*, such that its eigenvalue* $\widetilde{\lambda}$ *is discrete in* $\sigma(\widetilde{\Delta})$*, i.e., there is an open disc* $D$ *in* $\mathbb{C}$ *such that* $\sigma(\widetilde{\Delta}) \cap D = \{\widetilde{\lambda}\}$*. Then there exists a normalised eigenvector* $\Phi$ *of* $\Delta$ *with* $\Phi \in \operatorname{ran} \mathbb{1}_D(\Delta)$ *and a universal constant* $C$ *depending only on* $\widetilde{\lambda}$ *(and the radius of* $D$*) such that*

$$\|J^1\Phi - \widetilde{\Phi}\|_{\widetilde{\mathcal{E}}} \leq C\delta.$$

Note that the eigenvalue $\widetilde{\lambda}$ does not necessarily need to have finite multiplicity.

## 2. POST-CRITICALLY FINITE SELF-SIMILAR FRACTALS

In [11] the authors applied the quasi-unitary equivalence to the case of certain fractals called *post-critically finite self-similar fractals* (which supports a regular resistance form in the sense of [6]; see also [17]). Here, we will simply discuss two examples. For the general case, we refer to [11] and for a further generalisation to magnetic energy forms on finitely ramified fractals, we refer to [14].

### 2.1. The unit interval

At the first glance it might look a bit odd to call the unit interval $K = [0,1]$ a fractal, but it will turn out that this approach is quite elegant for the approximation. We begin by defining two contractions $F_1, F_2 \colon \mathbb{R} \to \mathbb{R}$ with contraction ratio $1/2$ and fixed points $0$ and $1$ by

$$F_1(t) = \frac{t}{2} \qquad \text{and} \qquad F_2(t) = \frac{1+t}{2}.$$

Then, we have $K = F_1(K) \cap F_2(K)$ and $K$ is the unique non-empty compact subset of $\mathbb{R}$ with that property. We call $K$ the *self-similar fractal* with respect to $F = \{F_1, F_2\}$. Moreover, the maps $F_j$ describe a cell structure on $K$ via

$$w \mapsto F_w(K) := (F_{w_1} \circ \cdots \circ F_{w_m})(K)$$

where $w = w_1 \ldots w_m \in W_m := \{1,2\}^m$ is a word of length $m$. We refer to $F_w(K)$ as an $m$-cell whenever $w \in W_m$.

Next, we define the *(vertex) boundary* by $V_0 = \{0,1\}$. Note that in the special case of the interval, the topological boundary and the vertex boundary coincide but that is not necessarily the case (see *e.g.* the Sierpiński gasket). Then, we define the approximating sequence of (finite weighted discrete) graphs as follows: Let $G_0 = (V_0, E_0)$ be the complete graph with two vertices $V_0 = \{0,1\}$ and one edge. Moreover, define $G_m = (V_m, E_m)$ inductively, where $V_m =$

$\{ k2^{-m} \mid k = 0, \ldots, 2^m \}$ are the $m$-dyadic numbers and where we have an edge between (distinct) vertices $x$ and $y$ in $V_m$ if and only if $|x - y| = 2^{-m}$.

Let us specify the Hilbert spaces and energy forms now. As a measure on $K = [0, 1]$ we fix the Lebesgue measure $\mu$ and our Hilbert space is the usual space of square integrable function with respect to the Lebesgue measure, *i.e.*, $\widetilde{\mathscr{H}} = \mathsf{L}_2(K, \mu)$.

The approximating measure $\mu_m = \{\mu_m(x)\}_{x \in V_m}$ on $G_m$ is defined by

$$\mu_m(x) = \int_0^1 \psi_{x,m}(t)\,\mathrm{d}\mu(t) = \frac{1}{2^m} \begin{cases} 1 & x \in V_m \setminus V_0 \\ 1/2 & x \in V_0, \end{cases} \tag{2.1}$$

where $\psi_{x,m} \colon K \to [0, 1]$ is given by $\mathbb{1}_{\{x\}}$ on $V_m$ and extended to $K$ by linear interpolation. Hence, our Hilbert space on the graph $G_m$ is $\mathscr{H}_m = \ell_2(V_m, \mu_m)$ with norm

$$\|f\|^2_{\ell_2(V_m,\mu_m)} = \sum_{x \in V_m} \mu_m(x) |f(x)|^2.$$

On each graph $G_m$ we now define a discrete energy form $\mathcal{E}_m$ in $\ell_2(V_m, \mu_m)$. For each $f \in \ell(V_m) := \{ f \mid f \colon V_m \to \mathbb{C} \}$, we set

$$\mathcal{E}_m(f) = \sum_{\{x,y\} \in E_m} c_{\{x,y\},m} |f(y) - f(x)|^2, \tag{2.2}$$

where the *conductances* $c_{\{x,y\},m} \geq 0$ are chosen such that

$$\mathcal{E}_m(\varphi) = \min\big\{ \mathcal{E}_{m+1}(f) \,\big|\, f \in \ell(V_{m+1}), f\restriction_{V_m} = \varphi \big\}, \tag{2.3}$$

for all $\varphi \in \ell(V_m)$. Working this out, we see that $c_{\{x,y\},m} = 2^m$. A sequence $\{\mathcal{E}_m\}_{m \in \mathbb{N}_0}$ of energy forms that satisfies (2.3) for all $m$ is called *compatible sequence.*

From the classical theory of calculus it is well-known that the limit form is given by

$$\mathcal{E}(u) = \int_0^1 |u'(t)|^2 \,\mathrm{d}\mu(t)$$

for each weakly differentiable $u \in \mathsf{L}_2(K, \mu)$ with $u' \in \mathsf{L}_2(K, \mu)$, *i.e.*, we have $\operatorname{dom} \mathcal{E} = \mathsf{H}^1(K, \mu)$.

THEOREM 2.1 ([11]). *The energy form $(\mathcal{E}, \mathsf{H}^1(K, \mu))$ in $\mathsf{L}_2(K, \mu)$ and the discrete energy form $\mathcal{E}_m$ in $\ell_2(V_m, \mu_m)$ are $\delta_m$-quasi-unitarily equivalent, where the error is*

$$\delta_m = (1 + \sqrt{2}) \cdot \frac{1}{2^m}.$$

Let us briefly discuss the idea of the proof: First, we need to choose the identification operators from Definition 1.4. On the Hilbert space level, we define $J_m \colon \mathscr{H}_m \to \widetilde{\mathscr{H}}$ and $J'_m = J^*_m \colon \widetilde{\mathscr{H}} \to \mathscr{H}_m$ by

$$J_m f := \sum_{x \in V_m} f(x) \psi_{x,m} \qquad \text{resp.} \qquad J'_m u(y) = \frac{1}{\mu_m(y)} \langle u, \psi_{y,m} \rangle_{\widetilde{\mathscr{H}}},$$

where $f \in \mathscr{H}_m$, $u \in \widetilde{\mathscr{H}}$ and $y \in V_m$. Moreover, we define $J^1_m \colon \mathscr{H}^1_m \to \widetilde{\mathscr{H}}^1$ by $J^1_m f := J_m f$. Note that this is well-defined because $\psi_{x,m} \in \operatorname{dom} \mathcal{E}$. The last operator $J'^1_m$ is chosen to be the evaluation in points of $V_m$, *i.e.*, $J'^1_m u(x) = u(x)$. Again, this choice makes sense because functions in the domain of $\mathcal{E}$ are continuous on $K$. That is because

$$|u(x) - u(y)|^2 \le \mathcal{E}(u) R(x,y), \tag{2.4}$$

for $x, y \in K$, where $R$ is the *resistance metric* associated with $\mathcal{E}$, given by

$$R(x,y) := \Big(\min\{\, \mathcal{E}(u) \,\big|\, u \in \operatorname{dom}\mathcal{E}, u(x) = 1 \text{ and } u(y) = 0 \,\}\Big)^{-1}$$

and since $R(x,y) = |x-y|$, the relative topology of $K$ coincides with the $R$-topology (see [6, Sec. 3.4] for a more general result).

Now, we need to verify the validity of the inequalities in Definition 1.4. This is done in [11, Sec. 4] in greater details but let us discuss the key steps here:

Applying the Cauchy-Young inequality in the first inequality of (1.5a) we see that $\|Jf\|_{\widetilde{\mathscr{H}}} \le \|f\|_{\mathscr{H}_m}$ for each $f \in \mathscr{H}_m$ and the second one is fulfilled because $J'_m = J^*_m$.

The inequalities in (1.5b) follow by applying the Cauchy-Schwarz inequality and by using the improved Hölder inequality (2.4). For the first one, we rewrite

$$f(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} f(x) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\widetilde{\mathscr{H}}}$$

using the fact that $\{\psi_{x,m}\}_{x \in V_m}$ is a partition of unity on $K$ and

$$J'_m J_m f(y) = \sum_{x \in V_m} f(x) J'_m \psi_{x,m}(y) = \frac{1}{\mu_m(y)} \sum_{x \in V_m} f(x) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\widetilde{\mathscr{H}}}.$$

Hence, by applying the above mentioned inequalities and some standard arguments, we can estimate $f - J'_m J_m f$ in norm.

Note, that the first inequality from (1.5c) is trivially fulfilled by the choice of $J_m$ and $J^1_m$. Instead of verifying the second one, we apply Lemma 1.7. This is particularly useful here because it helps us to skip a discussion about eigenvalues, we would otherwise have (see *e.g.* [12]).

The particular choice of the identification operators becomes clear now: The last inequality (1.5d) holds actually with equality because the $\{\psi_{x,m}\}$ minimise the energy, *i.e.* as above, $\mathcal{E}(\psi_{x,m}) = \mathcal{E}_m(\psi_{x,m}\restriction_{V_m}) = \mathcal{E}_m(\mathbb{1}_{\{x\}})$ where $\mathbb{1}_{\{x\}}$ is the characteristic function of the set $\{x\} \subset V_m$. Note that the letter expression can be computed explicitly using (2.2).

## 2.2. The Sierpiński gasket

A more illustrative example for a post-critically finite self-similar fractal is the Sierpiński gasket which is described by the family of contractions $F$, given by

$$F_j \colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad F_j(x) = \frac{1}{2}(x - p_j) + p_j \qquad (j = 1, 2, 3)$$

where the fixed points $p_j$ are chosen such that $\{p_1, p_2, p_3\}$ are the vertices of an equilateral triangle in $\mathbb{R}^2$. Then, as in the case of the unit interval, the Sierpiński gasket is defined as the unique non-empty compact subset $K$ of $\mathbb{R}^2$ that satisfies

$$K = F(K) := F_1(K) \cup F_2(K) \cup F_3(K).$$

Again, the family of contractions $\{F_1, F_2, F_3\}$ describes a cell structure on the Sierpiński gasket via the map $w \mapsto F_w(K)$ where $w \in W_m = \{1, 2, 3\}^m$. The vertex boundary is defined as $V_0 = \{p_1, p_2, p_3\}$. Note that in contrast to the situation in the example of the unit interval, $V_0$ does not coincide with the topological boundary of $K$ which is actually $K$ itself.

We define our approximating sequence of graphs in the same way as before: Let $G_0 = (V_0, E_0)$ be the complete graph and let $G_m = (V_m, E_m)$ be given by

$$V_m := \bigcup_{j=1}^{3} F_j(V_{m-1}) \qquad \text{and} \qquad E_m := \{\, e \,|\, e = x, y \subset V_m \text{ and } x \sim_m y \,\},$$

where we write $x \sim_m y$ if and only if $x$ and $y$ are two distinct vertices in $V_m$ and there exists a word $w \in W_m$ such that $x, y \in F_w(K)$. Moreover, we define an energy form on $G_m$ by

$$\mathcal{E}_m(f) = \sum_{x \sim_m y} \Big(\frac{5}{3}\Big)^m |f(x) - f(y)|^2,$$

for each $f \in \ell(V_m)$. Here we sum over all vertices $x \in V_m$ and their neighbours $y$ in the level $m$ graph and, as in the case of the interval, the conductances $c_{\{x,y\},m} = (5/3)^m$ are chosen such that the sequence of energy forms $\{\mathcal{E}_m\}_{m \in \mathbb{N}_0}$ is compatible.

Then the limit form exists and we define an energy form on the Sierpiński gasket by

$$\mathcal{E}(u) := \lim_{m \to \infty} \mathcal{E}_m(u\restriction_{V_m}), \qquad u \in \operatorname{dom} \mathcal{E} := \{\, u \,|\, u \colon K \to \mathbb{C}, \mathcal{E}(u) < \infty \,\}$$

(see [6, 17]).

As a (canonical) measure $\mu$ on the Sierpiński gasket we choose the homogeneous self-similar measure, *i.e.*, the uniquely determined probability measure

$\mu$ that satisfies

$$\mu = \frac{1}{3}\big(\mu \circ F_1^{-1} + \mu \circ F_2^{-1} + \mu \circ F_3^{-1}\big).$$

That is, $\mu$ is the Hausdorff measure of dimension $\log 3/\log 2$ and every $m$-cell $F_w(K)$ has measure $1/3^m$ (we would like to stress that our approach works for a general Borel regular probability measure on $K$; see [11] for details). As Hilbert space structure on the fractal $K$, we choose $\widetilde{\mathscr{H}} = \mathsf{L}_2(K,\mu)$. Then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a closed quadratic form in $\widetilde{\mathscr{H}}$.

On the graphs $G_m$ we define a measure as in (2.1) but here we choose the functions $\psi_{x,m} \colon K \to [0,1]$ to be the unique solution of the minimisation problem

$$\mathcal{E}_m(\mathbb{1}_{\{x\}}) = \min\big\{\, \mathcal{E}(u) \,\big|\, u \in \operatorname{dom}\mathcal{E}, u\restriction_{V_m} = \mathbb{1}_{\{x\}} \,\big\}.$$

These functions exist and are called $m$-harmonic functions with boundary values $\mathbb{1}_{\{x\}}$ on $V_m$ (cf. [6]). The values of $\psi_{x,m}$ can be computed explicitly by iteration: If the values in the vertices of $V_m$ are known, then, for each vertex $y \in V_{m+1} \setminus V_m$, there exists a unique $m$-cell that contains $y$; the value $\psi_{x,m}(y)$ is given by $1/5$ times the value at the vertex in $V_m$ opposite to $y$ in the $m$-cell plus $2/5$ times the values of $\psi_{x,m}$ at the vertices (of $V_m$) adjacent to $y$ in the same $m$-cell (cf. [17, Sec. 1.3]).

By the symmetry of the Sierpiński gasket and the functions $\{\psi_{x,m}\}_{x \in V_m}$, which define a partition of unity on $K$, we can specify $\mu_m$ also in this example as

$$\mu_m(x) = \int_K \psi_{x,m} \, \mathrm{d}\mu = \frac{1}{3^m} \begin{cases} 1/3 & x \in V_0 \\ 2/3 & x \in V_m \setminus V_0. \end{cases}$$

The Hilbert space, we consider on the approximating sequence of graphs is again given by $\mathscr{H}_m = \ell_2(V_m, \mu_m)$ and we conclude:

THEOREM 2.2 ([11]). *The energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\mathsf{L}_2(K,\mu)$ and the discrete energy form $\mathcal{E}_m$ in $\ell_2(V_m, \mu_m)$ are $\delta_m$-quasi-unitarily equivalent where the error is*

$$\delta_m = \frac{(1+\sqrt{3})\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{5^{m/2}}.$$

The idea of the proof is the same as described above in the case of the unit interval (see [11]).

## 3. NEUMANN OBSTACLES

In this section, we briefly present another class of examples. For details, we refer to [1]. Let $X$ be a complete Riemannian manifold of bounded geometry (*i.e.*, its Ricci curvature is bounded from below and its injectivity radius is bounded from below by a positive constant); a simple example is $X = \mathbb{R}^n$. We denote by $\mathsf{L}_2(X)$ the Hilbert space of square-integrable functions with respect to the standard volume measure, and by $\mathsf{H}^k(X)$ the Sobolev space of $k$-times weakly differentiable and square integrable functions. Denote its Laplacian by $\Delta_X \geq 0$ (defined via its quadratic form $\mathcal{E}_X(f) = \int_X |df|^2 \,\mathrm{d}\,\mathrm{vol}$). Under the above assumptions (completeness and bounded geometry) it can then be shown using the Bochner-Lichnerowicz-Weitzenböck formula that there is a constant $C_{\mathrm{ell.reg}} \geq 1$ such that

$$\|f\|_{\mathsf{H}^2(X)} \leq C_{\mathrm{ell.reg}} \|(\Delta_X + 1) f\|_{\mathsf{L}_2(X)} \tag{3.1}$$

for all $f \in \operatorname{dom} \Delta_X = \mathsf{H}^2(X)$, see *e.g.* [1, Proposition 3.2] and references therein.

We assume that $B \subset X$ is a closed subset such that the following holds:

(i) there is $\delta \geq 0$ such that

$$\|f\|_{\mathsf{L}_2(B)} \leq \delta \|f\|_{\mathsf{H}^1(X)}$$

for all $f \in \mathsf{H}^1(X)$;

(ii) there is a *bounded extension operator*, *i.e.*, there is $E \colon \mathsf{H}^1(X \setminus B) \longrightarrow \mathsf{H}^1(X)$ such that $Eu\restriction_{X\setminus B} = u$ with operator norm bounded by $C_{\mathrm{ext}} \geq 1$.

One can think of $B$ as the disjoint union of small balls or other obstacles. Denote by $\Delta^{\mathrm{N}}_{X\setminus B}$ the Neumann Laplacian defined via its quadratic form $\mathcal{E}^{\mathrm{N}}_{X\setminus B}(u) := \int_{X\setminus B} |du|^2 \,\mathrm{d}\,\mathrm{vol}$. It can be seen that the first estimate extends to

$$\|df\|_{\mathsf{L}_2(B)} \leq \delta \|f\|_{\mathsf{H}^2(X)}$$

for $f \in \mathsf{H}^2(X)$ without any assumption on the manifold (cf. [1, Proposition 3.7]). We have the following result:

Theorem 3.1 ([1, Theorem 4.3]). *Under the above assumptions, the Laplacian $\Delta_X$ and the Neumann Laplacian $\Delta^{\mathrm{N}}_{X\setminus B}$ are $C_{\mathrm{ext}} C_{\mathrm{ell.reg}} \delta$-quasi-unitarily equivalent.*

*Proof.* We are showing a slightly modified version of quasi-unitary equivalence for the corresponding energy forms. We first define the following identification operators as follows:

$$J \colon \mathscr{H} := \mathsf{L}_2(X) \longrightarrow \widetilde{\mathscr{H}} := \mathsf{L}_2(X \setminus B), \qquad f \mapsto f\restriction_{X\setminus B},$$

$J^1 \colon \mathscr{H}^1 := \mathsf{H}^1(X) \longrightarrow \widetilde{\mathscr{H}}^1 := \mathsf{H}^1(X \setminus B)$, $f \mapsto f\restriction_{X\setminus B}$, $J' = J^*$ (hence $J'u$ is the extension of $u$ by 0 onto $B$) and

$$J'^1 \colon \widetilde{\mathscr{H}} := \mathsf{H}^1(X \setminus B) \longrightarrow \mathscr{H}^1 = \mathsf{H}^1(X), \qquad u \mapsto Eu.$$

In particular, $J^1 f = Jf$ for $f \in \mathsf{H}^1(X)$, $JJ'u = u$, $\|J\| = 1$ and $J' = J^*$. It remains to check the first inequality of (1.5b), the second of (1.5c) and a modified version of (1.5d): the first estimate is fulfilled with $\delta$ since

$$\|f - J'Jf\|_{\mathsf{L}_2(X)} = \|f\|_{\mathsf{L}_2(B)} \le \delta \|f\|_{\mathsf{H}^1(X)}$$

by our assumption. For the second, we argue

$$\|J'^1 u - J'u\|_{\mathsf{L}_2(X)} = \|Eu\|_{\mathsf{L}_2(B)} \le \delta \|Eu\|_{\mathsf{H}^1(X)} \le C_{\mathrm{ext}} \delta \|u\|_{\mathsf{H}^1(X \setminus B)},$$

hence the estimate is fulfilled with $C_{\mathrm{ext}}\delta \ge \delta$. Instead of (1.5d), we show the slightly stronger estimate

$$\big|\widetilde{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| \le \delta \|(\Delta_X + 1) f\|_{\mathsf{L}_2(X)} \|u\|_{\widetilde{\mathcal{E}}} \qquad \text{for } (f \in \mathscr{H}^2 := \operatorname{dom} \Delta, u \in \widetilde{\mathscr{H}}^1), \tag{3.2}$$

*i.e.*, we use the operator graph norm instead of the energy norm for $f$. In particular, we have

$$\begin{aligned}
\big|\widetilde{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| &= \Big|\int_B \langle df, d(Eu)\rangle \,\mathrm{d}\,\mathrm{vol}\Big| \\
&\le \|df\|_{\mathsf{L}_2(B)} \|d(Eu)\|_{\mathsf{L}_2(B)} \le \|f\|_{\mathsf{H}^2(X)} \|Eu\|_{\mathsf{H}^1(X)} \\
&\le C_{\mathrm{ell.reg}} \|(\Delta_X + 1) f\|_{\mathsf{L}_2(X)} C_{\mathrm{ext}} \|u\|_{\mathsf{H}^1(X \setminus B)}
\end{aligned}$$

using the assumptions. The quasi-unitary equivalence for the operators follows then similarly as in Proposition 1.5. $\square$

Note that if $B = \bigcup_{x \in I_\varepsilon} B_\varepsilon(x)$ is the disjoint union of balls of radius $\varepsilon > 0$ with centres $x \in I_\varepsilon$ separated by $2\varepsilon^\alpha$ (*i.e.*, $x, y \in I_\varepsilon$ and $x \ne y$, implies $d(x,y) > 2\varepsilon^\alpha$) with $0 < \alpha < 1$, then one can show that the extension operator is *uniformly* bounded, *i.e.*, $C_{\mathrm{ext}}$ can be chosen to be independent of $\varepsilon$. Moreover, one can choose $\delta = \delta_\varepsilon$ to be of order $\varepsilon^{1-\alpha}$ in dimension $n \ge 3$ (resp. $\varepsilon^{1-\alpha} \log|\varepsilon|$ in dimension $n = 2$). Note that the sets $I_\varepsilon$ for different values of $\varepsilon$ may be totally unrelated, see [1, Sec. 4.2] for details.

## REFERENCES

[1] C. Anné and O. Post, *Wildly perturbed manifolds: norm resolvent and spectral convergence.* (2018). ArXiv:1802.01124.

[2] S. Bögli, *Local convergence of spectra and pseudospectra.* J. Spectr. Theory **8** (2018), 1051–1098.

[3] J. Brasche and R. Fulsche, *Approximation of eigenvalues of Schrödinger operators* (2017). arXiv:1712.03754.

[4] P. Exner and O. Post, *Approximation of quantum graph vertex couplings by scaled Schrödinger operators on thin branched manifolds.* J. Phys. A **42** (2009), 415305.

[5] T. Kato, *Perturbation theory for linear operators.* Springer-Verlag, Berlin, 1966.

[6] J. Kigami, *Analysis on fractals.* Cambridge Tracts in Math. **143**, Cambridge Univ. Press, Cambridge, 2001.

[7] A. Khrabustovskyi and O. Post, *Operator estimates for the crushed ice problem.* Asymptot. Anal. **110** (2018), 137–161.

[8] D. Mugnolo, R. Nittka and O. Post, *Norm convergence of sectorial operators on varying Hilbert spaces.* Oper. Matrices **7** (2013), 955–995.

[9] O. Post, *Spectral convergence of quasi-one-dimensional spaces.* Ann. Henri Poincaré **7** (2006), 933–973.

[10] O. Post, *Spectral analysis on graph-like spaces.* Lecture Notes in Math. **2039**, Springer, Heidelberg, 2012.

[11] O. Post and J. Simmer, *Approximation of fractals by discrete graphs: norm resolvent and spectral convergence.* Integral Equations Operator Theory **90** (2018), *6*, Paper No. 68.

[12] O. Post and J. Simmer, *Approximation of fractals by manifolds and other graph-like spaces.* (2018). ArXiv:1802.02998.

[13] O. Post and J. Simmer, *Approximation of magnetic Laplacians on the Sierpiński gasket by discrete magnetic Laplacians.* Preprint, 2018.

[14] O. Post and J. Simmer, *Convergence of finite dimensional approximations of magnetic Laplacians on finitely ramified fractals.* Preprint, 2019.

[15] O. Post and J. Simmer, *A distance of operators acting in different Hilbert spaces and operator convergence.* Preprint, 2019.

[16] M. Reed and B. Simon, *Methods of modern mathematical physics. I: Functional analysis.* Academic Press, New York, 1980.

[17] R.S. Strichartz, *Differential equations on fractals*, Princeton Univ. Press, Princeton, NJ, 2006.

[18] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil 1.* Mathematische Leitfäden., B. G. Teubner, Stuttgart, 2000, Grundlagen.

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