WEAK LIMIT THEOREM FOR A ONE-DIMENSIONAL SPLIT-STEP QUANTUM WALK

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This paper proves a weak limit theorem for a one-dimensional split-step quantum walk and investigates the limit density function. In the density function, the difference between two Konno's functions appears.

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1. INTRODUCTION

A large amount of work has been devoted to the study of discrete-time quantum walks, which are viewed as quantum counterparts of random walks (see [1, 20, 41] and references therein). One of the most interesting topics in quantum walks is a weak limit theorem, which was first proved by Konno [18, 19] for a homogeneous quantum walk on $\mathbb{Z}$ and which was extended to more general situations by many authors [11,14,23,25,26,34,37] (see also [39]). The weak limit theorem has also been proved for quantum walks on the half line [27], trees [3], joined half lines [4], higher-dimensional lattices [8,29,38,42], crystal lattices [12], and several graphs [13, 31]. A random environment case and temporally inhomogeneous case were studied in [21, 28, 30]. Recently, the weak limit theorem for a nonlinear quantum walk was established in [32, 33].

A spatially inhomogeneous discrete-time quantum walk on $\mathbb{Z}$ is described by a unitary evolution operator

$$\tilde{U} = \tilde{S}C \tag{1.1}$$

which is the product of a shift operator $\tilde{S}$ and a coin operator C on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$. Here the shift operator $\tilde{S}$ is defined as $\tilde{S} = L \oplus L^*$, where $\mathcal{H}$ is identified with $\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ and L is the left-shift on $\ell^2(\mathbb{Z})$. The coin operator C depends on a position $x \in \mathbb{Z}$ and is defined as the multiplication operator by a family of unitary matrices $\{C(x)\}_{x\in\mathbb{Z}} \subset U(2)$. In particular, in the case where $C(x) = C'$ $(x \in \mathbb{Z} \setminus \{0\})$ and $C(x) = C(0)$ $(x = 0)$ with

$C', C(0) \in U(2)$, the quantum walk is called a one-defect model, for which the weak limit theorem was proved in [6, 22]. In [40], the weak limit theorem was extended to the short-range case, where $C_0 := \lim_{|x|\to\infty} C(x)$ and

$$\|C(x) - C_0\| = O(|x|^{-1-\epsilon}). \tag{1.2}$$

It is clear that (1.2) covers the homogeneous case with $C(x) \equiv C_0$. In the case where $C(x) = C_+$ $(x \geq 0)$ and $C(x) = C_-$ $(x < 0)$ with C_+, $C_- \in U(2)$, the quantum walk is called a complete two-phase model, for which the weak limit theorem was proved in [7]. See also [5] for a two-phase model with one defect. An anisotropic quantum walk was introduced in [35], where $C(x)$ is assumed to satisfy

$$\|C(x) - C_\pm\| = O(|x|^{-1-\epsilon}) \quad \text{for } \pm x > 0.$$

This condition covers the two-phase models and allows us to prove the weak limit theorem [36].

In this paper, we consider a split-step quantum walk on $\mathbb{Z}$ introduced in previous papers [9,10], whose evolution operator is given by $U = SC$. The shift operator S of the split-step quantum walk is given by

$$S = \begin{pmatrix} p & qL \\ \bar{q}L^* & -p \end{pmatrix}, \tag{1.3}$$

where $p \in \mathbb{R}$ and $q \in \mathbb{C}$ satisfy $p^2 + |q|^2 = 1$. We suppose that $C(x)$ satisfies (1.2), prove the weak limit theorem, and calculate the limit density function. The difference between Konno's functions [18] appears in the limit density function. As put into evidence in [36, 40], Konno's function always appears in the limit density function for the one-dimensional quantum walk with the evolution $\tilde{U} = \tilde{S}C$. However, to the authors' best knowledge, this is the first work where the difference of Konno's functions appears.

This paper is organized as follows. In Section 2, we compare the split-step quantum walk with the other models. In Section 3, we present the weak limit theorem for the split-step quantum walk. We give the proof in the final section.

2. SPLIT-STEP QUANTUM WALK AND THE OTHER WALKS

As pointed out in [10], the split-step quantum walk unifies the spatially inhomogeneous quantum walk described by (1.1) and Kitagawa's quantum walk [17]. Indeed, if we take $p = 0$, then $S = \tilde{S}\sigma_1$ with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and hence U becomes the evolution of a spatially inhomogeneous quantum walk (1.1) with a different coin operator. If we take $p = \sin(\theta/2)$ and $q = \cos(\theta/2)$, then

$S = \sigma_1 S_- R(\theta) S_+$, where $S_+ = L^* \oplus 1$, $S_- = 1 \oplus L$, and $R(\theta)$ is a rotation matrix. Taking $C(x) = R(\theta')\sigma_1$, we get

$$U = \sigma_1 U_{\mathrm{ss}}(\theta', \theta)\sigma_1,$$

where $U_{\mathrm{ss}}(\theta', \theta) = S_- R(\theta) S_+ R(\theta')$ is the evolution of Kitagawa's split-step quantum walk [17]. Thus the split-step quantum walk is unitarily equivalent to Kitagawa's one. For more details, the reader can consult [10, Examples 2.1–2.2].

The split-step quantum walk is viewed as a lazy quantum walk, *i.e.*, at each time, the walker does not only move to the left and right but also stays at the same position. Let $\Psi_0 \in \mathcal{H}$ be the initial state and $\Psi_t = U^t \Psi_0$ the state of the walker at time $t = 0, 1, 2, \ldots$. Then the state evolution is given by

$$\Psi_{t+1}(x) = P(x+1)\Psi_t(x+1) + Q(x-1)\Psi_t(x-1) + R(x)\Psi_t(x), \quad x \in \mathbb{Z}$$

with some 2×2 matrices $P(x)$, $Q(x)$, and $R(x)$. The existence of the third term in the right-hand side indicates that the split-step quantum walk is lazy. Note that $R(x) \equiv 0$ if and only if $p = 0$. In [14, 24], a lazy quantum walk on $\mathbb{Z}$ is defined as a three-state quantum walk on the Hilbert space $\ell^2(\mathbb{Z}; \mathbb{C}^3)$, whereas the split-step quantum walk is defined as a two-state quantum walk on $\mathcal{H} = \ell^2(\mathbb{Z}; \mathbb{C}^2)$.

3. WEAK LIMIT THEOREM

In quantum walks, the position X_t of a walker at time t with an initial state $\Psi_0 \in \mathcal{H}$ ($\|\Psi_0\| = 1$) is a random variable with the distribution

$$P(X_t = x) = \|\Psi_t(x)\|^2, \quad x \in \mathbb{Z},$$

where $\Psi_t := U^t \Psi_0$ is the state of the walker at time t with the evolution $U = SC$ and the shift operator S defined in (1.3). As shown in [36, 40] for the short-range cases, X_t/t converges in law to a random variable V as $t \to \infty$. This assertion is called the weak limit theorem. To show the weak limit theorem, we suppose the following.

(A.1) There exists a matrix $C_0 \in U(2)$ such that

$$\|C(x) - C_0\| \le \kappa |x|^{-1-\epsilon}, \quad x \in \mathbb{Z} \setminus \{0\}$$

with some $\kappa, \epsilon > 0$ independent of x.

For the simplicity of the presentation, the following simplifying conditions are assumed.

(A.2) $p, q > 0, \quad C_0 = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a, b > 0.$

Let $U_0 = SC_0$ be a homogeneous evolution, whose coin operator is the limit of $C(x)$ at spatial infinity. The assumption (A.1) ensures that the wave operator $W = \text{s-}\lim_{t\to\infty} U^{-t}U_0^t$ exists and $W^* = \text{s-}\lim_{t\to\infty} U_0^{-t}U^t\Pi_{\mathrm{ac}}(U)$. Here $\Pi_{\mathrm{ac}}(U)$ is the projection onto the absolutely continuous spectral subspace of U. Indeed, this can be proved by a discrete analog of the Kato-Rosenblum Theorem [40, Proposition 3.1] (see also [15,16]), because (A.1) guarantees that $U - U_0$ is trace class. Moreover, (A.1) allows us to construct a conjugate operator A for U such that U and A satisfy a Mourre estimate in a way similar to [35, Proposition 4.5, Lemma 4.9, and Proposition 4.11] (see also [2]). This proves that U has no singular continuous spectrum. Let $\Pi_{\mathrm{p}}(U)$ be the projection onto the direct sum of all eigenspaces of U and $f_K(v;r)$ be Konno's function defined as

$$f_K(v;r) = \frac{\sqrt{1-r^2}}{\pi(1-v^2)\sqrt{r^2-v^2}} I_{(-r,r)}(v), \quad v \in \mathbb{R},\ 0<r<1,$$

where I_A is the characteristic function of a set $A \subset \mathbb{R}$. We use F to denote the Fourier transform, which maps $\mathcal{H}$ to $\mathcal{K} = L^2(\mathbb{T};\mathbb{C}^2;dk/(2\pi))$ with $\mathbb{T} = [0,2\pi)$. Let

$$\hat{U}_0(k) = \begin{pmatrix} p & qe^{ik} \\ qe^{-ik} & -p \end{pmatrix} C_0, \quad k \in \mathbb{T} \tag{3.1}$$

and use $u_j(k)$ to denote its normalized eigenvectors corresponding to eigenvalues

$$\lambda_j(k) = \exp\left((-1)^{j+1} i \arccos \tau(k)\right), \quad j = 1,2,$$

where $\tau(k) = pa + qb\cos k$ ($k \in \mathbb{T}$). We are now in a position to state our main result.

Theorem 3.1. *Let X_t be as above. Then X_t/t converges in law to a random variable V, whose distribution is given by*

$$\mu_V(\mathrm{d}v) = w_0\delta_0(\mathrm{d}v) + w_+(v)f_+(v)\mathrm{d}v + w_-(v)f_-(v)\mathrm{d}v,$$

where $w_0 = \|\Pi_{\mathrm{p}}(U)\Psi_0\|^2$ is a nonnegative constant, and $f_\pm(v)$ and $w_\pm(v)$ are nonnegative functions given by

$$f_\pm(v) = \frac{|f_K(v;q) \mp f_K(v;b)|}{2} I_{(-q,q)\cap(-b,b)}(v), \tag{3.2}$$

$$w_\pm(v) = \begin{cases} w_1(2\pi - \arccos g_\pm(v)) + w_2(\arccos g_\pm(v)), & v \geq 0, \\ w_1(\arccos g_\pm(v)) + w_2(2\pi - \arccos g_\pm(v)), & v < 0 \end{cases} \tag{3.3}$$

with

$$g_\pm(v) = \frac{pav^2 \pm \sqrt{(q^2-v^2)(b^2-v^2)}}{qb(1-v^2)} \tag{3.4}$$

and

$$w_j(k) = \langle u_j(k), (FW^*\Psi_0)(k)\rangle \ (j=1,2). \tag{3.5}$$

This theorem says that if $b \neq q$, we have the difference between two Konno's functions $f_K(v;q)$ and $f_K(v;b)$ in the density function. To the best of the authors' knowledge, this is the first work where such a difference appears.

4. PROOF OF THE WEAK LIMIT THEOREM

Throughout this section, we assume (A.1) and (A.2), and prove Theorem 3.1. The proof proceeds along the same lines as the proof of the weak limit theorem in [36, 40]. A key ingredient is that the Heisenberg operator $\hat{x}(t) = U^{-t}\hat{x}U^t$ for the position operator $\hat{x}$ divided by t converges to the asymptotic velocity operator $\hat{v}$ for U, *i.e.*,

$$\text{s-}\lim_{t\to\infty} e^{i\xi\hat{x}(t)/t} = \Pi_{\mathrm{p}}(U) + e^{i\xi\hat{v}}\Pi_{\mathrm{ac}}(U), \quad \xi \in \mathbb{R}. \tag{4.1}$$

This gives the limit distribution in terms of the projection $\Pi_{\mathrm{p}}(U)$ and the spectral measure of $\hat{v}$. A direct calculation of the spectral measure yields the density function of the limit distribution.

Following the above procedure, we first define the velocity operator $\hat{v}$. We remark that the Fourier transform $F : \mathcal{H} \to \mathcal{K}$ is unitary and satisfies

$$(F\Psi)(k) = \sum_{x\in\mathbb{Z}} e^{-ikx}\Psi(x), \quad k \in \mathbb{T}$$

for all $\Psi \in \mathcal{H}$ with a finite support. The Fourier transform FU_0F^* of U_0 is the multiplication operator on $\mathcal{K}$ by $\hat{U}_0(k)$ defined in (3.1). The velocity operator $\hat{v}_0$ for the homogeneous evolution U_0 is defined so that the Fourier transform $F\hat{v}_0F^*$ is given by the multiplication operator on $\mathcal{K}$ by the 2×2 hermitian matrix

$$(F\hat{v}_0F^*)(k) = \sum_{j=1,2} v_j(k)|u_j(k)\rangle\langle u_j(k)|, \quad k \in \mathbb{T} \tag{4.2}$$

with

$$v_j(k) := \frac{i\lambda_j'(k)}{\lambda_j(k)} = \frac{(-1)^{j+1}}{\sqrt{1-\tau^2(k)}}\frac{\mathrm{d}\tau}{\mathrm{d}k}. \tag{4.3}$$

The velocity operator $\hat{v}$ for the inhomogeneous evolution U is defined as $\hat{v} = W\hat{v}_0W^*$. We use $E_{\hat{v}}(\cdot)$ to denote the spectral measure of $\hat{v}$. The following proposition can be proved by (4.1) in a way similar to [40, Corollary 2.4].

PROPOSITION 4.1. *Suppose (A.1) and (A.2). Let V be a random variable with the distribution*

$$\mu_V(\mathrm{d}v) = \|\Pi_{\mathrm{p}}(U)\Psi_0\|^2\delta_0(\mathrm{d}v) + \|E_{\hat{v}}(\mathrm{d}v)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2.$$

Then, X_t/t converges to V in law as $t \to \infty$.

It suffices from Proposition 4.1 to calculate the density function of the continuous part $\|E_{\hat{v}}(\mathrm{d}v)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2$. To this end, we calculate the Fourier transform of $\|E_{\hat{v}}(\cdot)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2$. Because $E_{\hat{v}}(v)\Pi_{\mathrm{ac}}(U) = WE_{\hat{v}_0}(v)W^*$ and $W : \mathcal{H} \to \mathrm{Ran}\Pi_{\mathrm{ac}}(U)$ is unitary, $\|E_{\hat{v}}(\cdot)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2 = \|E_{\hat{v}_0}(\cdot)W^*\Psi_0\|^2$. Combining this with (3.5) and (4.2), we have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{i\xi v}\mathrm{d}\|E_{\hat{v}}(v)\Pi_{\mathrm{ac}}(U)\Psi_0\|^2 &= \langle W^*\Psi_0, e^{i\xi\hat{v}_0}W^*\Psi_0\rangle \\ &= \int_{\mathbb{T}} \frac{\mathrm{d}k}{2\pi}\left(e^{i\xi v_1(k)}w_1(k) + e^{-i\xi v_1(k)}w_2(k)\right),\end{aligned} \tag{4.4}$$

where we have used $v_1(k) = -v_2(k)$ in the last equation. In what follows, we make the substitutions $v = v_1(k)$ in RHS of (4.4). To do so, we calculate the inverse function of $v = v_1(k)$ and the Jacobian $\frac{1}{2\pi}\frac{\mathrm{d}k}{\mathrm{d}v}$. By (4.3), we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}k} = \frac{ap(\tau - \frac{a}{p})(\tau - \frac{p}{a})}{(1-\tau^2)^{\frac{3}{2}}}.$$

For the moment, we assume $p > a$. Because $\frac{a}{p} < 1 < \frac{p}{a}, \tau < 1$ and $0 < \frac{aq}{bp} < 1$, the condition $\tau = \frac{a}{p}$ is equivalent to the following condition:

$$k = \arccos\frac{aq}{bp} \quad \text{or} \quad k = 2\pi - \arccos\frac{aq}{bp},$$

where $0 < \arccos\frac{aq}{bp} < \pi$. By these facts, we have

$$\begin{cases} \frac{\mathrm{d}v}{\mathrm{d}k} < 0, & k \in [0, \arccos\frac{aq}{bp}) \cup (2\pi - \arccos\frac{aq}{bp}, 2\pi), \\ \frac{\mathrm{d}v}{\mathrm{d}k} > 0, & k \in (\arccos\frac{aq}{bp}, 2\pi - \arccos\frac{aq}{bp}). \end{cases}$$

Therefore, the function $k \mapsto v$ is injective on each domain of $[0, \arccos\frac{aq}{bp})$, $[\arccos\frac{aq}{bp}, \pi), [\pi, 2\pi - \arccos\frac{aq}{bp})$ and $[2\pi - \arccos\frac{aq}{bp}, 2\pi)$. Observe that $v\left([0, \arccos\frac{aq}{bp})\right) = v\left([\arccos\frac{aq}{bp}, \pi)\right) = [-q, 0]$ and $v\left([\pi, 2\pi - \arccos\frac{aq}{bp})\right) = v\left([2\pi - \arccos\frac{aq}{bp}, 2\pi)\right) = [0, q]$. Because $v^2 = (\tau')^2\left(1-\tau^2\right)^{-1}$, we know that $\tau(k) = \frac{pa \pm\sqrt{(q^2-v^2)(b^2-v^2)}}{1-v^2}$. Since $\tau(k) = pa + qb\cos k$, we obtain

$$k = \begin{cases} \arccos g_+(v), & k \in [0,\ \arccos\frac{aq}{pb}), \\ \arccos g_-(v), & k \in [\arccos\frac{aq}{pb},\ \pi), \\ 2\pi - \arccos g_-(v), & k \in [\pi,\ 2\pi - \arccos\frac{aq}{pb}), \\ 2\pi - \arccos g_+(v), & k \in [2\pi - \arccos\frac{aq}{pb},\ 2\pi), \end{cases}$$

where $g_\pm(v)$ has been defined in (3.4). By direct calculation, we have

$$\frac{\mathrm{d}}{\mathrm{d}v}\arccos g_\pm(v) = \pm 2\pi\,\mathrm{sgn}(v) f_\pm(v),$$

where $f_\pm(v)$ has been defined in (3.2). Hence,

$$\text{(4.5)} \quad \frac{1}{2\pi}\frac{\mathrm{d}k}{\mathrm{d}v} = \begin{cases} -f_+(v), & k \in [0,\ \arccos\frac{aq}{pb}) \cup [2\pi - \arccos\frac{aq}{pb},\ 2\pi), \\ f_-(v), & k \in [\arccos\frac{aq}{pb},\ 2\pi - \arccos\frac{aq}{pb}). \end{cases}$$

Substituting $v = v_1(k)$ with (4.5), we have

$$\text{RHS of (4.4)} = \sum_{\#\in\{+,-\}} \int_{-q}^{0} \mathrm{d}v f_\#(v) \left\{e^{i\xi v} w_1(\arccos g_\#(v)) + e^{-i\xi v} w_2(\arccos g_\#(v))\right\}$$

$$+ \sum_{\#\in\{+,-\}} \int_{0}^{q} \mathrm{d}v f_\#(v) \left\{e^{i\xi v} w_1(2\pi - \arccos g_\#(v)) + e^{-i\xi v} w_2(2\pi - \arccos g_\#(v))\right\}$$

$$= \int_{-\infty}^{\infty} e^{i\xi v} \left(f_+(v)w_+(v) + f_-(v)w_-(v)\right) \mathrm{d}v,$$

where $w_\pm$ has been defined in (3.3). This completes the proof for the case of $p > a$. The same proof works for $p < a$. In the case of $p = a$, the proof is immediate, because $\mathrm{d}v/\mathrm{d}k = ap(1-\tau)^2/(1+\tau)^{\frac{3}{2}} \geq 0$, *i.e.*, $v = v_1(k)$ is monotonically increasing.

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