ON THE HARTLE-HAWKING-ISRAEL STATES FOR SPACETIMES WITH STATIC BIFURCATE KILLING HORIZONS

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We revisit the construction by K. Sanders [19] of the Hartle-Hawking-Israel state for a free quantum Klein-Gordon field on a spacetime with a static, bifurcate Killing horizon and a wedge reflection. Using the notion of the Calderón projector for elliptic boundary value problems and pseudodifferential calculus on manifolds, we give a short proof of its Hadamard property.

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1. INTRODUCTION

Quantum Field Theory on curved spacetimes describes quantum fields in an external gravitational field, represented by the Lorentzian metric on the ambient spacetime. It is used in situations when both the quantum nature of the fields and the effect of gravitation are important, but the quantum nature of gravity can be neglected in first approximation. Its most important areas of application are the study of phenomena occurring in the early universe and in the vicinity of black holes, and its most celebrated result is the discovery by Hawking [5] that quantum particles are created near the horizon of a black hole.

The symmetries of the Minkowski spacetime, which play such a fundamental role, are absent in curved spacetimes, except in some simple situations, like stationary or static spacetimes. Therefore the traditional approach to quantum field theory has to be modified: one has first to perform an algebraic quantization, which for free theories amounts to introduce an appropriate phase space, which is either a symplectic or an Euclidean space, in the bosonic or fermionic case. From such a phase space one can construct CCR or CAR $*$-algebras,

The second step consists in singling out, among the many states on these $*$-algebras, the physically meaningful ones, which should resemble the Minkowski vacuum, at least in the vicinity of any point of the spacetime. This leads to the notion of *Hadamard states*, which are substitutes for the *vacuum state*, which plays a fundamental role in Quantum Field Theory on the Minkowski spacetime.

Hadamard states were originally defined by requiring that their two-point functions have a specific asymptotic expansion near the diagonal, called the *Hadamard expansion*. A very important progress was made by Radzikowski [18] who introduced the characterization of Hadamard states by the wavefront set of their two-point functions.

Hadamard states are nowadays widely accepted as possible physical states of non-interacting quantum fields on a curved spacetime. One of the main reasons for the importance of Hadamard states is their applicability to the renormalization of the stress-energy tensor, a necessary step in the formulation of semi-classical Einstein equations. Moreover, the Hadamard condition plays an essential role in the perturbative construction of interacting quantum field theory [2].

1.1. Black hole spacetimes

One of the early successes of QFT on curved spacetimes was Hawking's discovery [5] of black hole radiation, produced by a spherically symmetric star collapsing to a black hole. A related line of research was initiated by Hartle and Hawking [6] and Israel [8], who conjectured the existence of a 'ground state' for a Klein-Gordon field propagating in a spacetime containing a static black hole. Let us now describe in more details the precise geometrical framework.

One considers a globally hyperbolic spacetime (M, g), with a *bifurcate Killing horizon*, see [13, 19] or Subsection 2.1 for precise definition. The bifurcate Killing horizon $\mathscr{H}$ is generated by the *bifurcation surface* $\mathscr{B} = \{x \in M : V(x) = 0\}$, where V is the Killing vector field. It allows to split (M, g) into four globally hyperbolic regions, the *right/left wedges* $\mathscr{M}^+$, $\mathscr{M}^-$ and the *future/past cones* $\mathscr{F}$, $\mathscr{P}$, each invariant under the flow of V. An important object related with the Killing horizon $\mathscr{H}$ is its *surface gravity* κ, which is a scalar, constant over all of $\mathscr{H}$.

Let us consider on (M, g) a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\Box_g \phi(x) + m(x)\phi(x) = 0,$$

where $m \in C^\infty(M, \mathbb{R})$, $m(x) > 0$ is invariant under V, and its associated free field algebra.

If V is *time-like* in $(\mathscr{M}^+, g)$, *i.e.* if $(\mathscr{M}^+, g, V)$ is a stationary spacetime, there exists (see [20]) for any $\beta > 0$ a *thermal state* ω_β^+ at temperature β^{-1} with respect to the group of Killing isometries of $(\mathscr{M}^+, g)$ generated by V.

It was conjectured by Hartle and Hawking [6] and Israel [8] that if $\beta = 2\pi\kappa^{-1}$ is the *inverse Hawking temperature*, denoted by $\beta_{\rm H}$ in the sequel, then ω_β^+ can be extended to the whole of M as a pure state, invariant under V, the *Hartle-Hawking-Israel state*, denoted in the sequel by $\omega_{\rm HHI}$.

The rigorous construction of the HHI state was first addressed by Kay in [12], who constructed the HHI state in the Schwarzschild double wedge of the Kruskal spacetime. In such a double wedge, the HHI state is a *double KMS state*, see [10, 11]. Later Kay and Wald [13] considered the more general case of spacetimes with a bifurcate Killing horizon, and study general properties of stationary states on this class of spacetimes. They emphasized in particular the importance of the *Hadamard condition*. They proved that a specific sub-algebra of the free field algebra has at most one state invariant under V and Hadamard. They also showed that if M admits a *wedge reflection* (see Subsection 2.2) the restriction of such a state to $\mathscr{M}^+$ will necessarily be a $\beta_{\rm H}-$KMS state. These results were later improved in [9].

The existence of such a state, *i.e.* of the HHI state, was however not proved in [6]. The first proof of the existence of $\omega_{\rm HHI}$ was given by Sanders in the remarkable paper [19], if the bifurcate Killing horizon is static, *i.e.* if V is static in $\mathscr{M}^+$, assuming also the existence of a wedge reflection. Sanders showed that there exists a unique Hadamard state $\omega_{\rm HHI}$ on M extending the *double $\beta_{\rm H}-$KMS state ω_β* on $\mathscr{M}^+ \cup \mathscr{M}^-$. The double β_H-KMS state ω_β is a pure state on $\mathscr{M}^+ \cup \mathscr{M}^-$ which is the natural extension of ω_β^+ defined using the wedge reflection, see [10, 11]. It is an exact geometrical analog of the Fock vacuum vector in the Araki-Woods representation of a thermal state.

1.2. **Content of the paper**

In this paper, we revisit the construction in [19] of the Hartle-Hawking-Israel state in a spacetime with a static bifurcate Killing horizon. Using the notion of the *Calderón projector* (see Section 5), which is a standard tool in elliptic boundary value problems, we significantly shorten the proof of the Hadamard property of $\omega_{\rm HHI}$.

In [19] the fact that $\omega_{\rm HHI}$ is Hadamard was proved by a careful comparison of the Hadamard parametrix construction for the D'Alembertian $-\Box_g + m$ associated to the Lorentzian metric g and for the Laplacian $-\Delta_{\hat g} + m$ associated to the Riemannian metric $\hat g$ obtained from g by Wick rotation in the Killing time coordinate.

In our paper, we avoid working with the spacetime covariances of states and instead systematically work with the *Cauchy surface covariances* (see

Subsection 3.3) associated with a Cauchy surface Σ containing the bifurcation surface $\mathscr{B}$.

It turns out that the Cauchy surface covariances $\lambda^{\pm}$ of the double $\beta-$KMS state ω_{β} are related to a Calderón projector D.

Let us informally recall what is the Calderón projector associated to a elliptic boundary value problem, see Section 5 for more details:

let $(N, \hat{g})$ be a complete Riemannian manifold and $P = -\Delta_{\hat{g}} + m(x)$ for $m \in C^{\infty}(N)$, $m(x) > 0$ a Laplace-Beltrami operator. Let also $\Omega \subset M$ a smooth open set. To Ω is naturally associated the canonical surface density dS, defined by $\langle dS|u\rangle = \int_{\partial\Omega} u d\sigma$, for $u \in C_{\rm c}^{\infty}(M)$, where $d\sigma$ is the induced surface element on $\partial\Omega$.

If ∂_{ν} is the external normal derivative to $\partial\Omega$ and $\gamma u = \begin{pmatrix} u\restriction_{\partial\Omega} \\ \partial_{\nu}u\restriction_{\partial\Omega}\end{pmatrix}$ for $u \in C^{\infty}(\overline{\Omega})$ the Calderón projector D is a map from $C_{\rm c}^{\infty}(\partial\Omega)\otimes\mathbb{C}^2$ to $C^{\infty}(\partial\Omega)\otimes\mathbb{C}^2$ defined by:

$$Df := \gamma \circ G(f_1(dVol_{\hat{g}})^{-1}dS - f_0(dVol_{\hat{g}}^{-1})\partial_{\nu}^{*}dS),\ f = \begin{pmatrix} f_0 \\ f_1\end{pmatrix} \in C_{\rm c}^{\infty}(\partial\Omega)\otimes\mathbb{C}^2,$$

where $G = P^{-1}$. It is easy to see that $f \in C^{\infty}(\Sigma)\otimes\mathbb{C}^2$ equals γu for some $u \in C^{\infty}(\overline{\Omega})$ solution of $Pu = 0$ in Ω if and only if $Df = f$.

In our case we take $N = \mathbb{S}_{\beta}\times\Sigma^{+}$, where $\mathbb{S}_{\beta}$ is the circle of length β and $\Sigma^{+} = \Sigma\cap\mathscr{M}^{+}$ is the right part of the Cauchy surface Σ. The Riemannian metric is $\hat{g} = v^2(y)d\tau^2 + h_{ij}(y)dy^i dy^j$, obtained by the *Wick rotation* $t =: {\rm i}\tau$ of the Lorentzian metric $g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j$ on $\mathscr{M}^{+}\sim\mathbb{R}\times\Sigma^{+}$ where $\mathscr{M}^{+}$ is identified to $\mathbb{R}\times\Sigma^{+}$ using the Killing time coordinate t.

The existence of an extension of $\omega_{\beta_{\rm H}}$ to M is then an almost immediate consequence of the fact that $(N, \hat{g})$ admits a smooth extension $(N_{\rm ext}, \hat{g}_{\rm ext})$ if and only if $\beta = \beta_{\rm H}$, a well-known result which plays also a role in [19].

In fact this geometrical fact implies that D, viewed as an operator defined on $C_{\rm c}^{\infty}(\Sigma\backslash\mathscr{B})\otimes\mathbb{C}^2$ uniquely extends to a Calderon projector $D_{\rm ext}$, defined on $C_{\rm c}^{\infty}(\Sigma)\otimes\mathbb{C}^2$. From $D_{\rm ext}$ one can then easily obtain a pure quasi-free state $\omega_{\rm HHI}$ on the whole of M.

The Hadamard property of $\omega_{\rm HHI}$ follows then from the well-known fact that $D_{\rm ext}$, being a Calderón projector, is a 2×2 matrix of pseudodifferential operators on Σ, and of the Hadamard property of ω_{β} in $\mathscr{M}^{+}\cup\mathscr{M}^{-}$.

Beside shortening the proof of the Hadamard property of $\omega_{\rm HHI}$, we think that our paper illustrates the usefulness of pseudodifferential calculus for the construction and study of Hadamard states, see also [14–17] for other applications. We believe that Calderón projectors could also be used to construct the Hartle-Hawking-Israel state in the still open case of spacetimes with a Killing horizon that is only stationary.

1.3. Plan of the paper

Let us now briefly give the plan of the paper. In Section 2, we recall the notion of a static bifurcate Killing horizon, following [19] and introduce the associated Klein-Gordon equation.

Section 3 is devoted to background material on CCR^*-algebras, bosonic quasi-free states and their spacetime and Cauchy surface covariances in the case of quantum Klein-Gordon fields. We use the framework of *charged fields*, which is in our opinion more elegant, even when considering only neutral field equations. We also recall the notion of *pseudodifferential operators* on a manifold, which will be useful later on and formulate a consequence of [15] which states that the Cauchy surface covariances of *any* Hadamard state for Klein-Gordon fields is given by a matrix of pseudodifferential operators.

In Section 4, we define various 'Euclidean' Laplacians, $K = -\Delta_{\hat{g}} + m$ acting on $N = \mathbb{S}_\beta \times \Sigma^+$ and a related operator $\tilde{K}$, obtained from Wick rotation of the Lorentzian metric on M in the Killing time coordinate, which are considered in [19]. It is sufficient for us to define these Laplacians by quadratic form techniques, which simplifies some arguments.

In Section 5, we recall the definition of the *Calderón projector*, which is a standard notion in elliptic boundary value problems. In Section 6, using the explicit expression for $\tilde{K}^{-1}$, we show that the projection associated to the double $\beta-$KMS state ω_β equals to the Calderón projector D associated to K and the open set $\Omega =]0, \beta/2[\times\Sigma^+$.

In Section 7, we recall the well-known fact that a smooth extension ($N_{\rm ext}$, $\hat{g}_{\rm ext}$) of $(N, \hat{g})$ exists iff $\beta = \beta_{\rm H}$. The extended Calderón projector $D_{\rm ext}$ generates a pure state on M, which is the Hartle-Hawking-Israel state $\omega_{\rm HHI}$. In Prop. 7.4, we show that such an extension is unique among quasi-free states whose spacetime covariances map $C^\infty_{\rm c}(M)$ into $C^\infty(M)$ continuously. Finally, we give in the proof of Thm. 7.5 a new and elementary proof of the Hadamard property of $\omega_{\rm HHI}$, using the pseudodifferential calculus on Σ.

2. SPACETIMES WITH A STATIC BIFURCATE KILLING HORIZON

2.1. Static bifurcate Killing horizons

We consider as in [19] a globally hyperbolic spacetime (M, g) with a *static bifurcate Killing horizon*. We recall, see [19, Def. 2.2], that this is a triple (M, g, V), such that

(1) the Lorentzian manifold (M, g) is globally hyperbolic,

(2) V is a complete Killing vector field for (M, g),
(3) $\mathscr{B} := \{x \in M : V(x) = 0\}$ is a compact, orientable submanifold of codimension 2,
(4) there exists a Cauchy hypersurface Σ containing $\mathscr{B}$,
(5) V is $g-$orthogonal to Σ,

see Fig. 1 below where the vector field V is represented by arrows.

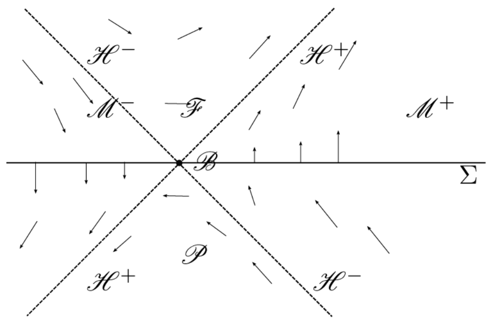

Fig. 1

For simplicity we will also assume that the bifurcation surface $\mathscr{B}$ is connected. Denoting by n the future pointing normal vector field to Σ one introduces the *lapse function*:

$$v(x) := -n(x)\cdot g(x)V(x), \ x \in \Sigma, \tag{2.1}$$

and Σ decomposes as

$$\Sigma = \Sigma^- \cup \mathscr{B} \cup \Sigma^+,$$

where $\Sigma^\pm = \{x \in \Sigma : \pm v(x) > 0\}$. The spacetime M splits as

$$M = \mathscr{M}^+ \cup \mathscr{M}^- \cup \overline{\mathscr{F}} \cup \overline{\mathscr{P}},$$

where the future cone $\mathscr{F} := I^+(\mathscr{B})$, the past cone $\mathscr{P} := I^-(\mathscr{B})$, the right/left wedges $\mathscr{M}^\pm := D(\Sigma^\pm)$, are all globally hyperbolic when equipped with g.

2.2. Wedge reflection

Additionally one has to assume the existence of a *wedge reflection*, see [19, Def. 2.6], *i.e.* a diffeomorphism R of $\mathscr{M}^+ \cup \mathscr{M}^- \cup U$ onto itself, where U is an open neighborhood of $\mathscr{B}$ such that:

(1) $R \circ R = \mathrm{Id}$,
(2) R is an isometry of $(\mathscr{M}^+ \cup \mathscr{M}^-, g)$ onto itself, which reverses the time orientation,

(3) $R = \mathrm{Id}$ on $\mathscr{B}$,
(4) $R^*V = V$ on $\mathscr{M}^+ \cup \mathscr{M}^-$.

It follows that R preserves Σ, see [19, Prop. 2.7], and we denote by r the restriction of R to Σ. Denoting by h the induced Riemannian metric on Σ one has:

$$r^*h = h, r^*v = -v. \tag{2.2}$$

2.3. Killing time coordinate

Denoting by $\Phi_s^V : M \to M$ the flow of the Killing vector field V, we obtain a diffeomorphism

$$\chi : \mathbb{R} \times (\Sigma \backslash \mathscr{B}) \ni (t, y) \mapsto \Phi_t^V(y) \in \mathscr{M}^+ \cup \mathscr{M}^-,$$

which defines the coordinate t on $\mathscr{M}^+ \cup \mathscr{M}^-$ called the *Killing time coordinate.* The metric g on $\mathscr{M}^+ \cup \mathscr{M}^-$ pulled back by χ takes the form (see [19, Subsection 2.1]):

$$g = -v^2(y)dt^2 + h_{ij}(y)dy^i dy^j, \tag{2.3}$$

where the Riemannian metric $h_{ij}(y)dy^i dy^j$ is the restriction of g to Σ.

2.4. Klein-Gordon operator

We fix a real potential $m \in C^\infty(M)$. As in [19] we assume that m is stationary w.r.t. the Killing vector field V and invariant under the wedge reflection, *i.e.*:

$$V^a \nabla_a m(x) = 0, \ m \circ R(x) = m(x), \ x \in \mathscr{M}^+ \cup \mathscr{M}^- \cup U. \tag{2.4}$$

For simplicity we also assume that

$$m(x) \geq m_0^2 > 0, \ x \in M, \tag{2.5}$$

i.e. we consider only massive fields. Note that in [19] the weaker condition $m(x) > 0$ was assumed. We consider the Klein-Gordon operator

$$P = -\Box_g + m. \tag{2.6}$$

3. FREE KLEIN-GORDON FIELDS

In this section, we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [15, Section 2] based on *charged fields.*

3.1. **Charged CCR algebra**

3.1.1. CHARGED BOSONIC FIELDS

Let $\mathcal{Y}$ a complex vector space, $\mathcal{Y}^*$ its anti-dual. Sesquilinear forms on $\mathcal{Y}$ are identified with elements of $L(\mathcal{Y},\mathcal{Y}^*)$ and the action of a sesquilinear form β is correspondingly denoted by $\overline{y}_1\cdot\beta y_2$ for $y_1,y_2\in\mathcal{Y}$. We fix $q\in L_{\rm h}(\mathcal{Y},\mathcal{Y}^*)$ a non degenerate hermitian form on $\mathcal{Y}$, *i.e.* such that ${\rm Ker}\, q=\{0\}$.

The *CCR* $^*-$*algebra* ${\rm CCR}(\mathcal{Y},q)$ is the complex $^*-$algebra generated by symbols $\mathbf{1},\psi(y),\psi^*(y),y\in\mathcal{Y}$ and the relations:

$$\begin{array}{l}\psi(y_1+\lambda y_2)=\psi(y_1)+\overline{\lambda}\psi(y_2),\ y_1,y_2\in\mathcal{Y},\lambda\in\mathbb{C},\\ \psi^*(y_1+\lambda y_2)=\psi^*(y_1)+\lambda\psi^*(y_2),\ y_1,y_2\in\mathcal{Y},\lambda\in\mathbb{C},\\ [\psi(y_1),\psi(y_2)]=[\psi^*(y_1),\psi^*(y_2)]=0,\ [\psi(y_1),\psi^*(y_2)]=\overline{y}_1\cdot qy_2\mathbf{1},\ y_1,y_2\in\mathcal{Y},\\ \psi(y)^*=\psi^*(y),\ y\in\mathcal{Y}.\end{array}$$

A state ω on ${\rm CCR}(\mathcal{Y},q)$ is *(gauge invariant) quasi-free* if

$$\omega(\prod_{i=1}^{p}\psi(y_i)\prod_{i=1}^{q}\psi^*(y_j))=\left\{\begin{array}{l}0 \text{ if } p\neq q,\\ \sum_{\sigma\in S_p}\prod_{i=1}^{p}\omega(\psi(y_i)\psi^*(y_{\sigma(i)}))\text{ if } p=q.\end{array}\right.$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see *e.g.* the discussion in [15, Section 2]. It is convenient to associate to ω its *(complex) covariances* $\lambda^{\pm}\in L_{\rm h}(\mathcal{Y},\mathcal{Y}^*)$ defined by:

$$\begin{array}{l}\omega(\psi(y_1)\psi^*(y_2))=:\overline{y}_1\cdot\lambda^+y_2,\\ \omega(\psi^*(y_2)\psi(y_1))=:\overline{y}_1\cdot\lambda^-y_2,\end{array}\quad y_1,y_2\in\mathcal{Y}.$$

The following results are well-known, see *e.g.* [4, Section 17.1] or [15, Section 2]:

– two hermitian forms $\lambda^{\pm}\in L_{\rm h}(\mathcal{Y},\mathcal{Y}^*)$ are the covariances of a quasi-free state ω iff

$$\lambda^{\pm}\geq 0,\ \lambda^+-\lambda^-=q. \tag{3.1}$$

– Let $\mathcal{Y}_\omega$ be the completion of $\mathcal{Y}$ for the Hilbertian scalar product $\lambda^++\lambda^-$. If there exist linear operators $c^{\pm}\in L(\mathcal{Y}_\omega)$ such that

$$c^++c^-=\mathbf{1},\ (c^{\pm})^2=c^{\pm},$$

(*i.e.* $c^{\pm}$ is a pair of complementary projections) and $\lambda^{\pm}=\pm q\circ c^{\pm}$, then ω is a *pure state*.

3.1.2. NEUTRAL BOSONIC FIELDS

We complete this subsection by explaining the relationship with the formalism of neutral fields, see *e.g.* [15, Subsection 2.5].

Let $\mathcal{X}$ a real vector space, $\mathcal{X}^{\#}$ its dual, and $\sigma \in L_{\rm a}(\mathcal{X}, \mathcal{X}^{\#})$ a symplectic form on $\mathcal{X}$. The $^*-$algebra ${\rm CCR}(\mathcal{X}, \sigma)$ is the complex $^*-$algebra generated by symbols $\mathbf{1}, \phi(x), x \in \mathcal{X}$ and relations:

$$\begin{aligned}
&\phi(x_1 + \lambda x_2) = \phi(x_1) + \lambda\phi(x_2),\ x_1, x_2 \in \mathcal{X}, \lambda \in \mathbb{R},\\
&[\phi(x_1), \phi(x_2] = {\rm i} x_1 \cdot \sigma x_2 \mathbf{1},\ x_1, x_2 \in \mathcal{X},\\
&\phi(x)^* = \phi(x),\ x \in \mathcal{X}.
\end{aligned}$$

To relate the neutral to the charged formalism one sets $\mathcal{Y} = \mathbb{C}\mathcal{X}$ and for $\beta \in L(\mathcal{X}, \mathcal{X}^{\#})$ denote by $\beta_{\mathbb{C}} \in L(\mathcal{Y}, \mathcal{Y}^*)$ its sesquilinear extension. $\mathcal{Y}_{\mathbb{R}} \sim \mathcal{X} \oplus \mathcal{X}$ is the *real form* of $\mathcal{Y}$, *i.e.* $\mathcal{Y}_{\mathbb{R}} = \mathcal{Y}$ as a real vector space. Then $(\mathcal{Y}_{\mathbb{R}}, {\rm Re}\sigma_{\mathbb{C}}) \sim (\mathcal{X}, \sigma) \oplus (\mathcal{X}, \sigma)$ is a real symplectic space and we denote by $\phi(y), y \in \mathcal{Y}_{\mathbb{R}}$ the selfadjoint generators of ${\rm CCR}(\mathcal{Y}_{\mathbb{R}}, {\rm Re}\sigma_{\mathbb{C}})$. Under the identification$\phi(y) \sim \phi(x) \otimes \mathbf{1} + \mathbf{1} \otimes \phi(x')$ for $y = x + {\rm i}x'$ we can identify ${\rm CCR}(\mathcal{Y}_{\mathbb{R}}, {\rm Re}\sigma_{\mathbb{C}})$ with ${\rm CCR}(\mathcal{X}, \sigma) \otimes {\rm CCR}(\mathcal{X}, \sigma)$ as $^*-$algebras.

Note also that under the identification

$$\psi(y) \sim \frac{1}{\sqrt{2}}(\phi(y) + {\rm i}\phi(iy)),\ \psi^*(y) \sim \frac{1}{\sqrt{2}}(\phi(y) - {\rm i}\phi(iy)),\ y \in \mathcal{Y}$$

we can identify ${\rm CCR}(\mathcal{Y}_{\mathbb{R}}, {\rm Re}\sigma_{\mathbb{C}})$ with ${\rm CCR}(\mathcal{Y}, q)$ for $q = {\rm i}\sigma_{\mathbb{C}}$.

A quasi-free state ω on ${\rm CCR}(\mathcal{X}, \sigma)$ is determined by its *real covariance* $\eta \in L_{\rm s}(\mathcal{X}, \mathcal{X}^{\#})$ defined by:

$$\omega(\phi(x_1)\phi(x_2)) =: x_1 \cdot \eta x_2 + \frac{\rm i}{2} x_1 \cdot \sigma x_2,\ x_1, x_2 \in \mathcal{X}.$$

A symmetric form $\eta \in L_{\rm s}(\mathcal{X}, \mathcal{X}^{\#})$ is the covariance of a quasi-free state iff

$$\eta \geq 0,\ |x_1 \cdot \sigma x_2| \leq 2(x_1 \cdot \eta x_1)^{\frac{1}{2}}(x_2 \cdot \eta x_2)^{\frac{1}{2}},\ x_1, x_2 \in \mathcal{X}.$$

To such a state ω we associate the quasi-free state $\tilde{\omega}$ on ${\rm CCR}(\mathcal{Y}_{\mathbb{R}}, {\rm Re}\sigma_{\mathbb{C}})$ with real covariance ${\rm Re}\eta_{\mathbb{C}}$. Then its complex covariances $\lambda^{\pm}$ are given by (see [15, Subsection 2.5]):

$$\lambda^{\pm} = \eta_{\mathbb{C}} \pm \frac{1}{2}{\rm i}\sigma_{\mathbb{C}}. \tag{3.2}$$

Applying complex conjugation, we immediately see that in this case

$$\lambda^{+} \geq 0 \ \Leftrightarrow\ \lambda^{-} \geq 0, \tag{3.3}$$

so it suffices to check for example that $\lambda^{+} \geq 0$.

3.2. **Free Klein-Gordon fields**

Let $P = -\Box_g + m(x)$, $m \in C^\infty(M, \mathbb{R})$ a Klein-Gordon operator on a globally hyperbolic spacetime (M, g) (we use the convention $(1, d)$ for the

Lorentzian signature). Let $E^{\pm}$ be the advanced/retarded inverses of P and $E := E^{+} - E^{-}$. We apply the above framework to

$$\mathcal{Y} = \frac{C_{\rm c}^{\infty}(M)}{PC_{\rm c}^{\infty}(M)},\ \overline{[u]}\cdot q[u] = \mathrm{i}(u|Eu)_{M},$$

where $(u|v)_{M} = \int_{M} \overline{u}v dVol_{g}$. The associated CCR $*$-algebra will be denoted by CCR(P).

One restricts attention to quasi-free states on CCR(P) whose covariances are given by distributions on $M \times M$, *i.e.* such that there exists $\Lambda^{\pm} \in \mathcal{D}'(M \times M)$ with

$$\begin{aligned} &\omega(\psi([u_{1}])\psi^{*}([u_{2}])) = (u_{1}|\Lambda^{+}u_{2})_{M}, \\ &\omega(\psi^{*}([u_{2}])\psi([u_{1}])) = (u_{1}|\Lambda^{-}u_{2})_{M}, \end{aligned} \quad u_{1}, u_{2} \in C_{\rm c}^{\infty}(M). \tag{3.4}$$

In the sequel, the distributions $\Lambda^{\pm} \in \mathcal{D}'(M \times M)$ will be called the *spacetime covariances* of the state ω.

In (3.4) we identify distributions on M with distributional densities using the density $dVol_{g}$ and use the notation $(u|\varphi)_{M}$, $u \in C_{\rm c}^{\infty}(M)$, $\varphi \in \mathcal{D}'(M)$ for the duality bracket. We have then

$$\begin{aligned} &P(x, \partial_{x})\Lambda^{\pm}(x, x') = P(x', \partial_{x'})\Lambda^{\pm}(x, x') = 0, \\ &\Lambda^{+}(x, x') - \Lambda^{-}(x, x') = \mathrm{i}E(x, x'). \end{aligned} \tag{3.5}$$

Before recalling the definition of Hadamard states (see [18] for the neutral case and [15] for the complex case), we need to introduce some more background notation. We recall that $T^{*}M$ is the cotangent bundle of M and we denote by $o \subset T^{*}M$ its zero section. One sets $\Gamma' = \{((x, \xi), (x', -\xi')) : ((x, \xi), (x', \xi')) \in \Gamma\}$ for $\Gamma \subset T^{*}M \times T^{*}M$.

The *characteristic manifold* $\mathcal{N}$ of P is

$$\mathcal{N} := \{(x, \xi) \in T^{*}M \setminus o : \xi_{\mu}g^{\mu\nu}(x)\xi_{\nu} = 0\}. \tag{3.6}$$

The time orientation of M induces a corresponding splitting of $\mathcal{N}$ into its two connected components $\mathcal{N}^{\pm}$ where

$$\mathcal{N}^{\pm} = \{(x, \xi) \in \mathcal{N} : \pm\xi \cdot v > 0,\ \forall v \in T_{x}M \text{ time-like future directed}\}.$$

Definition 3.1. A quasi-free state ω on CCR(P) is called a *Hadamard state* if its spacetime covariances $\Lambda^{\pm}$ satisfy:

$$\mathrm{WF}(\Lambda^{\pm})' \subset \mathcal{N}^{\pm} \times \mathcal{N}^{\pm}. \tag{3.7}$$

3.3. Cauchy surface covariances

Denoting by $Sol_{\rm sc}(P)$ the space of smooth space-compact solutions of $P\phi = 0$, it is well known that

$$[E] : \frac{C^\infty_{\rm c}(M)}{PC^\infty_{\rm c}(M)} \ni [u] \mapsto Eu \in Sol_{\rm sc}(P)$$

is bijective, with

$$\mathrm{i}(u_1|Eu_2) = \overline{Eu}_1 \cdot qEu_2, \ u_i \in C^\infty_{\rm c}(M),$$

for

$$\overline{\phi}_1 \cdot q\phi_2 := \mathrm{i}\int_\Sigma (\nabla_\mu \overline{\phi}_1 \phi_2 - \overline{\phi}_1 \nabla_\mu \phi_2) n^\mu d\sigma, \tag{3.8}$$

where Σ is any spacelike Cauchy hypersurface, n^μ is the future directed unit normal vector field to Σ and $d\sigma$ the induced surface density. Setting

$$\rho : C^\infty_{\rm sc}(M) \ni \phi \mapsto \begin{pmatrix} \phi\restriction_\Sigma \\ \mathrm{i}^{-1}\partial_\nu \phi\restriction_\Sigma \end{pmatrix} = f \in C^\infty_{\rm c}(\Sigma) \oplus C^\infty_{\rm c}(\Sigma)$$

Since the Cauchy problem

$$\begin{cases} P\phi = 0, \\ \rho u = f \end{cases}$$

has a unique solution $\phi \in Sol_{\rm sc}(P)$ for $f \in C^\infty_{\rm c}(\Sigma) \oplus C^\infty_{\rm c}(\Sigma)$ the map

$$\frac{C^\infty_{\rm c}(M)}{PC^\infty_{\rm c}(M)} \ni [u] \mapsto \rho Eu \in C^\infty_{\rm c}(\Sigma) \oplus C^\infty_{\rm c}(\Sigma)$$

is bijective, and

$$\mathrm{i}(u|Eu)_M = \overline{\rho Eu} \cdot q\rho Eu,$$

for

$$\overline{f} \cdot qf := \int_\Sigma \overline{f}_1 f_0 + \overline{f}_0 f_1 d\sigma_\Sigma, \ f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}. \tag{3.9}$$

It follows that to a quasi-free state with spacetime covariances $\Lambda^\pm$ one can associate its *Cauchy surface covariances* $\lambda^\pm$ defined by:

$$\Lambda^\pm =: (\rho E)^* \lambda^\pm (\rho E). \tag{3.10}$$

Using the canonical scalar product $(f|f)_\Sigma := \int_\Sigma \overline{f}_1 f_1 + \overline{f}_0 f_0 d\sigma_\Sigma$ we identify $\lambda^\pm$ with operators, still denoted by $\lambda^\pm$, belonging to $L(C^\infty_{\rm c}(\Sigma) \oplus C^\infty_{\rm c}(\Sigma), \mathcal{D}'(\Sigma) \oplus \mathcal{D}'(\Sigma))$.

A more explicit expression of $\lambda^\pm$ in terms of $\Lambda^\pm$ is as follows, see *e.g.* [14, Thm. 7.10]: let us introduce Gaussian normal coordinates to Σ

$$U \ni (t, y) \mapsto \chi(t, y) \in V,$$

where U is an open neighborhood of $\{0\}\times\Sigma$ in $\mathbb{R}\times\Sigma$ and V an open neighborhood of Σ in M, such that $\chi^* g = -dt^2 + h_{ij}(t,y)dy^i dy^j$. We denote by $\Lambda^\pm(t,y,t',y')\in\mathcal{D}'(U\times U)$ the restriction to $U\times U$ of the distributional kernel of $\Lambda^\pm$. By (3.5) and standard microlocal arguments, their restrictions to fixed times t,t', denoted by $\Lambda^\pm(t,t')\in\mathcal{D}'(\Sigma\otimes\Sigma)$ are well defined.

We know also that $\partial_t^k\partial_{t'}^{k'}\Lambda^\pm(0,0)\in\mathcal{D}'(\Sigma\times\Sigma)$ is well defined for $k,k'=0,1$. Then setting $\lambda^\pm =: \pm q\circ c^\pm$ we have:

$$c^\pm = \pm\begin{pmatrix} \mathrm{i}\partial_{t'}\Lambda^\pm(0,0) & \Lambda^\pm(0,0)\\ \partial_t\partial_{t'}\Lambda^\pm(0,0) & \mathrm{i}^{-1}\partial_t\Lambda^\pm(0,0)\end{pmatrix}. \tag{3.11}$$

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [14, 15] using pseudodifferential calculus on Σ, see below for a short summary.

3.4. **Pseudodifferential operators**

We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [21, Section 4.3] of [7, Section 18.1] for details.

For $m\in\mathbb{R}$ we denote by $\Psi^m(\mathbb{R}^d)$ the space of classical pseudodifferential operators of order m on $\mathbb{R}^d$, associated with poly-homogeneous symbols of order m see *e.g.* [21, Section 3.7].

Let N be a smooth, $d-$dimensional manifold. Let $U\subset N$ a precompact chart open set and $\psi: U\to\tilde{U}$ a chart diffeomorphism, where $\tilde{U}\subset\mathbb{R}^d$ is precompact, open. We denote by $\psi^*: C_\mathrm{c}^\infty(\tilde{U})\to C_\mathrm{c}^\infty(U)$ the map $\psi^* u(x) := u\circ\psi(x)$.

Definition 3.2. A linear continuous map $A: C_\mathrm{c}^\infty(N)\to C^\infty(N)$ belongs to $\Psi^m(N)$ if the following condition holds:

(C) Let $U\subset N$ be precompact open, $\psi: U\to\tilde{U}$ a chart diffeomorphism, $\chi_1,\chi_2\in C_\mathrm{c}^\infty(U)$ and $\tilde{\chi}_i=\chi_i\circ\psi^{-1}$. Then there exists $\tilde{A}\in\Psi^m(\mathbb{R}^d)$ such that

$$(\psi^*)^{-1}\chi_1 A\chi_2\psi^* = \tilde{\chi}_1\tilde{A}\tilde{\chi}_2. \tag{3.12}$$

Elements of $\Psi^m(N)$ are called *(classical) pseudodifferential operators* of order m on N.

The subspace of $\Psi^m(N)$ of pseudodifferential operators with *properly supported kernels* is denoted by $\Psi_\mathrm{c}^m(N)$.

Note that if $\Psi_{(\mathrm{c})}^\infty(N) := \bigcup_{m\in\mathbb{R}}\Psi_{(\mathrm{c})}^m(N)$, then $\Psi_\mathrm{c}^\infty(N)$ is an algebra, but $\Psi^\infty(N)$ is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

We denote by $T^*N\setminus o$ the cotangent bundle of N with the zero section removed.

To $A\in\Psi^m(N)$ one can associate its *principal symbol* $\sigma_{\rm pr}(A)\in C^\infty(T^*N\setminus o)$, which is homogeneous of degree m in the fiber variable ξ in T^*M, in $\{|\xi|\geq 1\}$. A is called *elliptic* in $\Psi^m(N)$ at $(x_0,\xi_0)\in T^*N\setminus o$ if $\sigma_{\rm pr}(A)(x_0,\xi_0)\neq 0$.

If $A\in\Psi^m(N)$ there exists (many) $A_{\rm c}\in\Psi^m_{\rm c}(N)$ such that $A-A_{\rm c}$ has a smooth kernel.

Finally one says that $(x_0,\xi_0)\notin {\rm essupp}(A)$ for $A\in\Psi^\infty(N)$ if there exists $B\in\Psi^\infty_{\rm c}(N)$ *elliptic* at (x_0,ξ_0) such that $A_{\rm c}\circ B$ is smoothing, where $A_{\rm c}\in\Psi^\infty_{\rm c}(N)$ is as above, *i.e.* $A-A_{\rm c}$ is smoothing.

3.5. The Cauchy surface covariances of Hadamard states

We now state a result which follows directly from a construction of Hadamard states in [15, Subsection 8.2].

THEOREM 3.3. *Let ω be any Hadamard state for the free Klein-Gordon field on (M,g) and Σ a spacelike Cauchy hypersurface. Then its Cauchy surface covariances $\lambda^\pm$ are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$.*

Proof. It is well known (see *e.g.* [18]) that if ω_1,ω_2 are Hadamard states, then $\Lambda_1^\pm-\Lambda_2^\pm$ are smoothing operators on M. Using (3.10) this implies that $\lambda_1^\pm-\lambda_2^\pm$ are matrices of smoothing operators on Σ. From the definition of $\Psi^\infty(\Sigma)$ it hence suffices to construct *one* Hadamard state ω whose Cauchy surface covariances $\lambda^\pm$ are matrices of pseudodifferential operators. The state constructed in [15, Subsection 8.2] has this property, as can be seen from [15, Equ. (8.2)]. $\Box$

4. EUCLIDEAN OPERATORS

The construction of the $\beta-$KMS state on $\mathscr{M}^+$ with respect to the Killing vector field V relies on the *Wick rotation*, where $(\mathbb{R}\times\Sigma^+,g)$ is replaced by $(\mathbb{S}_\beta\times\Sigma^+,\hat{g})$:

$$\hat{g}=v^2(y)d\tau^2+h_{ij}(y)dy^idy^j, \tag{4.1}$$

is the Riemannian metric obtained from (2.3) by setting $t={\rm i}\tau$ and $\mathbb{S}_\beta=[0,\beta[$ with endpoints identified is the circle of length β.

In this section, we recall various 'Euclidean' operators related to $\hat{g}$ appearing in [19, 20]. It will be convenient to construct them by quadratic form techniques.

We set

$$N := \mathbb{S}_\beta \times \Sigma^+,$$

whose elements are denoted by (τ, y). We equip N with the Riemannian metric $\hat{g}$ in (4.1) and the associated density $dVol_{\hat{g}} = |v|(y)|h|^{\frac{1}{2}}(y)d\tau dy$. The hypersurface Σ^+ is equipped with the induced density $dVol_h = |h|^{\frac{1}{2}}(y)dy$.

4.1. **Euclidean operator on N**

We consider the operator

$$K := -\Delta_{\hat{g}} + m(y),$$

for m as in Subsection 2.4. Note that m depends only on y since m is invariant under the Killing flow. We have

$$K = -v^{-2}(y)\overline{\partial}_\tau^2 - |v|^{-1}(y)|h|^{-\frac{1}{2}}(y)\overline{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\overline{\partial}_{y^j} + m(y).$$

K is well defined as a selfadjoint operator on $L^2(N, dVol_{\hat{g}})$ obtained from the quadratic form:

$$Q(u,u) := \int_N \left(|v|^{-2}|\partial_\tau u|^2 + \partial_i\overline{u}h^{ij}\partial_j u + m|u|^2\right) dVol_{\hat{g}}, \tag{4.2}$$

which is closeable on $C_c^\infty(N)$, since K is symmetric and bounded from below on this domain. Denoting its closure again by Q and the domain of its closure by $\mathrm{Dom}\, Q$, K is the selfadjoint operator associated to Q, *i.e.* the Friedrichs extension of its restriction to $C^\infty(\mathbb{S}_\beta) \otimes C_c^\infty(\Sigma^+)$. We know that $u \in \mathrm{Dom}\, K$, $Ku = f$ iff

$$u \in \mathrm{Dom}\, Q \text{ and } Q(w,u) = (w|f)_{L^2(N)}, \ \forall w \in C_c^\infty(N). \tag{4.3}$$

From (2.5) we know that $K \geq m_0^2$ hence is boundedly invertible and we set

$$G := K^{-1}.$$

4.2. **Change of volume form**

Let us set $\hat{Q}(u,u) = Q(vu, vu)$, $\mathrm{Dom}\,\hat{Q} = \{u \in L^2(N) : vu \in \mathrm{Dom}\, Q\}$. By (2.5) we have $\hat{Q}(u,u) \geq m_0^2\|vu\|^2$. If $u_n \in \mathrm{Dom}\,\hat{Q}$, $u \in L^2(N)$ with $\|u_n - u\| \to 0$ and $\hat{Q}(u_n - u_m, u_n - u_m) \to 0$ then from the inequality above we obtain that $vu \in L^2(N)$ and $\|v(u_n - u)\| \to 0$. Since Q is closed we obtain that $u \in \mathrm{Dom}\,\hat{Q}$ and $\hat{Q}(u_n - u, u_n - u) \to 0$, *i.e.* $\hat{Q}$ is closed.

Let $\hat{K}$ be the injective selfadjoint operator associated to $\hat{Q}$, (which is formally equal to vKv) and let $\hat{G} = \hat{K}^{-1}$. We claim that

$$G = v\hat{G}v, \text{ on } v^{-1}L^2(N). \tag{4.4}$$

This follows easily from the characterization (4.3) of G and similarly of $\hat{G}$.

Let now $U : L^2(N) \to L^2(\mathbb{S}_\beta) \otimes L^2(\Sigma^+)$ the unitary map given by $Uu = v^{\frac{1}{2}} u$. We set

$$\tilde{K} := U\hat{K}U^*.$$

We have

$$\tilde{K} = -\overline{\partial}_\tau^2 + \epsilon^2(y, \overline{\partial}_y),$$

where:

$$\epsilon^2(y, \overline{\partial}_y) = -|v|^{\frac{1}{2}}(y)|h|^{-\frac{1}{2}}(y)\overline{\partial}_{y^i}|v|(y)|h|^{\frac{1}{2}}(y)h^{ij}(y)\overline{\partial}_{y^j}|v|^{\frac{1}{2}}(y) + v^2(y)m(y),$$

is obtained as above from the quadratic form

$$\int_{\Sigma^+} \left(\partial_i|v|^{\frac{1}{2}}\overline{u}|v|h^{ij}\partial_i|v|^{\frac{1}{2}}u + |v|^2 m|u|^2\right)|h|^{\frac{1}{2}}dy. \tag{4.5}$$

If $\tilde{G} := \tilde{K}^{-1}$ we have by (4.4):

$$G = |v|^{1/2}\tilde{G}|v|^{3/2}, \text{ on } v^{-3/2}L^2(N). \tag{4.6}$$

We now recall a well known expression for $\tilde{G}$. Let

$$F(\tau) = \frac{\mathrm{e}^{-\tau\epsilon} + \mathrm{e}^{(\tau-\beta)\epsilon}}{2\epsilon(1 - \mathrm{e}^{-\beta\epsilon})}, \ \tau \in [0, \beta[,$$

extended to $\tau \in \mathbb{R}$ by $\beta-$periodicity. In particular, we have:

$$F(\tau) = \frac{\mathrm{e}^{-|\tau|\epsilon} + \mathrm{e}^{(|\tau|-\beta)\epsilon}}{2\epsilon(1 - \mathrm{e}^{-\beta\epsilon})}, \ \tau \in [-\beta, \beta] \tag{4.7}$$

The following expression for $\tilde{G}$ is well-known (see *e.g.* [4, Section 18.3.2]):

$$\tilde{G}\tilde{u}(\tau) = \int_{\mathbb{S}_\beta} F(\tau - \tau')\tilde{u}(\tau')d\tau', \ \tilde{u} \in L^2(S_\beta) \otimes L^2(\Sigma\backslash\mathscr{B}). \tag{4.8}$$

Note that since $\epsilon^2 \geq mv^2$ by (4.5), we have also $\epsilon^{-2} \leq m^{-1}v^{-2}$ by Kato-Heinz theorem hence $C_{\mathrm{c}}^\infty(\Sigma^+) \subset \mathrm{Dom}\, F(\tau)$.

5. CALDERÓN PROJECTORS

In this section, we recall some standard facts on *Calderón projectors.* We refer the reader to [3, Sects. 5.1–5.3] for details.

5.1. The Calderón projector

Let (N, h) a complete Riemannian manifold and $P = -\Delta_h + m$, where $m \in C^\infty(N)$ is a real potential with $m(x) \geq m_0^2 > 0$. As in Section 4, we construct P as a selfadjoint operator on $L^2(N, dVol_h)$ using the quadratic form

$$Q(u,u) = \int_N \partial_i \overline{u} h^{ij} \partial_j u + m(x)|u|^2(x) dVol_h. \tag{5.1}$$

We obtain that $0 \in \rho(P)$, hence $G := P^{-1}$ is a bounded operator on $L^2(N, dVol_h)$, defined by

$$Q(Gv, w) = (v|w)_{L^2(N)}, \ \forall w \in C_c^\infty(N). \tag{5.2}$$

Let $\Omega \subset N$ an open set such that $\partial\Omega = S = \bigcup_1^n S_i$, where S_i are the connected components of S and are assumed to be smooth hypersurfaces. We denote by $C^\infty(\overline{\Omega})$ the space of restrictions to Ω of functions in $C^\infty(N)$.

We associate to S_i the distribution density dS_i defined by:

$$\langle dS_i|u\rangle := \int_{S_i} u d\sigma_h^{(i)}, \ u \in C_c^\infty(N),$$

where $d\sigma_h^{(i)}$ is the induced Riemannian density on S_i and we set

$$dS = \sum_{i=1}^n dS_i.$$

We denote by ∂_ν the unit exterior normal vector field to S and set

$$\langle \partial_\nu^* dS|u\rangle := \langle dS|\partial_\nu u\rangle, \ u \in C_c^\infty(N).$$

For $u \in C^\infty(\overline{\Omega})$ we set

$$\gamma u := \begin{pmatrix} u\restriction_S \\ \partial_\nu u\restriction_S \end{pmatrix} =: \begin{pmatrix} \gamma_0 u \\ \gamma_1 u \end{pmatrix}.$$

For $v \in C_c^\infty(S)$ we denote by $\tilde{v} \in C_c^\infty(N)$ an extension of v to N such that $\tilde{u}\restriction_S = u, \partial_\nu \tilde{u}\restriction_S = 0$.

Definition 5.1. Let $f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in C_c^\infty(S) \oplus C_c^\infty(S)$. We set:

$$Df := \gamma \circ G(\tilde{f}_1 (dVol_h)^{-1} dS - \tilde{f}_0 (dVol_h^{-1}) \partial_\nu^* dS).$$

– The operator $D : C_c^\infty(S) \oplus C_c^\infty(S) \to C^\infty(S) \oplus C^\infty(S)$ is continuous and is called the *Calderón projector* associated to (P, S).

– The operator D is a 2×2 matrix of pseudodifferential operators on S.

Note that dS and $\partial_\nu^* dS$ are distributional densities, hence $(dVol_h)^{-1} dS$ and $(dVol_h)^{-1} \partial_\nu^* dS$ are distributions on N, supported on S.

Note also that the Calderón projector is obviously covariant under diffeomorphisms: if $\chi : (N, h) \to (N', h')$ is an isometric diffeomorphism with

$S' = \chi(S)$, $P = \chi^* P'$, then

$$D = \psi^* D',$$

where $\psi : S \to S'$ is the restriction of χ to S.

5.1.1. EXPRESSION IN GAUSSIAN NORMAL COORDINATES

Let U_i be a neighborhood of $\{0\} \times S_i$ in $\mathbb{R} \times S_i$ and V_i a neighborhood of S_i in N such that Gaussian normal coordinates to S_i induce a diffeomorphism:

$$\chi_i : U_i \ni x \mapsto (s, y) \in V_i$$

from U_i to V_i, and $ds^2 + k_s(y)dy^2 = \chi_i^* h$ on U_i. Then for $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$ we have

$$\begin{aligned} &\chi_i^* \left(\tilde{f}_1 (dVol_h)^{-1} dS - \tilde{f}_0 (dVol_h)^{-1} \partial_\nu^* dS \right) \\ = \; &\delta_0(s) \otimes (f_1(y) - r_0(y) f_0(y)) - \delta_0'(s) \otimes f_0(y), \end{aligned} \tag{5.3}$$

where $r_s(y) = |k_s|^{-\frac{1}{2}}(y) \partial_s |k_s|^{\frac{1}{2}}$.

If $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi \geq 0$, $\int \varphi(s)ds = 1$, setting $\varphi_n(s) = n\varphi(ns)$, we can compute Df for $f \in C_c^\infty(S_i) \otimes \mathbb{C}^2$ as

$$Df = \lim_{n \to +\infty} \gamma \circ G(\varphi_n(s) \otimes (f_1(y) - r_0(y) f_0(y)) - \varphi_n'(s) \otimes f_1(y)), \tag{5.4}$$

where the limit takes place in $C^\infty(S) \oplus C^\infty(S)$.

Note that it is not obvious that $Df \in C^\infty(S) \oplus C^\infty(S)$. To prove it one can first replace G by a properly supported pseudodifferential parametrix $P^{(-1)} \in \Psi_c^{-2}(N)$. Using then Gaussian normal coordinates near a point $x^0 \in S$, one is reduced locally to $N = \mathbb{R}^d$, $S = \{x_1 = 0\}$. The details can be found for example in [3, Sects. 5.1- 5.3].

Another useful identity is the following: for $u \in C^\infty(\overline{\Omega})$ let Iu be the extension of u by 0 in $N \backslash \overline{\Omega}$. Then

$$PIu = \tilde{f}_1 (dVol_h)^{-1} dS - \tilde{f}_0 (dVol_h^{-1}) \partial_\nu^* dS + IPu, \text{ for } f = \gamma u. \tag{5.5}$$

6. THE DOUBLE β–KMS STATE

In this section, we consider the *double β–KMS state* ω_β in $\mathscr{M}^+ \cup \mathscr{M}^-$. It is obtained as the natural extension to $\mathscr{M}^+ \cup \mathscr{M}^-$ of the state ω_β^+ in $\mathscr{M}^+$, which is a β–KMS state in $\mathscr{M}^+$ with respect to the Killing flow. Its construction, for the more general stationary case is given in [19, Thm. 3.5].

Since $\Sigma \backslash \mathscr{B}$ is a Cauchy surface for $\mathscr{M}^+ \cup \mathscr{M}^-$, we associate to ω_β its (complex) Cauchy surface covariances on $\Sigma \backslash \mathscr{B}$ $\lambda^\pm$, and (since ω_β is a pure state), the pair of complementary projections $c^{\pm} = \pm q^{-1} \circ \lambda^{\pm}$, see Subsection 3.1. We will study in details the projection c^{+}.

We identify $C_c^\infty(\Sigma\backslash\mathscr{B})$ with $C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2$ using the map

$$\begin{aligned} \hat{R}: \quad & C_c^\infty(\Sigma^+) \otimes \mathbb{C}^2 \to C_c^\infty(\Sigma^+) \oplus C_c^\infty(\Sigma^-) \\ & g = g^{(0)} \oplus g^{(\beta/2)} \mapsto f = g^{(0)} \oplus r^* g^{(\beta/2)}, \end{aligned} \tag{6.1}$$

where $r : \Sigma \to \Sigma$ is the restriction to Σ of the wedge reflection R, see Subsection 2.2.

We will show that

$$C := \hat{R}^{-1} \circ c^{+} \circ \hat{R}$$

is exactly the Calderón projector for the Euclidean operator K_+ acting on $(N, \hat{g})$, see Subsection 4.1, and the open set

$$\Omega := \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

6.1. The double β–KMS state

We recall now the expression of ω_β given by Sanders, see [19, Section 3.3].

There are some differences in signs and factors of i with the expression given by Sanders in [19, Section 3.3]. They come from two differences between our convention for quantized Klein-Gordon fields and the one of Sanders:

(1) our convention for Cauchy data of a solution of $Pu = 0$ is given the map $\rho u = \begin{pmatrix} u\restriction_\Sigma \\ \mathrm{i}^{-1}\partial_\nu u\restriction_\Sigma \end{pmatrix} =: f$, which is more natural for complex fields and leads to a more symmetric formulation of the Hadamard condition, while Sanders uses $\rho u = \begin{pmatrix} u\restriction_\Sigma \\ \partial_\nu u\restriction_\Sigma \end{pmatrix} = g$, so $f = \begin{pmatrix} 1 & 0 \\ 0 & -\mathrm{i} \end{pmatrix} g$.

(2) we use as complex symplectic form $\overline{f}\cdot\sigma f' = (f_1|f_0') - (f_0|f_1')$, while Sanders uses $g\cdot\sigma g' = (g_0|g_1') - (g_1|g_0')$. In terms of spacetime fields, we use $\mathrm{i}^{-1}E$, Sanders uses $\mathrm{i}E$.

Let us unitarily identify $L^2(\Sigma, |h|^{\frac12} dy)$ with $L^2(\Sigma^+, |h|^{\frac12} dy) \oplus L^2(\Sigma^-, |h|^{\frac12} dy)$, by

$$u \mapsto u_+ \oplus u_-, \ u_\pm = u\restriction_{\Sigma^\pm} .$$

Under this identification the action of the wedge reflection $r^* u = u \circ r$ will be denoted by T, with:

$$T(u_+ \oplus u_-) := r^* u_- \oplus r^* u_+. \tag{6.2}$$

A direct comparison with the formulas in [19, Section 3.3], using the identity (3.2) gives the following proposition.

Proposition 6.1. *The double $\beta-$KMS state on $\mathscr{M}^+\cup\mathscr{M}^-$ is given by the Cauchy surface covariance $\lambda^+=\begin{pmatrix}\lambda^+_{00} & \lambda^+_{01}\\ \lambda^+_{10} & \lambda^+_{11}\end{pmatrix}$ where:*

$$
\begin{aligned}
\lambda^+_{00} &= \tfrac{1}{2}|v|^{\frac12}\left(\epsilon^{-1}\coth(\tfrac{\beta}{2}\epsilon)+\epsilon^{-1}T\mathrm{sh}^{-1}(\tfrac{\beta}{2}\epsilon)\right)|v|^{\frac12},\\
\lambda^+_{11} &= \tfrac{1}{2}|v|^{-\frac12}\left(\epsilon\coth(\tfrac{\beta}{2}\epsilon)-\epsilon T\mathrm{sh}^{-1}(\tfrac{\beta}{2}\epsilon)\right)|v|^{-\frac12},\\
\lambda^+_{01} &= \lambda^+_{10}=\tfrac12\mathbf{1}.
\end{aligned}
\tag{6.3}
$$

As in Subsection 3.1 we have $\lambda^-=\lambda^+-q$, where the charge $q=\mathrm{i}\sigma$ is given by the matrix $q=\begin{pmatrix}0&1\\1&0\end{pmatrix}$. We introduce the operators $c^\pm:=\pm q^{-1}\lambda^\pm$ and obtain

$$
c^+=\begin{pmatrix}\frac12 & \lambda^+_{00}\\ \lambda^+_{11} & \frac12\end{pmatrix}.
\tag{6.4}
$$

Note that if

$$
b_0=\epsilon^{-1}\coth(\frac{\beta}{2}\epsilon)+\epsilon^{-1}T\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon),\ b_1=\epsilon\coth(\frac{\beta}{2}\epsilon)-\epsilon T\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon),
$$

then using that $[T,\epsilon]=0$ we obtain that

$$
b_0b_1=b_1b_0=\coth(\frac{\beta}{2}\epsilon)^2-\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon)^2=1,
$$

from which it follows easily that $c^\pm$ are (formally) projections. This is expected since the double $\beta-$KMS state ω_β is a pure state in $\mathscr{M}^+\cup\mathscr{M}^-$.

6.2. **Conjugation by $\hat{R}$**

The map $\hat{R}$ defined in (6.1) allows to unitarily identify $L^2(\Sigma^+)\otimes\mathbb{C}^2$ with $L^2(\Sigma^+)\oplus L^2(\Sigma^-)$. We have:

$$
\hat{R}^{-1}\epsilon\hat{R}=\epsilon_+\oplus\epsilon_+,\ \hat{R}^{-1}T\hat{R}=\begin{pmatrix}0&1\\1&0\end{pmatrix}.
\tag{6.5}
$$

Denoting by c^+_{ij} for $i,j\in\{0,1\}$ the entries of the matrix c^+ and setting

$$
C_{ij}:=\hat{R}^{-1}\circ c^+_{ij}\circ\hat{R},
$$

we obtain after an easy computation using (6.3), (6.1):

$$
\begin{aligned}
C_{00}g_0 &= \frac12 g_0^{(0)}\oplus\frac12 g_0^{(\beta/2)},\\
C_{11}g_1 &= \frac12 g_1^{(0)}\oplus\frac12 g_1^{(\beta/2)},
\end{aligned}
\tag{6.6}
$$

$$C_{01}g_1 = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}g_1^{(0)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}g_1^{(\beta/2)}$$

$$\oplus\frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}g_1^{(\beta/2)} + \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}g_1^{(0)},$$

$$C_{10}g_0 = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}g_0^{(0)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}g_0^{(\beta/2)}$$

$$\oplus\frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}g_0^{(\beta/2)} - \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}g_0^{(0)}.$$

In (6.6) the upper indices (0), $(\beta/2)$ refer to the two connected components $\{\tau = 0\}$ and $\{\tau = \beta/2\}$ of $\partial\Omega$, while the lower indices 0, 1 refer to the two components of g.

6.3. **The Calderón projector**

We now compute the Calderón projector for K_+, associated to the Riemannian manifold $(N, \hat{g})$. We choose

$$\Omega = \{(\tau, y) \in N : 0 < \tau < \beta/2\}.$$

We have $S = \partial\Omega = S_0 \cup S_{\beta/2}$ and we write $f \in C_c^\infty(S) \oplus C_c^\infty(S)$ as $f = f^{(0)} \oplus f^{(\beta/2)}$ for $f^{(i)} \in C_c^\infty(S_i) \oplus C_c^\infty(S_i)$.

We denote by $\gamma^{(i)}$, $i = 0, \beta/2$ the trace operators on S_i defined by $\gamma u = \gamma^{(0)}u \oplus \gamma^{(\beta/2)}u$ for $u \in C^\infty(\overline{\Omega})$. We have:

$$\text{(6.7)}\quad \begin{array}{l} \gamma^{(0)}u = \lim_{\tau\to 0^+}\begin{pmatrix} u(\tau, y) \\ -|v(y)|^{-1}\partial_\tau u(\tau, y)\end{pmatrix}, \\ \gamma^{(\beta/2)}u = \lim_{\tau\to(\beta/2)^-}\begin{pmatrix} u(\tau, y) \\ |v(y)|^{-1}\partial_\tau u(\tau, y)\end{pmatrix}. \end{array}$$

We denote similarly by $\partial_\nu^{(i)}$ the exterior normal derivatives on S_i.

We compute the Calderón projector D defined in Subsection 5.1 using the coordinates (τ, y). Since $dS_i = |h|^{\frac{1}{2}}(y)dy$ and $dVol_{\hat{g}} = |v|^{\frac{1}{2}}(y)|h|^{\frac{1}{2}}(y)dy$, we obtain:

$$\text{(6.8)}\quad Df = D^{(0)}f \oplus D^{(\beta/2)}f,$$

for

$$\text{(6.9)}\quad \begin{array}{rl} D^{(i)}f = & \gamma^{(i)} \circ G \circ |v|^{-1}\Big(\partial_\nu^{(0)}\delta_0(\tau) \otimes f_0^{(0)}(y) + \delta_0(\tau) \otimes f_1^{(0)}(y) \\ & +\partial_\nu^{(\beta/2)}\delta_{\beta/2}(\tau) \otimes f_0^{(\beta/2)}(y) + \delta_{\beta/2}(\tau) \otimes f_1^{(\beta/2)}(y)\Big). \end{array}$$

Since $G = |v|^{\frac{1}{2}}\tilde{G}|v|^{3/2}$ we have $G \circ |v|^{-1} = |v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}$. Denoting by $D_{kl}^{(i)(j)}$ for

$i, j \in \{0, \beta/2\}$ and $k, l \in \{0, 1\}$ the various entries of D, we obtain:

$$D_{kl}^{(i)(j)} v = \begin{cases} \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\partial_\nu^{(j)} \delta_j(\tau) \otimes v(y)), & l = 0, \\ \gamma_k^{(i)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} (\delta_j(\tau) \otimes v(y)), & l = 1. \end{cases} \tag{6.10}$$

We also set

$$\partial_\tau^{(i)} = \mp \partial_\tau, \text{ for } i = 0, \beta/2,$$

so that $\partial_\nu^{(i)} = |v|^{-1}(y) \partial_\tau^{(i)}$.

PROPOSITION 6.2. *We have $D = \hat{R}^{-1} \circ c^+ \circ \hat{R}$.*

Proof. We recall that C_{ij} are the entries of $\hat{R}^{-1} \circ c^+ \circ \hat{R}$. We compute $D_{kl}^{(i)(j)}$ using (6.10) and the explicit formulas (4.7), (4.8) for the kernel $\tilde{G}(\tau, \tau')$ of $\tilde{G}$.

Computation of D_{00}:

$$\begin{aligned} D_{00}^{(0)(0)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \lim_{\tau \to 0^+} \partial_{\tau'} \tilde{G}(\tau, 0) |v|^{-\frac{1}{2}} u = \frac{1}{2} u, \end{aligned}$$

$$\begin{aligned} D_{00}^{(0)(\beta/2)} u &= \gamma_0^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \lim_{\tau \to 0^+} \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = 0, \end{aligned}$$

$$\begin{aligned} D_{00}^{(\beta/2)(0)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u \\ &= |v|^{\frac{1}{2}} \lim_{\tau \to \beta/2^-} \partial_{\tau'} \tilde{G}(\tau, 0) u = 0, \end{aligned}$$

$$\begin{aligned} D_{00}^{(\beta/2)(\beta/2)} u &= \gamma_0^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(\beta/2)} \delta_{\beta/2} \otimes u \\ &= -|v|^{\frac{1}{2}} \lim_{\tau \to \beta/2^-} \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = \frac{1}{2} u. \end{aligned}$$

Hence

$$D_{00} g_0 = C_{00} g_0.$$

Computation of D_{11}:

$$\begin{aligned} D_{11}^{(0)(0)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_0 \otimes u \\ &= -|v|^{-\frac{1}{2}} \lim_{\tau \to 0^+} \partial_\tau \tilde{G}(\tau, 0) |v|^{\frac{1}{2}} u = \frac{1}{2} u, \\ D_{11}^{(0)(\beta/2)} u &= \gamma_1^{(0)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \delta_{\beta/2} \otimes u \end{aligned}$$

$$
\begin{aligned}
&= -|v|^{-\frac{1}{2}} \lim_{\tau\to 0^+} \partial_\tau \tilde{G}(\tau,\beta/2)|v|^{\frac{1}{2}}u = 0,\\
D_{11}^{(\beta/2)(0)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0\otimes u\\
&= |v|^{-\frac{1}{2}} \lim_{\tau\to \beta/2^-} \partial_\tau \tilde{G}(\tau,0)|v|^{\frac{1}{2}}u = 0,\\
D_{11}^{(\beta/2)(\beta/2)}u &= \gamma_1^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2}\otimes u\\
&= |v|^{-\frac{1}{2}} \lim_{\tau\to \beta/2^-} \partial_\tau \tilde{G}(\tau,\beta/2)|v|^{\frac{1}{2}}u = \frac{1}{2}u.
\end{aligned}
$$

Hence

$$D_{11}g_0 = C_{11}g_0.$$

Computation of D_{01}:

$$
\begin{aligned}
D_{01}^{(0)(0)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0\otimes u\\
&= |v|^{\frac{1}{2}} \lim_{\tau\to 0^+} \tilde{G}(\tau,0)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u,\\
D_{01}^{(0)(\beta/2)}u &= \gamma_0^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2}\otimes u\\
&= |v|^{\frac{1}{2}} \lim_{\tau\to 0^+} \tilde{G}(\tau,\beta/2)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u,\\
D_{01}^{(\beta/2)(0)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_0\otimes u\\
&= |v|^{\frac{1}{2}} \lim_{\tau\to \beta/2^-} \tilde{G}(\tau,0)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u,\\
D_{01}^{(\beta/2)(\beta/2)}u &= \gamma_0^{(\beta/2)}|v|^{\frac{1}{2}}\tilde{G}|v|^{\frac{1}{2}}\delta_{\beta/2}\otimes u\\
&= |v|^{\frac{1}{2}} \lim_{\tau\to \beta/2} \tilde{G}(\tau,\beta/2)|v|^{\frac{1}{2}}u = \frac{1}{2}|v|^{\frac{1}{2}}\epsilon_+^{-1}\coth(\frac{\beta}{2}\epsilon_+)|v|^{\frac{1}{2}}u.
\end{aligned}
$$

Hence

$$D_{01}g_1 = C_{01}g_1.$$

Computation of D_{10}:

$$
\begin{aligned}
D_{10}^{(0)(0)}u &= \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(0)}\delta_0\otimes u\\
&= -|v|^{-\frac{1}{2}} \lim_{\tau\to 0^+} \partial_\tau\partial_{\tau'}\tilde{G}(\tau,0)|v|^{-\frac{1}{2}}u = \frac{1}{2}|v|^{-\frac{1}{2}}\epsilon_+\coth(\frac{\beta}{2}\epsilon_+)|v|^{-\frac{1}{2}}u,
\end{aligned}
$$

$$D_{10}^{(0)(\beta/2)}u = \gamma_1^{(0)}|v|^{\frac{1}{2}}\tilde{G}|v|^{-\frac{1}{2}}\partial_\tau^{(\beta/2)}\delta_{\beta/2}u$$

$$= |v|^{-\frac{1}{2}} \lim_{\tau \to 0^+} \partial_\tau \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = -\frac{1}{2} |v|^{-\frac{1}{2}} \epsilon_+ \mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+) |v|^{-\frac{1}{2}} u,$$

$$D_{10}^{(\beta/2)(0)} u = \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{-\frac{1}{2}} \partial_\tau^{(0)} \delta_0 \otimes u$$

$$= |v|^{-\frac{1}{2}} \lim_{\tau \to \beta/2^-} \partial_\tau \partial_{\tau'} \tilde{G}(\tau, 0) |v|^{-\frac{1}{2}} u = -\frac{1}{2} |v|^{-\frac{1}{2}} \epsilon_+ \mathrm{sh}^{-1}(\frac{\beta}{2}\epsilon_+) |v|^{-\frac{1}{2}} u,$$

$$D_{10}^{(\beta/2)(\beta/2)} u = \gamma_1^{(\beta/2)} |v|^{\frac{1}{2}} \tilde{G} |v|^{\frac{1}{2}} \partial_\nu^{(\beta/2)} \delta_{\beta/2} \otimes u$$

$$= -|v|^{-\frac{1}{2}} \lim_{\tau \to \beta/2^-} \partial_\tau \partial_{\tau'} \tilde{G}(\tau, \beta/2) |v|^{-\frac{1}{2}} u = \frac{1}{2} |v|^{-\frac{1}{2}} \epsilon_+ \coth(\frac{\beta}{2}\epsilon_+) |v|^{-\frac{1}{2}} u.$$

Hence

$$D_{10} g_0 = C_{10} g_0.$$

This completes the proof of the proposition. □

7. THE HARTLE-HAWKING-ISRAEL STATE AND ITS PROPERTIES

7.1. The smooth extension of $(N, \hat{g})$ and the Hawking temperature

The existence of the Hartle-Hawking-Israel state and the definition of the Hawking temperature $T_H = \kappa(2\pi)^{-1}$ (where κ is the surface gravity) rely on the existence of an extension $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ of $(N, \hat{g})$ such that the two components $S_0, S_{\beta/2} \sim \Sigma^+$ of $\partial\Omega$ are smoothly glued together into $\Sigma \subset N_{\text{ext}}$.

The extended Riemannian metric $\hat{g}_{\text{ext}}$ is smooth iff $\beta = (2\pi)\kappa^{-1}$ (for other values of β $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ has a conic singularity on $\mathscr{B}$).

Let us embed $\Sigma \backslash \mathscr{B}$ into N by:

$$\hat{r} : \begin{cases} x \mapsto (0, x) \text{ for } x \in \Sigma^+, \\ x \mapsto (\beta/2, r(x)) \text{ for } x \in \Sigma^-, \end{cases}$$

Note that for $\hat{R}$ defined in (6.1) we have

$$\hat{R} = (\hat{r})^*. \tag{7.1}$$

We recall that the function $m : \Sigma \to \mathbb{R}^+$ was introduced in Subsection 2.4.

PROPOSITION 7.1 ([19, Subsection 2.2]). *Assume that $\beta = (2\pi)\kappa^{-1}$. Then there exists a smooth complete Riemannian manifold $(N_{\text{ext}}, \hat{g}_{\text{ext}})$ and*

1. *a smooth isometric embedding $\psi : \Sigma \to N_{\text{ext}}$,*
2. *a smooth isometric embedding $\chi : (N, \hat{g}) \to (N_{\text{ext}} \backslash \mathscr{B}_{\text{ext}}, \hat{g}_{\text{ext}})$ for $\mathscr{B}_{\text{ext}} = \psi(\mathscr{B})$,*

(3) *a smooth function* $m_{\rm ext} : N_{\rm ext} \to \mathbb{R}$ *with* $m_{\rm ext} \geq m_0^2 > 0$ *such that*

$$\psi\upharpoonright_{\Sigma\backslash\mathscr{B}} = \chi \circ \hat{r}, \ \ \psi^* m_{\rm ext} = m\upharpoonright_N \ .$$

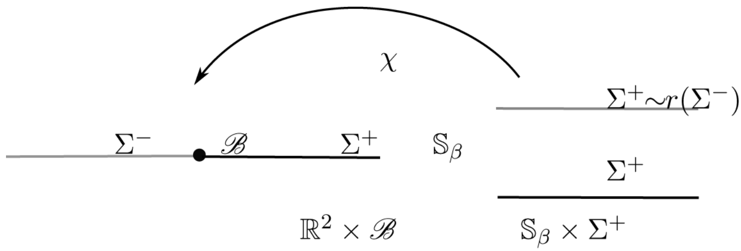

Fig. 2 – The embedding χ.

This fundamental fact is well explained in [19, Subsection 2.2]. Let us briefly recall the construction of χ following [19]: one introduces Gaussian normal coordinates to $\mathscr{B}$ in (Σ, h), where h is the Riemannian metric induced by g on Σ. We choose the unit normal vector field to $\mathscr{B}$ pointing towards Σ^+. Using these coordinates we can, since $\mathscr{B}$ is compact, identify a small neighborhood U of $\mathscr{B}$ in Σ with $]-\delta, \delta[\times\mathscr{B}$. Denoting by ω local coordinates on $\mathscr{B}$ we have a map

$$\begin{aligned}
&\phi :]-\delta, \delta[\times\mathscr{B} \ni (s, \omega) \mapsto y = \exp_\omega^h(s) \in U,\\
&U^+ = \phi(]0, \delta[\times\mathscr{B}),\\
&\phi^* h = ds^2 + k_{\alpha\beta}(s, \omega) d\omega^\alpha d\omega^\beta,
\end{aligned}$$

for $U^+ = \Sigma^+ \cap U$. In the local coordinates (τ, s, ω) on $\mathcal{S}_\beta \times U^+$ the embedding χ takes the form:

$$\chi : \begin{array}{l} \mathcal{S}_\beta \times]0, \delta[\times\mathscr{B} \to B_2(0, \delta) \times \mathscr{B} \\ (\tau, s, \omega) \mapsto (s\cos(\beta(2\pi)^{-1}\tau), s\sin(\beta(2\pi)^{-1}\tau), \omega) =: (X, Y, \omega), \end{array} \tag{7.2}$$

where $B_2(0, \delta) = \{(X, Y) \in \mathbb{R}^2 : 0 < X^2 + Y^2 < \delta^2\}$. A straightforward computation performed in [19, Subsection 2.2] shows that $\hat{g}$ admits a smooth extension $\hat{g}_{\rm ext}$ to $N_{\rm ext}$ iff $\beta = (2\pi)\kappa^{-1}$.

7.2. The extension of ω_β to M

We recall from Subsection 4.1 that K is defined from the closure $\overline{Q}$ of the quadratic form Q on $C_{\rm c}^\infty(N)$.

Similarly $K_{\rm ext} = -\Delta_{\hat{g}_{\rm ext}} + m_{\rm ext}$, acting on $N_{\rm ext}$ is defined using the corresponding quadratic form $Q_{\rm ext}$.

The following lemma is equivalent to [19, Prop. 5.2], for completeness we give a short proof using quadratic form arguments (note that we assume the stronger condition that $\inf m(x) > 0$).

LEMMA 7.2. *Let* $U : C_{\rm c}^\infty(N) \to C_{\rm c}^\infty(N_{\rm ext}\backslash\mathscr{B}_{\rm ext})$ *defined by*

$$Uu = u \circ \chi^{-1}.$$

Then U *extends as a unitary operator* $U : L^2(N) \to L^2(N_{\rm ext})$ *with* $K_{\rm ext} = UKU^*$.

Proof. U clearly extends as a unitary operator. To check the second statement it suffices, taking into account the way K and $K_{\rm ext}$ are defined, to prove that $C_{\rm c}^\infty(N_{\rm ext}\backslash\mathscr{B}_{\rm ext})$ is a form core for $Q_{\rm ext}$. The domain of $\overline{Q}_{\rm ext}$ is the Sobolev space $H^1(N_{\rm ext})$ associated to $\hat{g}_{\rm ext}$, so we need to show that $C_{\rm c}^\infty(N_{\rm ext}\backslash\mathscr{B}_{\rm ext})$ is dense in $H^1(N_{\rm ext})$. Using the coordinates (X, Y, ω) near $\mathscr{B}_{\rm ext} \sim \{0\} \times \mathscr{B}$, this follows from the fact that $C_{\rm c}^\infty(\mathbb{R}^2\backslash\{0\})$ is dense in $H^1(\mathbb{R}^2)$, see *e.g.* [1, Thm. 3.23]. □

We recall that the projection c^+ associated to the double β−KMS state ω_β was defined in (6.4). Let us identify in the sequel Σ with $\Sigma_{\rm ext} = \psi(\Sigma) \subset N_{\rm ext}$.

THEOREM 7.3. *Let* $D_{\rm ext}$ *the Calderón projector for* $(K_{\rm ext}, \Sigma)$. *Then for* $f, g \in C_{\rm c}^\infty(\Sigma\backslash\mathscr{B}) \otimes \mathbb{C}^2$ *we have:*

$$(g|c^+ g)_{L^2(\Sigma)} = (g_{\rm ext}|D_{\rm ext} f_{\rm ext})_{L^2(\Sigma)},$$

where $f_{\rm ext} = (\psi^*)^{-1} f$, $g_{\rm ext} = (\psi^*)^{-1} g$.

Proof. This follows from Prop. 6.2, the fact that $\hat{R}$ is implemented by the embedding $\hat{r}$ of $\Sigma\backslash\mathscr{B}$ into N, (see (7.1)) and Lemma 7.2. □

7.3. **Uniqueness of the extension**

We discuss now the uniqueness of extensions of ω_β to M. Other types of uniqueness results were obtained before in [13] and [9].

PROPOSITION 7.4. *There exists at most one quasi-free state* ω *for the Klein-Gordon field on* M *such that:*

1. *the restriction of* ω *to* $\mathscr{M}^+ \cup \mathscr{M}^-$ *equals* ω_β,
2. *the spacetime covariances* $\Lambda^\pm$ *of* ω *map* $C_{\rm c}^\infty(M)$ *into* $C^\infty(M)$.

Proof. Let ω a quasi-free state for the Klein-Gordon operator P in M, with spacetime covariances $\Lambda^\pm$. We assume that $\Lambda^\pm : C_{\rm c}^\infty(M) \to C^\infty(M)$.

Denoting by $\Lambda^\pm(x,x')$ their Schwartz kernels, we have $P(x,\partial_x)\Lambda^\pm(x,x') = P(x',\partial_{x'})\Lambda^\pm(x,x') = 0$, which implies that

$$\mathrm{WF}(\Lambda^\pm)' \subset \mathscr{N}\times\mathscr{N}, \tag{7.3}$$

where $\mathscr{N}$ is defined in (3.6). We claim that the entries $c^\pm_{k,k'}$, $k,k'=0,1$ of $c^\pm$ defined in (3.11) map $C^\infty_{\rm c}(\Sigma)$ into $C^\infty(\Sigma)$. In fact by (7.3) we have $\Lambda^\pm = \Lambda^\pm\circ A$ modulo smoothing, where $A\in\Psi^0(M)$ is a pseudodifferential operator with essupp(A) included in an arbitrary small conical neighborhood of $\mathscr{N}$. For $u\in C^\infty_{\rm c}(\Sigma)$ we have, modulo factors of i:

$$c^\pm_{k,k'}u = \partial_t^k\Lambda^\pm\circ A(-\partial_t^{k'}\delta_0\otimes u)\restriction_{t=0},$$

see (3.11). Since $\mathrm{WF}((-\partial_t^{k'}\delta_0)\otimes u)\subset N^*\Sigma$, where $N^*\Sigma\subset T^*M$ is the conormal bundle to Σ and Σ is spacelike, we have $N^*\Sigma\cap\mathscr{N}=\emptyset$, hence $A(-\partial_t^{k'}\delta_0\otimes u)\in C^\infty(M)$, which proves our claim.

Let now ω_i, $i=1,2$ be two quasi-free states as in the proposition. Since $(u|(\Lambda_1^+-\Lambda_2^+)v)_{L^2(M)}=0$ for $u,v\in C^\infty_{\rm c}(\mathscr{M}^+\cup\mathscr{M}^-)$ we obtain that $(f|(\lambda_1^+-\lambda_2^+)g)_{L^2(\Sigma)\otimes\mathbb{C}^2}=0$ for $f,g\in C^\infty_{\rm c}(\Sigma\backslash\mathscr{B})\otimes\mathbb{C}^2$ hence $\mathrm{supp}(\lambda_1^+-\lambda_2^+)g\subset\mathscr{B}$ for $g\in C^\infty_{\rm c}(\Sigma\backslash\mathscr{B})\otimes\mathbb{C}^2$. Since we have seen that $\lambda_i^+ : C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2\to C^\infty(\Sigma)\otimes\mathbb{C}^2$ this implies that $(\lambda_1^+-\lambda_2^+)f=0$ for $f\in C^\infty_{\rm c}(\Sigma\backslash\mathscr{B})\otimes\mathbb{C}^2$. Since λ_i^+ are selfadjoint for $L^2(\Sigma, dVol_h)\otimes\mathbb{C}^2$ this implies that $\mathrm{supp}(\lambda_1^+-\lambda_2^+)f\subset\mathscr{B}$ for $f\in C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2$, hence $(\lambda_1^+-\lambda_2^+)f=0$ using again that $\lambda_i^+ : C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2\to C^\infty(\Sigma)\otimes\mathbb{C}^2$. □

7.4. **The Hartle-Hawking-Israel state**

THEOREM 7.5 ([19]). *Let us set*

$$\lambda^+_{\rm HHI} := q\circ D_{\rm ext},\ \ \lambda^-_{\rm HHI} := \lambda^+_{\rm HHI} - q,$$

where $D_{\rm ext}$ is the Calderón projector for $(K_{\rm ext},\Sigma)$ and the charge quadratic form q is defined in (3.8)*. Then:*

(1) *$\lambda^\pm_{\rm HHI}$ are the Cauchy surface covariances for the Cauchy surface Σ of a quasi-free state $\omega_{\rm HHI}$ for the free Klein-Gordon field on M.*

(2) *the Hartle-Hawking-Israel state $\omega_{\rm HHI}$ is a pure Hadamard state and is the unique extension to M of the double $\beta-$KMS state ω_β with the property that its spacetime covariances $\Lambda^\pm_{\rm HHI}$ map continuously $C^\infty_{\rm c}(M)$ into $C^\infty(M)$.*

Proof. Let us first prove (1). By (3.3) it suffices to check the positivity of $\lambda^+_{\rm HHI}$. This was shown in [19, Thm. 5.3] using reflection positivity. For the reader's convenience, let us briefly repeat the argument:

for $u\in L^2(N)$ we set $Ru(\tau,y)=u(-\tau,y)$, for $\tau\in[-\beta/2,\beta/2]\sim\mathbb{S}_\beta$. The operator $G=K^{-1}$ is *reflection positive, i.e.*

(7.4) $$(Ru|Gu)_{L^2(N)}\geq 0,\ \forall u\in L^2(N),\ \mathrm{supp}u\subset[0,\beta/2]\times\Sigma^+.$$

In fact setting $\tilde{u}=|v|^{3/2}u$, (7.4) is equivalent to
(7.5)
$$(R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta)\otimes L^2(\Sigma^+)}\geq 0,\ \forall\tilde{u}\in L^2(\mathbb{S}_\beta)\otimes L^2(\Sigma^+),\ supp\tilde{u}\subset[0,\beta/2]\times\Sigma^+.$$

Using (4.7) we obtain

$$\begin{aligned}&(R\tilde{u}|\tilde{G}\tilde{u})_{L^2(\mathbb{S}_\beta)\otimes L^2(\Sigma^+)}\\ =\ &(u_0|\tfrac{1}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}u_0)_{L^2(\Sigma^+)}+(u_\beta|\tfrac{1}{2\epsilon(1-\mathrm{e}^{-\beta\epsilon})}u_\beta)_{L^2(\Sigma^+)},\end{aligned}$$

for

$$u_0=\int_{S_\beta}\mathrm{e}^{-\tau\epsilon}\tilde{u}(\tau)d\tau,\ u_\beta=\int_{S_\beta}\mathrm{e}^{(\tau-\beta/2)\epsilon}\tilde{u}(\tau)d\tau$$

where $\tilde{u}$ is identified with the map $\mathbb{S}_\beta\ni\tau\mapsto\tilde{u}(\tau)\in L^2(\mathbb{S}_\beta;L^2(\Sigma^+))$. This proves (7.4).

By Lemma 7.2 and using that $G_{\rm ext}=K_{\rm ext}^{-1}$ is bounded on $L^2(N_{\rm ext})$, we deduce from (7.4) that $G_{\rm ext}$ is also reflection positive, *i.e.*
(7.6)
$$(R_{\rm ext}u|G_{\rm ext}u_{\rm ext})_{L^2(N_{\rm ext})}\geq 0,\ u\in L^2(N_{\rm ext}),\ \mathrm{supp}u\subset N^+_{\rm ext}=\chi([0,\beta/2]\times\Sigma^+),$$

for $R_{\rm ext}=URU^*$. By the remark before [19, Thm. 5.3], if (s,y) are Gaussian normal coordinates to Σ in $N_{\rm ext}$ we have $R_{\rm ext}u(s,y)=u(-s,y)$, *i.e.* $R_{\rm ext}$ is given by the reflection in Gaussian normal coordinates. This map is a isometry of $(N_{\rm ext},\hat{g}_{\rm ext})$, which implies that if $\hat{g}_{\rm ext}=ds^2+h_{\rm ext}(s,y)dy^2$ near Σ, we have $h_{\rm ext}(s,y)=h_{\rm ext}(-s,y)$ hence if $r_s(y)=|h_{\rm ext}(s,y)|^{-\frac12}\partial_s|h_{\rm ext}(s,y)|^{\frac12}$ we have $r_0(y)\equiv 0$.

If $f\in C^\infty_{\rm c}(\Sigma)\otimes\mathbb{C}^2$ it follows from (5.3) that $D_{\rm ext}f=\gamma G_{\rm ext}Tf$ for

$$Tf=\delta_0(s)\otimes f_1-\delta_0'(s)\otimes f_0.$$

We have $R_{\rm ext}Tf=\delta_0(s)\otimes f_1+\delta_0'(s)\otimes f_0$. Applying the reflection positivity (7.6) to $u=Tf$ we obtain that:

$$(R_{\rm ext}Tf|G_{\rm ext}Tf)_{L^2(N_{\rm ext})}=(f|qD_{\rm ext}f)_{L^2(\Sigma)}\geq 0,$$

which proves the positivity of $\lambda^+_{\rm HHI}$. To make the argument rigorous is suffices to approximate δ_0 by a sequence φ_n as in (5.4). This completes the proof of (1).

Let us now prove (2). The fact that $\omega_{\rm HHI}$ is the unique extension of ω_β to M with the stated properties has been proved in Prop. 7.4. It remains to prove that $\omega_{\rm HHI}$ is a pure Hadamard state in M.

The fact that $\omega_{\rm HHI}$ is pure follows from the fact that $D_{\rm ext}$ is a projection. To prove the Hadamard property let us fix a reference Hadamard state $\omega_{\rm ref}$ for the Klein-Gordon field in M. By Thm. 3.3 its Cauchy surface covariances on Σ $\lambda^{\pm}_{\rm ref}$ are matrices of pseudodifferential operators on Σ. The same is true of $c^{\pm}_{\rm ref} = \pm q^{-1}\circ\lambda^{+}_{\rm ref}$ and of $c^{\pm}_{\rm HHI}$, since Calderón projectors are given by matrices of pseudodifferential operators on Σ.

Moreover we know that the restriction of $\omega_{\rm HHI}$ to $\mathscr{M}^{+}\cup\mathscr{M}^{-}$ is a Hadamard state. The same is obviously true of the restriction of $\omega_{\rm ref}$ to $\mathscr{M}^{+}\cup\mathscr{M}^{-}$. Going to Cauchy surface covariances, this implies that if $\chi\in C^{\infty}_{\rm c}(\Sigma^{\pm})$ then

$$\chi\circ(c^{\pm}_{\rm HHI}-c^{\pm}_{\rm ref})\circ\chi \text{ is a smoothing operator on } \Sigma.$$

We claim that this implies that $c^{\pm}_{\rm HHI}-c^{\pm}_{\rm ref}$ is smoothing, which will imply that $\omega_{\rm HHI}$ is a Hadamard state.

If fact let a be one of the entries of $c^{\pm}_{\rm HHI}-c^{\pm}_{\rm ref}$, which is a scalar pseudodifferential operator belonging to $\Psi^{m}(\Sigma)$ for some $m\in\mathbb{R}$. We know that $\chi\circ a\circ\chi$ is smoothing for any $\chi\in C^{\infty}_{\rm c}(\Sigma\backslash\mathscr{B})$. Then its principal symbol $\sigma_{\rm pr}(a)$ vanishes on $T^{*}(\Sigma\backslash\mathscr{B})$ hence on $T^{*}\Sigma$ by continuity, so $a\in\Psi^{m-1}(\Sigma)$. Iterating this argument we obtain that a is smoothing, which proves our claim and completes the proof of the theorem. □

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