QUANTUM OBSERVABLES AS MAGNETIC ΨDO

VIOREL IFTIMIE, MARIUS MĂNTOIU and RADU PURICE

In this paper, we shall present a review of some basic results concerning the *twisted* Weyl quantization [12] with some modified proofs that allow a special focus on the dependence on the behaviour of the magnetic field. The main new result of this paper is contained in Theorem 2.11 and states that *the symbol of the evolution group of the self-adjoint operator defined by a real elliptic symbol of strictly positive order in a smooth bounded magnetic field leaves invariant the space of Schwartz test functions and its dual.*

AMS 2010 Subject Classification: Primary 35SO5, 47L15; Secondary 47L90, 47A60, 81Q10.

Key words: magnetic field, pseudodifferential operator, self-adjoint extension, functional calculus, dynamical system, Moyal product.

1. INTRODUCTION

When dealing with theoretical models for physical systems in *magnetic fields* one faces the well-known problem that some important physical observables are described in terms of the *vector potential* generating the magnetic field, although its choice is highly non-unique! In fact, the basic idea in proposing a theoretical framework for the description of the interaction between a physical system defined by a given Hamiltonian and a magnetic field is the so-called *minimal coupling hypothesis*. It states that the coupling is described by replacing the *canonical momenta* of the physical system by some *magnetic momenta* defined in terms of a vector potential generating the magnetic field. Some theoretical arguments allow then to prove that the description of the dynamics of the system is not modified by changing the vector potential and thus any choice is acceptable in principle. This aspect rises the interesting question of formulating a general abstract mathematical procedure to deal in an elegant and efficient way with this *minimal coupling hypothesis*. For classical systems, an interesting proposal is presented in [5], considering the magnetic field as a perturbation of the canonical symplectic form on the cotangent bundle of the configuration space. For quantum systems, we have proposed in some previous papers ([11–14], see also the monograph [3]) a related procedure based on the replacement of the usual Weyl system with a *twisted Weyl system* defined in terms of a cocycle associated to the flux of the magnetic fields through triangles (a similar construction appears also in [15] but for some different purposes). In [8–10] we have developed the *magnetic pseudodifferential calculus* associated to this *twisted Weyl system*, proving that it behaves very much like the usual one.

In this paper, we concentrate on the *magnetic symbolic* calculus for operators describing observables of quantum system in a bounded smooth magnetic field. A large part of the paper is devoted to a review of some previous results concerning some properties important in the study of quantum observables: composition of symbols, L^2-boundedness, inferior semi-boundedness for elliptic symbols, existence of self-adjoint extensions. Although these results have been presented in our previous papers, we concentrate now on a smaller class of Hörmander type symbols (see Definition 1.1), that is of interest for the study of quantum systems and offer a better perspective on their proofs. Moreover, we take benefit of this simplification of the proofs and obtain an explicit control on the seminorms of the magnetic field that control the different estimates, having in view possible extensions towards problems with unbounded magnetic fields. In the last subsection we present a new result, Theorem 2.11, concerning the evolution group of a quantum Hamiltonian in magnetic fields. This result extends our previous one (Lemma 7.7 in [9]) from Hörmander type elliptic symbols of order at most 1 to those of any order $p > 0$. This is important because one can cover the case of Schrödinger Hamiltonians with magnetic fields, having order 2.

In the first subsection of the introduction, we fix the framework of our analysis and some notations. The following two subsections present a brief reminder of the main facts about *the magnetic quantization* and *the magnetic Moyal algebra* introduced in [11]. The second section contains the main body of the paper analysing the properties of a large class of physical observables defined in terms of *the magnetic pseudodifferential calculus*.

1.1. **Notations**

We consider only systems having an *affine configuration space* $\mathcal{X} \cong \mathbb{R}^d$ for some $d \in \mathbb{N}$ with $d \geq 2$. We shall denote by $\bigwedge^k \mathcal{X}$ the space of smooth k-forms on $\mathcal{X}$. *The magnetic field* is then described by a *closed 2-form* $B \in \bigwedge^2 \mathcal{X}$, thus satisfying $dB = 0$ (see lecture 13 in [3]). Due to the topological triviality of the configuration space $\mathcal{X}$ we can find a 1-form $A \in \bigwedge \mathcal{X}$, called *a vector potential*, such that $B = dA$. Clearly the choice of $A \in \bigwedge \mathcal{X}$ is highly non-unique, different choices being related by *gauge transformations* $A \mapsto A' = A + df$ for some $f \in C^2(\mathcal{X};\mathbb{R})$. A usual choice of the vector potential is *the transversal gauge*:

$$A_j(x) = -\sum_{k=1}^{d} \int_0^1 ds\; B_{jk}(sx) s x_k, \tag{1.1}$$

verifying $x \cdot A(x) = 0$.

A Hamiltonian system is described by a *smooth, lower bounded function* $h : \Xi \to \mathbb{R}$, where $\Xi := \mathcal{X} \times \mathcal{X}^*$ is *the phase space* of the system, with $\mathcal{X}^*$ the dual of $\mathcal{X}$ (as a finite dimensional real vector space), with the duality map $\langle \cdot, \cdot \rangle : \mathcal{X}^* \times \mathcal{X} \to \mathbb{R}$ (see [13] and the references therein).

We shall use the notation $[t] \in \mathbb{Z}$ for the *integer part* of $t \in \mathbb{R}$, defined as $[t] := \max\big\{k \in \mathbb{Z} \mid\ k \leq t\big\}$.

For any Euclidean space $\mathcal{V} \cong \mathbb{R}^N$ we denote by $\mathscr{S}(\mathcal{V})$ the Fréchet space of Schwartz test functions and by $\mathscr{S}'(\mathcal{V})$ its dual, the space of tempered distributions on $\mathcal{V}$. We shall denote by $C^\infty(\mathcal{V})$ the space of smooth functions on $\mathcal{V}$ and by $C^\infty_{\mathsf{pol}}(\mathcal{V})$, resp. by $C^\infty_{\mathsf{pol,u}}(\mathcal{V})$ and resp. by $BC^\infty(\mathcal{V})$ its subspaces of smooth functions that are polynomially bounded together with all their derivatives, resp. those with uniform polynomial growth on all the derivatives, resp. those smooth and bounded together with all their derivatives. We use the notation $< v > := \sqrt{1 + |v|^2}$ for any $v \in \mathcal{V}$.

When working in a Hilbert space $L^2(\mathcal{V})$ over a Euclidean space $\mathcal{V} \cong \mathbb{R}^N$ with the Lebesgue measure, we shall denote by $F(Q)$ the operator of multiplication with the measurable function $F : \mathcal{V} \to \mathbb{C}$, i.e.

$$\big(F(Q)f\big)(v) := F(v)f(v),\ \forall v \in \mathcal{V}, \quad \forall f \in L^2(\mathcal{V}). \tag{1.2}$$

Moreover, we shall denote by $\mathbb{B}\big(\mathcal{H}\big)$ and $\mathcal{U}\big(\mathcal{H}\big)$ the algebra of bounded linear operators, respectively, the group of unitary linear operators on the Hilbert space $\mathcal{H}$.

For k-forms on $\mathcal{X}$, with $k \in \mathbb{N}$, we shall consider the spaces $\mathfrak{L}^k_{\mathsf{pol}}(\mathcal{X})$ and $\mathfrak{L}^k_{\mathsf{bc}}(\mathcal{X})$ defined as the spaces of k-forms with components of class $C^\infty_{\mathsf{pol}}(\mathcal{X})$, resp. $BC^\infty(\mathcal{X})$. Clearly $\mathfrak{L}^0_{\mathsf{pol}}(\mathcal{X}) = C^\infty_{\mathsf{pol}}(\mathcal{X})$ and $\mathfrak{L}^0_{\mathsf{bc}}(\mathcal{X}) = BC^\infty(\mathcal{X})$.

On $\mathfrak{L}^k_{\mathsf{bc}}(\mathcal{X})$ we shall use the following two families of semi-norms, indexed by $m \in \mathbb{N}$:

$$\mu_m(F) := \max_{(j_1,\dots,j_k)\in\{1,\dots,d\}^k} \Big(\max_{|\alpha|=m} \sup_{x\in\mathcal{X}} \big|\big(\partial^\alpha F_{j_1,\dots,j_k}\big)(x)\big| \Big), \tag{1.3}$$

$$\rho_m(F) := \max_{n\leq m} \mu_n(F). \tag{1.4}$$

We shall call *weight* a positive function, verifying the properties of a semi-norm but being allowed to take also the value $+\infty$. On $\mathfrak{L}^k_{\mathsf{pol}}(\mathcal{X})$ we consider *weight functions* indexed by $(p,m)\in\mathbb{R}\times\mathbb{N}$:

$$(1.5)\qquad \nu^p_m(F):=\max_{(j_1,\dots,j_k)\in\{1,\dots,d\}^k}\Big(\sup_{x\in\mathcal{X}}<x>^{-p}\max_{|\alpha|=m}\big|\big(\partial^\alpha F_{j_1,\dots,j_k}\big)(x)\big|\Big).$$

An important role will be played by a specific imaginary exponential of the magnetic flux through some triangles (see (1.30)) and the following family of weights:

$$(1.6)\qquad \forall B\in\mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X}),\quad \mathfrak{w}_m(B):=$$

$$:=\max\Big\{\mu_0(B),\max_{1\le l\le m}\max_{\underline{p}\in\mathscr{P}_l(m)}\prod_{s=1}^{l}\big(\mu_{p_s}(B)+\mu_{p_s-1}(B)\big)\Big\},$$

where $\mathscr{P}_l(m):=\big\{\underline{p}=(p_1,\dots,p_l)\mid p_j\ge 1\,\forall j\in\{1,\dots,d\},\ p_1+\dots+p_l=m\big\}$.

For the points of $\Xi=\mathcal{X}\times\mathcal{X}^*$ we shall use notations of the form $X=(x,\xi),Y=(y,\eta),Z=(z,\zeta)$. On Ξ we have a *canonical symplectic form*:

$$(1.7)\qquad \sigma(Y,Z):=\langle\eta,z\rangle-\langle\zeta,y\rangle.$$

For the space $C^\infty_{\mathsf{pol}}(\Xi)$ we shall use a family of *weight functions* of the form (1.5), but with four indices $(p_1,p_2,m_1,m_2)\in\mathbb{R}^2\times\mathbb{N}^2$:

$$(1.8)\qquad \nu^{p_1,p_2}_{m_1,m_2}(F):=\sup_{(x,\xi)\in\Xi}<x>^{-p_1}<\xi>^{-p_2}\max_{|a|=m_1}\max_{|\alpha|=m_2}\big|\big(\partial^a_x\partial^\alpha_\xi F\big)(x,\xi)\big|,$$

$$\nu^p_{m_1,m_2}(F)\equiv\nu^{0,p}_{m_1,m_2}(F).$$

Definition 1.1. For any $p\in\mathbb{R}$ we denote by:

(1.9)

$$S^p(\Xi):=$$

$$:=\left\{F\in C^\infty_{\mathsf{pol}}(\Xi)\mid \sup_{(x,\xi)\in\Xi}<\xi>^{-p+|\beta|}\big|\big(\partial^\alpha_x\partial^\beta_\xi F\big)(x,\xi)\big|<\infty,\ \forall\,\alpha,\beta\in\mathbb{N}^d\right\},$$

with the topology defined by the semi-norms $\{\nu^{p-m_2}_{m_1,m_2}\}$ with $(m_1,m_2)\in\mathbb{N}^2$. We also set $S^\infty(\Xi):=\bigcup_{p\in\mathbb{R}}S^p(\Xi)$, $S^{-\infty}(\Xi):=\bigcap_{p\in\mathbb{R}}S^p(\Xi)$ and $S^-(\Xi):=\bigcap_{p<0}S^p(\Xi)$.

Let us point out that (1.9) is the class $S^p_{1,0}(\Xi)$ with the notations from [7, 8]. Noticing that $\nu^p_{m_1,m_2}(F)\le\nu^{p-m_2}_{m_1,m_2}(F)$ for any $F\in S^p(\Xi)$ and any $(m_1,m_2)\in\mathbb{N}^2$, we shall also use the following semi-norms on $S^p(\Xi)$:

$$(1.10)\qquad \rho^p_{m_1;m_0,m_2}(F):=\max_{n_1\le m_1}\max_{m_0\le n_2\le m_0+m_2}\nu^p_{n_1,n_2}(F),\quad \forall(m_0,m_1,m_2)\in\mathbb{N}^3.$$

Remark 1.2. We shall often consider the spaces $C^\infty_{\mathsf{pol}}(\mathcal{X})$ and $C^\infty_{\mathsf{pol}}(\mathcal{X}^*)$ as subspaces of $C^\infty_{\mathsf{pol}}(\Xi)$ containing functions constant in the directions in $\mathcal{X}^*$ or resp. of $\mathcal{X}$.

Definition 1.3. We say that a symbol $F \in S^p(\Xi)$ is *elliptic* if there exist two constants $(R, C) \in \mathbb{R}_+ \times \mathbb{R}_+$ such that

$$|F(x,\xi)| \geq C < \xi >^p, \ \forall (x,\xi) \in \mathcal{X} \times \{\xi \in \mathcal{X}^* \mid |\xi| \geq R\}.$$

For any $s \in \mathbb{R}$, we shall use the notation $\mathfrak{p}_s(x,\xi) :=< \xi >^s$, defining an elliptic symbol of order $s \in \mathbb{R}$ and $\mathfrak{q}_s(x,\xi) :=< x >^s$ (that for $s > 0$ is not a Hörmander type symbol).

We shall use the following Fourier transforms ($\forall (x,\xi) \in \mathcal{X} \times \mathcal{X}^*$):

$$\big(\mathcal{F}\phi\big)(\xi) := (2\pi)^{-d/2} \int_{\mathcal{X}} e^{-i\langle \xi, x\rangle} \phi(x)\, dx, \quad \forall \phi \in \mathscr{S}(\mathcal{X}) \tag{1.11}$$

$$\big(\mathcal{F}^-\phi\big)(\xi) := (2\pi)^{-d/2} \int_{\mathcal{X}} e^{i\langle \xi, x\rangle} \phi(x)\, dx, \quad \forall \phi \in \mathscr{S}(\mathcal{X}) \tag{1.12}$$

$$\big(\mathcal{F}_*\psi\big)(x) := (2\pi)^{-d/2} \int_{\mathcal{X}^*} e^{-i\langle \xi, x\rangle} \psi(\xi)\, d\xi, \quad \forall \phi \in \mathscr{S}(\mathcal{X}^*) \tag{1.13}$$

$$\big(\widetilde{\mathcal{F}}F\big)(x,\xi) := (2\pi)^{-d} \int_{\Xi} e^{i\langle \xi, y\rangle - i\langle \eta, x\rangle} F(y,\eta)\, dy\, d\eta, \quad \forall F \in \mathscr{S}(\Xi). \tag{1.14}$$

1.2. **The magnetic quantization**

Given a magnetic field $B \in \bigwedge^2 \mathcal{X}$ and an associated vector potential $A \in \bigwedge \mathcal{X}$, let us consider the following *invariant integrals*, whose significance in constructing a *gauge covariant functional calculus* has been noticed in [2,15]:

$$\Lambda^A(x,y) := e^{-i\int_{[x,y]} A}, \qquad \Omega^B(x,y,z) := e^{-i\int_{<x,y,z>} B}, \tag{1.15}$$

where $[x, y]$ is the oriented line segment from $x \in \mathcal{X}$ to $y \in \mathcal{X}$ and $< x, y, z >$ is the oriented triangle in $\mathcal{X}$ having the vertices $\{x, y, z\} \subset \mathcal{X}$.

Definition 1.4. Given a magnetic field $B \in \bigwedge^2 \mathcal{X}$ and an associated vector potential $A \in \bigwedge \mathcal{X}$, we call *the Magnetic Weyl system* on $\Xi = \mathcal{X} \times \mathcal{X}^*$ associated to $A \in \bigwedge \mathcal{X}$ the application

$$W^A : \Xi \to \mathcal{U}\big(L^2(\mathcal{X})\big),$$

$$\big(W^A(z,\zeta)f\big)(x) := \Lambda^A(x, x+z) e^{-(i/2)\langle \zeta, z\rangle} e^{-i\langle \zeta, x\rangle} f(x+z), \tag{1.16}$$

$$\forall f \in L^2(\mathcal{X}), \ \forall x \in \mathcal{X}.$$

We shall sometimes use the notation $U^A(x) = W^A(-x, 0)$. We refer to [11] for the connection of these operators with the *minimal coupling hypothesis.*

Definition 1.5. Given a magnetic field $B \in \bigwedge^2 \mathcal{X}$ and an associated vector potential $A \in \bigwedge \mathcal{X}$, we define *the magnetic Weyl quantization* as the application

$\mathfrak{Op}^A : \mathscr{S}(\Xi) \to \mathbb{B}\big(L^2(\mathcal{X})\big)$ defined as a bounded sesquilinear form, in the sense that for any pair $(u, v) \in L^2(\mathcal{X})$ we have the equality: by

$$\big\langle u\,,\,\mathfrak{Op}^A(F)v\big\rangle_{L^2(\mathcal{X})} := (2\pi)^{-d}\int_\Xi \big(\widetilde{\mathcal{F}}F\big)(x,\xi)\langle u\,,\,W^A(x,\xi)v\rangle_{L^2(\mathcal{X})}\,dx\,d\xi.$$

Remark 1.6. For $A = 0$ the above formula gives precisely the usual Weyl quantization, denoted by $\mathfrak{Op} \equiv \mathfrak{Op}^0$.

Inserting the definition (1.16) in the above formula (in Definition (1.5)) one obtains the more familiar formula

$$\big(\mathfrak{Op}^A(F)\phi\big)(x) = \tag{1.17}$$
$$= (2\pi)^{-d/2}\int_{\mathcal{X}} dy\,\Lambda^A(x,y)\int_{\mathcal{X}^*} d\xi\, e^{i\langle\xi, x-y\rangle}F\big((x+y)/2,\xi\big)\phi(y)$$

for any $F \in \mathscr{S}(\Xi)$ and any $\phi \in \mathscr{S}(\mathcal{X})$.

PROPOSITION 1.7 (Proposition 3.4 in [11]). *Given two gauge equivalent vector potentials $A' = A + df$, the corresponding magnetic quantizations are unitarily equivalent; more precisely we have*

$$\mathfrak{Op}^{A'}(F) = e^{if(Q)}\mathfrak{Op}^A(F)e^{-if(Q)}, \qquad \forall F \in \mathscr{S}(\Xi). \tag{1.18}$$

PROPOSITION 1.8 (A Diamagnetic Inequality for symbols). *Suppose given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{pol}}(\mathcal{X})$. Then, for any distribution $F \in \mathscr{S}'(\mathcal{X}^*)$ such that $\mathcal{F}_*F$ is a non-negative measure we have:*

$$\big|\mathfrak{Op}^A(F)\phi\big| \le \mathfrak{Op}(F)|\phi|, \quad \forall\phi \in \mathscr{S}(\mathcal{X}), \tag{1.19}$$

with $\mathfrak{Op}(F)$ the usual Weyl quantization of F introduced in Remark 1.6.

Proof. Here we are evidently using Remark 1.2 and consider F as element of $C^\infty_{\mathsf{pol}}(\Xi)$. Thus, $\forall(\phi,\psi) \in \mathscr{S}(\mathcal{X})^2$

$$\big[\mathfrak{Op}^A(F)\phi\big](\overline{\psi}) = (2\pi)^{-d/2}\int_{\mathcal{X}} \mu_F(dx)\big\langle\psi, U^A(-x)\phi\big\rangle_{L^2(\mathcal{X})} =$$

$$= (2\pi)^{-d/2}\int_{\mathcal{X}}\big(< x >^{-N}\,\mu_F(dx)\big) < x >^N \big\langle\psi, U^A(-x)\phi\big\rangle_{L^2(\mathcal{X})}, \tag{1.20}$$

where for $N \in \mathbb{N}$ large enough the measure $\mu_{F,N}(dx) :=< x >^{-N}\,\mu_F(dx)$ is a finite positive measure with total mass $M_{F,N} < \infty$. We have that

$$\big\langle\psi, U^A(-x)\phi\big\rangle_{L^2(\mathcal{X})} = \int_{\mathcal{X}} dz\,\Lambda^A(z, z+x)\overline{\psi(z)}\phi(z+x) \tag{1.21}$$

and we deduce that, for any $N \in \mathbb{N}$, there exists some $C_N > 0$ and some $(n_1, n_2) \in \mathbb{N}\times\mathbb{N}$ such that

$$\Big|\big\langle\psi, U^A(-x)\phi\big\rangle_{L^2(\mathcal{X})}\Big| \le \frac{C_N}{< x >^N}\Big(\sup_{z\in\mathcal{X}} < z >^{n_1} |\psi(z)|\Big)\Big(\sup_{z\in\mathcal{X}} < z >^{n_2} |\phi(z)|\Big).$$

Using the results in [17], we conclude that $\mathfrak{Op}^A(F)\phi$ is a tempered complex measure on $\mathcal{X}$. Thus its absolute value is a well defined positive tempered measure and we can write

$$(1.22) \qquad \left|\mathfrak{Op}^A(F)\phi\right| \leq (2\pi)^{-d/2} \int_{\mathcal{X}} \mu_F(dx) \left|U^A(-x)\phi\right| = \mathfrak{Op}(F)|\phi|. \quad \square$$

From (1.17) we obtain the integral kernel of the operator $\mathfrak{Op}^A(F)$ as

$$(1.23) \qquad \mathfrak{K}_F^A(x,y) = (2\pi)^{-d/2}\Lambda^A(x,y)\big((\mathbb{1}\otimes\mathscr{F}_*)F\big)\big((x+y)/2, y-x\big).$$

Remark 1.9. The map $\Upsilon : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ defined by $\Upsilon(x,y) := \big((x+y)/2, y-x\big)$ is a linear bijection with Jacobian 1 and its inverse has the explicit formula $\Upsilon^{-1}(u,v) = \big(u-v/2, u+v/2\big)$. Using it one can write for any $F\in\mathscr{S}(\Xi)$

$$\mathfrak{K}_F^A = (2\pi)^{-d/2}\Lambda^A \cdot \Big[\Big(\Upsilon\circ\big(\mathbb{1}\otimes\mathscr{F}_*\big)\Big)F\Big].$$

For two vectorial topological spaces $\mathscr{V}_1$ and $\mathscr{V}_2$ we denote by $\mathcal{L}\big(\mathscr{V}_1;\mathscr{V}_2\big)$ the space of linear continuous maps from $\mathscr{V}_1$ to $\mathscr{V}_2$ with the topology of uniform convergence on bounded subsets. The following two statements are proved in [11].

PROPOSITION 1.10. *If the vector potential A is in $\mathfrak{C}^1_{\mathsf{pol}}(\mathcal{X})$, then the application $\mathfrak{Op}^A$ defines a linear and topological isomorphism*

$$(1.24) \qquad \mathfrak{Op}^A : \mathscr{S}(\Xi) \tilde{\to} \mathcal{L}\big(\mathscr{S}'(\mathcal{X});\mathscr{S}(\mathcal{X})\big).$$

Using the *Kernel Theorem* of L. Schwartz ([18]) and Remark 1.9, we can extend the magnetic quantization to the space of tempered distributions.

PROPOSITION 1.11. *If the vector potential $A\in\mathfrak{C}^1_{\mathsf{pol}}(\mathcal{X})$, then the application $\mathfrak{Op}^A$ defines a linear and topological isomorphism*

$$(1.25) \qquad \mathfrak{Op}^A : \mathscr{S}'(\Xi) \tilde{\to} \mathcal{L}\big(\mathscr{S}(\mathcal{X});\mathscr{S}'(\mathcal{X})\big)$$

Using Proposition 1.10, we notice, that given a magnetic field $B\in\mathfrak{C}^2_{\mathsf{pol}}(\mathcal{X})$, formula (1.1) allows us to fix an associated vector potential $A\in\mathfrak{C}^1_{\mathsf{pol}}(\mathcal{X})$, and thus, for any pair of test functions $(\phi,\psi)\in\mathscr{S}(\Xi)\times\mathscr{S}(\Xi)$, the product $\mathfrak{Op}^A(\phi)\mathfrak{Op}^A(\psi)$ belongs to $\mathcal{L}\big(\mathscr{S}'(\mathcal{X});\mathscr{S}(\mathcal{X})\big)$ and there exists a unique test function $\rho^B(\phi,\psi)\in\mathscr{S}(\Xi)$ such that

$$(1.26) \qquad \mathfrak{Op}^A(\phi)\mathfrak{Op}^A(\psi) \;=\; \mathfrak{Op}^A\big(\rho^B(\phi,\psi)\big).$$

Definition 1.12. For any magnetic field $B\in\mathfrak{C}^2_{\mathsf{pol}}(\mathcal{X})$, we define the following composition map:

$$(1.27) \qquad \mathscr{S}(\Xi)\times\mathscr{S}(\Xi) \ni (\phi,\psi) \mapsto \phi\sharp^B\psi := \rho^B(\phi,\psi)\in\mathscr{S}(\Xi),$$

with $\rho^B(\phi,\psi)$ satisfying (1.26). We call this composition defined above *the magnetic Moyal product.*

Clearly $\rho^B(\phi,\psi)\in\mathscr{S}(\Xi)$ depends linearly and continuously on both variables $(\phi,\psi)\in\mathscr{S}(\Xi)\times\mathscr{S}(\Xi)$ (due also to Proposition 1.10). A straightforward computation allows us to prove that

$$(1.28)\quad \big(\phi\sharp^B\psi\big)(X)=\pi^{-2d}\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}\varpi^B_x(y,z)\phi(X-Y)\psi(X-Z)=$$

$$(1.29)\qquad =\pi^{-2d}\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(X-Y,X-Z)}\omega^B(x,y,z)\phi(Y)\psi(Z),$$

where

$$(1.30)\quad \varpi^B_x(y,z):=\exp\left\{-i\int_{\mathscr{T}_x(y,z)}B\right\},\quad \omega^B(x,y,z):=\exp\left\{-i\int_{\mathscr{T}(x,y,z)}B\right\},$$

with $\mathscr{T}_x(y,z)$ the oriented triangle in $\mathcal{X}$ with vertices $x-y-z, x+y-z, x-y+z$ and $\mathscr{T}(x,y,z)$ the oriented triangle in $\mathcal{X}$ with vertices $y+z-x, z+x-y, x+y-z$. In the first Appendix to this paper, we prove a number of needed estimates on the function $\varpi^B\in C^\infty_{\mathsf{pol}}\big(\mathcal{X};C^\infty_{\mathsf{pol}}(\mathcal{X}\times\mathcal{X})\big)$.

In [11] (Lemma 4.14 and Corollary 4.15) the following statement is proved.

Proposition 1.13. *Given a magnetic field $B\in\mathfrak{L}^2_{\mathsf{pol}}(\mathcal{X})$ we have:*

$$(1.31)\qquad \forall(\phi,\psi)\in\mathscr{S}(\Xi)^2:\quad \int_\Xi dX\,\big(\phi\sharp^B\psi\big)(X)=\int_\Xi dX\,\phi(X)\psi(X);$$

$$(1.32)\quad \forall(\phi,\psi,\chi)\in\mathscr{S}(\Xi)^3:$$
$$\int_\Xi dX\,\big(\phi\sharp^B\psi\big)(X)\chi(X)=\int_\Xi dX\,\phi(X)\big(\psi\sharp^B\chi\big)(X)=\int_\Xi dX\,\psi(X)\big(\chi\sharp^B\phi\big)(X)).$$

The above result allows us to extend the magnetic Moyal product by duality and define two continuous bilinear maps

$$(1.33)\qquad \sharp^B:\mathscr{S}'(\Xi)\times\mathscr{S}(\Xi)\to\mathscr{S}'(\Xi);\quad \sharp^B:\mathscr{S}(\Xi)\times\mathscr{S}'(\Xi)\to\mathscr{S}'(\Xi).$$

1.3. **The magnetic Moyal algebra**

We shall briefly recall some definitions and results from [11]. Let us define

$$(1.34)\quad \mathfrak{M}^B(\Xi):=\left\{F\in\mathscr{S}'(\Xi)\mid F\sharp^B\phi\in\mathscr{S}(\Xi),\ \phi\sharp^B F\in\mathscr{S}(\Xi),\ \forall\phi\in\mathscr{S}(\Xi)\right\}.$$

Remark 1.14 (Proposition 4.20 in [11]). $\mathfrak{M}^B(\Xi)$ is a unital *-algebra (with the *-involution given by the complex conjugation) containing $\mathscr{S}(\Xi)$ as a two-sided *-ideal.

Definition 1.15. We call $\mathfrak{M}^B(\Xi)$ *the magnetic Moyal algebra* associated to the magnetic field $B \in \mathfrak{L}^2_{\mathsf{pol}}(\mathcal{X})$.

Using Proposition 1.13 we can extend $\sharp^B : \mathfrak{M}^B(\Xi) \times \mathfrak{M}^B(\Xi) \to \mathfrak{M}^B(\Xi)$ by duality to the following applications:

$$\sharp^B : \mathfrak{M}^B(\Xi) \times \mathscr{S}'(\Xi) \to \mathscr{S}'(\Xi), \quad \sharp^B : \mathscr{S}'(\Xi) \times \mathfrak{M}^B(\Xi) \to \mathscr{S}'(\Xi). \tag{1.35}$$

PROPOSITION 1.16. *If the vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$, then the application $\mathfrak{Op}^A$ defines a linear and topological isomorphism*

$$\mathfrak{Op}^A : \mathfrak{M}^B(\Xi) \xrightarrow{\sim} \mathcal{L}\big(\mathscr{S}(\mathcal{X}); \mathscr{S}(\mathcal{X})\big) \bigcap \mathcal{L}\big(\mathscr{S}'(\mathcal{X}); \mathscr{S}'(\mathcal{X})\big). \tag{1.36}$$

Definition 1.17. Given a vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$, we define *the algebra of bounded magnetic symbols* associated to the magnetic field $B = dA$ as

$$\mathfrak{C}^B(\Xi) := \left\{F \in \mathscr{S}'(\Xi) \mid \mathfrak{Op}^A(F)\big[L^2(\mathcal{X})\big] \subset L^2(\mathcal{X})\right\}. \tag{1.37}$$

By the Uniform Boundedness Principle ([16]) $F \in \mathfrak{C}^B(\Xi)$ if and only if $\mathfrak{Op}^A(F) \in \mathbb{B}\big(L^2(\mathcal{X})\big)$ and using Proposition 1.7 we see that this condition only depends on the magnetic field $B = dA$.

PROPOSITION 1.18 (Lemma 2.1 in [8]). *Given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{pol}}(\mathcal{X})$ we have that $S^p(\Xi) \subset \mathfrak{M}^B(\Xi)$, $\forall p \in \mathbb{R}$.*

Remark 1.19. We may transport on $\mathfrak{C}^B(\Xi)$ the operator norm from $\mathbb{B}\big(L^2(\mathcal{X})\big)$ and denote it by $\|F\|_B := \|\mathfrak{Op}^A(F)\|_{\mathbb{B}(L^2(\mathcal{X}))}$. This norm only depends on B, due to Proposition 1.7. Then $\mathfrak{C}^B(\Xi)$ becomes a C^*-algebra isomorphic with $\mathbb{B}\big(L^2(\mathcal{X})\big)$.

2. OBSERVABLES IN BOUNDED SMOOTH MAGNETIC FIELDS

In the estimates that follow, we shall often use operators of the form $< \nabla >^s$ (Fourier transforms of symbols of type $\mathfrak{p}_s$) and we shall frequently prefer to work with differential operators, so that we shall, when possible, consider orders of the form $s = 2N$ with $N \in \mathbb{N}$ even if this will give slightly weaker results. We shall use the notation $\tilde{d} := 2[d/2] + 2$ and for any $p \in \mathbb{R}_+$ we set $\tilde{p} := 2[(d+p)/2] + 2$.

2.1. Composition of Hörmander type symbols

In Proposition 2.6 in [8], we have proven a result concerning the composition of Hörmander type symbols that is very similar to the known result for

Weyl calculus. We present in the following Theorem a new proof of a simplified version (which is enough for the applications to quantum mechanics that we have in view), emphasizing the dependence of the estimates on the behaviour of the magnetic field. In fact, this Theorem is a direct consequence of the Proposition 3.2 that we prove in the second Appendix to this paper.

THEOREM 2.1. *Given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$, $\forall (p_1, p_2) \in \mathbb{R}^2$ the restriction of the Moyal product to $S^{p_1}(\Xi) \times S^{p_2}(\Xi)$ defines a continuous bilinear application $S^{p_1}(\Xi) \times S^{p_2}(\Xi) \ni (F, G) \mapsto F \sharp^B G \in S^{p_1+p_2}(\Xi)$. More precisely, for any pair $(q_1, q_2) \in \mathbb{N} \times \mathbb{N}$ there exists a constant $C := C(d, p_1, p_2, q_1, q_2) > 0$ such that*

$$\nu^{p_1+p_2-q_2}_{q_1,q_2}\big(F\sharp^B G\big) \leq$$
$$\leq C\, \mathfrak{w}_{q_1+\tilde{p}_1+\tilde{p}_2}(B) \sum_{0\leq k\leq q_2} \rho^{p_1-k}_{q_1+\tilde{p}_2;k,m_2}(F)\, \rho^{p_2-q_2+k}_{q_1+\tilde{p}_1;q_2-k,m_1}(G),$$

where $m_1 = 2[\tilde{p}_2 + (q_1 + \tilde{p}_1)/2] + 2$ and $m_2 = 2[\tilde{p}_1 + (q_1 + \tilde{p}_2)/2] + 2$.

We formulate separately a consequence of the above result that will be used several times in this paper.

PROPOSITION 2.2. *Given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$, for any $(p_1, p_2) \in \mathbb{R}^2$ and $(F, G) \in S^{p_1}(\Xi) \times S^{p_2}(\Xi)$, we have $F\sharp^B G - FG \in S^{p_1+p_2-1}(\Xi)$. Moreover, for any $(q_1, q_2) \in \mathbb{N} \times \mathbb{N}$, there exists a constant $C > 0$, depending on d, p_1, p_2, q_1, q_2, such that*

$$\nu^{p_1+p_2-1-q_2}_{q_1,q_2}\big(F\sharp^B G - FG\big) \leq C \mathfrak{w}_{q_1+\tilde{p}_1+\tilde{p}_2}(B)\mathfrak{w}_{2+q_1+\tilde{p}_1+\tilde{p}_2}(B) \times$$
$$\times \sum_{0\leq k\leq q_2} \Big[\Big(\rho^{p_1-k}_{q_1+\tilde{p}_2;k,m_2}(\nabla_x F) + \rho^{p_1-k-1}_{q_1+\tilde{p}_2;k,m_2}(\nabla_\xi F)\Big) \times$$
$$\times \Big(\rho^{p_2-q_2+k-1}_{q_1+\tilde{p}_1;q_2-k,m_1}(\nabla_\xi G) + \rho^{p_2-q_2+k}_{q_1+\tilde{p}_1;q_2-k,m_1}(\nabla_x G)\Big)\Big]$$

where $m_1 = 2[\tilde{p}_2 + (q_1 + \tilde{p}_1)/2] + 2$ and $m_2 = 2[\tilde{p}_1 + (q_1 + \tilde{p}_2)/2] + 2$.

Proof. We go back to formula (1.28) and notice that

$$\big(F\sharp^B G\big)(X) - F(X)G(X) =$$
$$= \pi^{-2d} \int_{\Xi\times\Xi} dY\, dZ\, e^{-2i\sigma(Y,Z)} \omega^B_x(y, z)\, F(X - Y)\, G(X - Z) - F(X)G(X),$$

$$\big(F\sharp^B G\big)(X) - F(X)G(X) = \pi^{-2d} \int_{\Xi\times\Xi} dY\, dZ\, e^{-2i\sigma(Y,Z)} \omega^B_x(y, z) \times$$
$$\times \left[\int_0^1 ds\big((Y\cdot\nabla)F\big)(X - sY)\right] \left[\int_0^1 dt\big((Z\cdot\nabla)G\big)(X - tZ)\right]$$

and using the usual integration by parts method one transforms the factors linear in Y and Z in derivations obtaining in the end a sum of products of derivatives of order 1 and 2 of the symbols F and G and of derivatives of order at most two of the factor ϖ^B. $\square$

2.2. A criterion for L^2-boundedness

From the Schur-Holmgren criterion for boundedness of integral operators on L^2 (see for example Theorem 5.2 in [6]), we easily conclude that

$$\left\|\mathfrak{Op}^A(F)\right\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq \sup_{x\in\mathcal{X}}\left\|\mathfrak{K}^A_F(x,\cdot)\right\|_{L^1(\mathcal{X})}. \tag{2.1}$$

Using then Remark 1.9 above together with Proposition 1.3.6 in [1] and Lemma A.4 in [14], we obtain the following statement.

Proposition 2.3. *Suppose that $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$. Then $S^-(\Xi) \subset \mathfrak{C}^B(\Xi)$, i.e. $\mathfrak{Op}^A(F)$ is a bounded operator on $L^2(\mathcal{X})$ for any $F \in S^-(\Xi,\mathcal{X})$ and*

$$\|F\|_B \equiv \left\|\mathfrak{Op}^A(F)\right\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq \nu^{-s}_{0,\tilde{d}}(F) \tag{2.2}$$

for $s > 0$ such that $F \in S^{-s}(\Xi)$.

2.3. Inferior semi boundedness

Proposition 2.4. *Suppose given $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and a real symbol F in $S^p(\Xi)$, with $p \geq 0$, elliptic if $p > 0$, and verifying $F \geq a_F > 0$ for some $a_F \in \mathbb{R}_+$. Then there exist $G \in S^{p/2}(\Xi)$ and $X \in S^0(\Xi)$ such that $F = G\sharp^B G + X$ and $\left\|\mathfrak{Op}^A(X)\right\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq \mathscr{W}_p(B)\mathscr{N}_p(F)$ where*

- *$\mathscr{W}_p(B)$ is a polynomial of maximum degree $2[p] + 2$ in the weights $\{\mathfrak{w}_m(B)\}_{m\leq M}$ for some $M = M(d,p) \in \mathbb{N}$, having positive coefficients depending only on d and p,*
- *$\mathscr{N}_p(F)$ a polynomial of maximum degree $2[p] + 2$ in the seminorms $\nu^{P_1}_{M_1,M_2}(F)$ with P_1, M_1 and M_2 depending only on d and p, and with positive coefficients depending only on d, p and $(a_F)^{-j} < \infty$, with $0 \leq j \leq 2[p] + 2$.*

Proof. We can define $G_0 := \sqrt{F} \in S^{p/2}(\Xi)$ and use Proposition 2.2 to get:

$$X^B_1(F) := F - G_0\sharp^B G_0 = G_0^2 - G_0\sharp^B G_0 \ \in\ S^{p-1}(\Xi), \tag{2.3}$$

with $\nu^{p-1-q_2}_{q_1,q_2}\big(X^B_1(F)\big)$ satisfying an estimate of the form obtained in Proposition 2.2 with $p_1 = p_2 = p/2$:

$$\begin{aligned} &\nu^{p-1-q_2}_{q_1,q_2}\big(X^B_1(F)\big) \leq \\ &\leq C\mathfrak{w}_{q_1+2n}(B)\mathfrak{w}_{2+q_1+2n}(B)\nu^{p/2}_{q_1+n,q_2+m}(\nabla_x\sqrt{F})\nu^{p/2-1}_{q_1+n,q_2+m}(\nabla_\xi\sqrt{F}). \end{aligned} \tag{2.4}$$

But $G_0 = \sqrt{F} \geq \sqrt{a_F} > 0$ so that we can define

$$G_1 := (1/2) G_0^{-1} X_1^B(F) \in S^{p/2-1}(\Xi) \tag{2.5}$$

and notice that

$$\begin{aligned} F - \big(G_0 + G_1\big)\sharp^B\big(G_0 + G_1\big) &= F - G_0\sharp^B G_0 - G_1\sharp^B G_1 - \big(G_0\sharp^B G_1 + G_1\sharp^B G_0\big) \\ &= -G_1\sharp^B G_1 - \big(G_0\sharp^B G_1 + G_1\sharp^B G_0 - 2G_0G_1\big) =: X_2^B(F) \in S^{p-2}(\Xi) \end{aligned}$$

with $\nu^{p-2-q_2}_{q_1,q_2}\big(X_2^B(F)\big)$ satisfying an estimate obtained from Proposition 2.2 and Theorem 2.1.

$$\begin{aligned} \nu^{p-2-q_2}_{q_1,q_2}\big(X_2^B(F)\big) &\leq C\mathfrak{w}_{q_1+2n}(B)\nu^{p/2}_{q_1+n,q_2+m}(\nabla_x G_1)\nu^{p/2-1}_{q_1+n,q_2+m}(\nabla_\xi G_1) \\ &+ C\mathfrak{w}_{q_1+2n}(B)\mathfrak{w}_{2+q_1+2n}(B)\nu^{p/2}_{q_1+n,q_2+m}(\nabla_x G_0)\nu^{p/2-1}_{q_1+n,q_2+m}(\nabla_\xi G_1) \\ &+ \mathfrak{w}_{q_1+2n}(B)\mathfrak{w}_{2+q_1+2n}(B)\nu^{p/2-1}_{q_1+n,q_2+m}(\nabla_\xi G_0)\nu^{p/2}_{q_1+n,q_2+m}(\nabla_x G_1). \end{aligned} \tag{2.6}$$

Let us set $n_p := [p] + 1 \in \mathbb{N}$ and define recursively for $1 \leq k \leq n_p$

$$\begin{cases} X_k^B := F - \Big(\sum\limits_{j=0}^{k-1} G_j\Big)\sharp^B\Big(\sum\limits_{j=0}^{k-1} G_j\Big) \in S^{p-n_p}(\Xi) \subset S^0(\Xi), \\ G_k := (1/2)G_0^{-1}X_k^B \in S^{(p/2)-n_p}(\Xi). \end{cases} \tag{2.7}$$

Using the above results we obtain

$$\nu^{p-n_p-q_2}_{q_1,q_2}\big(X^B_{n_p}\big) \leq \mathscr{W}_p(B)\,\mathscr{N}_p(F), \tag{2.8}$$

where $\mathscr{W}_p(B)$ is a polynomial of maximum degree $2n_p$, with positive coefficients depending only on d, p, q_1 and q_2 in the weights $\mathring{\mathfrak{w}}_M(B)$ with $q_1 + 2\tilde{p} \leq M \leq 2 + q_1 + \tilde{p}n_p$ and $\mathscr{N}_p(F)$ is a polynomial of maximum degree $2n_p$, with positive coefficients depending only on d, p, q_1, q_2 and $(a_F)^{-j} < \infty$ (with $0 \leq j \leq n_p$) in the semi-norms $\nu^{P_1}_{M_1,M_2}(F)$ with $0 \leq P_1 \leq p$, $q_1 + \tilde{p} \leq M_1 \leq q_1 + \tilde{p}n_p$ and $q_2 + m \leq M_2 \leq q_2 + mn_p$. $\square$

Remark 2.5. The above Proposition implies that, for any $\phi \in \mathscr{S}(\mathcal{X})$, we have the estimate

$$\big\langle \phi, \mathfrak{Op}^A(F)\phi\big\rangle_{L^2(\mathcal{X})} \geq -\Big\|\mathfrak{Op}^A(X^B_{n_p})\Big\|_{\mathbb{B}(L^2(\mathcal{X}))} \|\phi\|^2_{L^2(\mathcal{X})}, \tag{2.9}$$

and one concludes that

$$\mathfrak{Op}^A(F) + \big(\mathscr{W}_p(B)\cdot\mathscr{N}_p(F) + a_F\big)\mathbb{1} \geq a_F\mathbb{1} > 0. \tag{2.10}$$

2.4. **A Calderon-Vaillancourt type Theorem**

In this subsection, we give another proof for Theorem 3.1 in [8] for symbols in our class $S^0(\Xi)$, following the idea of [7] for the case of the Weyl calculus.

THEOREM 2.6. *For any magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ we have $S^0(\Xi) \subset \mathfrak{C}^B(\Xi)$. More precisely, for any $F \in S^0(\Xi)$ let*

$$\mathscr{M}^B(F) := \sqrt{\nu^0_{0,0}(F)^2 + C(d)\mathscr{W}_0(B)\,\mathscr{N}_0(F) + \nu^{-1}_{0,\tilde{d}}(\overline{F}\sharp^B F - F^2)}. \tag{2.11}$$

Then:

$$\|F\|_B \equiv \left\|\mathfrak{Op}^A(F)\right\|_{\mathbb{B}(L^2(\mathcal{X}))} < \mathscr{M}^B(F). \tag{2.12}$$

Proof. From the definition of the class $S^0(\Xi)$ in Definition 1.1 we conclude that given any $F \in S^0(\Xi)$ there exists $M_F = \nu^0_{0,0}(F) > 0$ such that $|F(X)| \leq M_F$ for any $X \in \Xi$. Then $\widetilde{F} := (M_F + \delta)^2 - \overline{F}F$ is a strictly positive symbol of class $S^0(\Xi)$ for any $\delta > 0$. For any $\phi \in \mathscr{S}(\mathcal{X})$ we can compute

$$\left\|\mathfrak{Op}^A(F)\phi\right\|^2_{L^2(\mathcal{X})} = \left\langle\mathfrak{Op}^A(F)\phi, \mathfrak{Op}^A(F)\phi\right\rangle_{L^2(\mathcal{X})} = \left\langle\phi, \mathfrak{Op}^A(\overline{F}\sharp^B F)\phi\right\rangle_{L^2(\mathcal{X})} =$$

$$= \left\langle\phi, \mathfrak{Op}^A(\overline{F}\sharp^B F - |F|^2)\phi\right\rangle_{L^2(\mathcal{X})} + (M_F+\delta)^2\|\phi\|^2_{L^2(\mathcal{X})} - \left\langle\phi\,,\,\mathfrak{Op}^A(\widetilde{F})\phi\right\rangle_{L^2(\mathcal{X})}.$$

We can apply Proposition 2.4 above (with $p = 0$) and deduce that there exists some symbol $G^B_F \in S^0(\Xi)$ and some symbol $X^B_F \in S^{-1}(\Xi)$ such that

$$\widetilde{F} = G^B_F \sharp^B G^B_F + X^B_F, \qquad \left\|X^B_F\right\|_B \leq C(d)\mathscr{W}_0(B)\,\mathscr{N}_0(F).$$

Then Proposition 2.2 implies $|F|^2 - F\sharp^B F \in S^{-1}(\Xi)$ and the given estimates on its seminorms.

Thus for any $\phi \in \mathscr{S}(\mathcal{X})$ one has

(2.13)

$$\begin{aligned}\left\|\mathfrak{Op}^A(F)\phi\right\|^2_{L^2(\mathcal{X})} &= (M_F+\delta)^2\|\phi\|^2_{L^2(\mathcal{X})} - \\ &- \left\|\mathfrak{Op}^A(G^B_F)\phi\right\|^2_{L^2(\mathcal{X})} - \left\langle\phi\,,\,\mathfrak{Op}^A\left(X^B_F + (\overline{F}\sharp^B F - |F|^2)\right)\phi\right\rangle_{L^2(\mathcal{X})} \leq \\ &\leq \Big((M_F+\delta)^2 + C(d)\mathscr{W}_0(B)\,\mathscr{N}_0(F) + \nu^{-1}_{0,\tilde{d}}(\overline{F}\sharp^B F - |F|^2)\Big)\|\phi\|^2_{L^2(\mathcal{X})}, \ \forall\delta > 0. \quad \square\end{aligned}$$

2.5. **Self-adjointness**

It is well known that the physical observables of a quantum system with configuration space $\mathcal{X}$ are described by self-adjoint operators acting in the Hilbert space $L^2(\mathcal{X})$. We remark that *any real symbol in $\mathfrak{C}^B(\Xi)$ defines a bounded physical observable.*

In order to study *unbounded physical observables*, we have to pay attention to the domain of definition of magnetic quantized operators. A procedure to prove self-adjointness in $L^2(\mathcal{X})$ for an operator of the form $\mathfrak{Op}^A(F)$ for some real symbol $F \in \mathfrak{M}^B(\Xi)$ is to construct *a resolvent* for it. More precisely, to prove existence of two symbols

$$\mathfrak{r}^B_\pm(F) \in \mathfrak{C}^B(\Xi) \bigcap \mathfrak{M}^B(\Xi) \tag{2.14}$$

such that

$$(F \mp i)\sharp^B \mathfrak{r}^B_\pm(F) = 1 \qquad \mathfrak{r}^B_\pm(F)\sharp^B(F \mp i) = 1. \tag{2.15}$$

Definition 2.7. Given a vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$, for any $s \in \mathbb{R}_+$ we define *the magnetic Sobolev space* of order s as

$$\mathscr{H}^s_A(\mathcal{X}) := \left\{ f \in L^2(\mathcal{X}),\, \mathfrak{Op}^A(\mathfrak{p}_s) f \in L^2(\mathcal{X}) \right\} \tag{2.16}$$

endowed with the scalar product

$$\langle f, g\rangle_{\mathscr{H}^s_A} := \langle f, g\rangle_{L^2(\mathcal{X})} + \langle \mathfrak{Op}^A(\mathfrak{p}_s) f, \mathfrak{Op}^A(\mathfrak{p}_s) g\rangle_{L^2(\mathcal{X})}. \tag{2.17}$$

Remark 2.8.

- By Proposition 3.5 in [8], $\mathscr{H}^s_A(\mathcal{X})$ is a Hilbert space.
- Moreover for unbounded vector potentials, and thus for magnetic fields that do not vanish at infinity, these *magnetic Sobolev spaces* are different from the usual Sobolev spaces, their elements having also some decay properties as functions on $\mathcal{X}$.

For the rest of this section, we shall suppose that $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and make use of the notation and results in the first Appendix A.1. The following Theorem contains the main results in Theorem 5.1 in [8] and Proposition 6.31 in [9]. We present here a new proof of these results developing the ideas in the proof of Theorem 1.8 in [14].

THEOREM 2.9. *Given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and a vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$, for any real, lower semi bounded elliptic symbol $F \in S^p(\Xi)$ with $p > 0$ we have that:*

1. *there exist some symbols $\mathfrak{r}^B_\pm(F) \in \mathscr{S}'(\Xi)$ such that $(F \mp i)\sharp^B \mathfrak{r}^B_\pm(F) = 1$ and $\mathfrak{r}^B_\pm(F)\sharp^B(F \mp i) = 1$;*
2. *$\mathfrak{r}^B_\pm(F) \in S^{-p}(\Xi)$;*
3. *$\mathfrak{Op}^A(F)$ is self-adjoint in $L^2(\mathcal{X})$ with domain $\mathscr{H}^p_A(\mathcal{X})$ and essentially self-adjoint on $\mathscr{S}(\mathcal{X})$.*

Proof. By the hypothesis of the Theorem (see also Definition 1.3) there exist two constants $R > 0$ and $C > 0$ such that for $|\xi| \geq R$ we have the bound $C < \xi >^p \leq F(x, \xi)$. Then let us fix some $a > 0$ large enough such that $F + a > 0$, set $F_a := F + a$ and compute

$$\big(F_a \sharp^B F_a^{-1}\big)(X) - 1 = \pi^{-2d} \int_{\Xi\times\Xi} dY\, dZ\, e^{-2i\sigma(Y,Z)} \varpi^B_x(y,z) \frac{F(X-Y)+a}{F(X-Z)+a} - 1 =$$

$$= \pi^{-2d} \int_{\Xi\times\Xi} dY\, dZ\, e^{-2i\sigma(Y,Z)} \varpi^B_x(y,z)\times$$

(2.18)

$$\times \int_0^1 d\tau \frac{(Z-Y)\cdot\big(\nabla F\big)(X-Z+\tau(Z-Y))}{F(X-Z)+a}.$$

Integrating by parts the term containing the factor

$$(z_j - y_j)e^{-2i(\langle\eta,z\rangle-\langle\zeta,y\rangle)} = -(1/2i)\big(\partial_{\eta_j}+\partial_{\zeta_j}\big)e^{-2i(\langle\eta,z\rangle-\langle\zeta,y\rangle)}, \tag{2.19}$$

we obtain an oscillating integral of the following function:

$$\big(\partial_{\eta_j}+\partial_{\zeta_j}\big)\frac{\big(\partial_{x_j}F\big)(X-Z+\tau(Z-Y))}{F(X-Z)+a}. \tag{2.20}$$

We have to take into account that $\partial_{\zeta_j}F_a^{-1} = \big(\partial_{\xi_j}F\big)F_a^{-2}$, $\partial_{x_j}F\in S^p(\Xi)$, $\partial_{\xi_j}F\in S^{p-1}(\Xi)$, $\partial_{\xi_j}\partial_{x_j}F\in S^{p-1}(\Xi)$, $(F+a)^{-1}\in S^{-p}(\Xi)$ and $(F+a)^{-2}\in S^{-2p}(\Xi)$ and use the result in Proposition 3.2 to obtain the following upper bound for the seminorm $\nu^{-m}_{n,m}$ of the oscillating integral:

$$C(d,p,n,m)\mathfrak{w}_{n+n_1+n_2}(B)\sum_{0\le k\le m}\Big(\rho^{p-1-k}_{n+n_2+1;k+1,m_2}(F)\,\rho^{-(p-1-k+m)}_{n+n_1;m-k,m_1}(F_a^{-1})+$$

$$+\rho^{p-k}_{n+n_2;k,m_2}(F)\,\rho^{-(p-k+m)}_{n+n_1;m-k+1,m_1}(F_a^{-1})\Big), \tag{2.21}$$

where $n_1 = 2[(d+p)/2]+2$, $n_2 = 2[(d-p)/2]+2$, $m_1 = 2[(n+n_1)/2+n_2]+2$ and $m_2 = 2[(n+n_2)/2+n_1]+2$.

Let us study now the term containing the factor

$$(\zeta_j - \eta_j)e^{-2i(\langle\eta,z\rangle-\langle\zeta,y\rangle)} = (1/2i)\big(\partial_{y_j}+\partial_{z_j}\big)e^{-2i(\langle\eta,z\rangle-\langle\zeta,y\rangle)} \tag{2.22}$$

and leading to an oscillating integral of the function

$$\big(\partial_{y_j}+\partial_{z_j}\big)\Big(\varpi^B_x(y,z)\frac{\big(\partial_{\xi_j}F\big)(X-Z+\tau(Z-Y))}{F(X-Z)+a}\Big). \tag{2.23}$$

As in the previous analysis we obtain the following estimates on the seminorms $\nu^{-m}_{n,m}$ of the oscillating integral:

$$C(d,p,n,m)\mathfrak{w}_{n+n_1+n_2}(B)\sum_{0\le k\le m}\Big(\rho^{p-1-k}_{n+n_2+1;k+1,m_2}(F)\,\rho^{-(p-1-k+m)}_{n+n_1;m-k,m_1}(F_a^{-1})+$$

$$+\rho^{p-k-1}_{n+n_2;k,m_2}(\nabla_\xi F)\,\rho^{-(p-1-k+m)}_{n+n_1+1;m-k,m_1}(F_a^{-1})+$$

$$+\rho_1(B)\rho^{p-k-1}_{n+n_2+1;k+1,m_2}(F)\,\rho^{-(p-1-k+m)}_{n+n_1+1;m-k,m_1}(F_a^{-1})\Big), \tag{2.24}$$

where $n_1 = 2[(d+p)/2]+2$, $n_2 = 2[(d-p)/2]+2$, $m_1 = 2[(n+n_1)/2+n_2]+2$ and $m_2 = 2[(n+n_2)/2+n_1]+2$.

First we notice that

$$\rho_{n;r,m}^{-(p-1+r)}(F_a^{-1}) = \max_{|\gamma|=r} \rho_{n;0,m}^{-(p-1+r)}\big(\partial_\xi^\gamma F_a^{-1}\big),$$

$$\rho_{n;0,m}^{-(p-1+|\gamma|)}\big(\partial_\xi^\gamma F_a^{-1}\big) = \max_{|\gamma^1|\leq n} \max_{|\gamma^2|\leq m} \sup_{(x,\xi)\in\Xi} <\xi>^{p-1+|\gamma|} \big|\big(\partial_x^{\gamma^1}\partial_\xi^{\gamma^2+\gamma}F_a^{-1}\big)(x,\xi)\big|.$$

Let us denote by:

$$\mathfrak{F}_{(\beta^1,\dots,\beta^k)}^{(\alpha^1,\dots,\alpha^l)}(X) := \prod_{s=1}^{l}\frac{1}{\alpha^s!}\prod_{t=1}^{k}\frac{1}{\beta^t!}\left|\big(\partial_x^{\alpha^s}\partial_\xi^{\beta^t}F\big)(X)\right|, \tag{2.25}$$

$$\widetilde{\mathfrak{F}}_{(\beta^1,\dots,\beta^k)}^{(\alpha^1,\dots,\alpha^l)}(a;X) := \prod_{s=1}^{l}\frac{1}{\alpha^s!}\prod_{t=1}^{k}\frac{1}{\beta^t!}\frac{\left|\big(\partial_x^{\alpha^s}\partial_\xi^{\beta^t}F\big)(X)\right|}{F_a}, \tag{2.26}$$

so that using Faá di Bruno's formula [4] we can write

$$\begin{aligned} &<\xi>^{p-1+|\gamma|}\big|\big(\partial_x^{\gamma^1}\partial_\xi^{\gamma^2+\gamma}F_a^{-1}\big)(x,\xi)\big| \leq <\xi>^{p-1+|\gamma|}(\gamma^1!)((\gamma^2+\gamma)!)\times \\ &\times\sum_{1\leq l\leq n}\frac{(-1)^l}{l!}\sum_{1\leq k\leq m}\frac{(-1)^k}{k!}\big|F(X)+a\big|^{-(l+k+1)}\sum_{\substack{\alpha^1+\cdots+\alpha^l=\gamma^1\\ \beta^1+\cdots+\beta^k=\gamma^2+\gamma}}\mathfrak{F}_{(\beta^1,\dots,\beta^k)}^{(\alpha^1,\dots,\alpha^l)}(X)\leq \\ &\leq(\gamma^1!)((\gamma^2+\gamma)!)\frac{<\xi>^{p-1}}{F_a}\sum_{\substack{1\leq l\leq n\\ 1\leq k\leq m}}\frac{(-1)^{l+k}}{l!k!}<\xi>^{|\gamma|}\sum_{\substack{\alpha^1+\cdots+\alpha^l=\gamma^1\\ \beta^1+\cdots+\beta^k=\gamma^2+\gamma}}\widetilde{\mathfrak{F}}_{(\beta^1,\dots,\beta^k)}^{(\alpha^1,\dots,\alpha^l)}(a;X)\leq \end{aligned} \tag{2.27}$$

$$\leq C(F,n,m,r)\sup_{\xi\in\mathcal{X}^*}\frac{<\xi>^{p-1}}{<\xi>^p+a}. \tag{2.28}$$

Using the monotonicity and the concavity of the logarithm function one can prove that $\forall(a,b,q)\in\mathbb{R}_+\times\mathbb{R}_+\times(0,1)$:

$$a+b\geq\big(q^{-1}a\big)^q\big((1-q)^{-1}b\big)^{(1-q)}. \tag{2.29}$$

Taking $b=<\xi>^p$ we obtain that

$$\frac{<\xi>^{p-1}}{<\xi>^p+a} = \frac{b^{1-1/p}}{a+b} \leq q\left(\frac{1-q}{q}\right)^{1-q}\frac{<\xi>^{pq-1}}{a^q} \tag{2.30}$$

and thus, choosing some $q\in(0,p^{-1})\cap(0,1)$ we obtain that for any $r\in\mathbb{N}$ we have the estimate

$$\rho_{n;r,m}^{-(p-1+r)}(F_a^{-1}) \;\leq\; C(F,n,m,p,q)\,a^{-q}. \tag{2.31}$$

Using also (2.21), (2.24) and Proposition 2.2, and denoting by

$$\mathfrak{r}_F^B(a) := F_a\sharp^B F_a^{-1} - 1 \in S^{-1}(\Xi), \tag{2.32}$$

we obtain for any $q\in(0,p^{-1})\cap(0,1)$ that

$$\nu_{n,m}^{-1-m}\big(\mathfrak{r}_F^B(a)\big)\ \le\ C(F,n,m,p,q)\,a^{-q}\,\mu_1(B)\mathfrak{w}_{n+n_1+n_2}(B). \tag{2.33}$$

From Proposition 2.3 we know that:

$$\|F\|_B\ \le\ \nu_{0,\tilde d}^{-1}(F),\qquad \forall F\in S^{-1}(\Xi). \tag{2.34}$$

Thus, for $q=\min\{1,p^{-1}\}$ and $n_1=2[(d+p)/2]+2$, $n_2=2[(d-p)/2]+2$ we have

$$\big\|\mathfrak{r}_F^B(a)\big\|_B\ \le\ \nu_{0,\tilde d}^{-1}\big(F_a\sharp^B F_a^{-1}-1\big)\ \le\ C(d,F)\,a^{-q}\,\mu_1(B)\mathfrak{w}_{n_1+n_2}(B).$$

In conclusion, if we choose

$$a\ >\ \big[2C(d,F)\mu_1(B)\mathfrak{w}_{n_1+n_2}(B)\big]^{1/q} \tag{2.35}$$

we have $\big\|\mathfrak{r}_F^B(a)\big\|_B\le 1/2$. Moreover, we notice that

$$\begin{aligned} F_a\sharp^B\big[F_a^{-1}\sharp^B\big(1-\mathfrak{r}_F^B(a)\big)\big] &= 1+\mathfrak{r}_F^B(a)-\mathfrak{r}_F^B(a)-\big(\mathfrak{r}_F^B(a)\sharp^B\mathfrak{r}_F^B(a)\big) = \\ &= 1-\big(\mathfrak{r}_F^B(a)\sharp^B\mathfrak{r}_F^B(a)\big). \end{aligned} \tag{2.36}$$

From these we may conclude that the following limit exists in $\mathfrak{C}^B(\Xi)$ in the topology of the norm $\|\cdot\|_B$

$$\mathfrak{z}_F^B(a):=1+\lim_{N\nearrow\infty}\sum_{n=1}^{N}\big[-\mathfrak{r}_F^B(a)\big]^{\sharp^B n} \tag{2.37}$$

and the symbol $\mathfrak{r}_F^B(a):=F_a^{-1}\sharp^B\mathfrak{z}_F^B(a)$ satisfies the equality

$$\big(F+a\big)\sharp^B\mathfrak{r}_F^B(a)=1. \tag{2.38}$$

Starting then with the product $F_a^{-1}\sharp^B F_a$ and repeating exactly the above arguments we obtain a left inverse for $F+a$ for the magnetic Moyal product, and due to the well known abstract argument they have to be equal. We conclude that $\mathfrak{Op}^A(F)$ is a symmetric operator having the real number $-a$ in its resolvent set; as this set is open, we can find in its resolvent set points with strictly positive and strictly negative imaginary parts so that we conclude that it is self-adjoint. Moreover, we know that $\mathfrak{z}_F^B(a)\in\mathfrak{C}^B(\Xi)$ can be analytically continued to an analytic map

$$\{\mathfrak{z}\in\mathbb{C}\ |\ \Im m\,\mathfrak{z}\neq 0\}\bigcup\{x\in\mathbb{R}\ |\ x+a<\epsilon\}\ni\mathfrak{z}\mapsto\mathfrak{r}_F^B(\mathfrak{z})\in\mathfrak{C}^B(\Xi) \tag{2.39}$$

for some $\epsilon>0$ small enough and this map verifies the *resolvent equation*:

$$\mathfrak{r}_F^B(\mathfrak{z}_1)-\mathfrak{r}_F^B(\mathfrak{z}_2)=\big(\mathfrak{z}_2-\mathfrak{z}_1\big)\mathfrak{r}_F^B(\mathfrak{z}_1)\sharp^B\mathfrak{r}_F^B(\mathfrak{z}_2)=\big(\mathfrak{z}_2-\mathfrak{z}_1\big)\mathfrak{r}_F^B(\mathfrak{z}_2)\sharp^B\mathfrak{r}_F^B(\mathfrak{z}_1) \tag{2.40}$$

and also the defining relations for the inverse:

$$\big(F+\mathfrak{z}\big)\sharp^B\mathfrak{r}_F^B(\mathfrak{z})=\mathfrak{r}_F^B(\mathfrak{z})\sharp^B\big(F+\mathfrak{z}\big)=1. \tag{2.41}$$

Moreover, we notice that

$$\mathfrak{p}_p \sharp^B \mathfrak{r}_F^B(\mathfrak{z}) = \mathfrak{p}_p \sharp^B \mathfrak{r}_F^B(a) + (a-\mathfrak{z})\mathfrak{p}_p \sharp^B \mathfrak{r}_F^B(a) \sharp^B \mathfrak{r}_F^B(\mathfrak{z}) = \tag{2.42}$$

$$= \mathfrak{p}_p \sharp^B (F+a)^{-1} \sharp \mathfrak{z}_F^B(a) + (a-\mathfrak{z})\mathfrak{p}_p \sharp^B (F+a)^{-1} \sharp^B \mathfrak{z}_F^B(a) \sharp^B \mathfrak{r}_F^B(\mathfrak{z}) \in \mathfrak{C}^B(\Xi), \tag{2.43}$$

because $\mathfrak{p}_p \sharp^B (F+a)^{-1} \in S^0(\Xi) \subset \mathfrak{C}^B(\Xi)$, $\mathfrak{z}_F^B(a) \in \mathfrak{C}^B(\Xi)$, $\mathfrak{r}_F^B(\mathfrak{z}) \in \mathfrak{C}^B(\Xi)$ and we use the Theorem 2.1 for the composition of symbols and the fact that $\mathfrak{C}^B(\Xi)$ is a *-algebra for the magnetic Moyal product. We conclude by using Proposition 6.29 in [9] that in fact $\mathfrak{r}_F^B(\mathfrak{z}) \in S^{-p}(\Xi)$ for any $\mathfrak{z}$ in the resolvent set of $\mathfrak{Op}^A(F)$. From (2.41) we easily deduce that $\mathfrak{R}\mathsf{ange}\, \mathfrak{r}_F^B(\mathfrak{z}) = \mathscr{H}_A^p(\mathcal{X})$. Applying $\mathfrak{Op}^A(\mathfrak{r}_F^B(i))$ to $\mathscr{S}(\mathcal{X})$ and taking into account that $\mathfrak{r}_F^B(i) \in S^{-p}(\Xi) \subset \mathfrak{M}^B(\Xi)$, that $\|\mathfrak{Op}^A(\mathfrak{r}_F^B(i))\|_{\mathbb{B}(L^2(\mathcal{X}))} \leq 1$ and that $\mathscr{S}(\mathcal{X})$ is dense in $L^2(\mathcal{X})$, we obtain the essential self-adjointness of $\mathfrak{Op}^A(F)$ on $\mathscr{S}(\mathcal{X})$. $\square$

2.6. **The evolution group**

Suppose we are given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and a real, lower semi bounded elliptic symbol $h \in S^p(\Xi)$ for some $p > 0$. For some vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$ associated to B, we consider the self-adjoint operator $\mathfrak{Op}^A(h) : \mathscr{H}_A^p(\mathcal{X}) \to L^2(\mathcal{X})$ (as the one studied in the previous subsection) that we shall denote by $\mathfrak{Q}^A(h)$. Then, by Stone's Theorem, we can consider its associated one-parameter strongly continuous unitary group

$$\mathbb{R} \ni t \mapsto W_h^A(t) \in \mathcal{U}(L^2(\mathcal{X})). \tag{2.44}$$

It is defined as the unique solution of the Cauchy problem

$$\begin{cases} i\partial_t W_h^A(t) = \mathfrak{Q}^A(h) W_h^A(t), & \forall t \in \mathbb{R} \\ W_h^A(0) = \mathbb{1} \end{cases} \tag{2.45}$$

and given explicitly by the following formula (using the functional calculus with self-adjoint operators):

$$W_h^A(t) = \exp\big(-it\mathfrak{Q}^A(h)\big). \tag{2.46}$$

Remark 2.10. For any $t \in \mathbb{R}$, the unitary operator $W_h^A(t)$ leaves invariant the domain $\mathscr{H}_A^p(\mathcal{X})$ and by functional calculus with self-adjoint operators:

$$W_h^A(t)\mathfrak{Q}^A(h) f = \mathfrak{Q}^A(h) W_h^A(t) f, \qquad \forall f \in \mathscr{H}_A^p(\mathcal{X}). \tag{2.47}$$

Let us consider its distribution symbol defined by Proposition 1.11:

$$W_h^A(t) =: \mathfrak{Op}^A\big(w_h^B(t)\big). \tag{2.48}$$

A priori we know that $w_h^B(t) \in \mathfrak{C}^B(\Xi)$ for any $t \in \mathbb{R}$ and that it defines by magnetic quantization an invertible operator with the inverse having the following symbol (usually we denote by F_B^- the inverse of $F \in \mathscr{S}'(\Xi)$ for the magnetic Moyal product $\sharp^B$, when this inverse exists):

$$\left[w_h^B(t)\right]_B^- = w_h^B(-t) = \overline{w_h^B(t)} \in \mathfrak{C}^B(\Xi). \tag{2.49}$$

We also know that the function $\mathbb{R} \ni t \mapsto w_h^B(t) \in \mathfrak{C}^B(\Xi)$ is a solution of the Cauchy problem

$$\begin{cases} i\partial_t w_h^B(t) &= h\sharp^B w_h^B(t), \quad \forall t \in \mathbb{R} \\ w_h^B(0) &= 1 \end{cases} \tag{2.50}$$

considered in the weak distribution topology.

Let $q_j(x,\xi) := x_j$ and $p_j(x,\xi) := \xi_j$, for $1 \leq j \leq d$. We notice that all the distributions $q_1, \ldots, q_d$ and $p_1, \ldots, p_d$ are elements of $\mathfrak{M}^B(\Xi)$ (see [11]) and the distributions $p_1, \ldots, p_d$ are also in $S^1(\Xi)$. Let us also denote by $Q_j := \mathfrak{Op}^A(q_j)$ and $\Pi_j^A := \mathfrak{Op}^A(p_j)$. Notice that the topology of $\mathscr{S}(\mathcal{X})$ may be defined by the following family of seminorms, indexed by $(p,q) \in \mathbb{N} \times \mathbb{N}$:

$$\mathscr{S}(\mathcal{X}) \ni \phi \mapsto \mathfrak{n}_{p,q}^2(\phi) := \max_{|\alpha| \leq p} \max_{|\beta| \leq q} \left\| Q^\alpha \left[\Pi^A\right]^\beta \phi \right\|_{L^2(\mathcal{X})} \in \mathbb{R}_+. \tag{2.51}$$

Given some distribution $F \in \mathscr{S}'(\Xi)$, we set

$$\mathfrak{ad}_{q_j}^B(F) := q_j \sharp^B F - F \sharp^B q_j, \qquad \mathfrak{ad}_{p_j}^B(F) := p_j \sharp^B F - F \sharp^B p_j. \tag{2.52}$$

THEOREM 2.11. *Suppose we are given a magnetic field $B \in \mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and a real elliptic symbol $h \in S^p(\Xi)$ for some $p > 0$. For some vector potential $A \in \mathfrak{L}^1_{\mathsf{pol}}(\mathcal{X})$ associated to B, let us consider the self-adjoint operator $\mathfrak{Q}^A(h) : \mathscr{H}_A^p(\mathcal{X}) \to L^2(\mathcal{X})$ and its associated unitary group $\left\{W_h^A(t)\right\}_{t \in \mathbb{R}}$. Then $W_h^A(t)\mathscr{S}(\mathcal{X}) \subset \mathscr{S}(\mathcal{X})$ for any $t \in \mathbb{R}$.*

Proof. From Remark 2.10 and the definition of $\mathfrak{Q}^A(h)$, we conclude that $W_h^A(t)\mathscr{S}(\mathcal{X}) \subset \mathscr{H}_A^p(\mathcal{X})$. In order to prove the Theorem, it is clearly enough to prove that (using usual multi-index notation and the symbols and operators introduced above)

$$Q^\alpha \left(\Pi^A\right)^\beta W_h^A(t)\phi \in L^2(\mathcal{X}), \ \forall \phi \in \mathscr{S}(\mathcal{X}), \ \forall (\alpha,\beta) \in \mathbb{N}^{2d}. \tag{2.53}$$

In dealing with these computations, we shall use some notation.

For $1 \leq j \leq d$ and for any $k \in \mathbb{N}$, we denote by $p_j^{\sharp k}$ the magnetic Moyal product of k factors p_j, and similarly, $p^{\sharp\gamma} := (p_1^{\sharp\gamma_1})\sharp^B \ldots \sharp^B (p_d^{\sharp\gamma_d})$ with multi-index notation. We also use similar notations for the symbols $\{q_1, \ldots, q_d\}$.

For any multi-index $\alpha \in \mathbb{N}^d$ we denote by $\{\alpha\}$ the ordered set with α_1 entries equal to 1, followed by α_2 entries equal to 2 and so on up to the last α_d entries equal to d. Reciprocally, for any subset $\mathfrak{m} \subset \{\gamma\}$ for some given $\gamma \in \mathbb{N}^d$ we denote by $\gamma_{\mathfrak{m}} \in \mathbb{N}^d$ its associated multi-index.

For any $\gamma \in \mathbb{N}^d$ we shall denote by $[\mathfrak{ad}_q^B]^\gamma(F)$ the multiple commutator

$$[\mathfrak{ad}_q^B]^\gamma(F) := \mathfrak{ad}_{q_{i_1}}^B \circ \ldots \circ \mathfrak{ad}_{q_{i_{|\gamma|}}}^B(F), \tag{2.54}$$

where $\{i_1, \ldots, i_{|\gamma|}\} = \{\gamma\}$, and similarly for $[\mathfrak{ad}_p^B]^\gamma$.

We shall use several times the following commutation formula

$$\begin{aligned} &\big(F_1 \sharp^B \ldots \sharp^B F_N\big) \sharp^B G - G \sharp^B \big(F_1 \sharp^B \ldots \sharp^B F_N\big) = \\ &= \sum_{1 \leq k \leq N} \sum_{\{j_1,\ldots,j_k\} \subset \{1,\ldots,N\}} \Big[F_{j_1}, \big[F_{j_2} \ldots [F_{j_k}, G]_B \ldots\big]_B\Big]_B \sharp^B F_{l_1} \sharp^B \ldots \sharp^B F_{l_{N-k}} \end{aligned} \tag{2.55}$$

where $[F, G]_B := F \sharp^B G - G \sharp^B F$ and for any subset $\{j_1, \ldots, j_k\} \subset \{1, \ldots, N\}$ we denote by $\{l_1, \ldots, l_{N-k}\} = \{1, \ldots, N\} \setminus \{j_1, \ldots, j_k\}$ all the sets being considered ordered by the natural order induced by $\mathbb{N}$.

For any pair $(\alpha, \beta) \in \mathbb{N}^{2d}$ we have that

$$Q^\alpha \big(\Pi^A\big)^\beta = \mathfrak{Op}^A\big(q^{\sharp\alpha} \sharp^B p^{\sharp\beta}\big) \in \mathcal{L}\big(\mathscr{S}(\mathcal{X}); \mathscr{S}(\mathcal{X})\big), \tag{2.56}$$

so that for any test function $\phi \in \mathscr{S}(\mathcal{X})$ the following tempered distribution is well defined:

$$\begin{aligned} \big[Q^\alpha \big(\Pi^A\big)^\beta W_h^A(t)\phi\big](\psi) &= \big[W_h^A(t)\phi\big]\big(\big(\Pi^A\big)^\beta Q^\alpha \psi\big) \\ &= \Big\langle \overline{\big(\Pi^A\big)^\beta Q^\alpha \psi}\,,\, W_h^A(t)\phi \Big\rangle_{L^2(\mathcal{X})} \in \mathbb{C}, \quad \forall \psi \in \mathscr{S}(\mathcal{X}). \end{aligned}$$

We shall prove that it defines in fact a continuous functional of $\psi \in \mathscr{S}(\mathcal{X})$ for the topology induced by $\mathscr{H}_A^p(\mathcal{X})$. The idea is to compute the commutator

$$\Big[Q^\alpha \big(\Pi^A\big)^\beta\,,\, W_h^A(t)\Big] = \mathfrak{Op}^A\Big(\big(q^{\sharp\alpha} \sharp^B p^{\sharp\beta}\big) \sharp^B w_h^B(t) - w_h^B(t) \sharp^B \big(q^{\sharp\alpha} \sharp^B p^{\sharp\beta}\big)\Big)$$

by computing its distribution kernel and having in mind the composition laws (1.35) and the fact that the symbol $q^{\sharp\alpha} \sharp^B p^{\sharp\beta}$ is in the magnetic Moyal algebra $\mathfrak{M}^B(\Xi)$. In order to deal with the commutators with $W_h^A(t)$ we notice that given some operator $X^A := \mathfrak{Op}^A(F_X)$ for some $F_X \in \mathfrak{M}^B(\Xi)$ we can write

$$\begin{aligned} \big[X^A\,,\, W_h^A(t)\big]\phi &= \big(X^A W_h^A(t) - W_h^A(t) X^A\big)\phi \\ &= \int_0^t ds \big[\partial_s \left(W_h^A(t-s) X^A W_h^A(s)\right)\big]\phi \\ &= -i \int_0^t ds W_h^A(t-s) \Big[X^A\,,\, \mathfrak{Q}^A(h)\Big] W_h^A(s)\phi. \end{aligned}$$

We are interested for the moment to study the case $X^A = \mathfrak{Op}^A\big(q^{\sharp\alpha} \sharp^B p^{\sharp\beta}\big)\big)$ and the commutator

$$\Big[X^A\,,\, \mathfrak{Q}^A(h)\Big] = \mathfrak{Op}^A\big((q^{\sharp\alpha} \sharp^B p^{\sharp\beta}) \sharp^B h - h \sharp^B (q^{\sharp\alpha} \sharp^B p^{\sharp\beta})\big). \tag{2.57}$$

We are going to proceed by induction on $N := |\alpha + \beta| \in \mathbb{N}$, starting with the following *induction hypothesis*:

H_N: *Suppose that for any $(\alpha,\beta)\in\mathbb{N}^{2d}$ with $|\alpha|+|\beta|\leq N$ we know that $Q^\alpha\big(\Pi^A\big)^\beta W_h^A(t)\phi\in\mathscr{H}_A^p(\mathcal{X})$, for any $\phi\in\mathscr{S}(\mathcal{X})$ and for any $t\in\mathbb{R}$.*

First of all let us notice that the above statement is clearly true for $N=0$ due to the fact that $\mathscr{S}(\mathcal{X})\subset\mathscr{H}_A^p(\mathcal{X})$ and the unitary group $W_h^A(t)$ leaves invariant the domain of its generator $\mathfrak{Q}^A(h):\mathscr{H}_A^p(\mathcal{X})\to L^2(\mathcal{X})$.

Suppose now that we increase N in the Hypothesis H_N above by 1 either by increasing some α_j or some β_k by 1 and let us consider some pair $(\alpha',\beta')\in\mathbb{N}^{2d}$ with $|\alpha'|+|\beta'|=N+1$. Using formula (2.55) we can compute the following commutator in the magnetic Moyal algebra (we use the notation $\mathfrak{R}_-^B:=\big(\mathfrak{Q}(h)+1\big)^{-1}$):

(2.58)
$$\Big[\mathfrak{Op}^A\big(q^{\sharp\alpha'}\sharp^B p^{\sharp\beta'}\big)\big)\,,\,\mathfrak{Q}^A(h)\Big]=$$
$$=\sum_{\mathfrak{m}\subset\{\alpha'\}}{}'\sum_{\mathfrak{n}\subset\{\beta'\}}{}'\mathfrak{Op}^A\Big([\mathfrak{ad}_q^B]^{\gamma_\mathfrak{m}}\big([\mathfrak{ad}_p^B]^{\gamma_\mathfrak{n}}(h)\big)\Big)Q^{\gamma_{\mathfrak{m}^\complement}}\big(\Pi^A\big)^{\gamma_{\mathfrak{n}^\complement}}=$$
$$=\sum_{\mathfrak{m}\subset\{\alpha'\}}{}'\sum_{\mathfrak{n}\subset\{\beta'\}}{}'\mathfrak{Op}^A\Big([\mathfrak{ad}_q^B]^{\gamma_\mathfrak{m}}\big([\mathfrak{ad}_p^B]^{\gamma_\mathfrak{n}}(h)\big)\Big)\,\mathfrak{R}_-^B\big(\mathfrak{Q}(h)+i\big)\,Q^{\gamma_{\mathfrak{m}^\complement}}\big(\Pi^A\big)^{\gamma_{\mathfrak{n}^\complement}}$$

where $\mathfrak{m}^\complement=\{\alpha\}\setminus\mathfrak{m}$ and $\mathfrak{n}^\complement=\{\beta\}\setminus\mathfrak{n}$ and we denote by $\sum'_{\mathfrak{m}\subset\{\alpha\}}$ the sum over all subsets different from the void set, so that $|\gamma_{\mathfrak{m}^\complement}|+|\gamma_{\mathfrak{n}^\complement}|\leq N$.

Finally, for any $\phi\in\mathscr{S}(\mathcal{X})$ we have obtained the following equality of tempered distributions on $\mathcal{X}$:

$$Q^{\alpha'}\big(\Pi^A\big)^{\beta'}W_h^A(t)\phi=W_h^A(t)Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\phi-\tag{2.59}$$
$$-i\sum_{\mathfrak{m}\subset\{\alpha\}}{}'\sum_{\mathfrak{n}\subset\{\beta\}}{}'\int_0^t dsW_h^A(t-s)\mathfrak{Op}^A\big(\mathfrak{D}^B_{\gamma_\mathfrak{m}\gamma_\mathfrak{n}}(h)\big)\big(\mathfrak{Q}(h)+i\big)\,Q^{\gamma_{\mathfrak{m}^\complement}}\big(\Pi^A\big)^{\gamma_{\mathfrak{n}^\complement}}W_h^A(s)\phi.$$

We notice that

$$\mathfrak{D}^B_{\gamma_\mathfrak{m}\gamma_\mathfrak{n}}(h):=[\mathfrak{ad}_q^B]^{\gamma_\mathfrak{m}}\big([\mathfrak{ad}_p^B]^{\gamma_\mathfrak{n}}(h)\big)\sharp^B\mathfrak{r}^B_(h)-\in S^0(\Xi)\subset\mathfrak{C}^B(\Xi)\tag{2.60}$$

due to the fact that by Proposition 2.2:

$$[\mathfrak{ad}_p^B]^{\gamma_\mathfrak{n}}(h)\in S^p(\Xi),\quad\forall\mathfrak{n}\subset\{\beta\},\emptyset\neq\mathfrak{n}\neq\{\beta\},\tag{2.61}$$

$$[\mathfrak{ad}_q^B]^{\gamma_\mathfrak{m}}\big([\mathfrak{ad}_p^B]^{\gamma_\mathfrak{n}}(h)\big)\in S^{p-|\gamma_\mathfrak{n}|}(\Xi),\quad\forall\mathfrak{n}\subset\{\beta\},\ \forall\mathfrak{m}\subset\{\alpha\}.$$

Thus $\big\|\mathfrak{Op}^A\big(\mathfrak{D}^B_{\gamma_\mathfrak{m}\gamma_\mathfrak{n}}(h)\big)\big\|_{\mathbb{B}(L^2(\mathcal{X}))}\leq C_{\alpha,\beta}(h)<\infty$. Moreover, we notice that

$$Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\phi\in\mathscr{S}(\mathcal{X})\subset\mathscr{H}_A^p(\mathcal{X})\implies W_h^A(t)Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\phi\in\mathscr{H}_A^p(\mathcal{X}),\tag{2.62}$$

$$\text{(2.63)} \qquad \mathbf{H}_N \implies \big(\mathfrak{Q}(h)+i\big)\, Q^{\gamma_{\mathfrak{m}^{\complement}}}\big(\Pi^A\big)^{\gamma_{\mathfrak{n}^{\complement}}} W^A_h(s)\phi \in L^2(\mathcal{X}), \quad \forall s\in[0,t],$$

for any subsets $\mathfrak{m}\subset\{\alpha'\}$ and $\mathfrak{n}\subset\{\beta'\}$ different from the empty set.

Finally, we conclude that

$$\text{(2.64)} \qquad Q^{\alpha'}\big(\Pi^A\big)^{\beta'} W^A_h(t)\phi \in L^2(\mathcal{X}).$$

We consider now the following equality of tempered distributions:

$$\text{(2.65)} \qquad \mathfrak{Q}^A(h) Q^{\alpha'}\big(\Pi^A\big)^{\beta'} W^A_h(t)\phi \;=$$
$$= Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\mathfrak{Q}^A(h) W^A_h(t)\phi \,+\, \Big[\mathfrak{Q}^A(h)\,,\, Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\Big] W^A_h(t)\phi \;=$$
$$= Q^{\alpha'}\big(\Pi^A\big)^{\beta'} W^A_h(t)\mathfrak{Q}^A(h)\phi \,+\, \Big[\mathfrak{Q}^A(h)\,,\, Q^{\alpha'}\big(\Pi^A\big)^{\beta'}\Big] W^A_h(t)\phi$$

and using the above result and once again formula (2.58) we conclude that it defines in fact an element in $L^2(\mathcal{X})$. This proves that $\mathbf{H}_{(N+1)}$ is also true and finishes the proof of the Theorem. $\square$

3. APPENDICES

A.1. Estimates on the derivatives of ϖ^B

We shall consider the multi-indices $\{\sigma^j\}_{1\leq j\leq d}$ with $(\sigma^j)_k := \delta_{jk}$. We shall use the notation $\varpi^B_x(y,z) := e^{-iF^B_x(y,z)}$ with the explicit expression:

$$\text{(3.1)} \qquad F^B_x(y,z) = 4\sum_{j\neq k} y_j z_k \int_0^1 ds \int_0^s dt\, B_{jk}\big(x+(2s-1)y+(2t-1)z\big).$$

We shall use the shorthand notation:

$$\text{(3.2)} \qquad r_{y,z}(s,t) := (2s-1)y+(2t-1)z, \qquad \forall (s,t)\in[0,1]\times[0,1].$$

Then we have the following formulas:

(3.3)

$$\big(\partial^\alpha_x F^B_x\big)(y,z) = 4\sum_{j\neq k} y_j z_k \int_0^1\!ds \int_0^s\!dt \big(\partial^\alpha B_{jk}\big)\big(x+r_{y,z}(s,t)\big) = \int_{\mathcal{T}_x(y,z)} \partial^\alpha B,$$

(3.4)

$$\big(\partial^\alpha_y F^B_x\big)(y,z) = 4\sum_{j\neq k} y_j z_k \int_0^1\!ds \int_0^s\!dt \big(\partial^\alpha B_{jk}\big)\big(x+r_{y,z}(s,t)\big)(2s-1)^{|\alpha|} +$$
$$+\sum_{j:\alpha_j\geq 1}\sum_{k\neq j} z_k \int_0^1\!ds \int_0^s\!dt \big(\partial^{\alpha-\sigma^j} B_{jk}\big)\big(x+r_{y,z}(s,t)\big)(2s-1)^{|\alpha|-1}$$

(3.5)
$$\big(\partial_z^\alpha F_x^B\big)(y,z)=4\sum_{j\neq k}y_jz_k\int_0^1\!ds\int_0^s\!dt\big(\partial^\alpha B_{jk}\big)\big(x+r_{y,z}(s,t)\big)(2t-1)^{|\alpha|}+$$
$$+\sum_{k:\alpha_k\geq 1}\sum_{j\neq k}y_j\int_0^1\!ds\int_0^s\!dt\big(\partial^{\alpha-\sigma^k}B_{jk}\big)\big(x+r_{y,z}(s,t)\big)(2t-1)^{|\alpha|-1}.$$

Let us use the Faà di Bruno's formula ([4]) for the case of the exponential of a given function $-iF^B$, with the notations (for any $m\in\mathbb{N}\setminus\{0\}$ and any $\alpha\in\mathbb{N}^d\setminus\{(0,\dots 0)\}$)

(3.6)
$$\mathscr{P}_l(m):=\big\{\underline{p}=(p_1,\dots,p_l)\ \mid\ p_j\geq 1\,\forall j\in\{1,\dots,d\},\ p_1+\dots+p_l=m\,\big\},$$
(3.7)
$$\mathscr{P}_l(\alpha):=\big\{\underline{\gamma}=(\gamma_1,\dots,\gamma_l)\ \mid\ |\gamma_j|\geq 1\,\forall j\in\{1,\dots,d\},\ \gamma_1+\dots+\gamma_l=\alpha\,\big\},$$

in order to write

$$\partial_x^\alpha\varpi_x^B(y,z)=\alpha!\,\varpi_x^B(y,z)\sum_{1\leq l\leq|\alpha|}\frac{1}{l!}\sum_{\underline{\gamma}\in\mathscr{P}_l(\alpha)}\prod_{s=1}^{l}\frac{1}{\gamma^s!}\big(\partial_x^{\gamma^s}F_x^B\big)(y,z)$$
$$=\alpha!\,\varpi_x^B(y,z)\sum_{1\leq l\leq|\alpha|}\frac{1}{l!}\sum_{\underline{\gamma}\in\mathscr{P}_l(\alpha)}\prod_{s=1}^{l}\frac{1}{\gamma^s!}\left(\int_{\mathcal{T}_x(y,z)}\partial^{\gamma^s}B\right).$$

It follows then

$$\left|\partial_x^\alpha\varpi_x^B(y,z)\right|\ \leq C(d,|\alpha|)\max_{1\leq l\leq|\alpha|}\ \max_{\underline{p}\in\mathscr{P}_l(|\alpha|)}\ \prod_{s=1}^{l}\big(|y\wedge z|^{p_s}\rho_{p_s}(B)\big)$$
$$\leq C(d,|\alpha|)|y\wedge z|^{|\alpha|}\max_{1\leq l\leq|\alpha|}\ \max_{\underline{p}\in\mathscr{P}_l(|\alpha|)}\ \prod_{s=1}^{l}\mu_{p_s}(B). \tag{3.8}$$

$$\partial_y^\alpha\varpi_x^B(y,z)=\alpha!\,\varpi_x^B(y,z)\sum_{1\leq l\leq|\alpha|}\frac{1}{l!}\sum_{\underline{\gamma}\in\mathscr{P}_l(\alpha)}\prod_{s=1}^{l}\frac{1}{\gamma^s!}\big(\partial_y^{\gamma^s}F^B\big)(x,y,z)$$
$$=\alpha!\,\varpi_x^B(y,z)\sum_{1\leq l\leq|\alpha|}\frac{1}{l!}\sum_{\underline{\gamma}\in\mathscr{P}_l(\alpha)}\prod_{s=1}^{l}\frac{1}{\gamma^s!}\times$$
$$\times\left(\int_{\mathcal{T}_x(y,z)}\partial^{\gamma^s}B+\sum_{j:\alpha_j\geq 1}\mathfrak{S}^{|\gamma^s|,1}_{\mathcal{T}_x(y,z)}\big[\big(\partial^{\gamma^s-\sigma^j}B\big)\llcorner z\big]_j\right) \tag{3.9}$$
$$\left|\partial_y^\alpha\varpi_x^B(y,z)\right|\leq C(d,|\alpha|)<y>^{|\alpha|}<z>^{|\alpha|}\ \mathfrak{w}_{|\alpha|}(B). \tag{3.10}$$

Evidently we obtain a similar estimate for $\partial_z^\alpha\varpi_x^B(y,z)$.

A.2. Estimating an oscillating integral

For $(N, M, n_1, n_2, m_1, m_2) \in \mathbb{N}^6$ and for any $\Theta \in C^\infty_{\mathsf{pol}}\big(\mathcal{X} \times \mathcal{X} \times \mathcal{X}\big)$ we define:

(3.11)
$$\mathcal{W}^{N,n_1,n_2}_{M,m_1,m_2}(\Theta) :=$$
$$= \sup_{(x,y,z)\in\mathcal{X}\times\mathcal{X}\times\mathcal{X}} < x >^{-N} < y >^{-n_1} < z >^{-n_2} \max_{|a|\le M} \max_{|b|\le m_1} \max_{|c|\le m_2} \left|\partial^a_x \partial^b_y \partial^c_z \Theta(x, y, z)\right|$$

that can take also the value $+\infty$.

Remark 3.1. Given $\Theta \in C^\infty_{\mathsf{pol}}\big(\mathcal{X} \times \mathcal{X} \times \mathcal{X}\big)$ and $(M, m_1, m_2) \in \mathbb{N}^3$ there exist $(N, q_1, q_2) \in \mathbb{N}^3$ such that $\mathcal{W}^{N,q_1,q_2}_{M,m_1,m_2}(\Theta) < \infty$.

Proposition 3.2. *Suppose given a magnetic field of class $\mathfrak{L}^2_{\mathsf{bc}}(\mathcal{X})$ and $(F, G) \in S^{p_1}(\Xi) \times S^{p_2}(\Xi)$ and let us denote by $\tilde{p}_1 := 2[(d + p_1)/2] + 2$, $\tilde{p}_2 := 2[(d+p_2)/2]+2$. Suppose also given a function $\Theta \in C^\infty_{\mathsf{pol}}\big(\mathcal{X} \times \mathcal{X} \times \mathcal{X}\big)$ and let us consider the tempered distribution defined by the oscillating integral*

$$\mathcal{M}^B_\Theta[F, G](X) := \int_{\Xi\times\Xi} dY\, dZ e^{-2i\sigma(Y,Z)} \varpi^B_x(y, z)\Theta(x, y, z)F(X - Y)G(X - Z).$$

Then, for any $(\alpha, \beta) \in \mathbb{N}^{2d}$ there exist $(N, q_1, q_2) \in \mathbb{N}^3$ such that:

$$\mathcal{W}^{N,q_1,q_2}_{|\alpha|,\tilde{p}_2,\tilde{p}_1}(\Theta) < \infty,$$

$$< x >^{-N} < \xi >^{-(p_1+p_2)+|\beta|} \left|\big(\partial^\alpha_x \partial^\beta_\xi \mathcal{M}^B_\Theta[F, G]\big)(X)\right| \le C(d, p_1, p_2, \alpha, \beta)\times$$
$$\times \mathfrak{w}_{|\alpha|+\tilde{p}_1+\tilde{p}_2}(B)\, \mathcal{W}^{N,q_1,q_2}_{|\alpha|,\tilde{p}_2,\tilde{p}_1}(\Theta) \sum_{0\le k\le|\beta|} \rho^{p_1-k}_{|\alpha|+\tilde{p}_2;k,m_2}(F)\, \rho^{p_2-|\beta|+k}_{|\alpha|+\tilde{p}_1;|\beta|-k,m_1}(G),$$

where $m_1 := 2[\tilde{p}_2 + (n + \tilde{p}_1 + q_1)/2] + 2$ and $m_2 := 2[\tilde{p}_1 + (n + \tilde{p}_2 + q_2)] + 2$.

Proof. Fixing some $(x, \xi) \in \Xi$, the oscillating integrals of the form

$$< \xi >^{-(p_1+p_2)+|\beta|} \big(\partial^\alpha_x \partial^\beta_\xi \mathcal{M}^B_\Theta[F, G]\big)(X)$$

can be written by the Leibniz rule as finite linear combinations of a number (depending only on $|\alpha| \in \mathbb{N}$ and $|\beta| \in \mathbb{N}$) of terms of the form

$$< \xi >^{-(p_1+p_2)+|\beta|} \int_{\Xi\times\Xi} dY\, dZ\, e^{-2i\sigma(Y,Z)} \left[\big(\partial^{\alpha_0}_x \varpi^B_x\big)(y, z)\right] \left[\partial^{a_0}_x \Theta(x, y, z)\right] \times$$

(3.12)
$$\times \left[\big(\partial^{\alpha_1}_x \partial^{\beta_1}_\xi F\big)(X - Y)\right] \left[\big(\partial^{\alpha_2}_x \partial^{\beta_2}_\xi G\big)(X - Z)\right]$$

with $\alpha_0 + a_0 + \alpha_1 + \alpha_2 = \alpha$ and $\beta_1 + \beta_2 = \beta$.

Let us begin with a rough estimate of the 'momentum integrals' with respect to $(\eta,\zeta)\in\left[\mathcal{X}^*\right]^2$ and notice that:

$$<\xi>^{-(p_1+p_2)+|\beta|}\left[\left(\partial_x^{\alpha_1}\partial_\xi^{\beta_1}F\right)(X-Y)\right]\left[\left(\partial_x^{\alpha_2}\partial_\xi^{\beta_2}G\right)(X-Z)\right]\leq \tag{3.13}$$

$$\leq C\nu^{p_1-|\beta|_1}_{|\alpha_1||\beta_1|}(F)\,\nu^{p_2-|\beta_2|}_{|\alpha_2||\beta_2|}(G)\leq C\nu^{p_1}_{|\alpha||\beta|}(F)\,\nu^{p_2}_{|\alpha||\beta|}(G), \tag{3.14}$$

Then, in order to control these integrals, some extra factors of convergence $<\eta>^{-d-\epsilon}<\zeta>^{-d-\epsilon}$ have to be introduced. We are then obliged to get rid of these growing factors, integrating by parts using the identities

$$<\eta>^{n_1}e^{-2i<\langle\eta,z\rangle}=<(i/2)\nabla_z>^{n_1}e^{-2i\langle\eta,z\rangle}, \tag{3.15}$$

$$<\zeta>^{n_2}e^{2i\langle\zeta,y\rangle}=<(1/2i)\nabla_y>^{n_2}e^{2i\langle\zeta,y\rangle}. \tag{3.16}$$

We choose $n_1=\tilde{p}_1$ and $n_2=\tilde{p}_2$ (in order to work with polynomials in the differential operators we have to take even exponents). Thus the oscillating integral $<\xi>^{-(p_1+p_2)+|\beta|}\left(\partial_x^\alpha\partial_\xi^\beta\mathcal{M}^B_\Theta[F,G]\right)(X)$ becomes a linear combination of a number (depending only on $\{p_1,p_2,|\alpha|,|\beta|,d\}$) of terms of the form

$$<\xi>^{-M}\int_{\Xi\times\Xi}dY\,dZ\,e^{-2i\sigma(Y,Z)}\left[\left(\partial_x^{\alpha_0}\partial_y^{\mu_0}\partial_z^{\nu_0}\varpi_x^B\right)(y,z)\right]\left[\partial_x^{a_0}\partial_y^{b_0}\partial_z^{c_0}\Theta(x,y,z)\right]\times$$
$$\times<\eta>^{-n_1}\left[\left(\partial_x^{\alpha_1}\partial_y^{\mu_1}\partial_\xi^{\beta_1}F\right)(X-Y)\right]<\zeta>^{-n_2}\left[\left(\partial_x^{\alpha_2}\partial_z^{\nu_1}\partial_\xi^{\beta_2}G\right)(X-Z)\right],$$

where $M=p_1+p_2+|\beta|$, $a_0+\alpha_0+\alpha_1+\alpha_2=\alpha$, $\beta_1+\beta_2=\beta$, $|b_0+\mu_0+\mu_1|=\tilde{p}_2$ and $|c_0+\nu_0+\nu_1|=\tilde{p}_1$. A maximum number of $N_0:=|\alpha|+\tilde{p}_1+\tilde{p}_2$ derivatives of the factor ϖ_x^B will appear. Considering the factor $\partial_x^{a_0}\partial_y^{b_0}\partial_z^{c_0}\Theta(x,y,z)$ in the oscillating integral above we notice that $|a_0|\leq|\alpha|$, $|b_0|\leq\tilde{p}_2$ and $|c_0|\leq\tilde{p}_1$. Using the above Remark 3.1 for our function $\Theta\in C^\infty_{\mathsf{pol}}\left(\mathcal{X}\times\mathcal{X}\times\mathcal{X}\right)$ we can choose $(N,q_1,q_2)\in\mathbb{N}^3$ such that $\mathcal{W}^{N,q_1,q_2}_{|\alpha|,\tilde{p}_2,\tilde{p}_1}(\Theta)<\infty$.

In order to obtain integrability in the variables $(y,z)\in\mathcal{X}^2$, we shall insert the factors $<y>^{-m_1}<z>^{-m_2}$ with $m_1=2[(N_0+\tilde{p}_2+q_1)/2]+2$ and $m_2=2[(N_0+\tilde{p}_1+q_2)/2]+2$ and apply once again integration by parts to transform the compensating factors in derivations with respect to $(\eta,\zeta)\in(\mathcal{X}^*)^2$. Finally, we obtain a linear combination of a number (depending on $\{p_1,p_2,|\alpha|,|\beta|,d\}$) of terms of the form:

$$<\xi>^{-M}\int_\Xi<y>^{-m_1}<\eta>^{-\tilde{p}_1}dY\int_\Xi<z>^{-m_2}<\zeta>^{-\tilde{p}_2}dZ\times \tag{3.17}$$
$$\times\,e^{-2i\sigma(Y,Z)}\left[\left(\partial_x^{\alpha_0}\partial_y^{\mu_0}\partial_z^{\nu_0}\varpi_x^B\right)(y,z)\right]\left[\partial_x^{a_0}\partial_y^{b_0}\partial_z^{c_0}\Theta(x,y,z)\right]\times$$
$$\times\left[\left(\partial_x^{\alpha_1}\partial_y^{\mu_1}\partial_\xi^{\beta_1}\partial_\eta^{\theta_1}F\right)(X-Y)\right]\left[\left(\partial_x^{\alpha_2}\partial_z^{\nu_1}\partial_\xi^{\beta_2}\partial_\zeta^{\theta_2}G\right)(X-Z)\right].$$

Due to the above remarks we can estimate each integral of the form (3.17) by

$$< x >^N \mathfrak{w}_{|\alpha|+\tilde{p}_1+\tilde{p}_2}(B)\, \mathscr{W}^{N,q_1,q_2}_{|\alpha|,\tilde{p}_2,\tilde{p}_1}(\Theta) \sum_{0\leq k\leq|\beta|} \rho^{p_1-k}_{|\alpha|+\tilde{p}_2;k,m_2}(F)\, \rho^{p_2-|\beta|+k}_{|\alpha|+\tilde{p}_1;|\beta|-k,m_1}(G)$$

and finish the proof of the Proposition. □

Acknowledgements. MM and RP have been supported by the Fondecyt Project 1160359 and a large part of this paper has been elaborated while RP was visiting the Universidad de Chile to which we thank very much for its hospitality.

REFERENCES

[1] W.O. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians.* Progr. Math. **135**, Birkhäuser Verlag, 1996.

[2] H.D. Cornean and G. Nenciu, *Two dimensional magnetic Schrodinger operators: width of mini bands in the tight binding approximation.* Ann. Henri Poincaré **1** (2000), 203–222.

[3] G. Dell'Antonio, *Lectures on the Mathematics of Quantum Mechanics.* I. Atlantis Stud. Math. Phys. Theory Appl. **1**, Atlantis Press, 2015.

[4] L. Hernández Encinas and J. Muñoz Masqué, *A short proof of the generalized Faà di Bruno's formula.* Appl. Math. Lett. **16** (2003), 975–979.

[5] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics.* Cambridge Univ. Press, 1984.

[6] P.R. Halmos and V.S. Sunder, *Bounded Integral Operators on L^2 spaces.* Ergeb. Math. Grenzgeb. **96**, Springer-Verlag-Berlin-New York, 1978.

[7] L. Hörmander, *The Analysis of Linear Partial Differential Operators.* I. *Distribution theory and Fourier analysis,* Second Ed. Grundlehren Math. Wiss. **256**, Springer-Verlag, 1990.

[8] V. Iftimie, M. Mantoiu and R. Purice, *Magnetic pseudodifferential operators.* Publ. Res. Inst. Math. Sci. **43** (2007), *3*, 585–623.

[9] V. Iftimie, M. Mantoiu and R. Purice, *Commutator criteria for magnetic pseudodifferential operators.* Comm. Partial Differential Equations **35** (2010), 1058–1094.

[10] V. Iftimie, M. Mantoiu and R. Purice, *The magnetic formalism; new results.* Contemp. Math. **500** (2009), American Mathematical Society (AMS), 123–138.

[11] M. Mantoiu and R. Purice, *The magnetic Weyl calculus.* J. Math. Phys. **45** (2004), *4*, 1394–1417.

[12] M. Mantoiu and R. Purice, *Strict deformation quantization for a particle in a magnetic field.* J. Math. Phys. **46** (2005), *5*, 052105, 15 pp.

[13] M. Mantoiu and R. Purice, *The mathematical formalism of a particle in a magnetic field.* Mathematical physics of quantum mechanics, 417–434. Lecture Notes in Phys. **690**, Springer, Berlin, 2006.

[14] M. Mantoiu, R. Purice and S. Richard, *Spectral and propagation results for magnetic Schodinger operators; a C*-Algebraic framework.* J. Funct. Anal. **250** (2007), 42–67.

[15] G. Nenciu, *On asymptotic perturbation theory for Quantum Mechanics: Almost invariant subspaces and gauge invariant magnetic perturbation theory.* J. Math. Phys. **43** (2002), 1273–1298.

[16] M. Reed and B. Simon, *Functional Analysis*, Academic Press, 1981.

[17] L. Schwartz, *Théorie des Distributions*, Hermann, 1978.

[18] F. Tréves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, London, 1967.

Received 30 May 2018

Institute of Mathematics Simion Stoilow
of the Romanian Academy
Bucharest
Viorel.Iftimie@imar.ro

Universidad de Chile
Las Palmeras 3425, Casilla 653
Santiago de Chile
mantoiu@uchile.cl

Institute of Mathematics Simion Stoilow
of the Romanian Academy
Bucharest
Centre Francophone
en Mathématique Bucarest
Radu.Purice@imar.ro