PSEUDODIFFERENTIAL OPERATORS IN INFINITE DIMENSIONAL SPACES: A SURVEY OF RECENT RESULTS

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Quantization or pseudodifferential analysis is a theory which applies to numerous domains (functional analysis, operator theory, mathematical physics etc.) The classical framework is that of finite dimensional configuration and phase spaces. To handle problems involving an arbitrary number of particles, one needs to work with operators defined on infinite dimensional spaces, like the symmetric Fock space, and one may wish to use the efficient tools provided by the Weyl calculus, hence to work with an infinite dimensional measure space. This survey exposes the construction of a quantization on an abstract Wiener space, lists the results of pseudodifferential analysis which were generalized and gives applications to mathematical physics.

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1. INTRODUCTION

This paper summarizes several articles aiming at developing an infinite dimensional calculus akin to Weyl's finite dimensional quantization, in view of applications in mathematical physics, for example in nuclear magnetic resonance. The first articles in this direction go back to Bleher, Lascar, Vishik [13, 34, 35].

The material exposed here is taken from articles of Amour, Nourrigat, and collaborators Jager, Lascar. It was the subject of several talks too. This review exposes the definition and the results already obtained but, for the sake of concision, the proofs are, at most, hinted at. On a few subjects it brings together points of view or domains which had not been compared before.

Section 2 recalls some definitions about the finite dimensional Weyl quantization, its construction, its aims and some often-used tools. In this part, which may appear as very classical to many readers, we state which results were extended.

Section 3 recalls the infinite dimensional spaces and extends some classical notions (Wigner function, Wick symbol, anti-Wick calculus). An intrinsic Hilbert space which is classical in mathematical physics is the symmetric Fock space on an initial Hilbert space $\mathcal{H}$, for which we refer to [41, 42] and [18, 19]. We give the link between Fock spaces and Gaussian Hilbert spaces (defined in [32]). But one may use an abstract Wiener space B, built on the same initial $\mathcal{H}$ and on which a measure theory has been constructed in [22–25, 33, 40]. This point of view seems to be less classical and is therefore more developed here.

The following parts then give the basis of the constructions of the Weyl calculus in an infinite dimensional setting, beginning by two kinds of symbol classes (Section 4.1). These ones generalize finite dimensional symbol classes but a part of their study relies on a particular version of the heat operator (paragraph 4.1.3). The different constructions of the operators (or, in the preliminary case, of the quadratic form) are exposed in Section 4.2, with results about the continuity of such operators on a convenient L^2 space. In Section 4.3, we give composition results and a characterization analogous to the Beals characterization, stating that, under conditions on commutations with "basic" operators, an operator A has a symbol in one of the symbol classes mentioned before. The last section (4.4) is concerned with applications to mathematical physics.

2. WEYL CALCULUS IN THE FINITE DIMENSIONAL SETTING

This section recalls classical facts about Weyl calculus and its applications to the study of operators in mathematical physics. This includes tools and notions that are necessary to construct the pseudodifferential operators, as well as example of results. We also mention here what has been extended to the infinite dimensional case.

A quantization or pseudodifferential calculus is a way of associating, with a convenient function called symbol, a linear operator. The Weyl calculus has been chosen as a starting point here, since it is the quantization for which the most numerous tools were developed. Among many other references one may consult [15, 21, 27, 28, 36], from which we took (up to normalization) all the facts recalled in this section. Other quantizations than the Weyl calculus exist in a finite dimensional context (see [44, 45]). They are adapted to precise problems and require the use of different kinds of symbols.

In this finite dimensional case, the symbol F is defined on $\mathbb{R}^{2n}$ (the phase space) and the test functions u are defined on $\mathbb{R}^n$ (the configuration space). When both symbol and test function are rapidly decreasing, the Weyl quantization is defined by the classical formula:

$$(1) \qquad (Op_h^{W,cl}(F)(u))(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot\xi} F\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi$$

and the pseudodifferential operator $Op_h^{W,cl}(F)$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

The Weyl quantization can be adapted to take into account a magnetic field and the magnetic momenta. This is the case in [30], where the authors introduce an additional factor in formula (1). This is not the quantization that is generalized here, although magnetic fields appear in some examples of Section 4.4.

Formula (1) above has been given a sense for more general test functions and symbols, belonging to classes of functions satisfying differentiability and boundedness conditions and adapted to various problems. Quantizations must also satisfy some physical conditions. For example, the symbol 1 gives the identity operator. The coordinate functions x_j and ξ_j correspond, respectively, to the multiplication by x_j and to the differentiation operator $\frac{h}{i}\frac{\partial}{\partial x_j}$ ([15]). If F is real valued, the operator is (formally) self-adjoint.

The Wigner function of φ, ψ belonging to $L^2(\mathbb{R}^n)$ is given by

$$W_{\varphi,\psi}(x,\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}u\cdot\xi} \varphi(x + \frac{u}{2}) \overline{\psi(x - \frac{u}{2})}\, du.$$

It is the Weyl symbol of the rank one projection operator $\eta \mapsto \langle \eta, \psi \rangle_{L^2(\mathbb{R}^n)} \varphi$ (we take the convention that the complex scalar products are antilinear with respect to the second variable).

This gives another expression of the Weyl calculus, valid for u, v and F rapidly decreasing on their respective spaces:

$$(2) \qquad \langle Op_h^{W,cl}(F) f, g \rangle_{L^2(\mathbb{R}^n, d\lambda(x))} = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} F(x,\xi) W_{f,g}(x,\xi) dx d\xi.$$

Contrary to (1), this formula has been generalized in the infinite dimensional case and constitutes the starting point of the constructions (see Definition 20 below).

A fundamental result is the Calderòn-Vaillancourt theorem (see [14] and [16], [29], who relax the condition on the number of derivatives), generalized in the infinite dimensional setting in Section 4.2:

THEOREM 1. *If F is infinitely differentiable on $\mathbb{R}^{2n}$ and if its derivatives of arbitrary order are bounded, the operator $Op_h^{W,cl}(F)$ extends as a bounded operator on $L^2(\mathbb{R}^n)$.*

The conditions satisfied by F in the previous theorem are a particular case of the Hörmander symbol classes, namely the class $S(1,1,1)$. One denotes by $\mathcal{S}(m, \varphi, \Phi)$ the set of all functions F satisfying the following conditions. For all indices α, β, there exists a positive constant $C_{\alpha,\beta}$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2n},\ |\partial_x^\alpha \partial_\xi^\beta F(x,\xi)| \leq C_{\alpha,\beta} m(x,\xi) \varphi(x,\xi)^{-|\alpha|} \Phi(x,\xi)^{-|\beta|}.$$

The positive functions m, φ, Φ are admissible (slowly varying, temperate). If Φ and ϕ are greater than 1, one considers that one "gains" with every differentiation of the symbol F. Since the constant function is admissible, one can take $m = \Phi = \phi = 1$. This gives a particular Hörmander symbol class, which contains the function F of Theorem 1.

There exist results about composition of pseudodifferential operators. One may give an explicit, integral expression for the symbol of $Op_h^{W,cl}(a) \circ Op_h^{W,cl}(b)$, or one can write an asymptotic expansion of the symbol according to symbol classes, which are better and better the further one develops. The latter point of view was generalized (Section 4.2).

Let us state the Beals characterization ([36], Theorem 2.6.6), generalized in Section 4.3.

THEOREM 2. *Let A be a linear operator from $\mathcal{S}(R^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. It can be written as $A = Op^W(a)$ with $a \in \mathcal{S}(1,1,1)$ if and only if A and its iterated commutators with the multiplication by x_j and the differentiation with respect to x_j are bounded on L^2.*

Now recall briefly the definitions linked with the coherent states: the Wick symbol and the anti-Wick calculus. These notions have analogues in the infinite dimensional setting (in the Fock space or in the Wiener space).

The classical coherent states are the following functions of $L^2(\mathbb{R}^n, d\lambda)$, indexed by $X = (x,\xi) \in \mathbb{R}^{2n}$ and a positive h:

$$\Psi_{X,h}^{cl}(u) = (\pi h)^{-n/4} e^{-\frac{|u-x|^2}{2h}} e^{\frac{i}{h} u.\xi - \frac{i}{2h} x.\xi}.$$

There, λ denotes the Lebesgue measure on $\mathbb{R}^n$. They satisfy, for $f, g \in L^2(\mathbb{R}^n, d\lambda)$, the identity:

$$\langle f, g \rangle_{L^2(\mathbb{R}^n, d\lambda)} = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} \langle f, \Psi_{X,h}^{cl} \rangle \ \langle \Psi_{X,h}^{cl}, g \rangle d\lambda(X).$$

One may then define, on the one hand the anti-Wick operator associated with a convenient symbol F and, on the other hand, the Wick symbol of an operator A bounded on $L^2(\mathbb{R}^n, d\lambda)$ ([12]). Let F be measurable and bounded on $\mathbb{R}^{2n}$. The anti-Wick or Berezin-Wick operator is defined, for $f, g \in L^2(\mathbb{R}^n, d\lambda)$, by:

$$(3) \qquad \langle Op_h^{AW,cl}(F) f, g \rangle = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} F(X) \langle f, \Psi_{X,h}^{cl} \rangle \langle \Psi_{X,h}^{cl}, g \rangle d\lambda(X).$$

The anti-Wick calculus has properties which are easier to derive than similar (or even non existing) properties of the Weyl calculus. For example, $||Op_h^{AW,cl}(F)||_{L(L^2(\mathbb{R}^n,d\lambda))} \leq ||F||_\infty$ and when $F = 1$, it immediately gives the identity operator. Moreover, if F is a positive function, the operator is positive, which is not the case for the Weyl calculus. Indeed, the Gårding inequality only assures that the Weyl operator is bounded below.

The Wick or covariant symbol of an operator A, bounded on $L^2(\mathbb{R}^n, d\lambda)$, is defined by:

$$\sigma_h^{Wick}(A)(X) = \langle A\Psi_{X,h}, \Psi_{X,h}\rangle.$$

Finally, mention the links between Weyl, Wick and anti-Wick symbols [21]. One has, thanks to the heat operator, for a bounded Borel function F,

$$\langle Op_h^{AW,cl}(F)f, g\rangle = \langle Op_h^{W,cl}(e^{\frac{h}{4}\Delta}F)f, g\rangle, \quad \sigma_h^{Wick}(Op_h^{AW,cl}(F)) = e^{\frac{h}{2}\Delta}F.$$

3. THE INFINITE DIMENSIONAL SPACES

We recall in this section the infinite dimensional spaces which appear in the construction of the infinite dimensional quantization: first the abstract Wiener spaces, then (less extensively) the Fock space and Gaussian Hilbert spaces.

The symmetric Fock space is a Hilbert space constructed on an initial Hilbert space $\mathcal{H}$ with the help of symmetrized tensor products, for example to model the appearance or disappearance of undistinguishable particles. Its use is classical in mathematical physics.

Abstract Wiener spaces are infinite dimensional measure spaces, constructed as suitable completions of the Hilbert space $\mathcal{H}$. They allow the use of integrals, since the measure theory on them is very well developed: see [22–25, 33, 40] etc. There is a correspondence between both kind of spaces, more precisely between the Fock space and L^2 spaces on an abstract Wiener space, as will be seen below.

Ideally, to describe an arbitrary number of photons, one would like to replace the phase space $\mathbb{R}^n$ by an infinite dimensional Hilbert space endowed with a suitable measure and to generalize formulas (1) and (2). But an infinite dimensional Hilbert space cannot be equipped with a measure, either invariant by translations or by rotations and taking finite positive values on the open balls (see [33]). Indeed, since the unit ball contains infinitely many disjoint balls of radius 1/4, these assumptions can't hold simultaneously. The "natural" pseudomeasure (4) defined on cylinders is therefore just a starting point.

3.1. **Abstract Wiener spaces**

In this part, $\mathcal{B}(X)$ denotes the Borel sets of a normed space X and $\mathcal{F}_{fin}(X)$, the set of finite dimensional subspaces of X.

We first present the abstract Wiener space, which is less classically used in these circumstances. Most of the facts recalled here are taken from [33] (chap. 1 par. 4) but one can refer to [22–25, 40]. Let $\mathcal{H}$ be a real, separable, infinite dimensional Hilbert space with norm $|\cdot|$ and scalar product $\cdot$. For a finite dimensional subspace E of $\mathcal{H}$, let π_E be the orthogonal projection on E. To generalize a Gaussian probability measure on $\mathcal{H}$, it is natural to define the pseudomeasure of a "cylinder" on E, that is of a set C of the form $C = \{x \in \mathcal{H} \ : \ \pi_E(x) \in A\}$, where A is a Borel set of E, setting:

$$\mu_{\mathcal{H},s}(C) = \int_A e^{-\frac{|y|^2}{2s}} (2\pi s)^{-\dim(E)/2} d\lambda_E(y) = \int_A d\mu_{E,s}(y). \tag{4}$$

Here λ_E is the Lebesgue measure on E and the positive parameter s represents the variance. In the same way, a cylindrical (or tame) function on $\mathcal{H}$ is a function f which can be written as $f = \hat{f} \circ \pi_E$ for a function $\hat{f}$ defined on E. Tame functions depend on a finite number of variables.

The pseudomeasure (4) does not extend as a measure on $\mathcal{H}$, since it lacks the property of σ-additivity on the σ-algebra generated by cylinders. Indeed, let $(e_n)_n$ be an orthonormal basis of $\mathcal{H}$ and $(d_n)_{n\in\mathbb{N}}$, an increasing sequence of positive integers. Set $C_n = \{x \in \mathcal{H} \ : \ |x \cdot e_k| \le n \ \forall k \le d_n\}$. The pseudomeasure of C_n is equal to

$$\mu_{\mathcal{H},s}(C_n) = \left(\int_{-n}^{n} e^{-x^2/2}(2\pi)^{-1/2} \, dx\right)^{d_n},$$

which is smaller than $\frac{1}{2^{n+1}}$ provided d_n is large enough. Then $\mathcal{H} = \bigcup C_n$, but the sum of the $\mu_{\mathcal{H},s}(C_n)$ is not equal to $1 = \mu_{\mathcal{H},s}(\mathcal{H})$ [33].

The solution is to endow $\mathcal{H}$ with a norm $\|\ \|$ satisfying the "measurability" condition: for all positive ε, there exists a finite dimensional subspace of $\mathcal{H}$ such that

$$\forall F \in \mathcal{F}_{fin}(\mathcal{H}), \quad F \perp E_\varepsilon, \quad \mu_{\mathcal{H},t}(\{x \in \mathcal{H} : \|\pi_F(x)\| > \varepsilon\}) < \varepsilon. \tag{5}$$

Whereas all d-dimensional spaces E have the same status with respect to the initial norm, this norm introduces a kind of anisotropy. One denotes by B the completion of $\mathcal{H}$ with respect to $\|\ \|$, by B' its topological dual space. Since $\mathcal{H}$ is identified with its dual space $\mathcal{H}'$, one has the sequence of continuous inclusions, where every space is densely embedded in the following one: $B' \subset \mathcal{H} \subset B$. One often takes a Hilbert basis of $\mathcal{H}$ whose elements are contained in B'. The triple $(i, \mathcal{H}, B)$, where i is the injection of $\mathcal{H}$ in B, is called an abstract Wiener space,

B is a Wiener extension of $\mathcal{H}$, $\mathcal{H}$ is the Cameron Martin space of B. The new norm is not necessarily hilbertian and is never equivalent to the initial one. The choice of the measurable norm and hence of B is not unique and it is sometimes useful to skip from one extension to another one, as in Proposition 12.

One can define a measure on the cylinder sets of B. For $y_1, \dots, y_n$ in B' and a Borel set A of $\mathbb{R}^n$, one sets

$$\mu_{B,s}(\{x \in B \ : \ ((y_i, x)_{B',B})_{1\leq i\leq n} \in A\}) = \mu_{\mathcal{H},s}(\{x \in H \ : \ (y_i \cdot x)_{1\leq i\leq n} \in A\}).$$

This time, this measure is a probability measure on the σ-algebra generated by the cylinder sets of B, which is – as a consequence of the separability of $\mathcal{H}$ and B – the Borel σ-algebra of B.

The most famous example of such a triple is the classical Wiener space, where $\mathcal{H}$ is the set of functions in $H^1([0,1])$ vanishing at 0, B is the set $C^0([0,1])$ of continuous functions, vanishing at 0. The scalar product on $\mathcal{H}$ is given by $\langle u, v\rangle = \int_0^1 u'v' \, dt$ and the norm on B is the supremum norm.

We now introduce functions which play an important part in the rest of this construction. Their definition, in formula (6) below, extends the duality between B' and B or the scalar product of $\mathcal{H}$. As elements of a Gaussian Hilbert space defined below, they appear in the Segal isomorphisms linking Wiener and Fock spaces. In the theory of Wiener spaces, they allow one to define the stochastic extensions of functions initially defined on the Hilbert space. They also furnish "linear" symbols for the first construction of the operators (Definition 20).

An element $a \in B'$ can be seen as a random variable on the probability space $(B, \mathcal{B}(B), \mu_{B,s})$. It is denoted by ℓ_a to stress the difference of status and it has a normal distribution: $\ell_a \sim \mathcal{N}(0, \sigma = \sqrt{s}|a|)$. By density, one can define an application

$$\ell : \mathcal{H} \rightarrow L^2(B, \mu_{B,s}), \quad a \mapsto \ell_a. \tag{6}$$

One notices that $s^{-1/2}\ell$ is isometric. For $a \in \mathcal{H}$, ℓ_a is not necessarily linear on B since it is not defined everywhere. Nevertheless, some properties still hold: $\ell_{a+y}(x) = \ell_a(x) + x \cdot y$ if $y \in \mathcal{H}$, $\ell_a(-x) = -\ell_a(x)$.

Replacing scalar products on $\mathcal{H}$ by functions ℓ allows one to generalize the projection operators π_E from $\mathcal{H}$ to E (where $E \in \mathcal{F}_{fin}(\mathcal{H})$), which yields finite rank operators $\tilde{\pi}_E$ from B to E. If (u_j) is an orthogonal basis of E, one sets:

$$\tilde{\pi}_E = \sum_{j=1}^{\dim(E)} \ell_{u_j} u_j, \quad \tilde{\pi}_E : B \rightarrow E. \tag{7}$$

These generalized projectors allow us to give a link between the Hilbert space $\mathcal{H}$, on which no convenient measure exists, and its Wiener extension

B, which is a measure space without a scalar product or symplectic form (at least, devoid of the original $\mathcal{H}$ scalar product). This is the following notion of stochastic extension [40]:

Definition 3. Let $(i, \mathcal{H}, B)$ be an abstract Wiener space and s be a positive real number. A function f, defined on $\mathcal{H}$, is said to admit a stochastic extension $\widetilde{f}$ in $L^p(B, \mu_{B,s})$ $(1 \leq p < \infty)$ if, for every increasing sequence (E_n) of finite dimensional subspaces of $\mathcal{H}$, whose union is dense in $\mathcal{H}$, $\widetilde{f}$ and the functions $f \circ \widetilde{\pi}_{E_n}$ are in $L^p(B, \mu_{B,h})$ and if the sequence $f \circ \widetilde{\pi}_{E_n}$ converges in $L^p(B, \mu_{B,s})$ to $\widetilde{f}$.

If the sequence $f \circ \widetilde{\pi}_{E_n}$ converges in probability to $\widetilde{f}$, one only speaks of a stochastic extension and it is implied by the condition of the preceding definition. Usually, a stochastic extension is not a continuity extension, but it may be (Theorem 6.3 [33]). Reciprocally, restricting a measurable function defined on B to the subspace $\mathcal{H}$ makes no sense, since $\mathcal{H}$ is negligible in B for any measure $\mu_{B,s}$.

A fundamental example is that, for all $a \in \mathcal{H}$, the random variable ℓ_a is the stochastic extension in $L^p(B, \mu_{B,s})$ of the function $x \mapsto a \cdot x$, defined on $\mathcal{H}$. Indeed, $a \cdot \tilde{\pi}_{E_n}(x) = \ell_{\pi_{E_n}(a)}(x)$ and the properties of the normal distribution imply that $\ell_{\pi_{E_n}(a)} - \ell_a = \ell_{\pi_{E_n}(a)-a} \to 0$ in $L^p(B, \mu_{B,s})$. Moreover, if α is a multiindex (a family of integers, of which only a finite number is different from zero), if one defines the function a^α on $\mathcal{H}$ by

$$a^\alpha(x) = \prod_{i=1}^{n} (a_i \cdot x)^{\alpha_i},$$

then, for $s > 0$ and any $p \in [1, +\infty[$, the function a^α admits the function $\prod_{i=1}^{n} \ell_{a_i}^{\alpha_i}$ as a stochastic extension in $L^p(B, \mu_{B,s})$ (see [31]).

In Section 4.1 about symbol classes, one sees that the symbols are defined so as to admit stochastic extensions. But to give simpler examples, one can check that the functions $x \in \mathcal{H} \mapsto |x|^2$, $x \mapsto e^{i|x|^2}$ have no stochastic extension, whereas $x \mapsto e^{-|x|^2}$ admits the null function. This can be seen in [33], Chap. 1, Sec. 4, for the first function, where this example justifies the definition of "measurability" (5). For the exponential functions, explicit computations on finite dimensional spaces prove that the sequence $(f \circ \widetilde{\pi}_{E_n})$ is not a Cauchy sequence (first case) or converges to 0 (second case).

A further example of a stochastic extension is the following result, which will be useful to define the Segal-Bargmann transform:

THEOREM 4. *Let $(i, \mathcal{H}, B)$ be an abstract Wiener space. Let f be a continuous function on $\mathcal{H}$. Suppose that the restrictions of f to the subspaces in*

$\mathcal{F}_{fin}(\mathcal{H})$ are harmonic functions. Suppose there exist $h > 0$ and $M > 0$ such that, for every subset E in $\mathcal{F}_{fin}(\mathcal{H})$, $||f \circ \widetilde{\pi}_E||_{L^2(B,\mu_{B,h})} \leq M$. Then f admits a stochastic extension in $L^2(B, \mu_{B,h})$.

We mention here some particular features of the abstract Wiener space: most of the translations and dilations yield mutually orthogonal measures. We state the result for the translations, since we need the corresponding change of variable ([33], Chap.2, par.5):

THEOREM 5. *For $s > 0$ and $A \in \mathcal{B}(B)$, set*

$$\mu_{B,s}(x, A) = \mu_{B,s}(A - x), \quad A - x = \{a - x,\ a \in A\}.$$

The measures $\mu_{B,s}(x, \cdot)$ and $\mu_{B,t}(y, \cdot)$ are absolutely continuous with respect to one another if and only if $s = t$ and $x - y \in \mathcal{H}$.

For $u \in \mathcal{H}$ and g measurable and bounded:

$$\int_B g(y) d\mu_{B,s}(y) = \int_B g(x-u) e^{-\frac{1}{2s}|u|^2 + \frac{1}{s}\ell_u(x)} d\mu_{B,s}(x).$$

3.2. Gaussian Hilbert spaces, Fock spaces

We recall here what is strictly necessary about Gaussian Hilbert spaces (treated in [32]), in particular the Segal isomorphism between Gaussian Hilbert spaces and Fock spaces. The Fock symmetric space constructed on the complexified of a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{F}_s(\mathcal{H})$. Fock spaces are constructed in [41,42] or [18,19,32], from which we took the definition, the creation and annihilation operators and the second quantization. We denote by S_n the orthogonal projection on the symmetrized tensor product of order n, given by

$$S_n(u_1 \otimes \cdots \otimes u_n) := (n!)^{-1} \sum_{\sigma \in \Sigma_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

where Σ_n is the symmetric group.

A *Gaussian Hilbert space* $\mathcal{M}$ is a real vector space of random variables ξ defined on a probability space $(\Omega, \mathcal{F}, P)$, such that every random variable is centered and Gaussian. Hence every ξ belongs to $L^2(\Omega, \mathcal{F}, P)$. Moreover, when it is endowed with the norm and scalar product of $L^2(\Omega, \mathcal{F}, P)$, the Gaussian Hilbert space is required to be complete.

An example of a Gaussian Hilbert space is given by the set of functions ℓ_a, $a \in \mathcal{H}$, previously defined, which we call $\mathcal{M}$. In this case, $\mathcal{M} \subset L^2(B, \mathcal{B}(B), \mu_{B,s})$.

Since every element of $\mathcal{M}$ belongs to L^p for any finite p, polynomials in elements of $\mathcal{M}$ belong to $L^2(\Omega, \mathcal{F}, P)$. One defines by $\mathcal{P}_n(\mathcal{M})$ the closure, in

$L^2(\Omega, \mathcal{F}, P)$, of the set $\{p(\xi_1, \dots, \xi_m), m \in \mathbb{N}^m, \deg(p) \leq n, \xi_1, \dots, \xi_m \in \mathcal{M}\}$. One sets, for $n \geq 1$,

$$\mathcal{M}^{:n:} = \mathcal{P}_n(\mathcal{M}) \cap \mathcal{P}_{n-1}^{\perp}(\mathcal{M}),$$

where $\mathcal{M}^{:1:}$ is the set of constant random variables and $\mathcal{M}^{:-1:} = \{0\}$.

If $\mathcal{F}(\mathcal{M})$ is the σ-algebra generated by all functions of $\mathcal{M}$, the sets $\mathcal{M}^{:n:}$ are mutually orthogonal and closed subspaces of $L^2(\Omega, \mathcal{F}(\mathcal{M}), P)$ and one has the Wiener chaos decomposition ([32], Theorem 2.6):

$$\bigoplus_0^{\infty} \mathcal{M}^{:n:} = L^2(\Omega, \mathcal{F}(\mathcal{M}), P).$$

Let π_n be the orthogonal projection of $L^2(\Omega, \mathcal{F}, P)$ onto $\mathcal{M}^{:n:}$. The Wick product of the elements $\xi_1, \dots, \xi_n$ is then defined as

$$:\ \xi_1 \dots \xi_n\ :=\ \pi_n(\xi_1 \dots \xi_n).$$

One can express the Wick products in terms of usual products and conversely ([32], Chap.3). Let us recall the following result. Let $(\xi_i)_{i \in I}$ be an orthonormal basis of $\mathcal{M}$, (countable or not), let $\alpha = (\alpha_i)_{i \in I}$ be a multiindex (a family of integers, of which only a finite number is different from 0). Then

$$:\ \prod_i \xi_i^{\alpha_i}\ :=\ \prod_i h_{\alpha_i}(\xi_i). \tag{8}$$

where the h_n are the Hermite polynomials with leading coefficient equal to 1, defined by

$$h_{n+1}(x) = x h_n(x) - n h_{n-1}(x), \quad h_0(x) = 1.$$

One needs to specify the relationship between the Gaussian Hilbert space $\mathcal{M} = \{\ell_a, a \in \mathcal{H}\} \subset L^2(B, \mu_{B,s})$ and the Fock space $\mathcal{F}_s(\mathcal{H})$, since its normalization depends on the variance parameter s. For elements $a_1, \dots, a_n \in \mathcal{H}$, the Segal isomorphism associates, with the element $S_n(a_1 \otimes \dots \otimes a_n)$ of the Fock space, the element $\frac{1}{\sqrt{n!}} \left(\frac{1}{s}\right)^{n/2} : \ell_{a_1} \dots \ell_{a_n} :$ of the Gaussian Hilbert space $\mathcal{M}$. It extends as an isometry of the Fock space onto $L^2_{\mathbb{R}}(B, \mathcal{B}(B), \mu_{B,s})$. This is proved in [32], Chap. 4, with $s = 1$. Note that $\mathcal{B}(B)$ is generated by the elements of $\mathcal{M}$.

Now give the correspondence between the classical operators in the Fock space and their Wiener counterpart. Later on, we shall see that these operators can be expressed thanks to the pseudodifferential calculus and state their links with the Malliavin Gradient (Section 4.2).

Let $f, a_1, \dots, a_n$ belong to $\mathcal{H}$. The creation and annihilation operators act in the following way on Wick monomials:

$$a^*(\ell_f)(:\ell_{a_1}\ldots\ell_{a_n}:) = t^{-1/2} :\ell_f\ell_{a_1}\ldots\ell_{a_n}:$$

$$a(\ell_f)(:\ell_{a_1}\ldots\ell_{a_n}:) = t^{-1/2}\sum_{j=1}^{n}\langle\ell_{a_j},\ell_f\rangle_{L^2(B,\mu_{B,t})} :\ell_{a_1}\ldots\check{\ell_{a_j}}\ldots\ell_{a_n}:,$$

where $\check{\ell_{a_j}}$ indicates that the term is missing (see [32], Chap. 13, up to normalization). Hence the Fock Segal field $\Phi_F(f) = \frac{1}{\sqrt{2}}(a(f)+a^*(f))$ corresponds to

$$\Phi_S(\ell_f) :\ell_{a_1}\ldots\ell_{a_n} := \frac{1}{\sqrt{2t}}\left(:\ell_f\ell_{a_1}\ldots\ell_{a_n}: + \sum_{j=1}^{n}\langle\ell_{a_j},\ell_f\rangle_{L^2(\mu_{B,t})} :\ell_{a_1}\ldots\check{\ell_{a_j}}\ldots\ell_{a_n}:\right)$$

3.3. Coherent states, Wick, anti-Wick symbols

One can then define the coherent states indexed by $X=(x,\xi)\in\mathcal{H}^2$, as a family of functions defined on B and belonging to $L^2(B,\mu_{B,h/2})$ in the first place:

$$\Psi_{X,h}(u) = e^{\frac{1}{h}\ell_{x+i\xi}(u)-\frac{1}{2h}|x|^2-\frac{i}{2h}x\cdot\xi}$$

and as elements of the Fock space in the second place:

$$\Psi_{X,h} = e^{-\frac{|X|^2}{4h}}\sum_{n\geq 0}\frac{(x+i\xi)\otimes\cdots\otimes(x+i\xi)}{(2h)^{n/2}\sqrt{n!}},\quad \Psi_{0,h}=\Omega,$$

where Ω is the vacuum state. Coherent states satisfy

$$\langle\Psi_{X,h},\Psi_{Y,h}\rangle = e^{-\frac{1}{4h}|X-Y|^2+\frac{i}{2h}(\xi\cdot y - x\cdot\eta)},$$

with the scalar product of $L^2(B,\mu_{B,h/2})$ or of the Fock space. They are, in the Wiener space, the stochastic extensions of functions on $\mathcal{H}$ which have the same expression with a scalar product instead of the function ℓ [4].

As in the finite dimensional setting, they are useful to define the Wick symbol of a bounded operator as well as the Segal Bargmann transform of a function. Let A be a bounded operator over $L^2(B,\mu_{B,h/2})$. Then the Wick symbol of A is defined by

$$\sigma_h^{Wick}(A)(X) = \langle A\Psi_{X,h},\Psi_{X,h}\rangle.$$

Note that a Wick calculus has been developed in an infinite dimensional context (see [1, 2]). The authors associate, with a polynomial on a Fock space, an operator. This is not the point of view adopted here.

In the finite dimensional case, a formula of decomposition of the identity allowed one to define an operator called anti-Wick operator (3). Here we can't generalize it in this form, since one can't integrate on the Hilbert space. One has to use the Segal-Bargmann transformation. Let $f \in L^2(B, \mu_{B,h/2})$. We set $(T_h f)(X) = \frac{\langle f, \Psi_{X,h}\rangle}{\langle \Psi_{0,h}, \Psi_{X,h}\rangle}$ and define $\widetilde{T_h} f$ as its stochastic extension in $L^2(B^2, \mu_{B^2,h})$. This extension exists thanks to the antiholomorphy properties of $T_h f$, using Theorem 4. The map $f \mapsto \widetilde{T_h} f$ is the Segal-Bargmann transformation. It is a partial isometry from $L^2(B, \mu_{B,h/2})$ into $L^2(B^2, \mu_{B^2,h})$ (see [21] for the finite dimensional case).

We now may define the anti-Wick calculus. Let F be defined on $\mathcal{H}^2$, admitting a bounded Borel stochastic extension $\widetilde{F}$ in $L^2(B^2, \mu_{B^2,h})$. One sets

$$\langle Op_h^{AW}(F) f, g\rangle = \int_{B^2} \widetilde{F}(X) \widetilde{T}_h f(X) \overline{\widetilde{T}_h g(X)} d\mu_{B^2,h}(X).$$

As in the classical case, the anti-Wick calculus is easier to use and, for example, $||Op_h^{AW}(F)|| \leq ||\tilde{F}||_\infty$.

For $a, b \in H$, let us introduce the operator

$$\Sigma_{(a,b)} = \ell_{a+ib} + \frac{h}{i}\, b \cdot \frac{\partial}{\partial u} \tag{9}$$

defined on the set of tame and smooth functions. By $b \cdot \frac{\partial}{\partial u}$ we understand, if f is cylindrical on a finite dimensional subspace E of B' with orthonormal basis $(e_1, \ldots, e_n)$:

$$b \cdot \frac{\partial f}{\partial u}(u) = \sum_{i=1}^{n} \langle b, e_i\rangle_H \frac{\partial \varphi}{\partial y_i}(e_1(u), ..., e_n(u))$$

where $f(u) = \varphi(e_1(u), ..., e_n(u))$. The operator $\Sigma_{(a,b)}$ is the analogous of a Segal field. One can check this fact on simple elements of $L^2(B, \mu_{B,h/2})$ like the Wick product

$$f \;= : \prod_{j=1}^{n} (\ell_{e_j})^{\alpha_j} : = \left(\frac{h}{2}\right)^{|\alpha|/2} \prod_{j=1}^{n} h_{\alpha_j}(\sqrt{2/h}\; \ell_{e_j}),$$

where the family $(e_j)_j$ is orthonormal. Indeed, the multiplication part ℓ_{a+ib} can be treated by the following equality (a particular case of results of [32], Chap.3):

$$\xi : \xi_1 \ldots \xi_n : = : \xi\xi_1 \ldots \xi_n : + \sum_{i=1}^{n} E(\xi\xi_i) : \xi_1 \ldots \check{\xi}_i \ldots \xi_n : ,$$

where, as usual, $\check{\xi}_i$ does not appear. For the differentiation part one uses the correspondence (8). This finally gives

$$\Sigma_{(a,b)} f = \ell_{a+ib} f + \frac{h}{i}\, b \cdot \frac{\partial f}{\partial u} = \sqrt{h}\Phi(\ell_{a+ib}).$$

This can be extended by linearity to polynomials of functions ℓ_as.

Now, for $a, b \in \mathcal{H}$ and f defined on B set, according to [4]

$$(U_{a,b,h} f)(u) = e^{-\frac{h}{2}|b|^2 + i\frac{h}{2} a\cdot b + i\ell_{a+ib}(u)} f(u + hb). \tag{10}$$

One can check that, if f is tame and takes the form $\hat{f} \circ P_E$ with $\hat{f}$ polynomial, or if f is a linear combination of coherent states, then the following convergence holds in $L^2(B, \mu_{B,h/2})$:

$$\lim_{t\to 0} \frac{U_{t(\frac{a}{h},\frac{b}{h}),h} f - f}{t} = b \cdot \partial_u f + \frac{i}{h}\ell_{a+ib} f \text{ in } L^2(B, \mu_{B,h/2}).$$

This justifies the correspondence between the operators U and the Weyl operators [18, 41, 42].

$$U_{t(\frac{a}{h},\frac{b}{h}),h} = e^{i\frac{t}{\sqrt{h}}\Phi(\ell_{a+ib})}.$$

The operators $\Sigma_{(a,b)}, U_{a,b,h}$ can be expressed thanks to the quantization (see Section 4.2).

4. WEYL CALCULUS ON A WIENER SPACE

4.1. Symbol classes, heat operator

This paragraph presents the symbol classes introduced in the infinite dimensional setting. These are sets of functions defined on $\mathcal{H}^2$ and admitting stochastic extensions on B^2. One is led to define a heat operator which applies to such functions and hence differs from the classical one [24, 26, 33], acting on functions defined on B (or here on B^2). Further results about the symbol classes and the heat operator are mentioned here.

4.1.1. DEFINITIONS

The symbol classes recalled here are taken from [4, 7] and [31]. The first class satisfies conditions similar to the Calderòn Vaillancourt hypotheses and depends strongly of the choice of a Hilbert basis of $\mathcal{H}$. The second class, more intrinsic, resorts to a quadratic form. They are adapted to different problems, as will be seen in Section 4.4.

Let Γ be a countable set. A multi-index is a map (α, β) from Γ into $\mathbb{N} \times \mathbb{N}$ such that $\alpha_j = \beta_j = 0$ except for a finite number of indices. We denote by $\mathcal{M}_m$ the set of multi-indices of "depth" m, satisfying $\alpha_i, \beta_i \leq m$ for all $i \in \Gamma$.

Definition 6. Let $(i, \mathcal{H}, B)$ be an abstract Wiener space such that $B' \subset \mathcal{H} \subset B$. Let $\mathcal{B} = (e_j)_{(j\in\Gamma)}$ be a Hilbert basis of $\mathcal{H}$, each vector belonging to B' and set $u_j = (e_j, 0)$, $v_j = (0, e_j)$. Let m be a nonnegative integer and $\varepsilon = (\varepsilon_j)_{(j\in\Gamma)}$, a family of nonnegative real numbers. One denotes by $S_m(\mathcal{B}, \varepsilon)$ the set of bounded continuous functions $F : \mathcal{H}^2 \to \mathbb{C}$ satisfying the following condition. For every multi-index (α, β) of $\mathcal{M}_m$, the derivative

$$\partial_u^\alpha \partial_v^\beta F = \left[\prod_{j\in\Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j}\right] F$$

is well defined, continuous on $\mathcal{H}^2$ and there exists a real constant M such that

$$\forall (x,\xi) \in \mathcal{H}^2, \quad \left| \left[\prod_{j\in\Gamma} \partial_{u_j}^{\alpha_j} \partial_{v_j}^{\beta_j}\right] F(x,\xi) \right| \leq M \prod_{j\in\Gamma} \varepsilon_j^{\alpha_j+\beta_j} . \tag{11}$$

One then defines $||F||_{m,\varepsilon}$ as the smallest M such that (11) holds.

Remark that $S_m(\mathcal{B}, \varepsilon)$, equipped with $||\ ||_{m,\varepsilon}$, is a Banach space. Setting $S_\infty(\mathcal{B}, \varepsilon) = \bigcap_{m=0}^\infty S_m(\mathcal{B}, \varepsilon)$, one can define a distance by $d(F, G) = \sum_{m=0}^{\infty} 2^{-m} \dfrac{||F - G||_{m,\varepsilon}}{1 + ||F - G||_{m,\varepsilon}}$, for which $(S_\infty(\mathcal{B}, \varepsilon), d)$ is complete.

One easily checks that, if $G \in S_m(\mathcal{B}, \delta)$, then $FG \in S_m(\mathcal{B}, \varepsilon + \delta)$ with $||FG||_{m,\varepsilon+\delta} \leq ||F||_{m,\varepsilon} ||G||_{m,\delta}$. Moreover, if $m \geq k \geq 1$ and if α, β are two multi-indexes of depth k, then $\partial_u^\alpha \partial_v^\beta F \in S_{m-k}(\mathcal{B}, \varepsilon)$ and

$$||\partial_u^\alpha \partial_v^\beta F||_{m-k,\varepsilon} \leq ||F||_{m,\varepsilon} \prod_{j\in\Gamma} \varepsilon_j^{\alpha_j+\beta_j}.$$

One can define matrix valued symbols as well, as is the case in Definition 3.1 of [7]. The definition makes no assumption about the sequence ε but square summability or summability is required in most of the results of the following paragraphs.

The class of symbols defined below was introduced to treat a problem in NMR (see Section 4.4). The use of the former class was possible but imposed a sharp condition on a cutoff function, which the new classes enable one to lift.

Definition 7. Let A be a linear, self-adjoint, nonnegative, trace class application on a Hilbert space $\mathcal{H}$. For all $x \in \mathcal{H}$, one sets $Q_A(x) = \langle Ax, x\rangle$. Let $S(Q_A)$ be the class of all functions $f \in C^\infty(\mathcal{H})$ such that there exists $C(f) > 0$

satisfying:

$$\forall x \in \mathcal{H},\ |f(x)| \leq C(f), \forall m \in \mathbb{N}^*,\ \forall x \in \mathcal{H},\ \forall (U_1, \dots, U_m) \in \mathcal{H}^m,$$

(12) $$|(d^m f)(x)(U_1, ..., U_m)| \leq C(f) \prod_{j=1}^{m} Q_A(U_j)^{\frac{1}{2}}.$$

The smallest constant $C(f)$ such that (12) holds is denoted by $\|f\|_{Q_A}$.

Notice that $S(Q_A)$, equipped with the norm $|| \; ||_{Q_A}$, is a Banach space. One can also check that, if A and B satisfy the conditions of Definition 7, a product of functions belonging to $S(Q_A), S(Q_B)$ is in $S(Q_{2(A+B)})$ with

(13) $$||fg||_{Q_{2(A+B)}} \leq ||f||_{Q_A} ||g||_{Q_B}.$$

4.1.2. PROPERTIES OF SYMBOLS

We state here properties of both kinds of symbols. The symbols in both classes admit stochastic extensions (provided the sequence ε is summable for the $S_m(\mathcal{B}, \varepsilon)$). The proofs, which are omitted, rely on a Lipschitz property for the Calderòn-Vaillancourt class, on orthogonality properties for the second one. We then mention Frechet differentiability properties for the Calderòn-Vaillancourt classes, initially defined by conditions on partial derivatives. We skip technical results about the behaviour of Taylor expansions, integrals of symbol classes and most of the intermediate results.

PROPOSITION 8 ([4, 31]). *Let F be a function in $S_1(\mathcal{B}, \varepsilon)$ with respect to a Hilbert basis $\mathcal{B} = (e_j)_{(j \in \Gamma)}$. Assume that the sequence $(\varepsilon_j)_{(j \in \Gamma)}$ is summable.*

1. *For every positive h and every $q \in [1, +\infty[$, F admits a stochastic extension in $L^q(B^2, \mu_{B^2,h})$. This extension depends on h and q in a loose way: for all $h_0 > 0$ and $q_0 \in]1, +\infty[$, there exists a function $\widetilde{F}$ which is the stochastic extension of F in $L^q(B^2, \mu_{B^2,h})$ for all $h \in]0, h_0]$ and $q \in [1, q_0]$.*
2. *There exists a constant $K(q)$ such that for any $E \in \mathcal{F}_{fin}(\mathcal{H}^2)$, we have the inequalities:* $\forall (h, q) \in]0, h_0] \times [1, q_0]$,

$$||F \circ \tilde{\pi}_E - \widetilde{F}||_{L^q(B^2, \mu_{B^2,h})}$$
$$\leq ||F||_{1,\varepsilon} K(q) h^{1/2} \sum_{j=1}^{\infty} \varepsilon_j \Big(|u_j - \pi_E(u_j)| + |v_j - \pi_E(v_j)| \Big).$$

3. *If $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 1$, if $\widetilde{F}$ is the stochastic extension mentioned above, if $Y \in \mathcal{H}^2$, then $\tau_Y F$ admits $\tau_Y \widetilde{F}$ as a stochastic extension in $L^p(B^2, \mu_{B^2,h})$ for $h > 0$ and $p \in [1, +\infty[$.*

The last point of the former proposition is valid for all globally Lipschitz functions which admit a stochastic extension for every positive h and finite p.

Proposition 9 ([31]). *Let A be a linear, self-adjoint, positive and trace-class operator with eigenvalues (λ_i). Let $h > 0$ and let $p \in [1, +\infty[$. Every function f belonging to $S(Q_A)$ admits a stochastic extension $\widetilde{f}$ in $L^p(B, \mu_{B,h})$. The function $\widetilde{f}$ is bounded $\mu_{B,h}$ almost everywhere by $||f||_{Q_A}$. Moreover, there exists a constant $C(p, A)$ such that, for all $E \in \mathcal{F}_{fin}(\mathcal{H})$,*

$$(14)\quad ||f \circ \tilde{\pi}_E - \widetilde{f}(x)||_{L^p(B,\mu_{B,h})} \leq C(p, A) h^{\frac{1}{2}} \|f\|_{Q_A} \left(\sum_{j\geq 0} \lambda_j |\pi_E(u_j) - u_j|^{\alpha(p)} \right)^{1/\max(2,p)}.$$

For the latter class, one can prove similar extension properties for the multilinear forms defined by the derivatives. If $k \in \mathbb{N}^*$ and $x \in \mathcal{H}$, the function $y \mapsto d^k f(x) \cdot y^k$ defined on $\mathcal{H}$ admits a stochastic extension in $L^p(B, \mu_{B,h})$ and one has an inequality similar to (14).

We now state the differentiability properties for the Calderòn-Vaillancourt classes.

Proposition 10. *Let F be in $S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε square summable. Then F is C^{m-1} on $\mathcal{H}^2$. For the first order of differentiability, one has*

$$DF(X) \cdot Y = \sum_{j\in\Gamma} \langle Y, u_j \rangle \frac{\partial F}{\partial u_j}(X) + \langle Y, v_j \rangle \frac{\partial F}{\partial v_j}(X).$$

Moreover, for all X and Y in $\mathcal{H}^2$,

$$|F(X + Y) - F(X) - DF(X) \cdot Y| \leq ||F||_{m,\varepsilon} \sum_{j\in\Gamma} \varepsilon_j^2 (1 + 2\sqrt{2})\ |Y|^2.$$

Finally, for all $Y \in \mathcal{H}^2$, $X \mapsto DF(X) \cdot Y$ is in $S_{m-1}(\mathcal{B}, \varepsilon)$, with $||X \mapsto DF(X) \cdot Y||_{m-1,\varepsilon} \leq 2||F||_{m,\varepsilon}|Y|\sqrt{\sum_{j\in\Gamma} \varepsilon_j^2}$.

The loss of one order of differentiability is due to the fact that one handles series and estimates a countable number of integral Taylor rests.

In the former proposition, nothing proves that the functions have stochastic extensions, since the series is only square summable. All the computations were conducted in the Hilbert space, where the sums took the form $\sum \varepsilon_j u_j \cdot x$ and $\sum (u_j \cdot x)^2$ converges.

The following result ensures the existence of a stochastic extension for the first order differential. One needs the summability of ε, since the sums that appear are $\sum \varepsilon_j \ell_{u_j}$, where all the ℓ_{u_j} have the same norm. Here, $\mathcal{P}$ denotes the passage to a stochastic extension.

PROPOSITION 11. *Let $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 2$ and ε summable. The application $X \mapsto DF(X) \cdot Y$ from $\mathcal{H}$ in $\mathbb{R}$ admits a stochastic extension in $L^p(B^2, \mu_{B^2,t})$, which is the application*

$$\sum_{\Gamma} \langle Y, u_j \rangle \mathcal{P}\left(\frac{\partial F}{\partial u_j}\right) + \langle Y, v_j \rangle \mathcal{P}\left(\frac{\partial F}{\partial v_j}\right).$$

The next result allows one to construct another completion B_A of $\mathcal{H}$ in the case when ε is summable. The advantage is that stochastic extension and continuity extension coincide. The result is stated for the Calderòn-Vaillancourt classes but exists in the classes of Definition 7.

PROPOSITION 12. *Let ε be a summable sequence such that $\varepsilon_j > 0$ for all $j \in \Gamma$. One defines a symmetric, definite positive and trace class operator A by setting*

$$\forall X \in B^2, \; AX = \sum_{j \in \Gamma} \varepsilon_j \langle X, u_j \rangle u_j + \varepsilon_j \langle X, v_j \rangle v_j.$$

Set $||X||_A = \langle AX, X \rangle^{1/2}$. Then $|| \; ||_A$ is a measurable norm on H (see (5)). One denotes by B_A the completion of $\mathcal{H}$ for this norm.

If $F \in S_m(\mathcal{B}, \varepsilon)$ for $m \geq 2$, then F is uniformly continuous on $\mathcal{H}^2$ with respect to the norm $|| \; ||_A$. It admits a uniformly continuous extension F_A on B_A and the stochastic extension $\widetilde{F}$ of F given by Proposition 8 is equal to F_A $\mu_{B_A,h}$- a.e.

Finally, there are relationships between both symbol classes. If F belongs to $S_\infty(\mathcal{B}, \varepsilon)$ and if there exists a constant M such that $||F||_{m,\varepsilon} \leq M$ for all m, then $F \in S(Q_B)$ with B defined by $B = 4(\sum_\Gamma \varepsilon_j)A$, A being as in the former proposition.

4.1.3. HEAT OPERATOR

We now introduce the heat operator. On abstract Wiener spaces, there exists the following classical notion: if f is a Borel bounded function on the Wiener space $(B, \mathcal{B}(B))$, one sets

$$\forall x \in B, \; \forall t > 0, \quad \widetilde{H_t} f(x) = \int_B f(x+y) \; d\mu_{B,t}(y).$$

It has properties exposed in [24–26,33]. For example, $\widetilde{H_t} f$ is infinitely derivable along the directions of $\mathcal{H}$, $(\widetilde{H_t})_t$ is a strongly continuous contraction semigroup on the Banach space of all bounded, uniformly continuous, complex valued functions. One can see [4] that it is bounded and with a norm smaller than 1 from $L^p(B^2, \mu_{B^2,t+h})$ in $L^p(B^2, \mu_{B^2,h})$ for finite p and positive t, h. In the finite dimensional case it corresponds to $e^{\frac{t}{2}\Delta}$. One may define partial heat operators too, by integrating on particular subspaces of B as in (16) below.

As the symbols are defined on the Hilbert space and not on a Wiener extension, one is led to define the following version of the heat operator:

Definition 13. Let F be defined on $\mathcal{H}$, admitting a stochastic extension $\widetilde{F}$ in $L^p(B, \mu_{B,t})$ for a finite p. One defines H_tF or $e^{\frac{t}{2}\Delta}F$ by

$$\forall X \in \mathcal{H}, \quad (H_tF)(X) = \int_B \widetilde{F}(X+Y)\, d\mu_{B,t}(Y) = \int_B \widetilde{F}(Y) e^{-\frac{|X|^2}{2t}} e^{\ell_X/t}\, d\mu_{B,t}(Y).$$

The second equality is a consequence of Theorem 5, since $X \in \mathcal{H}$.

This definition is independent of the measurable norm on $\mathcal{H}$, of the Wiener extension of $\mathcal{H}$ and of the stochastic extension of F. Indeed, the fact that a sequence $(F \circ \tilde{\pi}_{E_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(B, \mu_{B,h})$ is expressed by integrals on finite dimensional subspaces of $\mathcal{H}$. Similarly, the integral does not depend on the integration space B, since it is a limit of integrals on finite dimensional spaces of $\mathcal{H}$. This property allows one to choose, transitorily, an extension like that of Proposition 12 and to use results of the classical theory.

The heat operator has properties similar to its finite dimensional model:

Proposition 14. *Let F be in $S(Q_A)$ or in $S_m(\mathcal{B}, \varepsilon)$, with ε summable. For all positive s, t and all X in the Hilbert space $\mathcal{H}$,*

$$H_t(H_sF)(X) = H_{t+s}F(X).$$

Moreover, one has (according to whether $F \in S(Q_A)$ or $S_m(\mathcal{B}, \varepsilon)$),

$$\forall X \in \mathcal{H}^2,\ |(H_tF)(X)| \leq ||F||_{m,\varepsilon} \ \text{ or } \ \forall X \in \mathcal{H},\ |(H_tF)(X)| \leq ||F||_{Q_A}.$$

To justify the notation $e^{\frac{t}{2}\Delta}$, we need to introduce the Laplace operator on the symbol classes. If the operator A satisfies the assumptions of Definition 7, then, for $f \in S(Q_A)$, the Laplace operator is normally defined by $\Delta f(x) = \mathrm{Tr}(d^2 f(x))$.

On the Calderòn-Vaillancourt classes it is defined as in the finite dimensional setting, as a sum of partial derivatives (and thus depends, a priori, on the chosen basis): if $m \geq 2$ and if the sequence ε is square summable, the following series converges and one sets

$$\Delta_{\mathcal{B}} F = \left(\sum_{j\in\Gamma} \left(\frac{\partial}{\partial u_j}\right)^2 + \left(\frac{\partial}{\partial v_j}\right)^2 \right) F.$$

We then have $\Delta_{\mathcal{B}}F \in S_{m-2}(\mathcal{B}, \varepsilon)$, with $||\Delta_{\mathcal{B}}F||_{m-2,\varepsilon} \leq 2\sum_j \varepsilon_j^2 ||F||_{m,\varepsilon}$.

But if $F \in S_m(\mathcal{B}, \varepsilon)$ with $m \geq 3$ and ε summable, the function F is C^2 in the usual (Frechet) sense, according to Proposition 10. One can, then, define the Laplace operator intrisically, by $\Delta f(x) = \mathrm{Tr}(d^2 f(x))$ as for the $S(Q_A)$ classes and it does not depend on $\mathcal{B}$.

Laplace operator and heat operator have the properties stated in the following results. The first one concerns the Calderòn-Vaillancourt classes, the second one, the classes defined by a quadratic form [31]. Notice that there is a loss of derivability in the first classes.

THEOREM 15. *Let ε be summable.*

1. *For $m \geq 1$, the heat operator H_t is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-1}(\mathcal{B}, \varepsilon)$. For $m \geq 2$, the Laplace operator Δ is continuous from $S_m(\mathcal{B}, \varepsilon)$ to $S_{m-2}(\mathcal{B}, \varepsilon)$. For $m \geq 3$, H_t and Δ commute.*
2. *Let $m \geq 6$ and $F \in S_m(\mathcal{B}, \varepsilon)$. The application $t \mapsto H_t F$ is C^1 from $[0, +\infty[$ in $S_{m-6}(\mathcal{B}, \varepsilon)$ and its derivative is $t \mapsto \frac{1}{2} H_t \Delta F$.*

THEOREM 16. *Let A be a linear application on H satisfying the conditions of Definition 7 and let f be in $S(Q_A)$.*

1. *The function Δf belongs to $S(Q_A)$ with*
$$||\Delta f||_{Q_A} \leq \mathrm{Tr}(A) ||f||_{Q_A}.$$
For all $t > 0$, $H_t f$ belongs to $S(Q_A)$ and $||H_t f||_{Q_A} \leq ||f||_{Q_A}$. Moreover, H_t and Δ commute.
2. *The function $t \mapsto H_t f$ is C^∞ on $[0, \infty[$ with values in $S(Q_A)$, with* $\frac{d^m}{dt^m} H_t f = \left(\frac{1}{2}\Delta\right)^m H_t f.$

Remark 17. This gives the unusual fact that the Laplace operator is continuous and is due to the stronger than holomorphy properties of the $S(Q_A)$ classes. Consequences are the expression of H_t as a series:
$$H_t = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{t\Delta}{2}\right)^n.$$
and the invertibility of the heat operator:

PROPOSITION 18. *Let A and Q_A satisfy the assumptions of Definition 7, let F be in $S(Q_A)$. For all positive t, the operator H_t is an isomorphism of $S(Q_A)$ onto itself and its inverse satisfies*
$$\|(H_t)^{-1}\| \leq e^{(t/2)\mathrm{Tr} A}.$$

Despite their apparent restrictivity, the classes of Definition 7 are useful in an application presented in Section 4.4.

The heat operator also gives a link between the three kinds of symbols (Weyl, Wick and anti-Wick), as is the case in the finite dimensional setting.

4.2. **Pseudodifferential operators**

This section gives the definitions of the pseudodifferential operators: by means of a quadratic form first, then with the different symbol classes seen in the preceding section. To stress the differences between the various situations and present the tools, we shall sketch the general outline of some of the proofs.

The parameter $h > 0$ is fixed and $(i, \mathcal{H}, B)$ is an abstract Wiener space, with variance parameter h. Due to the use of the Segal-Bargmann transform defined in Section 3.3, the spaces which appear are mainly, on the one hand $L^2(B, \mu_{B,h/2})$, containing the "test" functions on which the operators act and, on the other hand, $L^2(B^2, \mu_{B^2,h})$, in which the convergences linked with the stochastic extensions take place.

The first construction does not give an operator but a quadratic form, acting on a space $\mathcal{D}$ defined below which replaces the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of the finite dimensional case. This definition allows one to use unbounded symbols, provided they satisfy convenient estimates. The symbols are functions defined on B^2, not on $\mathcal{H}^2$.

The second definitions really yield operators defined on $L^2(B, \mu_{B,h/2})$, associated with symbols belonging to the classes defined above. In particular, the symbols are bounded and defined on $\mathcal{H}^2$.

The results and notions about the quadratic form, Q_h^W, the Segal fields and Calderòn-Vaillancourt type symbols are taken from [4]. The second class of symbols has been defined in [31] to be exploited in [5, 7].

Quadratic form. Let us define the space $\mathcal{D}$ which replaces $\mathcal{S}(\mathbb{R}^n)$. For $E \in \mathcal{F}_{fin}(B')$, let $\mathcal{S}_E$ be the space of all functions φ such that $x \mapsto \varphi(x)e^{-\frac{|x|^2}{2h}}$ is rapidly decreasing. We denote by $\mathcal{D}_E$ the set of applications $f : B \to \mathbb{C}$ of the form $f = \varphi \circ \widetilde{\pi}_E$, where $\widetilde{\pi}_E : B \to E$ is defined by (7). Such functions are said to be based on E. The space $\mathcal{D}$ is defined as the union of the spaces $\mathcal{D}_E$, taken over all E in $\mathcal{F}_{fin}(B')$. It is dense in $L^2(B, \mu_{B,h/2})$ and contains smooth, cylindrical functions. Similarly, let $\mathcal{D}_{\mathcal{H}}$ be the space obtained when the union is taken on the $E \in \mathcal{F}_{fin}(\mathcal{H})$.

Definition 19. If f and g belong to $\mathcal{D}$ and are based on E, their Wigner function $W_h(f, g)$ is defined, for all $Z = (z, \zeta) \in B^2$ by

$$W_h(f,g)(Z) = e^{\frac{|\tilde{\pi}_E(\zeta)|^2}{2}} \int_E e^{-2\frac{i}{h}\tilde{\pi}_E(\zeta)\cdot t} \; \hat{f}(\tilde{\pi}_E(z) + t)\overline{\hat{g}(\tilde{\pi}_E(z) - t)} \; d\mu_{E,h/2}(t).$$

The space E on which f, g are based is not unique but the Wigner function does not depend on the choice of such a space. One can check that, for all $f, g \in \mathcal{D}$, $W_h(f, g)$ belongs to $L^1(B^2, \mu_{B^2,h/2})$. The same notion has been recalled, in the finite dimensional case, in Section 2. We do not, thus, generalize formula (1), but formula (2):

Definition 20. Let $\widetilde{F}$ be a bounded Borel function on B^2. One defines $Q_h^W(\widetilde{F})$ by

$$Q_h^W(\widetilde{F})(f,g) = \int_{B^2} \widetilde{F}(Z) W_h(f,g)(Z) d\mu_{B^2,h/2}(Z),$$

for f and g in $\mathcal{D}$.

If $\widetilde{F}$ is not bounded, assume that there exists $m \geq 0$ such that

$$N_m(\widetilde{F}) := \sup_{Y \in H^2} \frac{||\tau_Y \widetilde{F}||_{L^1(B^2,\mu_{B^2,h/2})}}{(1+|Y|_{H^2})^m} < +\infty. \tag{15}$$

Then $Q_h^W(\widetilde{F})$ can be defined as above.

The condition (15) says that the translated of F remain in L^1 and that the norm depends polynomially on the translation vector Y. The vector Y is in $\mathcal{H}^2$ for otherwise, the initial measure and the measure translated by Y would be mutually orthogonal.

It is useful to be able to define the operator (as a quadratic form) for unbounded symbols like ℓ_a or polynomials of such functions. Indeed, symbols like $\widetilde{\varphi}_{a,b}\ (x,\xi) \mapsto= \ell_a(x) + \ell_b(\xi)$, or $\ell_a \otimes 1 + 1 \otimes \ell_b$ give a multiplication by the variable and a differentiation, as in the finite dimensional case ([4], Section 8):

$$Q_h^W(\widetilde{\varphi}_{a,b})(f,g) = \langle \ell_{a+ib} f + \frac{h}{i}\ b \cdot \frac{\partial f}{\partial u}, g\rangle_{L^2(B,\mu_{B,h/2})},$$

for $f, g \in \mathcal{D}$. One recognizes the operator $\Sigma_{(a,b)}$ (9), which is linked with the Segal fields on the set of finite particle states. The operator $\Sigma_{(a,b)}$ is defined on $\mathcal{D}$ and takes values in $\mathcal{D}$ if $a, b \in B'$. Without this restriction, it only preserves $\mathcal{D}_{\mathcal{H}}$. Therefore, provided we restrict ourselves to $\mathcal{D}$, we may consider commutators of such operators with an operator A letting $\mathcal{D}$ invariant (which allows one to give a Beals characterization). One can also consider the commutation relations in their Weyl form or exponential form, to avoid unbounded operators.

The heat operator gives a link between the Weyl, Wick and anti-Wick symbols. Suppose that the operator A and the quadratic form Q_A are as in Definition 7. For $F \in S(Q_A)$ and for all f and g linear combinations of coherent states,

$$Q_h^W(H_{h/2}F)(f,g) = \langle Op_h^{AW}(F) f, g\rangle, \quad \sigma_h^{Wick}(Op_h^{AW}(F)) = H_h F.$$

The anti-Wick quantization and Wick symbol were defined in the paragraph 3.3.

Finally, although it is not directly linked with the construction of these quantizations, it may be relevant to give the relationship between the operators with a linear symbol and the Malliavin gradient ([32], Chap 14,15, [11, 37, 38]). Let f be cylindrical and based on $E \subset B'$, with

$$f(x) = \varphi(e_1(x), \dots, e_d(x)),$$

where $(e_1, \dots, e_d)$ is an orthonormal basis of E. Then one has, for $x \in B$ and $y \in \mathcal{H}$,

$$\partial_y f(x) = \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t} = \sum_{i=1}^{d} \frac{\partial \varphi}{\partial u_i}(e_1(x), \dots e_d(x)) e_i \cdot y = y \cdot \frac{\partial f}{\partial u}(x),$$

where the first equality defines $\partial_y f(x)$.

On the Gaussian Hilbert space $\mathcal{M} = \{\ell_x, x \in \mathcal{H}\} \subset L^2(B, \mu_{B,h/2})$, one can then define a Malliavin gradient for f, setting

$$\forall x \in B,\ \nabla f(x) = \frac{2}{h} \sum_{i=1}^{d} \frac{\partial \varphi}{\partial u_i}(e_1(x), \dots e_d(x)) \ell_{e_i}.$$

The gradient satisfies, as is required,

$$\forall x \in B,\ y \cdot \frac{\partial f}{\partial u}(x) = \partial_y f(x) = \langle \nabla f(x), \ell_y \rangle_{L^2(B, \mu_{B,h/2})}.$$

Calderòn-Vaillancourt classes. When the symbol belongs to the first, Calderòn-Vaillancourt class (Definition 6), it is possible to construct an operator continuous on $L^2(B, \mu_{B,h/2})$:

THEOREM 21. *Let $\mathcal{B} = (e_i)_{i \in \Gamma}$ be a Hilbert basis of $\mathcal{H}$, with $e_i \in B'$ for all i. Let $\varepsilon = (\varepsilon_j)_{(j \in \Gamma)}$ be square summable. Let F belong to $S_2(\mathcal{B}, \varepsilon)$. Suppose that F admits a stochastic extension $\widetilde{F}$ with respect to $\mu_{B^2,h}$ and $\mu_{B^2,h/2}$.*

There exists an operator $Op_h^W(F)$ bounded on $L^2(B, \mu_{B,h/2})$ and such that, for all $f, g \in \mathcal{D}$,

$$\langle Op_h^W(F) f, g \rangle = Q_h^W(\widetilde{F})(f, g).$$

Moreover, for all $h \in]0, 1]$,

$$\|Op_h^W(F)\| \leq ||F||_{2,\varepsilon} \prod_{j \in \Gamma} (1 + 81\pi h S_\varepsilon \varepsilon_j^2) \quad \text{with} \quad S_\varepsilon = \sup_{j \in \Gamma} \max(1, \varepsilon_j^2).$$

The existence of the stochastic extension is ensured, for example, if the sequence ε is summable as has been seen in Proposition 8.

Let us suggest the general idea of the proof. One introduces hybrid operators, defined thanks to partial heat operators. If $E \in \mathcal{F}_{fin}(B')$, one sets $E^\perp = \{x \in B : (y, x)_{B',B} = 0\ \forall y \in E\}$. It is a Wiener space and, following [40], one can decompose the measure on B as a product $\mu_{B,s} = \mu_{E,s} \times \mu_{E^\perp,s}$ for all positive s. One then defines a partial heat operator:

$$(\widetilde{H}_{E^\perp,s} F)(X) = \int_{(E^\perp)^2} F(X + Y_{E^\perp}) d\mu_{(E^\perp)^2,s}(Y_{E^\perp}), \tag{16}$$

and a hybrid quadratic form by:

$$Q_h^{hyb,E}(F)(f,g) = Q_h^W(\widetilde{H}_{E^\perp,h/2}F)(f,g).$$

The hybrid operators act as Weyl operators on a finite number of variables (namely, the variables in E) and as anti-Wick operators, easier to handle, on the rest of the variables. The operator of Theorem 21 is, therefore, not defined by an integral formula, but as a limit of a Cauchy sequence of hybrid operators.

One example of an operator with a symbol in a class $\mathcal{S}_m(\mathcal{B},\varepsilon)$ is the operator $U_{a,b,h}$ defined in (10) above. For a and b in $\mathcal{H}$, let $F_{a,b}$ be the function on $\mathcal{H}^2$ defined by: $F_{a,b}(x,\xi) = e^{i(a\cdot x + b\cdot\xi)}$. If $\mathcal{B} = (e_j)_{(j\geq 1)}$ is an arbitrary Hilbert basis of $\mathcal{H}$ and if m is a positive integer, the function $F_{a,b}$ is in the set $S_m(\mathcal{B},\varepsilon)$, with $\varepsilon_j = \max(|e_j\cdot a|, |e_j\cdot b|)$ and this sequence is square summable. Let $Op_h^W(F_{a,b})$ be the operator, bounded on $L^2(B,\mu_{B,h/2})$, associated with $F_{a,b}$ by Theorem 21. Then $Op_h^W(F_{a,b}) = U_{a,b,h}$. We already saw that

$$e^{i\Phi_S(\ell_{a+ib})} = U_{t(\frac{a}{\sqrt{h}},\frac{b}{\sqrt{h}})h}. \tag{17}$$

Symbol defined thanks to a quadratic form. Now let us state the results for the classes defined thanks to a quadratic form (Definition 7).

For $V=(a,b)$ in $\mathcal{H}^2$, denote by L_hV the operator $\Sigma_{(-b,a)} = \ell_{-b+ia} + \frac{h}{i}\, a\cdot\frac{\partial}{\partial u}$ defined thanks to (9).

Definition 22. For a nonnegative quadratic form Q on $\mathcal{H}^2$, $\mathcal{L}(Q)$ is the set of families (A_h) of bounded operators in $L^2(B,\mu_{B,h/2})$, depending on $h\in(0,1]$, such that there exists a real constant $C_h(A_h,Q)$ satisfying, $\forall m\in\mathbb{N}^*, V_1,\dots,$ $V_m\in\mathcal{H}^2$:

$$\|\mathrm{ad}(L_hV_1)\cdots\mathrm{ad}(L_hV_m)A_h\| \leq C_h(A_h,Q)\ h^m \prod_{j=1}^m Q(V_j)^{1/2}, \quad h\in(0,1].$$

Set $\|(A_h)\|_{\mathcal{L}(Q)} = \sup_{h\in(0,1]} C_h(A_h,Q)$.

The commutators can be expressed in an exponential form (with the Weyl operators) to avoid problems of definition.

We may now state a theorem about the existence of an operator with a symbol in a class $S(\mathcal{H}^2,Q)$ and a result akin to Calderòn-Vaillancourt Theorem [5].

THEOREM 23. *Let $(i,\mathcal{H},B)$ be a Wiener space, let A a nonnegative, trace class, self-adjoint operator. Set $Q(X)=\langle AX,X\rangle$ and let $F\in S(\mathcal{H}^2,Q)$. We then have the following results:*

1. *The family of operators $(OP_h^{AW}(F))_h$ is in $\mathcal{L}(Q)$ and*

$$\|(OP_h^{AW}(F))\|_{\mathcal{L}(Q)} \leq \|F\|_Q.$$

2. *For each h, the quadratic form $Q_h^W(F)$ is associated with an operator $OP_h^W(F) \in \mathcal{L}(L^2(B, \mu_{B,h/2}))$. This family of operators is in $\mathcal{L}(Q)$ and:*

$$C_h(OP_h^W(F), Q) \leq \|F\|_Q e^{(h/4)\mathrm{Tr}A}.$$

The second point comes from the first one and uses the invertibility of the heat operator.

4.3. **Composition, Beals characterization**

The results in this section are taken from [5, 6]. They amount to a Beals characterization for both kind of symbol classes. There are also composition results, for the Wick symbol first, then for the Weyl symbol. The composition results are given in the shape of an asymptotic expansion, not of an integral formula. The proofs (which we omit) rely, in the second case, on the use of the classes $S(\mathcal{H}^2, Q)$, in which the heat operator is invertible.

THEOREM 24. *Let $(i, \mathcal{H}, B)$ be an abstract Wiener space. Let A be a bounded operator in $L^2(B, \mu_{B,h/2})$, with $0 < h < 1$. Let $\mathcal{B} = (e_j)_{j\in\Gamma}$ be a hilbertian basis of $\mathcal{H}$, consisting of elements of B'. Let $M > 0$ and let ε be a summable sequence indexed by Γ. Let $m \geq 2$. Suppose that, for all multi-index $(\alpha, \beta) \in \mathcal{M}_{m+4}$ (meaning that $0 \leq \alpha_i, \beta_i \leq m+4$) the commutator $\prod_{j\in\Gamma}(\mathrm{ad}\Sigma_{(0,e_j)})^{\alpha_j}(\mathrm{ad}\Sigma_{(e_j,0)})^{\beta_j} A$ is bounded in $L^2(B, \mu_{B,h/2})$, with*

$$|| \prod_{j\in\Gamma}(\mathrm{ad}\Sigma_{(0,e_j)})^{\alpha_j}(\mathrm{ad}\Sigma_{(e_j,0)})^{\beta_j} A||_{\mathcal{L}(L^2(B,\mu_{B,h/2}))} \leq M \prod_{j\in\Gamma}(h\varepsilon_j)^{\alpha_j+\beta_j}.$$

Then there exists a function $F \in S_m(\mathcal{B}, \varepsilon)$ such that $Op_h^W(F) = A$. Moreover

$$||F||_{m,\varepsilon} \leq M \prod_{j\in\Gamma}(1 + KS_\varepsilon^2 h\varepsilon_j^2)$$

with $S_\varepsilon = \sum_{\mathrm{j}\in\Gamma} \max(1, \varepsilon_j^2)$ and K a universal constant.

Here the operator $\Sigma_{(0,e_j)}$ or P_j corresponds to the position operator and $\Sigma_{(e_j,0)}$ or Q_j, to the impulsion operator (see (9) and section 4.2).

THEOREM 25. *Let (A_h) be a family of operators in $\mathcal{L}(Q)$.*
The Wick symbol $\sigma_h^{wick}(A_h)$ is bounded in $S(\mathcal{H}^2, Q)$ and:

$$\|\sigma_h^{wick}(A_h)\|_Q \leq \|(A_h)\|_{\mathcal{L}(Q)}.$$

There exists a family of functions (F_h), bounded in $S(\mathcal{H}^2, Q)$, such that $A_h = OP_h^W(F_h)$, with:

$$\|F_h\|_Q \leq \|(A_h)\|_{\mathcal{L}(Q)} e^{(h/4)\mathrm{Tr}A}.$$

There exists a family of functions (G_h) *in* $S(\mathcal{H}^2, Q)$*, such that* $A_h = OP_h^{AW}(G_h)$*, with:*

$$\|G_h\|_Q \leq \|(A_h)\|_{\mathcal{L}(Q)} e^{(h/2)\mathrm{Tr} A}.$$

We now give composition results. The first result concerns the Wick symbol for bounded operators:

THEOREM 26. *Let* B_1, B_2 *be bounded on* $L^2(B, \mu_{B,h/2})$*. The Wick symbols* F_i *of* B_i *are smooth on* $\mathcal{H}^2$*. For all nonnegative* k*, the following series (defined with respect to a basis of* $\mathcal{H}$*) is absolutely convergent:*

$$C_k^{wick}(F_1, F_2) = 2^{-k} \sum_{|\alpha|=k} \left(\frac{1}{\alpha!}\right) (\partial_x - i\partial_\xi)^\alpha F_1 \ (\partial_x + i\partial_\xi)^\alpha F_2.$$

Moreover, its sum $C_k^{wick}(F_1, F_2)$ *is independent of the basis. Then one has:*

$$\forall X \in H^2, \quad \sigma_h^{wick}(B_1 \circ B_2)(X) = \sum_{k=0}^{\infty} h^k C_k^{wick}(F_1, F_2)(X),$$

where the series is absolutely convergent.

The second theorem concerns the composition for operators defined by their Weyl symbol.

When the series converges, one defines, for a function F defined on $\mathcal{H}^2$, the following differential operator, which appears in the expansion:

$$\sigma(\nabla_1, \nabla_2) F = \sum_j \frac{\partial^2 F}{\partial y_j \partial \xi_j} - \frac{\partial^2 F}{\partial x_j \partial \eta_j}.$$

When it makes sense, set

$$C_k^{Weyl}(F, G)(X) = \frac{1}{(2i)^k k!} \sigma(\nabla_1, \nabla_2)^k (F \otimes G)(X, X).$$

THEOREM 27. *Let* A *and* $Q = Q_A$ *be as in Definition 7 and let* F, G *belong to* $S(\mathcal{H}^2, Q)$*. Then for all nonnegative* k*:*

$$\|C_k^{Weyl}(F, G)\|_{4Q} \leq \|F\|_Q \|G\|_Q \frac{(\mathrm{Tr} A)^k}{2^k k!}.$$

The series

$$K_h = \sum_{k=0}^{\infty} h^k C_k^{Weyl}(F, G)$$

is absolutely convergent and defines a function K_h *in* $S(\mathcal{H}^2, 4Q)$*. One has:*

$$Op_h^W(F) \circ Op_h^W(G) = Op_h^W(K_h).$$

4.4. **Some applications**

Symbols associated with the exponential of linear functions give the Weyl operators, as has been seen in (17).

We now give the example of a symbol belonging to a Calderòn-Vaillancourt class [4] and representing a lattice where each particle interacts with its neighbors. There is an infinite number of particles but the interactions (modeled by the coefficients g_is) decrease. Let $\Gamma = \mathbb{Z}^d$, $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, let $V : \mathbb{R} \to \overline{\mathbb{R}}_+$ be smooth, with bounded derivatives of order ≥ 1. Let $(g_j)_{(j\in\Gamma)}$ be a sequence of positive real numbers such that, if $|j-k| \leq 1$, $g_j/g_k \leq K_0$ with $K_0 > 1$. For all $(x,\xi) \in \mathcal{H}^2$ and $t > 0$ set:

$$f(x,\xi) = \sum_{j\in\Gamma} g_j^2 \xi_j^2 + \sum_{\substack{(j,k)\in\Gamma\times\Gamma \\ |j-k|=1}} g_j g_k V(x_j - x_k) \text{ and } F_t(x,\xi) = e^{-tf(x,\xi)},$$

where $|\cdot|$ is the sup norm of $\mathbb{R}^d$.

For all $m \geq 1$, there exists a real constant C_m and a sequence ε satisfying

$$\varepsilon_j^{(m)} \leq C_m \max(g_j^2, g_j^{1/m}),$$

such that the symbol $F_t \in S_m(\mathcal{B}, \varepsilon^{(m)})$ with $||F_t||_{S(\mathcal{B},\varepsilon)} \leq 1$. The constant C_m depends only of m, t, d, K_0 and of the bounds of the derivatives of V, up to order $2m$.

Now turn to a model of quantum electrodynamics due to Reuse [43], for which there are two different implementations, using both kinds of symbol classes [5, 7]. The system consists of N fixed particles with spin 1/2 in an external constant magnetic field and of an arbitrary number of photons. One wants to describe the evolution operator $e^{-i\frac{t}{h}H(h)}$, where $H(h)$ is the Hamiltonian of the system. What motivates the use of a quantization on an infinite dimensional space is the Hilbert space representing the photons, a symmetrized Fock space. We present successively the spaces, the operators and then the results.

Spaces. The initial Hilbert space is the one photon space, namely:

$$\mathcal{H} = \{q = q(k) \in L^2(\mathbb{R}^3, \mathbb{R}^3) \ : \ k\cdot q(k) = 0 \quad \text{a.e.}\},$$

endowed with the norm

$$|q|^2 = \sum_{j=1}^{3} \int_{\mathbb{R}^3} |q_j(k)|^2 dk$$

where the measure is the Lebesgue measure. The symmetrized Fock space $\mathcal{H}_{ph} = \mathcal{F}_s(\mathcal{H}_\mathbb{C})$, represents the photons, whereas $\mathcal{H}_{sp} = (C^2)^{\otimes N}$ represents the state of the N fixed particles. There exists an isomorphism between $\mathcal{H}_{ph}$ and

$L^2(B, \mu_{B,h/2})$, where B is a Wiener extension of the initial space $\mathcal{H}$ (Section 3). The Hilbert space describing the whole system is then

$$\mathcal{H}_{ph} \otimes \mathcal{H}_{sp} = \mathcal{F}_s(\mathcal{H}_{\mathbb{C}}) \otimes (C^2)^{\otimes N}.$$

Operators. The initial operator defined on $\mathcal{H}$ is the (unbounded) operator M_ω defined by $M_\omega q(k) = |k|q(k)$. On the Fock space, one sets $H_{ph} = hd\Gamma(M_\omega)$ and one defines the free operator by $H_0 = H_{ph} \otimes I$.

The Hamiltonian of the system is obtained by adding an interaction term to the free operator. The external constant magnetic field β and the induced electric and magnetic fields are taken into account in this interaction term. The difference between the two implementations lies there, in an assumption about the induced fields. The magnetic field B is defined as follows. For the parameters $j = 1, 2, 3$ and $x \in \mathbb{R}^3$, let $B_{j,x}$ be defined by

$$\forall k \in \mathbb{R}^3 \setminus \{0\},\ B_{j,x}(k) = \frac{i\chi(|k|)|k|^{1/2}}{(2\pi)^{3/2}} e^{-ik\cdot x} \frac{k \wedge e_j}{|k|}. \tag{18}$$

The function χ is a cutoff function and the e_is are the vectors of the canonical basis of $\mathbb{R}^3$. We set, for $(q, p) \in \mathcal{H}^2$,

$$B_j(x, q, p) = (\Re(B_{j,x}), q) + (\Im(B_{j,x}), p)\,.$$

The operator $B_j(x)$ is the unbounded operator of symbol $B_j(x, q, p)$, that is to say a Segal field. So is the electric field:

$$E_j(x, q, p) = -B_j(x, J(q, p))$$

where $J : \mathcal{H}^2 \to \mathcal{H}^2$ is the helicity operator defined by

$$J(q, p)(k) = \left(\frac{k \wedge q(k)}{|k|}, \frac{k \wedge p(k)}{|k|}\right), \quad k \in \mathbb{R}^3 \setminus \{0\}.$$

The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One denotes by $\sigma_m^{[\lambda]}$ the operator in $\mathcal{H}_{sp} = (C^2)^{\otimes N}$ defined by:

$$\sigma_m^{[\lambda]} = I \otimes \cdots \sigma_m \cdots \otimes I,$$

where σ_m is located at the λ^{th} position. The interaction operator is defined by:

$$H_{int} = \sum_{\lambda=1}^{N} \sum_{j=1}^{3} (\beta_j + B_j(a_\lambda) \oplus \sigma_j^{[\lambda]}),$$

for N particles located at $a_1, \ldots a_N \in \mathbb{R}^3$. We then set

$$H(h) = H_0 + H_{int} = H_{ph} \otimes I + H_{int}$$

and denote by $H_{int}(q,p)$ the Wick symbol of the operator H_{int}.

In [7], the cutoff function χ appearing in (18) is rapidly decreasing in $\mathbb{R}$ and vanishes in a neighborhood of 0. This last assumption has disappeared in [5].

In both cases the operator $H(h)$ has a self-adjoint extension with the same domain as the free operator $H_0 = H_{ph} \otimes I$.

The two evolution operators associated with a self-adjoint operator A on $\mathcal{H}_{ph} \otimes \mathcal{H}_{sp}$ are the following

$$A^{free}(t,h) = e^{i\frac{t}{h}H_{ph}\otimes I} A e^{-i\frac{t}{h}H_{ph}\otimes I}, \quad A(t,h) = e^{i\frac{t}{h}H(h)} A e^{-i\frac{t}{h}H(h)}.$$

Results. A first result is concerned with the link between both evolution operators. We state the version of [7], (where the cutoff function vanishes near 0).

The space $S_\infty^{mat}(\mathcal{B}, |t|\varepsilon(t)))$ is defined as in Section 4.1 (Definition 6), but the functions take their values in $\mathcal{L}((\mathbb{C}^2)^{\otimes N})$. Basis and sequence are carefully chosen. The basis $\mathcal{B}$ referred to is constructed as follows. Let E denote the set of L^2 vector fields on the unit sphere S^2 (of $\mathbb{R}^3$):

$$E = \{f = (f_1, f_2, f_3) : S^2 \to \mathbb{R}^3 : \sum_{j=1}^{3} \omega_j f_j(\omega) = 0\}.$$

The norm on E is given by

$$||f||_E^2 = \sum_{j=1}^{3} \int_{S^2} |f_j(\omega)|^2 \, d\mu(\omega),$$

where μ is the surface measure on S^2. Then $\mathcal{H}$ can be viewed as $L^2(\mathbb{R}^+, r^2dr, E)$. If $(u_m)_m$ and $(v_n)_n$ are, respectively, Hilbert bases of $L^2(\mathbb{R}^+, r^2dr)$ and E, then the family $(f_{m,n})$ defined by

$$f_{m,n}(k) = u_m(|k|) v_n(\frac{k}{|k|})$$

is a Hilbert basis of $\mathcal{H}$. One chooses the basis $(v_n)_n$ of E consisting of the eigenvectors of the Laplace operator on S^2. The functions $(u_m)_m$ are the eigenvectors of the operator $L = -\dfrac{d^2}{dr^2} - \dfrac{2}{r}\dfrac{d}{dr} + r^2$ and can be expressed thanks to generalized Laguerre polynomials.

Finally, the sequence $\varepsilon(t) = (\varepsilon_{m,n}(t))$ is rapidly decreasing, such that $t \mapsto \varepsilon_{m,n}(t)$ is a non decreasing function of $|t|$. It satisfies a condition linked with the free evolution operator and the impulsion and position operators of the $f_{m,n}$:

$$|||[P_h(f_{m,n}), H_\int^{free}(t)]||| + |||[Q_h(f_{m,n}), H_\int^{free}(t)]||| \leq h\varepsilon_{m,n}(t).$$

The following theorem has been proved in [7]. The symbol of the reduced propagator admits an expansion in powers of h, too.

THEOREM 28. *There exists a function $U(t,q,p,h)$ defined on $\mathbb{R}\times\mathcal{H}^2$, with parameter $h>0$ and taking matrix values in $\mathcal{L}((C^2)^{\otimes N})$ such that the operator $U_h^{red}(t)$ defined by*

$$U_h^{red}(t) = e^{i\frac{t}{h}H_0}e^{-i\frac{t}{h}H(h)}$$

can be written as

$$U_h^{red}(t) = Op_h^W(U(t,\cdot,h)).$$

As a function of $X=(q,p)\in\mathcal{H}^2$, $U(t,\cdot,h)$ belongs to the space $S_\infty^{mat}(\mathcal{B},|t|\varepsilon(t))$. Moreover, it is bounded in this space as t belongs to a compact set and h runs over $]0,1[$.

A second result [5] is an asymptotic expansion in powers of h of the Wick symbol of an evolution operator $A(t,h)$. Here, the cutoff function χ is rapidly decreasing but does not necessarily vanish near the origin. One uses the second classes of symbols.

One defines the following quadratic form, as

$$Q_t(q,p) = |t|\int_0^t |dH_{int}^{free}(s,q,p)|^2 ds$$

where $H_{int}^{free}(s,q,p)$ is the Wick symbol of the free evolution operator. More explicitly,

$$Q_t(q,p) = 2^N|t|\sum_{\lambda=1}^N\sum_{j=1}^3\int_0^t \mathrm{Re}\left(\int_{\mathbb{R}^3}(p+iq)(k)\overline{e^{is|k|}B_{jx_\lambda}(k)}dk\right)^2 ds.$$

We are concerned with observables of the following form:

$$A = Op_h^W(F_A)\otimes I + I\otimes S_A$$

where F_A is a linear continuous form on $\mathcal{H}^2$ and S_A is in $\mathcal{L}(\mathcal{H}_{sp})$. According to Proposition 8.4 [5], if A has this form, the operator $A(t,h)$ is the sum of an operator with linear symbol and of an operator bounded in $\mathcal{L}(4Q_t)$

THEOREM 29. *Let A be an observable of the form (4.4). For each M, there exist functions in $A^{[j]}(t,q,p)$ in $S(\mathcal{H}^2, 16^{j+1}Q_t)$ and a rest $R_M(t,\cdot,h)$ in*

$S(\mathcal{H}^2, 16^{M+5}Q_t)$ such that one can write:

$$\sigma_h^{wick}(A(t,h))(q,p) = \sum_{j=0}^{M} h^j A^{[j]}(t,q,p) + h^{M+1} R_M(t,q,p,h)$$

Moreover, the norm of the $A^{[j]}$ and of R_M (in their respective symbol classes) are bounded independently of t and h, when t remains in a compact subset of $\mathbb{R}$ and h varies in $(0,1)$.

REFERENCES

[1] Z. Ammari and F. Nier, *Mean field limit for bosons and infinite dimensional phase-space analysis.* Ann. Henri Poincaré **9** (2008), *8*, 1503–1574.

[2] Z. Ammari and F. Nier, *Mean field propagation of infinite-dimensional Wigner measures with a singular two-body interaction potential.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **14** (2015), *1*, 155–220.

[3] L. Amour, L. Jager and J. Nourrigat, *On bounded pseudodifferential operators in a high-dimensional setting.* Proc. Amer. Math. Soc. **143** (2015), *5*, 2057–2068.

[4] L. Amour, L. Jager and J. Nourrigat, *On bounded pseudodifferential operators in Wiener spaces.* J. Funct. Anal. **269** (2015), *9*, 2747–2812.

[5] L. Amour, L. Jager and J. Nourrigat, *Infinite dimensional analysis and applications to a model in NMR.* Preprint, 2017. arXiv 1705.07097.

[6] L. Amour, R. Lascar and J. Nourrigat, *Beals characterization of pseudodifferential operators in Wiener spaces.* Appl. Math. Res. Express **2017** (2017), *1*, 242–270.

[7] L. Amour, R. Lascar and J. Nourrigat, *Weyl calculus in Wiener spaces and in QED.* Preprint, 2016. arXiv 1610.06379.

[8] L. Amour, R. Lascar and J. Nourrigat, *Weyl calculus in QED I. The unitary group.* J. Math. Phys. **58** (2017), *1*, 013501.

[9] R. Beals, *Characterization of pseudodifferential operators and applications.* Duke Math. J. **44** (1977), *1*, 45–57.

[10] R. Beals, *Correction to: "Characterization of pseudodifferential operators and applications".* (Duke Math. J. **44** (1977), *1*, 45–57). Duke Math. J. **46** (1979), *1*, 215.

[11] D.R. Bell, *The Malliavin Calculus.* Reprint of the 1987 Edition. Dover Publications, Inc., Mineola, NY, 2006.

[12] F.A. Berezin, *The Method of Second Quantization.* Translated from the Russian. Pure Appl. Phys. **24**, Academic Press, New York, London, 1966.

[13] P. Bleher and M. Vishik, *On a class of pseudodifferential operators with infinite number of variables and their applications.* Math. USSR Sbornik **15** (1971), *3*, 443–491.

[14] A.P. Calderón and R. Vaillancourt, *A class of bounded pseudo-differential operators.* Proc. Natl. Acad. Sci. USA **69** (1972), 1185–1187.

[15] M. Combescure and D. Robert, *Coherent states and applications in mathematical physics.* Theoret. and Math. Phys., Springer, Dordrecht, 2012.

[16] H.O. Cordes, *On compactness of commutators of multiplications and convolutions, and boundedness of pseudo-differential operators.* J. Funct. Anal. **18** (1975), 115–131.

[17] T. Deck, *Der Itô Kalkül.* Springer-Verlag, Berlin, Heidelberg, 2000.

[18] J. Dereziński and C. Gérard, *Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians.* Rev. Math. Phys. **11** (1999), *4*, 383–450.

[19] J. Dereziński and C. Gérard, *Mathematics of quantization and quantum fields.* Cambridge Monogr. Math. Phys., Cambridge Univ. Press, Cambridge, 2013.

[20] B. Driver and B. Hall, *Yang-Mills theory and the Segal Bargmann transform.* Comm. Math. Phys. **201** (1999), *2*, 249–290.

[21] G.B. Folland, *Harmonic analysis in phase space.* Ann. of Math. Stud. **122**. Princeton Univ. Press, Princeton, NJ, 1989.

[22] L. Gross *Measurable functions on Hilbert space.* Trans. Amer. Math. Soc. **105** (1962), 372–390.

[23] L. Gross, *Abstract Wiener spaces.* Proc. 5th Berkeley Sym. Math. Stat. Prob. **2** (1965), 31–42.

[24] L. Gross, *Potential theory on Hilbert space.* J. Funct. Anal. **1** (1967), 123–181.

[25] L. Gross, *Abstract Wiener measure and infinite dimensional potential theory.* In: *Lectures in modern Analysis and applications, II.* Lecture Notes in Math. **140**, 84-116, Springer, 1970.

[26] B.C. Hall, *The heat operator in infinite dimensions.* In: A.N. Sengupta et al. (Eds.), *Infinite Dimensional Analysis in Honor of H.-H. Kuo.* World Scientific, 161–174, 2008.

[27] B. Helffer, *Théorie spectrale pour des opérateurs globalement elliptiques.* Astérisque (1984), *112*.

[28] L. Hörmander, *The analysis of linear partial differential operators. III: Pseudo-differential operators.* Grundlehren Math. Wiss. **274**, Springer-Verlag, 1985.

[29] I.L. Hwang, *The L^2 boundedness of pseudo-differential operators.* Trans. Amer. Math. Soc. **302** (1987), 55–76.

[30] V. Iftimie, M. Măntoiu and R. Purice, *Commutator criteria for magnetic pseudodifferential operators.* Comm. Partial Differential Equations **35** (2010), *6*, 1058–1094.

[31] L. Jager, *Stochastic extensions of symbols in Wiener spaces and heat operator.* Preprint, 2016. arXiv 1607.02253.

[32] S. Janson, *Gaussian Hilbert spaces.* Cambridge Tracts in Math. **129**, Cambridge Univ. Press, 1997.

[33] H.H. Kuo, *Gaussian measures in Banach spaces.* Lecture Notes in Math. **463**, Springer, Berlin-New York, 1975.

[34] B. Lascar, *Opérateurs pseudo-différentiels en dimension infinie. Applications.* C.R. Math. Acad. Sci. Paris Sér. A-B **284** (1977), *13*, A767–A769.

[35] B. Lascar, *Opérateurs pseudo-différentiels en dimension infinie. Etude de l'hypoellipticité de la résolubilité dans des classes de fonctions holderiennes et de distributions pour des opérateurs pseudo-différentiels elliptiques.* J. Anal. Math. **33** (1978), 39–104.

[36] N. Lerner, *Metrics on the phase space and non self-adjoint pseudo-differential operators.* Birkhäuser Springer, 2010.

[37] P. Malliavin, *Stochastic Analysis.* Grundlehren Math. Wiss. **313**, Springer-Verlag, Berlin, 1997.

[38] D. Nualart, *Malliavin Ccalculus and Its Applications.* Springer-Verlag, 1995.

[39] B. Øksendal, *Stochastic Differential Equations.* Springer-Verlag, Berlin, 2003.

[40] R. Ramer, *On nonlinear transformations of Gaussian measures.* J. Funct. Anal. **15** (1974), 166–187.

[41] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I: Functional Analysis.* Academic Press, New York, London, 1978.

[42] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. II: Fourier Analysis, Self-Adjointness.* Academic Press, New York, London, 1978.

[43] F.A. Reuse, *Electrodynamique et Optique Quantiques.* Presses Polytechniques et Universitaires Romandes, Lausanne, 2007.

[44] A. Unterberger, *Analyse harmonique et analyse pseudo-différentielle du cône de lumière.* Astérisque (1987), *156.*

[45] A. Unterberger, *Quantification relativiste.* Mém. Soc. Math. Fr. (N.S.) (1991), *44–45.*

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