SOLUTIONS TO EXTENDED WDVV EQUATIONS; $ST34, E_8$ CASES

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Potential vector fields are solutions to extended WDVV equation and play an important role in the theory of flat structures. There is a $(\mathbf{C}^*)^n$-action on the set of potential vector fields of n-variables with a same weight system. It is a question whether under this action, the set of polynomial potential vector fields is a unique orbit or not. Among others, it is complicated to solve the question for the two cases: the real reflection group of type E_8 and the complex reflection group $ST34$. The purpose of this paper is to give an answer to this question for these two groups.

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1. INTRODUCTION

We start this introduction with explaining the definition of an extended WDVV equation. To an (n-dimensional) extended WDVV equation, there associates a weight system consisting of numbers $w_1, w_2, \ldots, w_n$. We consider a vector-valued function $\vec{h} = (h_1, \ldots, h_n)$ such that each of $h_j(x)$ is a weighted homogeneous function of the form (1) below. From $\vec{h}$, we define an $n \times n$ matrix C whose (i, j)-entry is $\partial_{x_i} h_j$. Then the commutativity of the matrices $\partial_{x_i} C$ $(i = 1, \ldots, n)$ induces a system of differential equations for $(h_1, \ldots, h_n)$. This is called an extended WDVV equation (cf. [7]). Extended WDVV equations depend on the n-tuple of numbers $(w_1, \ldots, w_n)$. A vector-valued function $\vec{h}$ is called a potential vector field if $(h_1, \ldots, h_n)$ is a solution to an extended WDVV equation. If $\vec{h}$ is a potential vector field and if there is a function $F(x)$ such that $\partial_{x_j} F = h_{n-j+1}$ $(j = 1, \ldots, n)$, then F is a solution to a system of differential equations which is induced by the extended WDVV equation. This turns out to be the so-called WDVV equation for $F(x)$ and $F(x)$ is a prepotential.

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It is B. Dubrovin [3] who constructed polynomial solutions to WDVV equations in the $n = 3$ case and recognized the existence of a deep relationship between WDVV equations and real reflection groups. Frobenius manifolds introduced by him includes both WDVV equation and flat structures (cf. [9, 11, 12]). The treatment by Dubrovin on WDVV equation leads us to the problem of constructing polynomial potential vector fields, namely, finding solutions $(h_1, \ldots, h_n)$ to extended WDVV equations such that each h_j is a polynomial. It is expected and is proved that there exist polynomial potential vector fields corresponding to well-generated complex reflection groups (cf. [5, 7]). It is underlined here that there exist polynomial potential vector fields which do not correspond to any complex reflection groups (cf. [6]).

We are going to explain the problem treated in this paper. As will be shown in the main text, the totality of polynomial potential vector fields with the same weight system $(w_1, \ldots, w_n)$ admits a $(\mathbf{C}^*)^n$-action. Two polynomial potential vector fields with the same weight system are equivalent if they are transformed to each other by the $(\mathbf{C}^*)^n$-action. Then it is a basic problem to determine equivalent classes of polynomial potential vector fields for a given weight system $(w_1, \ldots, w_n)$. In particular, it is interesting to treat this when the weight system comes from the degrees of basic invariants of a real or complex reflection group. The problem for reflection groups is related with the result by A. Arsie and P. Lorenzoni [1]. There it is shown that for the case of the reflection group of type B_3, there is a parameter which represents equivalence classes of polynomial vector fields. In this paper, we treat the two cases:

(A) the complex reflection group $ST34$

(B) the real reflection group of type E_8

To answer the problem on these cases is rather complicated compared with the remaining reflection groups. One of our interests is to construct a polynomial potential vector field which is not related with any reflection groups. As to the case (A), we construct two polynomial potential vector fields. One is related with the reflection group $ST34$ whereas the other is not. On the other hand, we show that there is a unique polynomial potential vector field related with the reflection group of type E_8. Since it is known (cf. [7]) that there is a potential vector field which comes from the prepotential of the Frobenius manifold for E_8, the potential vector field obtained in Theorem 2 coincides with this one.

We finally mention that the sofware Mathemathica is used to obtain the results in this paper.

2. DEFINITION OF A POTENTIAL VECTOR FIELD

In this section, we review the definition and its basic properties of a potential vector field. For the details of the argument in this section, refer to [7,9] and the references therein.

Let $x = (x_1, x_2, \ldots, x_n)$ be a coordinate system of $\mathbf{C}^n$. We define an Euler vector field

$$E = \sum_{k=1}^{n} w_k x_k \partial_{x_k}.$$

We assume a condition on w_k $(k = 1, 2, \ldots, n)$:

$$0 < w_1 < w_2 < \cdots < w_n.$$

If a function $f = f(x_1, \cdots, x_n)$ satisfies $Ef = df$ for a constant d, f is said to be weighted homogeneous with the weight system $(w_1, \cdots, w_n)$ in this paper. We consider weighted homogeneous functions $h_1(x), h_2(x), \ldots, h_n(x)$ such that

$$Eh_j = (w_j + w_n)h_j \quad (j = 1, 2, \ldots, n)$$

and that

$$h_j = \begin{cases} x_j x_n + h_j^{(0)}(x_1, \ldots, x_{n-1}) & (j = 1, 2, \ldots, n-1), \\ \frac{1}{2}x_n^2 + h_n^{(0)}(x_1, \ldots, x_{n-1}) & (j = n) \end{cases} \tag{1}$$

with functions $h_j^{(0)}(x_1, \ldots, x_{n-1})$ of $x' = (x_1, \ldots, x_{n-1})$. Using $h_j(x)$ $(j = 1, 2, \ldots, n)$, we define $\gamma_{ij} = \partial_{x_i} h_j$ and an $n \times n$ matrix $C = (\gamma_{ij})$. It is easy to see that $\gamma_{nj} = x_j$ $(j = 1, 2, \ldots, n)$. We define matrices

$$\tilde{B}^{(p)} = \partial_{x_p} C \quad (p = 1, 2, \ldots, n).$$

We denote by $b_{ij}^{(p)}$ the (i, j)-entry of $\tilde{B}^{(p)}$ and collect basic properties of $\tilde{B}^{(p)}$ $(p = 1, 2, \ldots, n)$:

1. $\partial_{x_p} \tilde{B}^{(q)} = \partial_{x_q} \tilde{B}^{(p)} \quad (\forall p,\ q)$,
2. $b_{pq}^{(r)} = b_{rq}^{(p)} \quad (\forall p,\ q,\ r)$,
3. $b_{nq}^{(p)} = \delta_{pq} \quad (\forall p,\ q)$,
4. $\tilde{B}^{(n)} = I_n$,
5. $\partial_{x_n} \tilde{B}^{(p)} = O \quad (p = 1, 2, \ldots, n-1)$,

where δ_{pq} is Kronecker's delta and I_n is the identity matrix.

Definition 1. If $\tilde{B}^{(p)}\tilde{B}^{(q)} = \tilde{B}^{(q)}\tilde{B}^{(p)}$ $(\forall p,\ q = 1, 2, \ldots, n)$, then $\vec{h} = (h_1, h_2, \ldots, h_n)$ is called a potential vector field.

We define the $n \times n$ matrix T by

$$T = \frac{1}{w_n} EC = \sum_{p=1}^{n} (w_p/w_n) x_p \tilde{B}^{(p)}.$$

Since the (i, j)-entry of T coincides with that of C up to a non-zero constant factor. It follows from the definition that $\det(T)$ is a polynomial of degree n with respect to the variable x_n. If $\det(T)$ has no multiple factor, it can be shown (cf. [9]) that $\det(T) = 0$ defines a free divisor on $\mathbf{C}^n$ and that T is Saito matrix for the function $\det(T)$. For the general treatment of free divisors, see [10]. By this reason we restrict our attention to such a potential vector field that $\det(T)$ has no multiple factor. Moreover, we treat such a potential vector field that $\det(T)$ is not reduced to a function of m variables with $m < n$. We now introduce a $(\mathbf{C}^*)^n$-action on the set of potential vector fields. Let $\vec{h} = (h_1, \ldots, h_n)$ be a potential vector field. We define $h'_j(x_1, \ldots, x_n) = h_j(k_1x_1, \ldots, k_nx_n)/(k_jk_n)$ $(j = 1, \ldots, n)$, where $k_1, \ldots, k_n \in \mathbf{C}^*$. Then $\vec{h}' = (h'_1, \ldots, h'_n)$ is also a potential vector field. This means the existence of a $(\mathbf{C}^*)^n$-action on the set of potential vector fields.

Definition 2. Two potential vector fields with the same weight system are equivalent if they are transformed with each other by this action.

A potential vector field $\vec{h} = (h_1, \ldots, h_n)$ is a polynomial potential vector field (="PPVF" for short) if all of $h_1, \ldots, h_n$ are polynomials. In spite that concrete forms of potential vector fields are hard to obtain, it is hopeful to construct PPVFs for extended WDVV equations with appropriate weight systems.

We give here an example of solutions to the extended WDVV equation for $n = 3$ with the weight system $(1/4, 1/2, 1)$.

Example 1. Let x_1, x_2, x_3 be variables of weights $1/4, 1/2, 1$, respectively. Then the PPVFs below are representatives of equivalence classes under the $(\mathbf{C}^*)^3$-action. (We eliminate such a PPVF that $\det(T)$ has a multiple factor or it is reduced a polynomial of two variables or one variable.)

1. Case 1 ($t \in \mathbf{C}$, $t \neq -\frac{3}{16}$)
$$\begin{aligned} h_1 &= \tfrac{9+4t}{40}x_1^5 + x_1^3x_2 + x_1x_2^2 + x_1x_3, \\ h_2 &= \tfrac{3t}{10}x_1^6 + tx_1^4x_2 + x_2x_3, \\ h_3 &= \tfrac{9t+8t^2}{56}x_1^8 + tx_1^6x_2 + 2tx_1^4x_2^2 + \tfrac{1}{3}x_2^4 + \tfrac{1}{2}x_3^2. \end{aligned}$$
2. Case 2
$$\begin{aligned} h_1 &= \tfrac{1}{10}x_1^5 + x_1x_2^2 + x_1x_3, \\ h_2 &= x_1^4x_2 + x_2x_3, \\ h_3 &= \tfrac{1}{7}x_1^8 + 2x_1^4x_2^2 + \tfrac{1}{3}x_2^4 + \tfrac{1}{2}x_3^2. \end{aligned}$$
3. Case 3
$$\begin{aligned} h_1 &= x_1^5 + x_1x_3, \\ h_2 &= x_2^3 + x_2x_3, \\ h_3 &= \tfrac{1}{2}x_3^2. \end{aligned}$$

4. Case 4

$$\begin{aligned} h_1 &= -\tfrac{3}{4}x_1x_2^2 + x_1x_3, \\ h_2 &= x_1^6 + x_2^3 + x_2x_3, \\ h_3 &= -\tfrac{15}{2}x_1^6x_2 + \tfrac{15}{16}x_2^4 + \tfrac{1}{2}x_3^2. \end{aligned}$$

5. Case 5

$$\begin{aligned} h_1 &= \tfrac{3}{16}x_1^5 + x_1^3x_2 + x_1x_2^2 + x_1x_3, \\ h_2 &= \tfrac{3}{64}x_1^6 + \tfrac{3}{8}x_1^4x_2 + x_1^2x_2^2 + \tfrac{4}{9}x_2^3 + x_2x_3, \\ h_3 &= -\tfrac{9}{256}x_1^8 - \tfrac{9}{32}x_1^6x_2 - \tfrac{3}{4}x_1^4x_2^2 - \tfrac{2}{3}x_1^2x_2^3 - \tfrac{1}{9}x_2^4 + \tfrac{1}{2}x_3^2. \end{aligned}$$

This example suggests that the classification of equivalence classes of PPVFs is complicated to solve.

3. POTENTIAL VECTOR FIELDS ASSOCIATED WITH REAL AND COMPLEX REFLECTION GROUPS

For the definition and basic properties of reflection groups, refer to [13].

Let G be an irreducible real or complex reflection group acting on an n-dimensional real or complex vector space. Then the ring of invariant polynomials is generated by n number of algebraically independent homogeneous polynomials. Let $d_1, d_2, \dots, d_n$ be the degrees of such generators. In this paper, we only treat such groups that $d_1, \dots, d_n$ are mutually different. Then we may assume that $0 < d_1 < d_2 < \cdots < d_n$. It is known that the discriminant of G is regarded as a weighted homogeneous polynomial of the weight system $(d_1, \dots, d_n)$, taking the generators of the invariant ring as variables.

Definition 3. If $\det(T)$ coincides with the discriminant of an irreducible (real or complex) reflection group G, the potential vector field $\vec{h}$ is associated with G.

We are going to explain the problem which is treated in this paper. The totality of PPVFs with the same weight system $(w_1, \dots, w_n)$ admits a $(\mathbf{C}^*)^n$-action. Then it is a basic problem to determine equivalence classes of PPVFs for a given weight system $(w_1, \dots, w_n)$. In particular, it is interesting to treat this when the weight system comes from the degrees of basic invariants of a real or complex reflection group. This problem is easy to solve for the low rank group case. But similarly to many problems concerning reflection groups, it becomes difficult if the rank becomes higher. In this sense, the treatment of E_8 for the real reflection group case and $ST34$ group for the complex reflection group case are the most difficult cases among "exceptional" groups. In the subsequent sections, we treat these cases separately.

Remark 1. We considered extended WDVV equations for $n = 3$ such that the weight systems are $(1/k, 2/k, 1)$ $(k = 5, 6, 7)$ in [5]. In particular, the case $k = 5$ contains a PPVF which is associated with the complex reflection group $ST27$ and a PPVF which is not associated with any reflection group but is related with an algebraic solution to Painlev'e VI equation first obtained by A. Kitaev [8]. The case $k = 7$ contains a PPVF which is related with an algebraic solution to Painlevé VI equation equivalent to Klein solution obtained by Boalch [2].

4. *ST*34 CASE

The complex reflection group $ST34$ denoted by G in this section is a finite subgroup of $GL(6, \mathbf{C})$ generated by six pseudo-reflections. The degrees of the basic invariants are $6, 12, 18, 24, 30, 42$. Therefore the discriminant of G is regarded as a weighted homogeneous polynomial of the variables $x_1, x_2, \ldots, x_6$ of the weight system $(1, 2, 3, 4, 5, 7)/7$. We put $w_j = \dfrac{j}{7}$ $(j = 1, 2, \ldots, 5)$, $w_6 = 1$.

The purpose of this section is to obtain polynomial solutions to the extended WDVV equation of $(x_1, x_2, \ldots, x_6)$ of the weight system $(w_1, w_2, \ldots, w_6)$ $(= (1/7, 2/7, 3/7, 4/7, 5/7, 1))$. At the end of this section, we will mention that the potential vector field associated with the group G is obtained in this manner.

As a preparation, we introduce weighted homogeneous polynomials

$$h_j(x_1, \ldots, x_5, x_6) = \begin{cases} x_j x_6 + h_j^{(0)}(x_1, \ldots, x_5) & (j = 1, \ldots, 5), \\ \frac{1}{2}x_6^2 + h_6^{(0)}(x_1, \ldots, x_5) & (j = 6). \end{cases}$$

It follows from the definition that the weight of $h_j^{(0)}$ is $w_j + 1$ for $j = 1, \ldots, 5$ and that of $h_6^{(0)}$ is $2 (= 2w_6)$. Noting this, we expand $h_j^{(0)}$ as a polynomial of x_5. Then

$$h_j^{(0)} = S_j x_5^2 + M_j x_5 + P_j,$$

where S_j, M_j, P_j are weighted homogeneous polynomials of x_1, x_2, x_3, x_4. The polynomials S_j, M_j are given as follows by use of constants $r_1, r_2, \cdots, r_{52}$, $s_1, \ldots, s_9$:

$$S_1 = S_2 = 0,\ S_3 = s_1,\ S_4 = s_2 x_1,\ S_5 = s_3 x_1^2 + s_4 x_2,$$
$$S_6 = s_5 x_1^4 + s_6 x_1^2 x_2 + s_7 x_2^2 + s_8 x_1 x_3 + s_9 x_4,$$

$$\begin{aligned} M_1 &= r_1 x_1^3 + r_2 x_1 x_2 + r_3 x_3, \\ M_2 &= r_4 x_1^4 + r_5 x_1^2 x_2 + r_6 x_2^2 + r_7 x_1 x_3 + r_8 x_4, \\ M_3 &= r_9 x_1^5 + r_{10} x_1^3 x_2 + r_{11} x_1 x_2^2 + r_{12} x_1^2 x_3 + r_{13} x_2 x_3 + r_{14} x_1 x_4, \end{aligned}$$

$$\begin{aligned}
M_4 &= r_{15}x_1^6 + r_{16}x_1^4x_2 + r_{17}x_1^2x_2^2 + r_{18}x_2^3 + r_{19}x_1^3x_3 + r_{20}x_1x_2x_3 \\
&\quad + r_{21}x_3^2 + r_{22}x_1^2x_4 + r_{23}x_2x_4, \\
M_5 &= r_{24}x_1^7 + r_{25}x_1^5x_2 + r_{26}x_1^3x_2^2 + r_{27}x_1x_2^3 + r_{28}x_1^4x_3 + r_{29}x_1^2x_2x_3 \\
&\quad + r_{30}x_2^2x_3 + r_{31}x_1x_3^2 + r_{32}x_1^3x_4 + r_{33}x_1x_2x_4 + r_{34}x_3x_4, \\
M_6 &= r_{35}x_1^9 + r_{36}x_1^7x_2 + r_{37}x_1^5x_2^2 + r_{38}x_1^3x_2^3 + r_{39}x_1x_2^4 + r_{40}x_1^6x_3 \\
&\quad + r_{41}x_1^4x_2x_3 + r_{42}x_1^2x_2^2x_3 + r_{43}x_2^3x_3 + r_{44}x_1^3x_3^2 + r_{45}x_1x_2x_3^2 \\
&\quad + r_{46}x_3^3 + r_{47}x_1^5x_4 + r_{48}x_1^3x_2x_4 + r_{49}x_1x_2^2x_4 + r_{50}x_1^2x_3x_4 \\
&\quad + r_{51}x_2x_3x_4 + r_{52}x_1x_4^2.
\end{aligned}$$

Assume that $\vec{h} = (h_1, \ldots, h_6)$ is a potential vector field. We define $h'_j(x_1, \ldots, x_6) = h_j(k_1x_1, \ldots, k_6x_6)/(k_jk_6)$ $(j = 1, \ldots, 6)$, where $k_1, \ldots, k_6 \in \mathbf{C}^*$. Then $\vec{h}' = (h'_1, \ldots, h'_6)$ is also a potential vector field. This action induces that on the constants $(r_3, r_8, s_1, s_2, s_4, s_9)$:

$$(r_3, r_8, s_1, s_2, s_4, s_9) \longrightarrow \left(\frac{k_3k_5}{k_1k_6} \cdot r_3, \frac{k_4k_5}{k_2k_6} \cdot r_8, \frac{k_5^2}{k_3k_6} \cdot s_1, \frac{k_1k_5^2}{k_4k_6} \cdot s_2, \frac{k_2k_5^2}{k_5k_6} \cdot s_4, \frac{k_4k_5^2}{k_6^2} \cdot s_9 \right).$$

As a consequence, we may assume from the first that r_3, r_8, s_1, s_2, s_4 are appropriate numbers. It is underlined here that $\{r_3, r_8, s_1, s_2, s_4\}$ is taken as an initial datum of the extended WDVV equation for the weight system $\{w_1, \ldots, w_6\}$, namely if the datum is given, then $\vec{h}$ is "almost" uniquely determined if $\vec{h}$ is a solution to the extended WDVV equation. The meaning of "almost" will be clarified later.

We put

$$g_j = h_j - P_j \ (j = 1, 2, \ldots, 6)$$

and $C_A = (\gamma_{ij}^{(A)})$ and $C_B = (\gamma_{ij}^{(B)})$ are 6×6 matrices such that

$$\gamma_{ij}^{(A)} = \partial_{x_i} g_j, \quad \gamma_{ij}^{(B)} = \partial_{x_i} P_j.$$

Then $C = C_A + C_B$ is the matrix introduced before. It follows that C_B is independent of x_5, x_6. As before we put $\tilde{B}^{(k)} = \partial_{x_k} C$ $(k = 1, \ldots, 6)$. Then $\tilde{B}^{(k)} = \partial_{x_k} C_A + \partial_{x_k} C_B$ $(k = 1, 2, 3, 4)$, $\tilde{B}^{(k)} = \partial_{x_k} C_A$ $(k = 5, 6)$. By definition, the (i, j)-entry of $\partial_{x_k} C_B$ is $\partial_{x_k}\partial_{x_i} P_j$. Noting this, we put

$$Q_{kij} = \partial_{x_k}\partial_{x_i} P_j \quad (i, k = 1, 2, 3, 4,\ j = 1, 2, \cdots, 6). \tag{2}$$

Then Q_{kij} is weighted homogeneous and $Q_{ikj} = Q_{kij}$.

To construct PPVF in this case, we formulate the problem precisely. Let r_j $(j = 1, 2, \ldots, 52)$, and s_k $(k = 1, 2, \ldots, 9)$ be constants to be determined and let $Q_{\alpha\beta\gamma}$ be an unknown weighted homogeneous polynomial of x_1, x_2, x_3, x_4 with the weight $d_\gamma + d_7 - d_\alpha - d_\beta$ ($\alpha, \beta = 1, 2, 3, 4, \gamma = 1, \ldots, 6$). Assume that $Q_{\beta\alpha\gamma} = Q_{\alpha\beta\gamma}$ for all α, β, γ. The system of matrix equations

$$(3) \qquad [\tilde{B}^{(i)}, \tilde{B}^{(j)}] = O \quad (i, j = 1, \ldots, 6)$$

turns out to be a system of algebraic equations for $\{r_j, s_k, Q_{\alpha\beta\gamma}\}$ by computing the matrix entries. (It is underlined here that it is possible to eliminate P_j ($j = 1, 2, \cdots, 6$) in the matrices $\tilde{B}^{(i)}$ by using the equation (2).) Then the problem is to solve the system of equations (obtained in this manner) for $\{r_j, s_k, Q_{\alpha\beta\gamma}\}$ under the condition

$$(4) \qquad r_3 = 1, r_8 = 1, s_1 = 1, s_2 = 1, s_4 = \frac{2}{3}, \quad s_9 \neq 0.$$

This condition comes from the fact that if all of r_3, r_8, s_1, s_2, s_4 are not 0, then $(r_3, r_8, s_1, s_2, s_4)$ turns out to be $(1, 1, 1, 1, 2/3)$ under the $(\mathbf{C}^*)^6$-action.

Remark 2. (1) It is not necessary to assume that $s_9 \neq 0$. To make the computation simpler, we attach the condition $s_9 \neq 0$.

(2) Since $\tilde{B}^{(6)}$ is the identity matrix, it is sufficient to treat the equations $[\tilde{B}^{(i)}, \tilde{B}^{(j)}] = O$ for $i, j = 1, \ldots, 5$.

Theorem 1. *Under the condition (4) above, there are two solutions* $\{r_j, s_k, Q_{\alpha\beta\gamma}\}$ *to the system of algebraic equations obtained from (3) corresponding to the solutions of the equation*

$$(5) \qquad (3s_9 - 4)(9s_9 + 2) = 0.$$

If s_9 *is a solution of (5), then the remaining constants* r_j, s_k *and polynomials* $Q_{\alpha\beta\gamma}$ *are uniquely determined. Moreover, there exist weighted homogeneous polynomials* $P_1, \ldots, P_6$ *of* x_1, x_2, x_3, x_4 *uniquely which satisfy (2).*

Proof. We always assume the condition (4) in this proof. The proof is decomposed into several steps. Its idea is first to determine r_j, s_k by solving the equations containing only these variables and next to solve the remaining variables. This is not done completely because of the insufficiency of the equations which contain r_j, s_k only. For this reason, the argument below becomes complicated.

- The 1$^{\text{st}}$ step: Solve $\partial_{x_5}^2([\tilde{B}^{(5)}, \tilde{B}^{(i)}]) = O \quad (i = 1, 2, 3, 4)$.

  It is easy to see that $\partial_{x_5}^3([\tilde{B}^{(5)}, \tilde{B}^{(i)}]) = O$ $(i = 1, 2, 3, 4)$.
  Computing the matrix entries of

$$\partial_{x_5}^2([\tilde{B}^{(5)}, \tilde{B}^{(i)}]) = O \quad (i = 1, 2, 3, 4),$$

containing $r_1, r_2, \ldots, r_{52}, s_1, \ldots, s_9$ but not containing $Q_{\alpha\beta\gamma}$, we obtain a system of algebraic equation for $r_1, r_2, \ldots, r_{52}, s_1, \ldots, s_9$. Solving this system, we conclude that

$$r_1, r_2, r_5, r_6, r_7, r_j\ (11 \le j \le 14,\ 16 \le j \le 23, 32 \le j \le 34, 47 \le j \le 52),$$
$$s_j\ (5 \le j \le 8)$$

are expressed as polynomials of the remaining constants r_j and s_3, s_9.

- The 2[nd] step: Solve the system of equations obtained from the 5th-row of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(4)}]) = O$, the 5th-row of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(3)}]) = O$, the 5th, 3rd, 2nd-rows of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(2)}]) = O$, the 5th, 3rd, 2nd-rows of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(1)}]) = O$.

After this procedure, we conclude that the constants r_j $(1 \le j \le 52,\ j \ne 15, 24, 35, 36, 37, 38, 39)$ and s_3, s_5, s_6, s_7, s_8 are expressed as polynomials of s_9.

- The 3[rd] step: Solve the system of equations obtained from $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(j)}]) = O$ $(j = 1, 2, 3, 4)$.

Comparing both sides of the $(1, j)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(k)}]) = O$ $(j = 1, 2, 3,\ k = 1, 2, 3, 4)$, we conclude that each of Q_{4kj} $(j = 1, 2, 3,\ k = 1, 2, 3, 4)$ is expressed as a polynomial of x_1, x_2, x_3, x_4 whose coefficients are dependent on s_9 and r_j $(j = 15, 24, 35, 36, 37, 38, 39)$. Moreover, comparing the both sides of the $(1, 4)$, $(1, 5)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(4)}]) = O$, we conclude that Q_{444}, Q_{445} are expressed as polynomials of x_1, x_2, x_3, x_4 whose coefficients are dependent on s_9 and r_j $(j = 15, 24, 35, 36, 37, 38, 39)$.

Similarly, comparing the both sides of the $(2, 4), (2, 5), (3, 4), (3, 5)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(1)}]) = O$, those of the $(2, 4), (2, 5), (3, 4), (3, 5)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(2)}]) = O$, those of the $(2, 4), (2, 5)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(3)}]) = O$, those of the $(2, 6), (3, 6)$-entries of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(4)}]) = O$, we conclude that

$$Q_{121},\ Q_{122},\ Q_{221},\ Q_{222},\ Q_{231},\ Q_{232},\ Q_{244},\ Q_{131},\ Q_{132},\ Q_{331},\ Q_{332},\ Q_{344}$$

are expressed as polynomials of x_1, x_2, x_3, x_4 whose coefficients are dependent on s_9 and r_j $(j = 15, 24, 35, 36, 37, 38, 39)$.

After these computations, we observe that $r_{37}, r_{38}, r_{39}, r_{15}$ are expressed as polynomials of s_9. This follows from the comparison of both sides of the $(4, 6)$-entry of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(2)}]) = O$, and those of $\partial_{x_5}([\tilde{B}^{(5)}, \tilde{B}^{(3)}]) = O$.

- The 4[th] step: Solve the system of equations obtained from the 1st row of $[\tilde{B}^{(5)}, \tilde{B}^{(4)}] = O$ under the condition $x_5 = 0$.

Computing the (1,1), (1,2), (1,3)-entries of $[\tilde{B}^{(5)}, \tilde{B}^{(4)}] = O$ under the condition $x_5 = 0$, we obtain an equation containing r_{24}, that containing Q_{144} and that containing Q_{145}. As a consequence, we conclude that r_{24}, Q_{144}, Q_{145} are expressed as polynomials of s_9, r_{35}, r_{36}.

From the (1,4)-entry of $[\tilde{B}^{(5)}, \tilde{B}^{(4)}] = O$, we obtain an equation containing r_{35}, that containing r_{36}, and the equation $(3s_9 - 4)(9s_9 + 2) = 0$.

Therefore, we observe that all the constants r_j $(j = 1, 2, \dots, 52)$ and s_3, s_5, s_6, s_7, s_8 are expressed as polynomials of s_9 and s_9 satisfies the equation (5).

From now on, we treat the cases $s_9 = \frac{4}{3}$ and $s_9 = -\frac{9}{2}$ separately.

• The 5th step (for the case $s_9 = \frac{4}{3}$): Solve the system of equations obtained from $[\tilde{B}^{(5)}, \tilde{B}^{(j)}] = O$ $(j = 1, 2, 3, 4)$ under the condition $x_5 = 0$.

There are still many polynomials $Q_{\alpha\beta\gamma}$ to be determined. It is tedious but not difficult to determine these polynomials by computing matrix entries of $[\tilde{B}^{(5)}, \tilde{B}^{(j)}] = O$ $(j = 1, 2, 3, 4)$ under the condition $x_5 = 0$.

It is not clear whether the polynomials $Q_{\alpha\beta\gamma}$ determined so far satisfy all the equations $[\tilde{B}^{(j)}, \tilde{B}^{(k)}] = O$ $(j, k = 1, 2, 3, 4)$ or not. But it can be shown by direct computation that its answer is affirmative.

• The 6th step (for the case $s_9 = \frac{4}{3}$): Show the existence of weighted homogeneous polynomials $P_1, \dots, P_6$ satisfying the condition (2).

The existence of weighted homogeneous polynomials $P_1, \dots, P_6$ satisfying (2) is not straightforward. But this is answered affirmatively by solving the equations $\partial_{x_k}\partial_{x_i}P_j = Q_{kij}$ $(i, k = 1, 2, 3, 4)$ for P_j by using the polynomials Q_{kij} which are already determined. Moreover, the weighted homogeneity of $P_1, \dots, P_6$ implies the uniqueness of them.

In this manner, we have constructed a potential vector field $\vec{h}$ under the condition

$$r_3 = 1, r_8 = 1, s_1 = 1, s_2 = 1, s_4 = \frac{2}{3}, s_9 = \frac{4}{3}.$$

• The 5th step and the 6th step (for the case $s_9 = -\frac{2}{9}$):

By an argument parallel to the case $s_9 = \frac{4}{3}$, we can solve the system of algebraic equations obtained from (3) and show the existence of a potential vector field.

The existence of weighted homogeneous polynomials $P_1, \dots, P_6$ and their uniqueness are also solved affirmatively as in the case $s_9 = \frac{4}{3}$.

In this manner, we have constructed a potential vector field $\vec{h}$ under the condition

$$r_3 = 1, r_8 = 1, s_1 = 1, s_2 = 1, s_4 = \frac{2}{3}, s_9 = -\frac{2}{9}.$$

Therefore, we proved the theorem completely. □

Remark 3. (1) Since the potential vector field for the case $s_9 = \frac{4}{3}$ obtained above is, same as that shown in [5], we don't write down its concrete form here.

(2) The concrete form of the potential vector field $\vec{h} = (h_1, h_2, \cdots, h_6)$ for

the case $s_9 = -\frac{2}{9}$ is given as follows:

$$
\begin{aligned}
h_1 &= (2x_1^8 + 28x_1^6x_2 + 120x_1^4x_2^2 + 160x_1^2x_2^3 + 32x_2^4 + 54x_1^5x_3 + 360x_1^3x_2x_3 \\
&\quad +432x_1x_2^2x_3 + 243x_1^2x_3^2 + 243x_2x_3^2 + 90x_1^4x_4 + 432x_1^2x_2x_4 \\
&\quad +216x_2^2x_4 + 486x_1x_3x_4 + 162x_4^2 + 324x_1^3x_5 + 972x_1x_2x_5 \\
&\quad +729x_3x_5 + 729x_1x_6)/729, \\
h_2 &= (8x_1^9 + 120x_1^7x_2 + 624x_1^5x_2^2 + 1280x_1^3x_2^3 + 768x_1x_2^4 + 180x_1^6x_3 \\
&\quad +1620x_1^4x_2x_3 + 3888x_1^2x_2^2x_3 + 1440x_2^3x_3 + 972x_1^3x_3^2 + 3402x_1x_2x_3^2 \\
&\quad +729x_3^3 + 216x_1^5x_4 + 1728x_1^3x_2x_4 + 3024x_1x_2^2x_4 + 1944x_1^2x_3x_4 \\
&\quad +3888x_2x_3x_4 + 972x_1x_4^2 + 486x_1^4x_5 + 3888x_1^2x_2x_5 + 3888x_2^2x_5 \\
&\quad +4374x_1x_3x_5 + 4374x_4x_5 + 4374x_2x_6)/4374, \\
h_3 &= (8x_1^{10} + 124x_1^8x_2 + 688x_1^6x_2^2 + 1600x_1^4x_2^3 + 1344x_1^2x_2^4 + 192x_2^5 \\
&\quad +180x_1^7x_3 + 1800x_1^5x_2x_3 + 5184x_1^3x_2^2x_3 + 3744x_1x_2^3x_3 + 1296x_1^4x_3^2 \\
&\quad +5832x_1^2x_2x_3^2 + 2916x_2^2x_3^2 + 2187x_1x_3^3 + 180x_1^6x_4 + 1584x_1^4x_2x_4 \\
&\quad +3456x_1^2x_2^2x_4 + 1152x_2^3x_4 + 2592x_1^3x_3x_4 + 7776x_1x_2x_3x_4 + 3645x_3^2x_4 \\
&\quad +1296x_1^2x_4^2 + 1944x_2x_4^2 + 324x_1^5x_5 + 2592x_1^3x_2x_5 + 3888x_1x_2^2x_5 \\
&\quad +5832x_1^2x_3x_5 + 8748x_2x_3x_5 + 5832x_1x_4x_5 + 6561x_5^2 \\
&\quad +6561x_3x_6)/6561, \\
h_4 &= (48x_1^{11} + 800x_1^9x_2 + 4960x_1^7x_2^2 + 13760x_1^5x_2^3 + 16000x_1^3x_2^4 + 5376x_1x_2^5 \\
&\quad +1080x_1^8x_3 + 12240x_1^6x_2x_3 + 43200x_1^4x_2^2x_3 + 48960x_1^2x_2^3x_3 \\
&\quad +8640x_2^4x_3 + 8100x_1^5x_3^2 + 45360x_1^3x_2x_3^2 + 48600x_1x_2^2x_3^2 + 14580x_1^2x_3^3 \\
&\quad +14580x_2x_3^3 + 1080x_1^7x_4 + 11520x_1^5x_2x_4 + 34560x_1^3x_2^2x_4 \\
&\quad +25920x_1x_2^3x_4 + 17820x_1^4x_3x_4 + 77760x_1^2x_2x_3x_4 + 38880x_2^2x_3x_4 \\
&\quad +36450x_1x_3^2x_4 + 12960x_1^3x_4^2 + 38880x_1x_2x_4^2 + 29160x_3x_4^2 \\
&\quad +1620x_1^6x_5 + 16200x_1^4x_2x_5 + 38880x_1^2x_2^2x_5 + 12960x_2^3x_5 \\
&\quad +29160x_1^3x_3x_5 + 87480x_1x_2x_3x_5 + 32805x_3^2x_5 + 58320x_1^2x_4x_5 \\
&\quad +87480x_2x_4x_5 + 65610x_1x_5^2 + 65610x_4x_6)/65610, \\
h_5 &= (48x_1^{12} + 864x_1^{10}x_2 + 5952x_1^8x_2^2 + 19264x_1^6x_2^3 + 28800x_1^4x_2^4 \\
&\quad +16128x_1^2x_2^5 + 1536x_2^6 + 1152x_1^9x_3 + 14688x_1^7x_2x_3 + 62208x_1^5x_2^2x_3 \\
&\quad +97920x_1^3x_2^3x_3 + 41472x_1x_2^4x_3 + 9720x_1^6x_3^2 + 68040x_1^4x_2x_3^2 \\
&\quad +116640x_1^2x_2^2x_3^2 + 27216x_2^3x_3^2 + 23328x_1^3x_3^3 + 52488x_1x_2x_3^3 \\
&\quad +6561x_3^4 + 1080x_1^8x_4 + 12960x_1^6x_2x_4 + 47520x_1^4x_2^2x_4 + 55296x_1^2x_2^3x_4 \\
&\quad +10368x_2^4x_4 + 19440x_1^5x_3x_4 + 108864x_1^3x_2x_3x_4 + 116640x_1x_2^2x_3x_4 \\
&\quad +52488x_1^2x_3^2x_4 + 52488x_2x_3^2x_4 + 11664x_1^4x_4^2 + 46656x_1^2x_2x_4^2 \\
&\quad +23328x_2^2x_4^2 + 34992x_1x_3x_4^2 + 5832x_4^3 + 1944x_1^7x_5 + 23328x_1^5x_2x_5 \\
&\quad +77760x_1^3x_2^2x_5 + 62208x_1x_2^3x_5 + 43740x_1^4x_3x_5 + 209952x_1^2x_2x_3x_5 \\
&\quad +104976x_2^2x_3x_5 + 118098x_1x_3^2x_5 + 69984x_1^3x_4x_5 + 209952x_1x_2x_4x_5 \\
&\quad +157464x_3x_4x_5 + 157464x_1^2x_5^2 + 236196x_2x_5^2 + 354294x_5x_6)/354294, \\
h_6 &= (-528x_1^{14} - 11088x_1^{12}x_2 - 92736x_1^{10}x_2^2 - 389760x_1^8x_2^3 - 852992x_1^6x_2^4 \\
&\quad -903168x_1^4x_2^5 - 365568x_1^2x_2^6 - 24576x_2^7 - 15120x_1^{11}x_3 - 235872x_1^9x_2x_3
\end{aligned}
$$

$$-1338624x_1^7x_2^2x_3 - 3330432x_1^5x_2^3x_3 - 3386880x_1^3x_2^4x_3 - 967680x_1x_2^5x_3$$
$$-156492x_1^8x_3^2 - 1555848x_1^6x_2x_3^2 - 4762800x_1^4x_2^2x_3^2 - 4536000x_1^2x_2^3x_3^2$$
$$-653184x_2^4x_3^2 - 591948x_1^5x_3^3 - 2857680x_1^3x_2x_3^3 - 2449440x_1x_2^2x_3^3$$
$$-597051x_1^2x_3^4 - 413343x_2x_3^4 - 15120x_1^{10}x_4 - 221760x_1^8x_2x_4$$
$$-1128960x_1^6x_2^2x_4 - 2338560x_1^4x_2^3x_4 - 1693440x_1^2x_2^4x_4 - 193536x_2^5x_4$$
$$-317520x_1^7x_3x_4 - 2721600x_1^5x_2x_3x_4 - 6531840x_1^3x_2^2x_3x_4$$
$$-3773952x_1x_2^3x_3x_4 - 1530900x_1^4x_3^2x_4 - 5388768x_1^2x_2x_3^2x_4$$
$$-1959552x_2^2x_3^2x_4 - 1285956x_1x_3^3x_4 - 181440x_1^6x_4^2 - 1270080x_1^4x_2x_4^2$$
$$-2177280x_1^2x_2^2x_4^2 - 508032x_2^3x_4^2 - 1306368x_1^3x_3x_4^2 - 2939328x_1x_2x_3x_4^2$$
$$-734832x_3^2x_4^2 - 326592x_1^2x_4^3 - 326592x_2x_4^3 - 27216x_1^9x_5$$
$$-381024x_1^7x_2x_5 - 1741824x_1^5x_2^2x_5 - 2903040x_1^3x_2^3x_5 - 1306368x_1x_2^4x_5$$
$$-612360x_1^6x_3x_5 - 4490640x_1^4x_2x_3x_5 - 7838208x_1^2x_2^2x_3x_5$$
$$-1959552x_2^3x_3x_5 - 2571912x_1^3x_3^2x_5 - 5511240x_1x_2x_3^2x_5 - 826686x_3^3x_5$$
$$-816480x_1^5x_4x_5 - 4572288x_1^3x_2x_4x_5 - 4898880x_1x_2^2x_4x_5$$
$$-4408992x_1^2x_3x_4x_5 - 4408992x_2x_3x_4x_5 - 1469664x_1x_4^2x_5$$
$$-1102248x_1^4x_5^2 - 4408992x_1^2x_2x_5^2 - 2204496x_2^2x_5^2 - 3306744x_1x_3x_5^2$$
$$-1653372x_4x_5^2 + 3720087x_6^2)/7440174.$$

By direct calculation, we find that $\det(T)$ for $s_9 = \frac{4}{3}$ is an irreducible polynomial whereas $\det(T)$ for $s_9 = -\frac{2}{9}$ has a linear factor concerning x_6.

Before closing this section, we show that $\det(T)$ for $s_9 = \frac{4}{3}$ is regarded as a discriminant of G and $\{x_1, x_2, \ldots, x_6\}$ is regarded as a generator system of the ring of invariants of G. This is written as *a Conjecture* in [5]. For this purpose, we need the result on Saito matrix of the discriminant for G obtained by Bessis and Michel (unpublished). (The author abandons to write down its concrete form, since it takes more than ten pages.) Let x, y, z, t, u, v be the basic invariants used by Bessis and Michel. Their homogeneous degrees are 6,12,18,24,30,42, respectively. The author was informed by J. Michel that Saito matrix for $ST34$ was obtained by Terao and Enta [14] and Bessis and he confirmed the computation in [14], and that x, y, z, t, u, v coincide with $f_1, f_2, f_3, f_4, f_5, f_6$ in [14] up to non-zero constant factors, respectively. We only give here the relationship between $\{x, y, z, t, u, v\}$ and $\{x_1, \ldots, x_6\}$.

$$x = k_1x_1,$$
$$y = 3k_1^2(484x_1^2 + 125x_2)/56,$$
$$z = 3k_1^3(3840232x_1^3 + 1647300x_1x_2 + 466785x_3)/3136,$$
$$t = -k_1^4(10x_1^2x_2 - 6x_2^2 - 3x_1x_3 + 3x_4)/175616,$$
$$u = 27k_1^5(8037231129928x_1^5 + 7056011495960x_1^3x_2 + 923992672120x_1x_2^2$$
$$+4020065168070x_1^2x_3 + 320350835370x_2x_3 + 1029935376420x_1x_4$$
$$+170567816805x_5)/9834496,$$
$$v = (415352913869256405 12x_1^7 + 5540226826396953 2360x_1^5x_2$$
$$+1713476442242398 2880x_1^3x_2^2 + 1037814693245737680x_1x_2^3$$

$$+383640193831162795580x_1^4x_3 + 1384716990663416592 0x_1^2x_2x_3$$
$$+40215695286477186 0x_2^2x_3 + 18206957183728020 75x_1x_3^2$$
$$+1436169703447901604 0x_1^3x_4 + 2535507816645416760x_1x_2x_4$$
$$+322231741664151375x_3x_4 + 5379672850162662240x_1^2x_5$$
$$+306822518227402695x_2x_5 + 606436594603405 65x_6)/120472576,$$

where $k_1 = (64/27)^{1/7}$.

Remark 4. The result on the case of the weight system $(1/4, 1/2, 1)$ (cf. Example 1) suggests that a classification of the extended WDVV equation for $n = 6$ with the weight system $(1/7, 2/7, 3/7, 4/7, 5/7, 1)$ is complicated to answer. To answer the question of constructing a PPVF related with the group $ST34$, it is sufficient to treat the extended WDVV equation under the condition (4).

5. E_8 CASE

The real reflection group of type E_8 denoted by G in this section is a finite subgroup of $GL(8, \mathbf{R})$ generated by eight reflections. The degrees of the basic invariants are $2, 8, 12, 14, 18, 20, 24, 30$. Therefore the discriminant of G is regarded as a weighted homogeneous polynomial of the variables $x_1, x_2, \ldots, x_8$ of the weight system $(1, 4, 6, 7, 9, 10, 12, 15)/15$. We put

$$w_1 = \frac{1}{15}, w_2 = \frac{4}{15}, w_3 = \frac{6}{15}, w_4 = \frac{7}{15}, w_5 = \frac{9}{15}, w_6 = \frac{10}{15}, w_7 = \frac{12}{15}, w_8 = 1.$$

The purpose of this section is to obtain polynomial solutions to the extended WDVV equation of $(x_1, x_2, \ldots, x_8)$ of the weight system $(w_1, w_2, \ldots, w_8)$.

As a preparation we introduce weighted homogeneous polynomials

$$h_j(x_1, \ldots, x_7, x_8) = \begin{cases} x_jx_8 + h_j^{(0)}(x_1, \ldots, x_7) & (j = 1, \ldots, 7), \\ \frac{1}{2}x_8^2 + h_8^{(0)}(x_1, \ldots, x_7) & (j = 8). \end{cases}$$

It follows from the definition that the weight of $h_j^{(0)}$ is $w_j + 1$ for $j = 1, \ldots, 7$ and that of $h_8^{(0)}$ is 2. Noting this, we expand $h_j^{(0)}$ as a polynomial of x_7. Then

$$h_j^{(0)} = S_jx_7^2 + M_jx_7 + P_j,$$

where S_j, M_j, P_j are weighted homogeneous polynomials of $x_1, x_2, \ldots, x_6$. In particular, introducing constants $r_1, r_2 \ldots, r_{92}, s_1, \ldots, s_6$, we write S_j, M_j as follows:

$$S_1 = S_2 = S_3 = S_4 = 0, \quad S_5 = s_1, \quad S_6 = s_2x_1, \quad S_7 = s_3x_1^3,$$
$$S_8 = s_4x_3 + s_5x_2x_1^2 + s_6x_1^6,$$

$$\begin{aligned}
M_1 &= r_1x_1^4+r_2x_2,\\
M_2 &= r_3x_1^7+r_4x_1^3x_2+r_5x_1x_3+r_6x_4,\\
M_3 &= r_7x_1^9+r_8x_1^5x_2+r_9x_1x_2^2+r_{10}x_1^3x_3+r_{11}x_1^2x_4+r_{12}x_5,\\
M_4 &= r_{13}x_1^{10}+r_{14}x_1^6x_2+r_{15}x_1^2x_2^2+r_{16}x_1^4x_3+r_{17}x_2x_3+r_{18}x_1^3x_4\\
&\quad +r_{19}x_1x_5+r_{20}x_6,\\
M_5 &= r_{21}x_1^{12}+r_{22}x_1^8x_2+r_{23}x_1^4x_2^2+r_{24}x_2^3+r_{25}x_1^6x_3+r_{26}x_1^2x_2x_3+r_{27}x_3^2\\
&\quad +r_{28}x_1^5x_4+r_{29}x_1x_2x_4+r_{30}x_1^3x_5+r_{31}x_1^2x_6,\\
M_6 &= r_{32}x_1^{13}+r_{33}x_1^9x_2+r_{34}x_1^5x_2^2+r_{35}x_1x_2^3+r_{36}x_1^7x_3+r_{37}x_1^3x_2x_3\\
&\quad +r_{38}x_1x_3^2+r_{39}x_1^6x_4+r_{40}x_1^2x_2x_4\\
&\quad +r_{41}x_3x_4+r_{42}x_1^4x_5+r_{43}x_2x_5+r_{44}x_1^3x_6,\\
M_7 &= r_{45}x_1^{15}+r_{46}x_1^{11}x_2+r_{47}x_1^7x_2^2+r_{48}x_1^3x_2^3+r_{49}x_1^9x_3+r_{50}x_1^5x_2x_3\\
&\quad +r_{51}x_1x_2^2x_3+r_{52}x_1^3x_3^2+r_{53}x_1^8x_4+r_{54}x_1^4x_2x_4+r_{55}x_2^2x_4\\
&\quad +r_{56}x_1^2x_3x_4+r_{57}x_1x_4^2+r_{58}x_1^6x_5+r_{59}x_1^2x_2x_5+r_{60}x_3x_5\\
&\quad +r_{61}x_1^5x_6+r_{62}x_1x_2x_6,\\
M_8 &= r_{63}x_1^{18}+r_{64}x_1^{14}x_2+r_{65}x_1^{10}x_2^2+r_{66}x_1^6x_2^3+r_{67}x_1^2x_2^4+r_{68}x_1^{12}x_3\\
&\quad +r_{69}x_1^8x_2x_3+r_{70}x_1^4x_2^2x_3+r_{71}x_2^3x_3+r_{72}x_1^6x_3^2+r_{73}x_1^2x_2x_3^2\\
&\quad +r_{74}x_3^3+r_{75}x_1^{11}x_4+r_{76}x_1^7x_2x_4+r_{77}x_1^3x_2^2x_4+r_{78}x_1^5x_3x_4\\
&\quad +r_{79}x_1x_2x_3x_4+r_{80}x_1^4x_4^2+r_{81}x_2x_4^2+r_{82}x_1^9x_5+r_{83}x_1^5x_2x_5\\
&\quad +r_{84}x_1x_2^2x_5+r_{85}x_1^3x_3x_5+r_{86}x_1^2x_4x_5+r_{87}x_5^2+r_{88}x_1^8x_6\\
&\quad +r_{89}x_1^4x_2x_6+r_{90}x_2^2x_6+r_{91}x_1^2x_3x_6+r_{92}x_1x_4x_6.
\end{aligned}$$

In this case, there is a $(\mathbf{C}^*)^8$-action on potential vector fields and this action induces that on the coefficients $(r_2, r_6, r_{12}, r_{20}, s_1, s_2, s_3)$ by

(6) $(r_2, r_6, r_{12}, r_{20}, s_1, s_2, s_3)$

$$\longrightarrow \left(\frac{k_2k_7}{k_1k_8}\cdot r_2, \frac{k_4k_7}{k_2k_8}\cdot r_6, \frac{k_5k_7}{k_3k_8}\cdot r_{12}, \frac{k_6k_7}{k_4k_8}\cdot r_{20}, \frac{k_7^2}{k_5k_8}\cdot s_1, \frac{k_1k_7^2}{k_6k_8}\cdot s_2, \frac{k_1^3k_7}{k_8}\cdot s_3\right).$$

As in the case of $ST34$, we may assume from the first that $r_2, r_6, r_{12}, r_{20}, s_1, s_2, s_3$ are appropriate numbers.

For a moment, we take the constants $r_2, r_6, r_{12}, r_{20}, s_1, s_2, s_3$ as follows:

(7) $$r_2 = r_6 = r_{12} = r_{20} = s_1 = s_2 = s_3 = 1.$$

We put

$$g_j = h_j - P_j \ (j = 1, 2, \ldots, 8)$$

and $C_A = (\gamma_{ij}^{(A)})$ and $C_B = (\gamma_{ij}^{(B)})$ are 8×8 matrices such that

$$\gamma_{ij}^{(A)} = \partial_{x_i} g_j, \quad \gamma_{ij}^{(B)} = \partial_{x_i} P_j.$$

Then $C = C_A + C_B$ is the matrix introduced before. It follows that C_B is independent of x_7, x_8. As before we put $\tilde{B}^{(k)} = \partial_{x_k} C$ $(k = 1, \ldots, 8)$

Then $\tilde{B}^{(k)} = \partial_{x_k} C_A + \partial_{x_k} C_B$ $(k = 1, \ldots, 6)$, $\tilde{B}^{(k)} = \partial_{x_k} C_A$ $(k = 7, 8)$. By definition, the (i,j)-entry of $\partial_{x_k} C_B$ is $\partial_{x_k}\partial_{x_i} P_j$. Noting this, we put

(8) $$Q_{kij} = \partial_{x_k}\partial_{x_i} P_j \quad (i,k = 1,2,\cdots,6,\ j = 1,2,\cdots,8).$$

Then Q_{kij} is weighted homogeneous and $Q_{ikj} = Q_{kij}$. Moreover, we put $Q_k = \partial_{x_k} C_B$.

To construct PPVF in this case, we formulate the problem precisely as in the case of $ST34$. Let r_j $(j = 1,2,\ldots,92)$, and s_k $(k = 1,2,\ldots,6)$ be constants to be determined and let $Q_{\alpha\beta\gamma}$ be an unknown weighted homogeneous polynomial of $x_1, x_2, \ldots, x_6$ with the weight $d_\gamma + d_8 - d_\alpha - d_\beta$ $(\alpha,\beta = 1,2,3,4, \gamma = 1,\ldots,6)$. Assume that $Q_{\beta\alpha\gamma} = Q_{\alpha\beta\gamma}$ for all α,β,γ. (We don't assume the existence of $P_1,\ldots,P_8$ for the moment.) The system of matrix equations

(9) $$[\tilde{B}^{(i)}, \tilde{B}^{(j)}] = O \quad (i,j = 1,\ldots,8)$$

turns out to be a system of algebraic equations for $\{r_j, s_k, Q_{\alpha\beta\gamma}\}$ by computing the matrix entires. Then the problem is to solve the system under the condition (7).

THEOREM 2. *Under the condition (7), there is a unique solution* $\{r_j, s_k, Q_{\alpha\beta\gamma}\}$ *to the system of algebraic equations obtained from (9). Moreover, there exist weighted homogeneous polynomials* $P_1,\ldots,P_8$ *of* $x_1, x_2, \ldots, x_6$ *uniquely which satisfy (8).*

Proof. The construction of the proof is similar to that of Theorem 1.

- The 1st step: $\partial^2_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(j)}]) = O \quad (i = 1,\ldots,6)$.

It is easy to see that $\partial^3_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(j)}]) = O$ $(i = 1,\ldots,6)$.
Computing the both sides of the matrix entries of

$$\partial^2_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(j)}]) = O \quad (i = 1,\ldots,6),$$

we obtain a system of algebraic equations among $r_1, r_2, \ldots, r_{91}, s_4, s_5, s_6$. Solving this system, we conclude that

$$r_1, r_4, r_5, r_{10}, r_{31}, r_{44}, r_{61}, r_{88}, r_{89}, r_{90}, r_{91}, r_{92}, s_5$$

are expressed as polynomials of the remaining constants r_j and s_4, s_6.

- The 2nd step: Solve the system of equations obtained from the entries of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(j)}]) = O$ $(j = 1,2,\ldots,6)$ containing only constants r_j $(1 \le j \le 92,\ j \ne 1,4,5,10,31,44,61,88,89,90,91,92)$ and s_4, s_6.

By direct computation, we conclude that if

$$r_3, r_7, r_{13}, r_{21}, r_{32}, r_{45}, r_{63}, r_{64}, \ldots, r_{81}$$

are determined, the remaining constants r_k and s_4, s_6 are determined uniquely. At this moment, the constants $r_3, r_7, r_{13}, r_{21}, r_{32}, r_{45}, r_{63}, r_{64}, \ldots, r_{81}$ remain to be determined.

• The 3rd step: Solve the system of equations obtained from the entries of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(j)}]) = O$ $(j = 1, 2, \ldots, 6)$ containing polynomials $Q_{\alpha\beta\gamma}$.

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(1)}]) = O$

the 1st row $\Longrightarrow$ Q_{16k} $(k = 1, 2, \ldots, 5)$
the 6th row $\Longrightarrow$ $r_3, r_{75}, r_{76}, \ldots, r_{80}$
the 5th row $\Longrightarrow$ Q_{151}, r_7
the 4th row $\Longrightarrow$ Q_{141}, r_{13}
the 3rd row $\Longrightarrow$ $Q_{131}, r_{68}, r_{69}, r_{70}, r_{72}, r_{73}$
the 2nd row $\Longrightarrow$ $Q_{121}, r_{64}, r_{65}, r_{66}, r_{67}$

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(2)}]) = O$

the 1st row $\Longrightarrow$ Q_{26k} $(k = 1, 2, \ldots, 7)$
the 6th row $\Longrightarrow$ r_{81}
the 5th row $\Longrightarrow$ Q_{251}
the 4th row $\Longrightarrow$ Q_{241}
the 3rd row $\Longrightarrow$ Q_{231}, r_{71}
the 2nd row $\Longrightarrow$ Q_{221}

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(3)}]) = O$

the 1st row $\Longrightarrow$ Q_{36k} $(k = 1, 2, \ldots, 7)$
the 5th row $\Longrightarrow$ Q_{351}
the 4th row $\Longrightarrow$ Q_{341}
the 3rd row $\Longrightarrow$ Q_{331}

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(4)}]) = O$

the 1st row $\Longrightarrow$ Q_{46k} $(k = 2, \ldots, 7)$
the 5th row $\Longrightarrow$ Q_{451}
the 4th row $\Longrightarrow$ Q_{441}

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(5)}]) = O$

the 1st row $\Longrightarrow$ Q_{56k} $(k = 2, \ldots, 7)$

Computation of $\partial_{x_7}([\tilde{B}^{(7)}, \tilde{B}^{(6)}]) = O$

the 1st row $\Longrightarrow$ Q_{66k} $(k = 3, \ldots, 7, 8)$

It is underlined here that at this moment almost all constants r_k are determined but still $r_{21}, r_{32}, r_{45}, r_{63}$ remain to be determined.

• The 4th step: Solve the system of equations obtained from the 1st row of $[\tilde{B}^{(7)}, \tilde{B}^{(6)}] = O$ under the condition $x_7 = 0$.

From the (1,2), (1,3)-entries, we find that $r_{21} = 0, r_{32} = 0, r_{45} = 0$. Then we conclude that all the constants r_k are determined uniquely. This is the difference from the case $ST34$.

From the remaining matrix entries, we obtain a system of equations and solving it, we determine the polynomials

$$Q_{166}, Q_{167}, Q_{168}, Q_{268}, Q_{238}, Q_{468}, Q_{568}$$

and r_{63} which is the last constant which we want to determine.

• The 5^{th} step: Solve the system of equations obtained from the 1st row of $[\tilde{B}^{(7)}, \tilde{B}^{(j)}] = O$ $(j = 1, 2, \ldots, 5)$ under the condition $x_7 = 0$.

By the computation at this step, we find that the polynomials $Q_{\alpha\beta\gamma}$ determined uniquely.

• The 6^{th} step: Confirm $[\tilde{B}^{(j)}, \tilde{B}^{(k)}] = O$ $(j, k = 1, \ldots, 6)$.

It is not clear whether the polynomials $Q_{\alpha\beta\gamma}$ determined so far satisfy all the equations $[\tilde{B}^{(j)}, \tilde{B}^{(k)}] = O$ $(j, k = 1, \ldots, 6)$ or not. But by direct computation, it can be shown that its answer is affirmative.

• The 7^{th} step: Show the existence of weighted homogeneous polynomials $P_1, \ldots, P_8$ such that $Q_{\alpha\beta\gamma} = \partial_{x_\alpha}\partial_{x_\beta} P_\gamma$ $(\alpha, \beta = 1, \ldots, 6,\ \gamma = 1, \ldots, 6)$.

Since the argument above doesn't guarantee the existence of weighted homogeneous polynomials $P_1, P_2, \ldots, P_8$ such that

$$Q_{kij} = \partial_{x_k}\partial_{x_i} P_j \quad (i, k = 1, 2, \ldots, 6,\ j = 1, 2, \ldots, 8),$$

the problem to be solved is whether $\{Q_{\alpha\beta\gamma}\}$ are integrable or not.

This is also answered affirmatively.

In this manner, we have constructed the PPVF $\vec{h}$ uniquely under the condition

$$r_2 = r_6 = r_{12} = r_{20} = s_1 = s_2 = s_3 = 1.$$

Therefore the theorem is proved completely.[]

We consider the potential vector field corresponding to the initial condition

$$r_2 = 1,\ r_6 = 2,\ r_{12} = 1,\ r_{20} = 1,\ s_1 = 1,\ s_2 = \frac{1}{2},\ s_3 = \frac{1}{8},\ s_1 = 1.$$

This is realized by the substitution $k_1 = k_2 = \frac{1}{2}, k_3 = \cdots = k_8 = 1$ in (6). In this case there is a prepotential. Namely there is a polynomial $P = P(x_1, \cdots, x_8)$ such that $h_j = \partial_{x_{9-j}} P$ $(j = 1, 2, \cdots, 8)$ define a PPVF. The concrete form of P is as follows:

$$\begin{aligned}
P \;=\; & \frac{27}{471334912000000} x_1^{31} + \frac{27}{30146560000} x_1^{23} x_2^2 + \frac{3}{65536000} x_1^{19} x_2^3 \\
& + \frac{3}{1024000} x_1^{15} x_2^4 + \frac{117}{1638400} x_1^{11} x_2^5 + \frac{33}{40960} x_1^7 x_2^6 + \frac{1}{512} x_1^3 x_2^7 \\
& + \frac{9}{16384000} x_1^{17} x_2^2 x_3 + \frac{21}{819200} x_1^{13} x_2^3 x_3 + \frac{77}{81920} x_1^9 x_2^4 x_3 + \frac{21}{2560} x_1^5 x_2^5 x_3 \\
& + \frac{1}{128} x_1 x_2^6 x_3 + \frac{9}{311296000} x_1^{19} x_3^2 + \frac{9}{40960} x_1^{11} x_2^2 x_3^2 + \frac{21}{5120} x_1^7 x_2^3 x_3^2 \\
& + \frac{17}{512} x_1^3 x_2^4 x_3^2 + \frac{3}{409600} x_1^{13} x_3^3 + \frac{21}{1280} x_1^5 x_2^2 x_3^3 + \frac{1}{32} x_1 x_2^3 x_3^3 + \frac{3}{2560} x_1^7 x_3^4 \\
& + \frac{1}{80} x_1 x_3^5 + \frac{9}{8192000} x_1^{16} x_2^2 x_4 + \frac{39}{409600} x_1^{12} x_2^3 x_4 + \frac{27}{10240} x_1^8 x_2^4 x_4 \\
& + \frac{21}{1024} x_1^4 x_2^5 x_4 + \frac{1}{160} x_2^6 x_4 + \frac{9}{409600} x_1^{14} x_2 x_3 x_4 + \frac{99}{102400} x_1^{10} x_2^2 x_3 x_4 \\
& + \frac{21}{640} x_1^6 x_2^3 x_3 x_4 + \frac{33}{256} x_1^2 x_2^4 x_3 x_4 + \frac{27}{5120} x_1^8 x_2 x_3^2 x_4 + \frac{3}{32} x_1^4 x_2^2 x_3^2 x_4 \\
& + \frac{1}{8} x_2^3 x_3^2 x_4 + \frac{3}{16} x_1^2 x_2 x_3^3 x_4 + \frac{27}{69632000} x_1^{17} x_4^2 + \frac{9}{409600} x_1^{13} x_2 x_4^2
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{27}{10240}x_1^9x_2^2x_4^2+\tfrac{63}{1280}x_1^5x_2^3x_4^2+\tfrac{15}{128}x_1x_2^4x_4^2+\tfrac{9}{102400}x_1^{11}x_3x_4^2\\
&+\tfrac{9}{640}x_1^7x_2x_3x_4^2+\tfrac{21}{64}x_1^3x_2^2x_3x_4^2+\tfrac{27}{640}x_1^5x_3^2x_4^2+\tfrac{3}{8}x_1x_2x_3^2x_4^2\\
&+\tfrac{3}{12800}x_1^{10}x_4^3+\tfrac{21}{1280}x_1^6x_2x_4^3+\tfrac{9}{32}x_1^2x_2^2x_4^3+\tfrac{3}{64}x_1^4x_3x_4^3+\tfrac{1}{2}x_2x_3x_4^3\\
&+\tfrac{3}{32}x_1^3x_4^4+\tfrac{11}{102400}x_1^{10}x_2^3x_5+\tfrac{7}{2560}x_1^6x_2^4x_5+\tfrac{3}{256}x_1^2x_2^5x_5\\
&+\tfrac{3}{102400}x_1^{12}x_2x_3x_5+\tfrac{9}{5120}x_1^8x_2^2x_3x_5+\tfrac{5}{128}x_1^4x_2^3x_3x_5+\tfrac{1}{32}x_2^4x_3x_5\\
&+\tfrac{7}{640}x_1^6x_2x_3^2x_5+\tfrac{3}{32}x_1^2x_2^2x_3^2x_5+\tfrac{1}{12}x_2x_3^3x_5+\tfrac{9}{51200}x_1^{11}x_2x_4x_5\\
&+\tfrac{3}{320}x_1^7x_2^2x_4x_5+\tfrac{1}{8}x_1^3x_2^3x_4x_5+\tfrac{3}{2560}x_1^9x_3x_4x_5+\tfrac{3}{64}x_1^5x_2x_3x_4x_5\\
&+\tfrac{9}{16}x_1x_2^2x_3x_4x_5+\tfrac{1}{8}x_1^3x_3^2x_4x_5+\tfrac{9}{64}x_1^4x_2x_4^2x_5+\tfrac{3}{8}x_2^2x_4^2x_5\\
&+\tfrac{3}{8}x_1^2x_3x_4^2x_5+\tfrac{1}{4}x_1x_4^3x_5+\tfrac{3}{665600}x_1^{13}x_5^2+\tfrac{1}{5120}x_1^9x_2x_5^2\\
&+\tfrac{9}{640}x_1^5x_2^2x_5^2+\tfrac{1}{16}x_1x_2^3x_5^2+\tfrac{1}{1280}x_1^7x_3x_5^2+\tfrac{1}{16}x_1^3x_2x_3x_5^2+\tfrac{1}{8}x_1x_3^2x_5^2\\
&+\tfrac{1}{160}x_1^6x_4x_5^2+\tfrac{3}{16}x_1^2x_2x_4x_5^2+\tfrac{1}{4}x_3x_4x_5^2+\tfrac{1}{12}x_2x_5^3+\tfrac{1}{10240}x_1^9x_2^3x_6\\
&+\tfrac{21}{5120}x_1^5x_2^4x_6+\tfrac{3}{320}x_1x_2^5x_6+\tfrac{3}{640}x_1^7x_2^2x_3x_6+\tfrac{3}{64}x_1^3x_2^3x_3x_6\\
&+\tfrac{3}{5120}x_1^9x_3^2x_6+\tfrac{3}{16}x_1x_2^2x_3^2x_6+\tfrac{1}{16}x_1^3x_3^3x_6+\tfrac{9}{25600}x_1^{10}x_2x_4x_6\\
&+\tfrac{21}{1280}x_1^6x_2^2x_4x_6+\tfrac{3}{16}x_1^2x_2^3x_4x_6+\tfrac{3}{16}x_1^4x_2x_3x_4x_6+\tfrac{3}{8}x_2^2x_3x_4x_6\\
&+\tfrac{9}{1280}x_1^7x_4^2x_6+\tfrac{3}{16}x_1^3x_2x_4^2x_6+\tfrac{3}{4}x_1x_3x_4^2x_6+\tfrac{1}{2}x_4^3x_6\\
&+\tfrac{3}{1280}x_1^8x_2x_5x_6+\tfrac{3}{64}x_1^4x_2^2x_5x_6+\tfrac{1}{8}x_2^3x_5x_6+\tfrac{3}{8}x_1^2x_2x_3x_5x_6\\
&+\tfrac{3}{160}x_1^5x_4x_5x_6+\tfrac{3}{4}x_1x_2x_4x_5x_6+\tfrac{1}{16}x_1^3x_5^2x_6+\tfrac{9}{140800}x_1^{11}x_6^2\\
&+\tfrac{3}{32}x_1^3x_2^2x_6^2+\tfrac{9}{320}x_1^5x_3x_6^2+\tfrac{3}{4}x_2x_4x_6^2+\tfrac{1}{4}x_1x_6^3+\tfrac{1}{256}x_1^3x_2^4x_7\\
&+\tfrac{3}{640}x_1^5x_2^2x_3x_7+\tfrac{1}{16}x_1x_2^3x_3x_7+\tfrac{1}{16}x_1^3x_2x_3^2x_7+\tfrac{3}{64}x_1^4x_2^2x_4x_7\\
&+\tfrac{1}{8}x_2^3x_4x_7+\tfrac{3}{320}x_1^6x_3x_4x_7+\tfrac{3}{8}x_1^2x_2x_3x_4x_7+\tfrac{1}{2}x_3^2x_4x_7\\
&+\tfrac{9}{320}x_1^5x_4^2x_7+\tfrac{3}{4}x_1x_2x_4^2x_7+\tfrac{1}{160}x_1^6x_2x_5x_7+\tfrac{3}{16}x_1^2x_2^2x_5x_7\\
&+\tfrac{1}{16}x_1^4x_3x_5x_7+\tfrac{1}{2}x_2x_3x_5x_7+\tfrac{1}{8}x_1^3x_4x_5x_7+\tfrac{1}{4}x_1x_5^2x_7\\
&+\tfrac{3}{80}x_1^5x_2x_6x_7+\tfrac{3}{8}x_1x_2^2x_6x_7+\tfrac{3}{4}x_1^2x_4x_6x_7+x_5x_6x_7+\tfrac{3}{1120}x_1^7x_7^2\\
&+\tfrac{1}{8}x_1^3x_2x_7^2+\tfrac{1}{2}x_1x_3x_7^2+x_4x_7^2+x_4x_5x_8+x_3x_6x_8\\
&+x_2x_7x_8+\tfrac{1}{2}x_1x_8^2.
\end{aligned}
$$

This combined with the result by C. Hertling (cf. [4]) implies that the PPVF obtained in this section is associated with the real reflection group of type E_8.

Remark 5. At the last part of [5], it was written "if G is an irreducible finite complex reflection group and it is well-generated in the sense of [2], there is a unique flat structure for the discriminant for G." This claim is not correct as was pointed out by P. Lorenzoni (cf. [1]). This shows that a modification is necessary to"the uniqueness" of flat structures associated with not only complex reflection group case but also real reflection group case. This is the reason why we showed Theorem 2. Theorem 2 proves the uniqueness of flat structures for the case of E_8 which is an analogue of the result of C. Hertling on polynomial Frobenius manifolds for E_8 case.

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