INVERSION OF "MODULO p REDUCTION" AND A PARTIAL DESCENT FROM CHARACTERISTIC 0 TO POSITIVE CHARACTERISTIC

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In this paper, we focus on pairs consisting of the affine N-space and multiideals with a positive exponent. We introduce a method "lifting to characteristic 0" which is a kind of the inversion of "modulo p reduction". By making use of it, we prove that Mustaţă-Nakamura's conjecture and some uniform bound of divisors computing log canonical thresholds descend from characteristic 0 to certain classes of pairs in positive characteristic. We also pose a problem whose affirmative answer gives the descent of the statements to the whole set of pairs in positive characteristic.

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1. INTRODUCTION

For studies of singularities in characteristic 0, there are many tools; resolutions of the singularities, Bertini's theorem (generic smoothness), many kinds of vanishing theorems, etc., which are not available for singularities in positive characteristic. So, in order to avoid these difficulties, one way is to reduce our problems in positive characteristic to the analogous problem in characteristic 0. In this paper, we introduce "lifting to characteristic 0" which is a kind of inversion of "modulo p reduction" and show that some statements in characteristic 0 descend to a certain class of pairs in positive characteristic.

In this paper, we focus on a special object that is a pair $(A, \mathfrak{a}^e)$ consisting of the affine space $A = \mathbb{A}_k^N$ over an algebraically closed field k and a "multiideal" $\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}$ on A with the exponent $e = \{e_1, \ldots, e_s\} \subset \mathbb{R}_{>0}$. When we say "a pair", we always mean this object. Sometimes we treat the case $s = 1, e_1 = 1$.

Definition 1.1. Let k be an algebraically closed field of characteristic $p > 0$ and $\mathcal{S}_k$ a class of pairs $(\mathbb{A}_k^N, \mathfrak{a}^e)$. Let $\mathcal{T}_{\mathbb{C}}$ be a class of pairs $(\mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}^e)$. We say that a statement P in $\mathcal{T}_{\mathbb{C}}$ *descends to* $\mathcal{S}_k$, when the following holds:

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$$\text{If } P \text{ holds in } \mathcal{T}_{\mathbb{C}}, \text{ then } P \text{ also holds in } \mathcal{S}_k.$$

Note that we do not discuss whether P actually holds in $\mathcal{T}_{\mathbb{C}}$ but only whether P in $\mathcal{T}_{\mathbb{C}}$ descends to $\mathcal{S}_k$. The statements P we are interested in here are Mustaţă-Nakamura's conjecture, the ACC conjecture for minimal log discrepancies and a uniform bound of divisors computing log canonical thresholds.

Mustaţă-Nakamura's conjecture is as follows:

Conjecture 1.2 ($M_{N,e}$). *Let $A = \mathbb{A}_k^N$ be defined over an algebraically closed field k and let $0 \in A$ be the origin. Given a finite subset $e \subset \mathbb{R}_{>0}$, there is a positive integer $\ell_{N,e}$ (depending on N and e) such that for every multiideal $\mathfrak{a}^e$ on A with the exponent e, there is a prime divisor E that computes* $\mathrm{mld}(0; A, \mathfrak{a}^e)$ *and satisfies $k_E \leq \ell_{N,e}$.*

This conjecture is very useful, for example it guarantees the lower semi continuity of the map

$$\mathbb{A}_k^N \to \mathbb{R}_{\geq 0} \cup \{-\infty\}; \quad x \mapsto \mathrm{mld}(x; \mathbb{A}_k^N, \mathfrak{a}^e),$$

and also stability of log canonicity under deformations, which are not known in positive characteristic.

The ascending chain condition (ACC) conjecture, which is important in birational geometry, is as follows:

Conjecture 1.3 (A_N). *Let A, N and 0 be as above. For every fixed DCC set $J \subset \mathbb{R}_{>0}$, the set*

$$\{\mathrm{mld}(0; A, \mathfrak{a}^e) \mid e \subset J, (A, \mathfrak{a}^e) \textit{ is log canonical at } 0\}$$

satisfies the ascending chain condition (ACC).

Note that, in characteristic 0, $M_{N,e}$ holds for any finite exponent e if and only if A_N holds by [13] and [10].

Another statement we would like to let descend from characteristic 0 to positive characteristic is the following which is proved by Shibata (for the proof, see the appendix of this paper).

Proposition 1.4. *Let $\mathfrak{m}$ be the maximal ideal defining the origin $0 \in \mathbb{A}_{\mathbb{C}}^N$. Then, for a positive integer μ, there is a positive integer $L_{N,\mu}$ depending only on N and μ such that for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ on $\mathbb{A}_{\mathbb{C}}^N$ with $\mathfrak{m}^\mu \subset \mathfrak{a}$, there is some prime divisor E over $\mathbb{A}_{\mathbb{C}}^N$ that computes* $\mathrm{lct}(0; \mathbb{A}_{\mathbb{C}}^N, \mathfrak{a})$ *and satisfies $k_E \leq L_{N,\mu}$.*

Ideally, these three statements would descend from the set of pairs $(\mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}^e)$ to the whole set of pairs $(\mathbb{A}_k^N, \mathfrak{a}^e)$ for a field k of positive characteristic. However, what we obtain in this paper is restricted to some pairs $(\mathbb{A}_k^N, \mathfrak{a}^e)$. Now we give the restriction for pairs.

Definition 1.5. Let k be an algebraically closed field of arbitrary characteristic. For a pair $(\mathbb{A}_k^N, \mathfrak{a}^e)$, we say that the minimal log discrepancy $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e)$ *has a toric approximation* if either

for every $\epsilon > 0$, there exists a toric prime divisor E_ϵ over $\mathbb{A}_k^N$ such that

$$|a(E_\epsilon; \mathbb{A}_k^N, \mathfrak{a}^e) - \mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e)| < \epsilon,$$

or there exists a toric prime divisor over $\mathbb{A}_k^N$ computing $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e)$.

Here, a toric prime divisor means a prime divisor over $\mathbb{A}_k^N$ such that the corresponding valuation is a discrete monomial valuation (see Definition 3.1 for details).

Definition 1.6. Let k be an algebraically closed field of arbitrary characteristic. Let $\mathfrak{a}$ be a coherent ideal sheaf on $\mathbb{A}_k^N$. For a pair $(\mathbb{A}_k^N, \mathfrak{a})$, we say that the log canonical threshold $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$ *has a toric approximation* if for every $\epsilon > 0$ there exists a toric prime divisor E_ϵ over $\mathbb{A}_k^N$ such that

$$\left| \frac{k_{E_\epsilon} + 1}{\mathrm{val}_{E_\epsilon} \mathfrak{a}} - \mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a}) \right| < \epsilon.$$

Definition 1.7. For an algebraically closed field k, we define the sets $\mathcal{M}_k$ and $\mathcal{L}_k$ of pairs as follows:

$$\mathcal{M}_k = \{(\mathbb{A}_k^N, \mathfrak{a}^e) \mid N \geq 1, e \subset \mathbb{R}_{>0}, \mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e) \text{ has a toric approximation}\},$$
$$\mathcal{L}_k = \{(\mathbb{A}_k^N, \mathfrak{a}) \mid N \geq 1, \mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a}^e) \text{ has a toric approximation}\}.$$

The results of this paper are the following:

THEOREM 1.8. *Let k be an algebraically closed field of characteristic $p > 0$. Mustaţă-Nakamura's conjecture for all pairs in characteristic* 0 *descends to* $\mathcal{M}_k$. *And in this case there is a prime divisor over* $\mathbb{A}_k^N$ *computing* $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e)$.

Here, we note that existence of a prime divisor computing the minimal log discrepancy is not known in general in positive characteristic, since existence of resolutions of singularities is not known.

As Mustaţă-Nakamura's conjecture holds for any pair $(\mathbb{A}_{\mathbb{C}}^2, \mathfrak{a}^e)$, we obtain the following:

COROLLARY 1.9. *Let k be an algebraically closed field of characteristic $p > 0$. Mustaţă-Nakamura's conjecture holds for pairs* $(\mathbb{A}_k^2, \mathfrak{a}^e)$ *in* $\mathcal{M}_k$.

Since Mustaţă-Nakamura's conjecture holds for any pair $(\mathbb{A}_{\mathbb{C}}^N, \mathfrak{a}^e)$ such that the $\mathfrak{a}_i$'s are all monomial ideals, we obtain the following:

COROLLARY 1.10. *For an algebraically closed field of characteristic $p > 0$, for the set of pairs* $(\mathbb{A}_k^N, \mathfrak{a}^e)$ *such that* $\mathfrak{a}_i$*'s are all monomial ideals, Mustaţă-Nakamura's conjecture and the ACC conjecture hold.*

About log canonical threshold we obtain the following:

THEOREM 1.11. *Proposition 1.4 holds in $\mathcal{L}_k$ for an algebraically closed field of positive characteristic. I.e., let k be an algebraically closed field of positive characteristic. Let $\mathfrak{m}$ be the maximal ideal defining the origin $0 \in \mathbb{A}_k^N$. Then, for a positive integer μ, there is a positive integer $L_{N,\mu}$ depending only on N and μ such that for every $\mathfrak{m}$-primary ideal $\mathfrak{a}$ such that $\mathfrak{m}^\mu \subset \mathfrak{a}$ and $(\mathbb{A}_k^N, \mathfrak{a}) \in \mathcal{L}_k$, there is some prime divisor E over $\mathbb{A}_k^N$ that computes $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$ and satisfies $k_E \leq L_{N,\mu}$.*

As a corollary, we obtain the following:

COROLLARY 1.12. *Let $\mathfrak{m}$ be as above. Then for a pair $(\mathbb{A}_k^N, \mathfrak{a}) \in \mathcal{L}_k$ with $\mathfrak{m}$-primary ideal $\mathfrak{a}$ there exists a prime divisor over $\mathbb{A}_k^N$ computing $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$.*

Note that in positive characteristic, existence of prime divisors computing the log canonical threshold is not known in general, as existence of resolutions of singularities is not known.

This paper is organized as follows in the second section, we establish the basic notions of lifting of polynomials to characteristic 0. In the third section, we interpret Mustaţă-Nakamura's conjecture in terms of jet schemes. In the fourth section, as applications of lifting to characteristic 0, we prove the theorems and also pose a problem whose affirmative answer would give descent of the statements to the whole set of the pairs of positive characteristic. We also show some applications of the affirmative answer of the problem. In the appendix, the proof of Proposition 1.4 for characteristic 0 is given by Kohsuke Shibata. Our result on log canonical threshold is based on this.

2. PRELIMINARIES ON LIFTING TO CHARACTERISTIC 0

In this section, we do not assume the algebraic closedness of the base fields.

We introduce a method "lifting to characteristic 0" which constructs objects in characteristic 0 from objects in positive characteristic.

Definition 2.1. Let S be an integral domain of characteristic 0, i.e., the canonical homomorphism $\mathbb{Z} \to S$ of rings is injective. For a prime number $p \in \mathbb{Z}$, we denote the canonical projection by $\Phi_p : S \to S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$.

(1) (Basic case) For $\tilde{f} \in S$ and $f \in S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$, we say that $\tilde{f}$ is a *lifting to characteristic* 0 *(or just a lifting)* of f, if $\Phi_p(\tilde{f}) = f$. In this case, we also write $\tilde{f}(\mathrm{mod}\ p) = f$.

(2) Let $\mathbf{f} = \{f_1, f_2, \ldots, f_r\}$ be a set of elements of an integral domain R of characteristic $p > 0$. Let $\tilde{\mathbf{f}} = \{\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_r\}$ be a set of elements of an integral domain $\widetilde{R}$ of characteristic 0.
We say that $\tilde{\mathbf{f}}$ is a *lifting to characteristic* 0 *(or just a lifting)* of $\mathbf{f}$ and write $\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}$, if the following holds:
   (a) there exists a subring $S \subset \widetilde{R}$ and an injective homomorphism $\iota : S \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \hookrightarrow R$ of rings;
   (b) Identify the ring $S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ and its image by the injection ι. The inclusions $\tilde{\mathbf{f}} \subset S$ and $\mathbf{f} \subset S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ hold with the following relations:

$$\tilde{f}_i(\text{mod } p) = f_i \quad \text{for every} \quad i = 1, 2, \ldots, r.$$

In this case, we call S a *subring supporting* the lifting $\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}$.

(3) Let $\mathfrak{a}$ be an ideal of an integral domain R of characteristic $p > 0$. Let $\widetilde{\mathfrak{a}}$ be an ideal of an integral domain $\widetilde{R}$ of characteristic 0. We say that $\widetilde{\mathfrak{a}}$ is a *lifting to characteristic* 0 *(or just a lifting)* of $\mathfrak{a}$ and write

$$\widetilde{\mathfrak{a}}(\text{mod } p) = \mathfrak{a}$$

if there exist systems of generators $\mathbf{f} = \{f_1, \ldots, f_r\}$ of $\mathfrak{a}$ and $\tilde{\mathbf{f}} = \{\tilde{f}_1, \ldots, \tilde{f}_r\}$ of $\widetilde{\mathfrak{a}}$, respectively, such that

$$\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}.$$

If S is a subring supporting the lifting $\tilde{\mathbf{f}}(\text{mod } p) = \mathbf{f}$, we also call S a subring supporting the lifting $\widetilde{\mathfrak{a}}(\text{mod } p) = \mathfrak{a}$.

Remark 2.2. (1) Note that a lifting of an ideal is not unique and the height of the ideal is not preserved by a lifting. We also should note that a lifting of a prime ideal is not necessarily prime.

(2) When we define $\Phi_p : S \to S \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ as the canonical projection, it is obvious that the inclusion $\Phi_p(\widetilde{\mathfrak{a}} \cap S) \supset \mathfrak{a} \cap (S \otimes_{\mathbb{Z}} \mathbb{Z}/(p))$ holds. In general, the equality does not hold.

We have the following result about lifting to characteristic 0.

PROPOSITION 2.3. *Let $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a finite set of elements of a field k of characteristic $p > 0$. Then, there is a subset $\widetilde{\alpha} = \{\widetilde{\alpha}_1, \widetilde{\alpha}_2, \ldots, \widetilde{\alpha}_n\} \subset \mathbb{C}$ such that $\widetilde{\alpha}$ (mod $p) = \alpha$.*

Proof. Considering the subring $\mathbb{Z}/(p)[\alpha_1, \ldots, \alpha_n] \subset k$, we obtain canonical surjections:

$$R := \mathbb{Z}[Y_1, \ldots, Y_n] \xrightarrow{\psi} S := \mathbb{Z}/(p)[Y_1, \ldots, Y_n] \xrightarrow{\varphi} \mathbb{Z}/(p)[\alpha_1, \ldots, \alpha_n]$$

with $Y_i \mapsto \alpha_i$. Let $P := Ker\varphi \subset S$ and $Q := Ker\varphi \circ \psi \subset R$, then these are prime ideals in regular rings. Therefore R_Q and S_P are also regular local rings. Hence we obtain

$\tilde{f}_1, \ldots, \tilde{f}_c, p$ $(\in Q)$ which form a regular system of parameters of R_Q,
$f_1, \ldots, f_c$ $(\in P)$ which form a regular system of parameters of S_P and
$\psi(\tilde{f}_i) = f_i$ for $i = 1, \ldots, c$.

Then, $R_Q/(\tilde{f}_1, \ldots, \tilde{f}_c)R_Q$ is also a regular local ring, in particular it is an integral domain. By considering the homomorphism $\mathbb{Z}_{(p)} \to R_Q$ and the regular sequence $f_1, \ldots, f_c \subset R_Q \otimes_{\mathbb{Z}} \mathbb{Z}/(p) = R_Q \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_{(p)}/(p)\mathbb{Z}_{(p)}$, we obtain that $R_Q/(\tilde{f}_1, \ldots, \tilde{f}_c)R_Q$ is flat over $\mathbb{Z}_{(p)}$ by [11, Corollary, p.177]. In particular, the homomorphism $\mathbb{Z} \to R_Q/(\tilde{f}_1, \ldots, \tilde{f}_c)R_Q$ is injective, which implies the ring $R_Q/(\tilde{f}_1, \ldots, \tilde{f}_c)R_Q$ is of characteristic 0.

As $PS_P = (f_1, \ldots, f_c)S_P$, there exists $h \in S \setminus P$ such that $PS_h = (f_1, \ldots, f_c)S_h$. Take $\tilde{h} \in R \setminus Q$ such that $\psi(\tilde{h}) = h$ and let

$$\Sigma := R_{\tilde{h}}/(\tilde{f}_1, \ldots, \tilde{f}_c)R_{\tilde{h}}.$$

This, as well, is an integral domain and of characteristic 0. Now noting that $S/P = \mathbb{Z}/(p)[\alpha_1, \ldots, \alpha_c]$, we obtain

$$\begin{aligned}\Sigma \otimes_{\mathbb{Z}} \mathbb{Z}/(p) &= \frac{R_{\tilde{h}}/(\tilde{f}_1, \ldots, \tilde{f}_c)R_{\tilde{h}}}{p(R_{\tilde{h}}/(\tilde{f}_1, \ldots, \tilde{f}_c)R_{\tilde{h}})} = S_h/(f_1, \ldots, f_c)S_h \\ &= (S/P)_h \subset \mathbb{Z}/(p)(\alpha_1, \ldots, \alpha_c) \subset k.\end{aligned}$$

By the surjection $\Phi_p : \Sigma \to \Sigma \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ we can take $\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n \in \Sigma$ corresponding to $\alpha_1, \ldots, \alpha_n \in \Sigma \otimes \mathbb{Z}/(p)$.

By the definition of Σ, the field $K_0 := Q(\Sigma)$ of fractions of Σ is a finitely generated field extension of $\mathbb{Q}$. Then, by Baby Lefschetz Principle (see, for example, [14]), there is an isomorphism into the subring:

$$\phi : K_0 \xrightarrow{\sim} \phi(K_0) \subset \mathbb{C}.$$

Then we obtain $\widetilde{\alpha} = \{\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n\} \subset \mathbb{C}$ and $\widetilde{\alpha} \pmod p = \alpha$. As is seen, $\Sigma \subset \mathbb{C}$ is a subring supporting the lifting. $\square$

Proposition 2.4. *Let k be a field of characteristic $p > 0$. Then, the following hold:*

(i) *For a subset $\mathbf{f} = \{f_1, \ldots, f_n\} \subset k[X_1, X_2, \ldots, X_N]$, there exists a set $\widetilde{\mathbf{f}} = \{\widetilde{f}_1, \ldots, \widetilde{f}_n\}$ in $\mathbb{C}[X_1, X_2, \ldots, X_N]$ such that $\widetilde{\mathbf{f}} \pmod p = \mathbf{f}$.*

(ii) *For an ideal $\mathfrak{a} \subset k[X_1, X_2, \ldots, X_N]$, there exists an ideal $\widetilde{\mathfrak{a}} \subset \mathbb{C}[X_1, X_2, \ldots, X_N]$ such that $\widetilde{\mathfrak{a}} \subset (X_1, X_2, \ldots, X_N)$ and*

$$\widetilde{\mathfrak{a}} \pmod p = \mathfrak{a}.$$

In this case, we have in general

$$\mathrm{ht}\widetilde{\mathfrak{a}} \geq \mathrm{ht}\mathfrak{a}.$$

Here, we define the height of the unit ideal of a ring R as $\dim R + 1$.

Proof. For a proof of (i), let $\alpha = \{\alpha_i\}_{i\in I} \subset k$ be a finite set containing all nonzero coefficients of $f_1, \dots, f_n$. By Proposition 2.3 we can take a lifting $\widetilde{\alpha} = \{\widetilde{\alpha}_i\}_{i\in I} \subset \mathbb{C}$ to characteristic 0 of α. When f_j $(j = 1, \dots, n)$ is represented as

$$f_j = \sum_{e\in\mathbb{Z}^N_{\geq 0}} \alpha_{j_e} X^e, \quad \text{where} \quad X^e = X_1^{e_1} \cdots X_N^{e_N},$$

we define

$$\widetilde{f}_j = \sum_{e\in\mathbb{Z}^N_{\geq 0}} \widetilde{\alpha}_{j_e} X^e.$$

Then we have $\widetilde{\mathbf{f}}$ (mod p) $= \mathbf{f}$. When a subring Σ is supporting $\widetilde{\alpha}$ (mod p) $= \alpha_i$, then we can take $\Sigma[X_1, \dots, X_N]$ as a subring supporting the lifting $\widetilde{\mathbf{f}}$ (mod p) $= \mathbf{f}$.

For a proof of (ii), let $\mathbf{f} = \{f_1, \dots, f_n\} \subset k[X_1, X_2, \dots, X_N]$ be a system of generators of $\mathfrak{a}$. Then, by (i) we obtain a subset $\widetilde{\mathbf{f}} = \{\widetilde{f}_1, \dots, \widetilde{f}_n\}$ in $\mathbb{C}[X_1, X_2, \dots, X_N]$ such that $\widetilde{\mathbf{f}}$ (mod p) $= \mathbf{f}$. Then by definition, we have a lifting $\widetilde{\mathfrak{a}}$(mod p) $= \mathfrak{a}$.

Under the notation in the proof of (i), a subring supporting the lifting $\widetilde{\mathfrak{a}}$(mod p) $= \mathfrak{a}$ is $S := \Sigma[X_1, \dots, X_N]$.

Denote the fields of fractions of $\Sigma \otimes \mathbb{Z}/(p)$ and of Σ by Q and $\widetilde{Q}$, respectively. Let $\mathfrak{a}_Q \subset Q[X_1, \dots, X_N]$ be the ideal generated by $f_1, \dots, f_n$. Let $\widetilde{\mathfrak{a}}_\Sigma \subset \Sigma[X_1, \dots, X_N]$ and $\widetilde{\mathfrak{a}}_{\widetilde{Q}} \subset \widetilde{Q}[X_1, \dots, X_N]$ be the ideals generated by $\widetilde{f}_1, \dots, \widetilde{f}_n$ in each ring. Then $\mathfrak{a}$ is the extension of $\mathfrak{a}_Q$ and $\widetilde{\mathfrak{a}}$ is the extension of $\widetilde{\mathfrak{a}}_{\widetilde{Q}}$. As the field extensions $Q \hookrightarrow k$ and $\widetilde{Q} \hookrightarrow \mathbb{C}$ are faithfully flat, the ring extensions $Q[X_1, \dots, X_N] \hookrightarrow k[X_1, \dots, X_N]$ and $\widetilde{Q}[X_1, \dots, X_N] \hookrightarrow \mathbb{C}[X_1, \dots, X_N]$ are also faithfully flat, in particular flat. Then, by [11, Theorem 15.1], we have

$$\mathrm{ht}\mathfrak{a}_Q = \mathrm{ht}(\mathfrak{a}_Q(k[X_1, \dots, X_N])) = \mathrm{ht}\mathfrak{a},$$
$$\mathrm{ht}\widetilde{\mathfrak{a}}_{\widetilde{Q}} = \mathrm{ht}(\widetilde{\mathfrak{a}}_{\widetilde{Q}}(\mathbb{C}[X_1, \dots, X_N])) = \mathrm{ht}\widetilde{\mathfrak{a}}.$$

Therefore, for the proof of the lemma, we have only to show

$$\mathrm{ht}\mathfrak{a}_Q \leq \mathrm{ht}\widetilde{\mathfrak{a}}_{\widetilde{Q}}.$$

Actually let $\pi : Z(\widetilde{\mathfrak{a}}_\Sigma) \to \mathrm{Spec}\ \Sigma$ be the restriction of the flat morphism $\mathrm{Spec}\ \Sigma[X_1 \dots, X_N] \to \mathrm{Spec}\ \Sigma$. Let

$$\Phi'_p : \Sigma[X_1, \dots, X_N] \to \Sigma \otimes \mathbb{Z}/(p)[X_1, \dots, X_N] \hookrightarrow Q[X_1, \dots, X_N]$$

be the composite of the canonical projection and the inclusion. Then, the ideal $(\Phi'_p(\widetilde{\mathfrak{a}}_\Sigma)) \subset Q[X_1, \dots, X_N]$ generated by the image $\Phi'_p(\widetilde{\mathfrak{a}}_\Sigma)$ defines a special fiber of π over $\mathrm{Spec}\ Q$. On the other hand, $\widetilde{\mathfrak{a}}_{\widetilde{Q}}$ defines the generic fiber of π over $\mathrm{Spec}\ \widetilde{Q}$. Then, by [11, Theorem 15.1], we have

$$\dim Z(\widetilde{\mathfrak{a}}_{\widetilde{Q}}) \leq \dim Z(\Phi'_p(\widetilde{\mathfrak{a}}_\Sigma)).$$

Here, as Spec $\Sigma[X_1, \ldots, X_N] \to$ Spec Σ is flat, we have the inequality of codimensions, i.e.,

$$\mathrm{ht}\widetilde{\mathfrak{a}}_{\widetilde{Q}} \geq \mathrm{ht}(\Phi'_p(\widetilde{\mathfrak{a}}_\Sigma)),$$

where the right hand side is not less than $\mathrm{ht}\mathfrak{a}_Q$ because of the inclusion $(\Phi'_p(\widetilde{\mathfrak{a}}_\Sigma)) \supset \mathfrak{a}_Q$. (Note that the inclusion is not an equality in general.) Now we obtain

$$\mathrm{ht}\widetilde{\mathfrak{a}}_{\widetilde{Q}} \geq \mathrm{ht}\mathfrak{a}_Q,$$

which completes the proof of the equality

$$\mathrm{ht}\widetilde{\mathfrak{a}} \geq \mathrm{ht}\mathfrak{a}. \quad \square$$

3. THE CONJECTURES AND THEIR RELATIONS

In this section, we study a pair $(A, \mathfrak{a}^e)$ consisting of a nonsingular affine variety A of dimension N defined over an algebraically closed field of arbitrary characteristic and a "multiideal" $\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}$ on A with the exponent $e = \{e_1, \ldots, e_s\} \subset \mathbb{R}_{>0}$. First we give a basic definition.

Definition 3.1. We say that E is a *prime divisor over* a variety A, if there is a birational morphism $A' \to A$ with normal A' such that E is a prime divisor on A'.

We say that a prime divisor E over $A = \mathbb{A}_k^N$ is a *toric prime divisor*, if there is a vector $w = (w_1, \ldots, w_N) \in \mathbb{Z}_{\geq 0}^N$ such that

$$\mathrm{val}_E(f) = \min\{\langle w, u\rangle \mid X^u \in f\},$$

where $X^u \in f$ means that the monomial X^u appears in f with a nonzero coefficient and $\langle w, u\rangle = \sum_i w_i u_i$.

Note that in this case, E is a divisor on a toric variety A' birational to $\mathbb{A}_k^N$ such that E corresponds to the one dimensional cone $\mathbb{R}_{\geq 0} w$ in the defining fan of A'.

Definition 3.2. The log discrepancy of a pair $(A, \mathfrak{a}^e)$ at a prime divisor E over A is defined as

$$a(E; A, \mathfrak{a}^e) := k_E - \sum_{i=1}^{s} e_i \mathrm{val}_E \mathfrak{a}_i + 1,$$

where k_E is the coefficient of the relative canonical divisor $K_{\overline{A}/A}$ at E.

Here $\varphi : \overline{A} \to A$ is a birational morphism with normal $\overline{A}$ such that E appears on $\overline{A}$.

Definition 3.3. Let $(A, \mathfrak{a}^e)$ be a pair and $x \in A$ a closed point. Then the *minimal log discrepancy* is defined as follows:

(1) When $\dim A \geq 2$,

$$\mathrm{mld}(x; A, \mathfrak{a}^e) = \inf\{a(E; A, \mathfrak{a}^e) \mid E: \text{ prime divisor with the center } x\}.$$

(2) When $\dim A = 1$, define $\mathrm{mld}(x; A, \mathfrak{a}^e)$ by the same definitions as above if the right hand side of the above definition is non-negative and otherwise define $\mathrm{mld}(x; A, \mathfrak{a}^e) = -\infty$.

Definition 3.4. Let A, N, and e as above and $x \in A$ a closed point. We say that a prime divisor E over A with the center $\{x\}$ computes $\mathrm{mld}(x; A, \mathfrak{a}^e)$, if

$$\begin{cases} a(E; A, \mathfrak{a}^e) = \mathrm{mld}(x; A, \mathfrak{a}^e), & \text{when } \mathrm{mld}(x; A, \mathfrak{a}^e) \geq 0 \\ a(E; A, \mathfrak{a}^e) < 0, & \text{when } \mathrm{mld}(x; A, \mathfrak{a}^e) = -\infty \end{cases}$$

Remark 3.5. If there is a log resolution of $(A, \mathfrak{a}^e)$ in a neighborhood of x, or if e is a set of rational numbers, then a prime divisor computing $\mathrm{mld}(x; A, \mathfrak{a}^e)$ exists. Otherwise, the existence of such a divisor is not known in general.

Now we are going to interpret the conjecture $(M_{N,e})$ in terms of jet schemes. For that we introduce the notion of jet schemes briefly. For basic properties on jet schemes we refer to [3, 7, 8].

Definition 3.6. Let X be a scheme of finite type over a field k and $k' \supset k$ a field extension. For $m \in \mathbb{Z}_{\geq 0}$ a k-morphism $\mathrm{Spec}\ k'[t]/(t^{m+1}) \to X$ is called an *m-jet* of X and a k-morphism $\mathrm{Spec}\ k'[[t]] \to X$ is called an *arc* of X.

Let X_m be the *space of m-jets* or the *m-jet scheme* of X. It is well known that X_m has a scheme structure of finite type over k. There exists the projective limit

$$X_\infty := \varprojlim_m X_m$$

and it is called the *space of arcs* or the *space of ∞-jet* of X. Every point $\mathrm{Spec}\ k' \to X_\infty$ corresponds to an arc $\mathrm{Spec}\ k'[[t]] \to X$.

Definition 3.7. Denote the canonical truncation morphisms induced from $k[[t]] \to k[t]/(t^{m+1})$ and $k[t]/(t^{m+1}) \to k$ by $\psi_m : X_\infty \to X_m$ and $\pi_m : X_m \to X$, respectively. In particular, we denote the morphism $\psi_0 = \pi_\infty : X_\infty \to X$ by π. We also denote the canonical truncation morphism $X_{m'} \to X_m$ $(m' > m)$ induced from $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$ by $\psi_{m',m}$.

Definition 3.8. We define the subset *contact locus* in the space of arcs as follows:

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \{\gamma \in X_\infty \mid \mathrm{ord}_\gamma(\mathfrak{a}) := \mathrm{ord}_t \gamma^*(\mathfrak{a}) \geq m\},$$

where $\gamma^* : \mathcal{O}_X \to k'[[t]]$ is the homomorphism of rings corresponding to the arc $\gamma : \mathrm{Spec}\ k'[[t]] \to X$.

By this definition, we can see that

$$\mathrm{Cont}^{\geq m}(\mathfrak{a}) = \psi_{m-1}^{-1}(Z(\mathfrak{a})_{m-1}),$$

where $Z(\mathfrak{a})$ is the closed subscheme defined by the ideal $\mathfrak{a}$ on X.

Example 3.9. Let $Z \subset \mathbb{A}^N =: A$ be a closed subscheme of affine N-space $\mathbb{A}^N = \mathrm{Spec}\ k[X_1, \ldots, X_N]$ defined over a field k with the defining ideal $\mathfrak{a} \subset k[X_1, \ldots, X_N]$. Assume $\mathfrak{a}$ is generated by $f_1, \ldots, f_r \in k[X_1, \ldots, X_N]$. We define polynomials

$$F_i^{(j)} \in k[X_\ell^{(q)} \mid 1 \leq \ell \leq N, 0 \leq q \leq j]$$

so that

$$f_i\left(\sum_{q\geq 0} X_1^{(q)} t^q, \sum_{q\geq 0} X_2^{(q)} t^q, \ldots, \sum_{q\geq 0} X_N^{(q)} t^q\right) = F_i^{(0)} + F_i^{(1)} t + \cdots + F_i^{(j)} t^j + \cdots.$$

Then the m-jet scheme Z_m is defined in $(\mathbb{A}^N)_m \simeq \mathbb{A}^{N(m+1)}$ by the ideal of $k[X_\ell^{(q)} \mid 1 \leq \ell \leq N, 0 \leq q \leq m]$ generated by

$$F_i^{(j)} \quad (i = 1, \ldots, r, j = 0, 1, \ldots, m).$$

Here, we note that if all coefficients of f_i's are in a subring $\Sigma \subset k$, then all coefficients of $F_i^{(j)}$'s are also in Σ. (This fact will be used in the proof of Lemma 3.13.)

The fiber $\pi_m^{-1}(0)$ over the origin $0 \in \mathbb{A}^N = A$ by the truncation morphism $\pi_m : A_m = \mathrm{Spec}\ k[X_\ell^{(q)} \mid 1 \leq \ell \leq N, 0 \leq q \leq m] \to A$ is defined by

$$X_1^{(0)}, \ldots, X_N^{(0)}.$$

Proposition 3.10 ([3] for characteristic 0, [8] for arbitrary characteristic). *Let A be a nonsingular variety defined over an algebraically closed field of arbitrary characteristic, $0 \in A$ a closed point, $e = \{e_1, \ldots, e_s\}$ a finite subset of $\mathbb{R}_{>0}$ and $\mathfrak{a}^e$ as in the beginning of this section. Then, we obtain the following formula:*

$$(1) \qquad \mathrm{mld}(0; A, \mathfrak{a}^e)$$
$$= \inf_{m\in\mathbb{Z}_{\geq 0}^s}\left\{\mathrm{codim}(\mathrm{Cont}^{\geq m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{\geq m_s}(\mathfrak{a}_s) \cap \pi^{-1}(0), A_\infty) - \sum_{i=1}^s e_i m_i\right\}.$$

Definition 3.11. Under the same notation as above, define the function $s_m(0; A, \mathfrak{a}^e)$ on $m = (m_1, m_2, \ldots, m_s) \in \mathbb{Z}_{\geq 0}^s$ as follows:

$$s_m(0; A, \mathfrak{a}^e)$$
$$= \mathrm{codim}\left(\mathrm{Cont}^{\geq m_1}(\mathfrak{a}_1) \cap \cdots \cap \mathrm{Cont}^{\geq m_s}(\mathfrak{a}_s) \cap \pi^{-1}(0), A_\infty\right) - \sum_{i=1}^s e_i m_i$$

Remark 3.12. For a given $m=(m_1,\dots,m_s)\in\mathbb{Z}^s_{\geq 0}$, denote $m'_i=m_i-1$ and assume that $m'_1=\max\{m'_i\}$ to simplify the notation. By the definition of $\operatorname{codim}(-,A_\infty)$ ([1, Definition 3.2]), we have the following:

$$
\begin{aligned}
(2)\quad & s_m(0;A,\mathfrak{a}^e)\\
&= Nm_1-\dim\Big(Z(\mathfrak{a}_1)_{m'_1}\cap\psi^{-1}_{m'_1,m'_2}(Z(\mathfrak{a}_2)_{m'_2})\cap\cdots\cap\psi^{-1}_{m'_1,m'_s}(Z(\mathfrak{a}_s)_{m'_s})\\
&\qquad\qquad\qquad\cap\pi^{-1}_{m'_1}(0)\Big)-\sum_{i=1}^{s}e_im_i,
\end{aligned}
$$

where, we define $Z(\mathfrak{a}_i)_{-1}=A$ as a convention. Later on, we will use this expression of $s_m(0;A,\mathfrak{a}^e)$.

If $A=\mathbb{A}^N_k$, $0\in\mathbb{A}^N_k$ is the origin and $\mathfrak{a}_i$ are generated by f_{ij} $(j=1,\dots,r_i)$ for each $i=1,\dots,s$, then by Example 3.9,

$$Z(\mathfrak{a}_1)_{m'_1}\cap\psi^{-1}_{m'_1,m'_2}(Z(\mathfrak{a}_2)_{m'_2})\cap\cdots\cap\psi^{-1}_{m'_1,m'_s}(Z(\mathfrak{a}_s)_{m'_s})\cap\pi^{-1}_{m'_1}(0)$$

is defined in $\mathbb{A}^{Nm_1}=(\mathbb{A}^N)_{m'_1}$ by the ideal generated by

$$F^{(q)}_{1,\ell}\ (\ell=1,\dots,r_1,q=0,\dots,m'_1),\dots,F^{(q)}_{s,\ell}\ (\ell=1,\dots,r_s,q=0,\dots,m'_s)\ \text{and}\ X^{(0)}_1,\dots,X^{(0)}_N.$$

Note that these are elements of $k[X^{(q)}_i\mid 1\leq i\leq N, 0\leq q\leq m'_1]$.

LEMMA 3.13. *Let k be an algebraically closed field of characteristic $p>0$. Let $\mathfrak{a}=\{\mathfrak{a}_1,\dots,\mathfrak{a}_s\}$ be a set of ideals of $k[X_1,\dots,X_N]$ such that $\mathfrak{a}_i\subset(X_1,\dots,X_N)$ for every $i=1,\dots,s$ and $e=\{e_1,\dots,e_s\}\subset\mathbb{R}_{>0}$. Let $\widetilde{\mathfrak{a}}=\{\widetilde{\mathfrak{a}}_1,\dots,\widetilde{\mathfrak{a}}_s\}$ be a lifting to characteristic 0 of $\mathfrak{a}$.*

Then for every $m\in\mathbb{Z}^s_{\geq 0}$ it follows

$$s_m(0;\mathbb{A}^N_k,\mathfrak{a}^e)\leq s_m(0;\mathbb{A}^N_{\mathbb{C}},\widetilde{\mathfrak{a}}^e).$$

Proof. Let $\mathfrak{a}_i$ be generated by f_{ij} $(j=1,\dots,r_i)$ and $\widetilde{\mathfrak{a}}_i$ be generated by g_{ij} $(j=1,\dots,r_i)$ such that

$$\{g_{ij}\mid i,j\}\ (\mathrm{mod}\ p)=\{f_{ij}\mid i,j\}. \tag{3}$$

Take $m\in\mathbb{Z}^s_{\geq 0}$ and let $m_1=\max\{m_i\}$ to simplify the notation. We use the notation in Remark 3.12. Then, for every $m'_i:=m_i-1$, the m'_i-th jet scheme $Z(\mathfrak{a}_i)_{m'_i}$ is defined by

$$F^{(q)}_{i,j}\ (j=1,\dots,r_1,q=1,\dots,m'_i),$$

where $F^{(q)}_{i,j}$ is defined from $f_{i,j}$ as in Example 3.9. In the same way, $Z(\widetilde{\mathfrak{a}}_i)_{m'_i}$ is defined by

$$G^{(q)}_{i,j}\ (j=1,\dots,r_1,q=1,\dots,m'_i),$$

where $G_{i,j}^{(q)}$ is defined from $g_{i,j}$ as well. Now by the definition of $F_{i,j}^{(q)}$ and $G_{i,j}^{(q)}$ and (3), we have

$$\{G_{i,j}^{(q)} \mid i,j,q\} \pmod p = \{F_{i,j}^{(q)} \mid i,j,q\}.$$

In the formula (2), we denote the defining ideal of

$$Z(\mathfrak{a}_1)_{m'_1} \cap \psi^{-1}_{m'_1,m'_2}(Z(\mathfrak{a}_2)_{m'_2}) \cap \cdots \cap \psi^{-1}_{m'_1,m'_s}(Z(\mathfrak{a}_s)_{m'_s}) \cap \pi^{-1}_{m'_1}(0)$$

in $(\mathbb{A}_k^N)_{m'_1}$ by $I_m(\mathfrak{a})$ and the defining ideal of

$$Z(\widetilde{\mathfrak{a}}_1)_{m'_1} \cap \psi^{-1}_{m'_1,m'_2}(Z(\widetilde{\mathfrak{a}}_2)_{m'_2}) \cap \cdots \cap \psi^{-1}_{m'_1,m'_s}(Z(\widetilde{\mathfrak{a}}_s)_{m'_s}) \cap \pi^{-1}_{m'_1}(0)$$

in $(\mathbb{A}_{\mathbb{C}}^N)_{m'_1}$ by $I_m(\widetilde{\mathfrak{a}})$. Since, $I_m(\mathfrak{a})$ is generated by

$$F_{1,j}^{(q)}\ (j=1,\dots,r_1, q=1,\dots,m'_1),\dots,F_{s,j}^{(q)}\ (j=1,\dots,r_s, q=1,\dots,m'_s),$$
$$X_1^{(0)},\dots,X_N^{(0)}$$

and $I_m(\mathfrak{a})$ is generated by

$$G_{1,j}^{(q)}\ (j=1,\dots,r_1, q=1,\dots,m'_1),\dots,G_{s,j}^{(q)}\ (j=1,\dots,r_s, q=1,\dots,m'_s),$$
$$X_1^{(0)},\dots,X_N^{(0)}.$$

We obtain that

$$I_m(\widetilde{\mathfrak{a}}) \pmod p = I_m(\mathfrak{a}).$$

Then by Proposition 2.4, (ii), we have

$$\mathrm{ht} I_m(\mathfrak{a}) \leq \mathrm{ht} I_m(\widetilde{\mathfrak{a}}). \tag{4}$$

On the other hand, (2) gives

$$s_m(0;\mathbb{A}_k^N,\mathfrak{a}^e) = Nm_1 - \dim Z(I_m(\mathfrak{a})) - \sum_{i=1}^{s} e_i m_i,$$

$$s_m(0;\mathbb{A}_{\mathbb{C}}^N,\mathfrak{a}^e) = Nm_1 - \dim Z(I_m(\widetilde{\mathfrak{a}})) - \sum_{i=1}^{s} e_i m_i,$$

Therefore, by (4) we obtain

$$s_m(0;\mathbb{A}_k^N,\mathfrak{a}^e) \leq s_m(0;\mathbb{A}_{\mathbb{C}}^N,\widetilde{\mathfrak{a}}^e). \quad \square$$

Definition 3.14. Let A, x, $\mathfrak{a}$ and e be as in the beginning of this section. We say that $m \in \mathbb{Z}^s_{\geq 0}$ (or $s_m(x;A,\mathfrak{a}^e)$) computes $\mathrm{mld}(x;A,\mathfrak{a}^e)$ if

$$\begin{cases} s_m(x;A,\mathfrak{a}^e) = \mathrm{mld}(x;A,\mathfrak{a}^e), & \text{when } \mathrm{mld}(x;A,\mathfrak{a}^e) \geq 0 \\ s_m(x;A,\mathfrak{a}^e) < 0, & \text{when } \mathrm{mld}(x;A,\mathfrak{a}^e) = -\infty \end{cases}$$

Now, we state the conjectures about the minimal log discrepancies.

CONJECTURE 3.15 ($D_{N,e}$). *Let $A = \mathbb{A}_k^N$ be defined over an algebraically closed field k of arbitrary characteristic, 0 the origin and e a finite subset of $\mathbb{R}_{>0}$. Then, there is a positive integer $\ell'_{N,e}$ (depending only on N and e) such that for every multiideal $\mathfrak{a}^e$ on A, there is $m \in \mathbb{Z}^s_{\geq 0}$ $(s = \#e)$ computing* $\mathrm{mld}(0; A, \mathfrak{a}^e)$ *satisfying $|m| := \sum m_i \leq \ell'_{N,e}$.*

We obtain the equivalence of the conjectures. The equivalence on $\mathfrak{a}^e$ of a special type is proved in [9] and its proof works essentially for the general case, but for the reader's convenience we exhibit the proof here.

PROPOSITION 3.16. *Let A be an N-dimensional nonsingular affine variety defined over an algebraically closed field k of arbitrary characteristic, $0 \in A$ a closed point and e a finite subset of $\mathbb{R}_{>0}$. Then, we obtain the following:*

(i) *For any multiideal $\mathfrak{a}^e$ with the exponent e on A, there exists a prime divisor E computing* $\mathrm{mld}(0; A, \mathfrak{a}^e)$ *if and only if there exists $m \in \mathbb{Z}^s_{\geq 0}$ computing* $\mathrm{mld}(0; A, \mathfrak{a}^e)$*;*

(ii) *In the set of pairs $(A, \mathfrak{a}^e)$ consisting of nonsingular affine varietie A and multiideals ideal $\mathfrak{a}^e$, $M_{N,e}$ holds if and only if $D_{N,e}$ holds.*

Proof. We will prove (i) and (ii) together in the following way:

• Step 1. Assume that a prime divisor E over A computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$, then there exists $m \in \mathbb{Z}^s_{\geq 0}$ computing $\mathrm{mld}(0; A, \mathfrak{a}^e)$.

If moreover $k_E \leq \ell_{N,e}$ holds, then we can take m computing mld such that

$$|m| \leq (\ell_{N,e} + 1 + \max\{e_i\}) / \min\{e_i\}.$$

• Step 2. Assume that m computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$, then there exists a prime divisor E over A computing $\mathrm{mld}(0; A, \mathfrak{a}^e)$.

If moreover m satisfies $|m| \leq \ell'_{N,e}$, then the E obtained above satisfies

$$k_E \leq N - 1 + \ell'_{N,e} \cdot \max\{e_i\}.$$

[Proof of Step 1.] For the divisor E in the statement of Step 1, let $m_i := \mathrm{val}_E \mathfrak{a}_i$. Then,

$$k_E + 1 - \sum_{i=1}^{s} e_i m_i = \mathrm{mld}(0; A, \mathfrak{a}^e) \quad \text{or} \quad < 0.$$

Consider the maximal divisorial set $C_A(\mathrm{val}_E) \subset A_\infty$ that is the closure of the irreducible subset

$$\{\gamma \in A_\infty \mid \mathrm{ord}_\gamma = \mathrm{val}_E\}$$

(see [1] or [6] for details). Then, as $m_i = \mathrm{val}_E \mathfrak{a}_i$ for every i, $C_A(\mathrm{val}_E)$ is contained in the closed subset

$$C(m) := \mathrm{Cont}^{\geq m_1}(\mathfrak{a}_1) \cap \mathrm{Cont}^{\geq m_2}(\mathfrak{a}_2) \cap \cdots \cap \mathrm{Cont}^{\geq m_s}(\mathfrak{a}_s) \cap \pi^{-1}(0)$$

of A_∞. Then,

$$\operatorname{codim}(C_A(\mathrm{val}_E), A_\infty) \geq \operatorname{codim}(C(m), A_\infty).$$

As $s_m(0; A, \mathfrak{a}^e) = \operatorname{codim}(C(m), A_\infty) - \sum e_i m_i$ and $\operatorname{codim}(C_A(\mathrm{val}_E), A_\infty) = k_E + 1$ (it is proved in [1] for characteristic 0 and in [8] for positive characteristic) we obtain

$$s_m(0; A, \mathfrak{a}^e) \leq k_E + 1 - \sum e_i m_i, \tag{5}$$

As E computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$, the right hand side of the above inequality is either $\mathrm{mld}(0; A, \mathfrak{a}^e)$ or negative. This shows that m computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$.

Now we assume $k_E \leq \ell_{N,e}$. First we consider the case $\mathrm{mld}(0; A, \mathfrak{a}^e) \geq 0$. Then by (5), we obtain

$$\ell_{N,e} + 1 - \mathrm{mld}(0; A, \mathfrak{a}^e) \geq k_E + 1 - \mathrm{mld}(0; A, \mathfrak{a}^e) = \sum e_i m_i \geq \min_i \{e_i\} |m|,$$

and it shows

$$|m| \leq \frac{\ell_{N,e} + 1}{\min_i \{e_i\}}.$$

Next we consider the case $\mathrm{mld}(0; A, \mathfrak{a}^e) = -\infty$. For a prime divisor E computing $\mathrm{mld}(0; A, \mathfrak{a}^e)$ we have

$$k_E + 1 - \sum_i e_i \mathrm{val}_E(\mathfrak{a}_i) < 0.$$

But even for some choices of $n = (n_1, \ldots, n_s) \in \mathbb{Z}_{\geq 0}$ with $n_i \leq \mathrm{val}_E(\mathfrak{a}_i)$ $(\forall i)$ the following inequality may hold:

$$k_E + 1 - \sum_i e_i n_i < 0.$$

Considering this fact, we define

$$\mathcal{D} := \left\{ E \;\middle|\; \begin{array}{l} \text{prime divisor computing } \mathrm{mld}(0; A, \mathfrak{a}^e) \\ \text{and satisfying } k_E \leq \ell_{N,e} \end{array} \right\}$$

and let $n \in \mathbb{Z}^s_{\geq 0}$ attain the minimal value:

$$\min \left\{ |n| \;\middle|\; \begin{array}{l} E \in \mathcal{D},\ n_i \leq \mathrm{val}_E(\mathfrak{a}_i)\ (\forall i) \\ k_E + 1 - \sum_i e_i n_i < 0 \end{array} \right\}.$$

Let $E \in \mathcal{D}$ give the minimal value $|n|$. Here, we may assume that $n_1 \geq 1$. Now, define $n^* = (n_1^*, \ldots, n_s^*)$, so that $n_1^* = n_1 - 1$ and $n_i^* = n_i$ $(i \neq 1)$.

Claim. $s_{n^*}(0; A, \mathfrak{a}^e) \geq 0$.

Considering the definition of $s_{n^*}(0; A, \mathfrak{a}^e)$, we have an irreducible component $C \subset C(n^*)$ such that

$$\operatorname{codim}(C, A_\infty) = \operatorname{codim}(C(n^*), A_\infty).$$

It is known that a finite codimensional irreducible component of the intersection of finite number of contact loci is a maximal divisorial set [8, Corollary 3.16], therefore $C = C_A(q \cdot \mathrm{val}_{E'})$ for some $q \in \mathbb{N}$ and a prime divisor E' over A with the center $\{0\}$. Therefore, as $\mathrm{codim}(C_A(q \cdot \mathrm{val}_{E'}), A_\infty) = q(k_{E'}+1)$ (it is proved in [1] for characteristic 0 and in [8] for positive characteristic) we obtain

$$q(k_{E'}+1) - \sum e_i n_i^* = s_{n^*}(0; A, \mathfrak{a}^e). \tag{6}$$

Assume that this value is negative. Then, by (6) and $q \cdot \mathrm{val}_{E'}(\mathfrak{a}_i) \geq n_i^*$, we have

$$q\left(k_{E'} + 1 - \sum e_i \mathrm{val}_{E'}(\mathfrak{a}_i)\right) \leq q(k_{E'}+1) - \sum e_i n_i^* = s_{n^*}(0; A, \mathfrak{a}^e) < 0,$$

which implies that E' computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$. On the other hand, by $n_i^* \leq n_i$ for every i, we have $C_A(\mathrm{val}_E) \subset C(n) \subset C(n^*)$, which yields

$$\begin{aligned}\ell_{N,e} + 1 \geq k_E + 1 = \mathrm{codim}(C_A(\mathrm{val}_E), A_\infty) &\geq \mathrm{codim}(C(n^*), A_\infty)\\ = \mathrm{codim}(C_A(q \cdot \mathrm{val}_{E'}), A_\infty) &= q(k_{E'}+1) \geq k_{E'} + 1.\end{aligned}$$

Thus E' computes $\mathrm{mld}(0; A, \mathfrak{a}^e) = -\infty$, $k_{E'} \leq \ell_{N,e}$, $k_{E'} + 1 - \sum_i e_i n_i^* < 0$ and $|n^*| < |n|$, which is a contradiction to the minimality of $|n|$.

Now, as we showed the claim $s_{n^*}(0; A, \mathfrak{a}^e) \geq 0$ in the above discussion, we obtain $q(k_{E'}+1) - \sum e_i n_i^* \geq 0$. Here, as we see above $k_E + 1 \geq q(k_{E'}+1)$, we obtain

$$k_E + 1 - \sum e_i n_i + e_1 = k_E + 1 - \sum e_i n_i^* \geq 0,$$

which yields

$$l_{N,e} + 1 + e_1 \geq \sum e_i n_i \geq \min\{e_i\}|n|.$$

Now we obtain that $n \in \mathbb{Z}^s_{\geq 0}$ computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$ and satisfies $|n| \leq (\ell_{N,e} + 1 + e_1)/\min\{e_i\}$. Hence, to conclude the proof of Step 1, we can take

$$\ell'_{N,e} := (\ell_{N,e} + 1 + \max\{e_i\})/\min\{e_i\}.$$

[Proof of Step 2.] Assume that m computes the minimal log discrepancy. Take a maximal divisorial set $C_A(q \cdot \mathrm{val}_E) \subset C(m)$ which gives $\mathrm{codim}(C(m), A_\infty)$. Then

$$\mathrm{codim}(C_A(q \cdot \mathrm{val}_E), A_\infty) = s_m(0; A, \mathfrak{a}^e) + \sum e_i m_i.$$

Here, we have $\mathrm{codim}(C_A(q \cdot \mathrm{val}_E), A_\infty) = q(k_E + 1)$. On the other hand, since $C_A(q \cdot \mathrm{val}_E) \subset C(m)$, it follows $q \cdot \mathrm{val}_E(\mathfrak{a}_i) \geq m_i$. Then, we obtain

$$q(k_E+1) - \sum e_i(q \cdot \mathrm{val}_E(\mathfrak{a}_i)) \leq q(k_E+1) - \sum e_i m_i = s_m(0; A, \mathfrak{a}^e), \tag{7}$$

which implies that E computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$ in both cases $\mathrm{mld} \geq 0$ and $\mathrm{mld} = -\infty$.

Now, assume $m \in \mathbb{Z}^s_{\geq 0}$ computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$ and satisfies $|m| \leq \ell'_{N,e}$. Take a prime divisor E over A as above.

First we consider the case $\mathrm{mld}(0; A, \mathfrak{a}^e) \geq 0$. Then (7) implies $q = 1$ and the following:

(8)
$$k_E + 1 - \sum e_i \mathrm{val}_E(\mathfrak{a}_i) \leq k_E + 1 - \sum e_i m_i = s_m(0; A, \mathfrak{a}^e) = \mathrm{mld}(0; A, \mathfrak{a}^e) \leq N,$$

which yields that E computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$ and

$$k_E \leq N - 1 + \ell'_{N,e} \cdot \max\{e_i\}.$$

Next we consider the case $\mathrm{mld}(0; A, \mathfrak{a}^e) = -\infty$, then by (7), we have

$$q \cdot (k_E + 1 - \sum e_i \mathrm{val}_E(\mathfrak{a}_i)) \leq q(k_E + 1) - \sum e_i m_i = s_m(0; A, \mathfrak{a}^e) < 0,$$

which also yields that E computes $\mathrm{mld}(0; A, \mathfrak{a}^e)$ and

$$k_E \leq \ell'_{N,e} \cdot \max\{e_i\} - 1.$$

In the both, cases we obtain a bound

$$k_E \leq N - 1 + \ell'_{N,e} \cdot \max\{e_i\}. \quad \square$$

Up to now we discussed about minimal log discrepancies of a pair. In the rest of this section, we discuss about log canonical threshold which also evaluates “goodness” of the singularities of a pair.

To show the statements on log canonical threshold, we recall the definition of the log canonical threshold $\mathrm{lct}(A, \mathfrak{a})$ for a nonsingular variety A and a coherent ideal sheaf $\mathfrak{a}$ on A, as well as some basic properties of this invariant.

Definition 3.17. Let A be a nonsingular variety defined over an algebraically closed field and $\mathfrak{a}$ a nonzero coherent ideal sheaf on A. The *log canonical threshold* $\mathrm{lct}(x; A, \mathfrak{a})$ at a closed point $x \in A$ is defined as follows:

$$\mathrm{lct}(x; A, \mathfrak{a}) := \sup\{r \in \mathbb{R}_{\geq 0} \mid (A, \mathfrak{a}^r) \text{ is log canonical at } x\}.$$

Note that

$$\mathrm{lct}(x; A, \mathfrak{a}) = \inf\left\{\frac{k_E + 1}{\mathrm{val}_E(\mathfrak{a})} \mid E \text{ prime divisor over } A \text{ with } c_A(E) \ni x\right\}.$$

Let E be a prime divisor over A with the center containing x. We say that E *computes* $\mathrm{lct}(x; A, \mathfrak{a})$ if

$$\frac{k_E + 1}{\mathrm{val}_E(\mathfrak{a})} = \mathrm{lct}(x; A, \mathfrak{a})$$

holds.

Remark 3.18. (1) The log canonical threshold is defined for any klt variety, but in this paper we only work at the origin 0 of the affine space $A = \mathbb{A}_k^N$.

(2) If $(A, \mathfrak{a})$ has a log resolution $\varphi : \widetilde{A} \to A$, then

$$\mathrm{lct}(0; A, \mathfrak{a}) = \min \left\{ \frac{k_E + 1}{\mathrm{val}_E(\mathfrak{a})} \mid E \text{ prime divisor on } \widetilde{A} \text{ with } \varphi(E) \ni 0 \right\}.$$

Therefore, in this case there is a prime divisor over A computing $\mathrm{lct}(0; A, \mathfrak{a})$. But, up to this moment, for the base field of characteristic $p > 0$ the existence of a prime divisor computing log canonical threshold is not known in general.

Proposition 3.19 ([12] for characteristic 0 and [15] for positive characteristic). *Let $A = \mathbb{A}_k^N$, 0 and $\mathfrak{a}$ as above, then there is an interpretation of log canonical threshold by jet schemes as follows:*

$$\mathrm{lct}(0; A, \mathfrak{a}) = \inf_{m \in \mathbb{N}} \frac{\mathrm{codim}_0(Z(\mathfrak{a})_m, A_m)}{m+1},$$

where $\mathrm{codim}_0(Z(\mathfrak{a})_m, A_m) = \mathrm{codim}((Z(\mathfrak{a}) \cap U)_m, A_m)$ for sufficiently small neighborhood U of 0. Note that the right hand side is constant for sufficiently small U and in particular if $\sqrt{\mathfrak{a}} = \mathfrak{m}$ for the maximal ideal $\mathfrak{m}$ defining 0 in A, we have

$$\mathrm{codim}_0(Z(\mathfrak{a})_m, A_m) = \mathrm{codim}(Z(\mathfrak{a})_m, A_m) = \mathrm{codim}(\mathrm{Cont}^{\geq m+1}(\mathfrak{a}), A_\infty).$$

Definition 3.20. We define a function $z_m(0; A, \mathfrak{a})$ on $m \in \mathbb{N}$ as follows:

$$z_m(0; A, \mathfrak{a}) := \frac{\mathrm{codim}_0(Z(\mathfrak{a})_m, A_m)}{m+1}.$$

Then, as in Proposition 3.19,

$$\mathrm{lct}(0; A, \mathfrak{a}) = \inf_{m \in \mathbb{N}} z_m(0; A, \mathfrak{a}).$$

Lemma 3.21. *Let k be an algebraically closed field of arbitrary characteristic and $A = \mathbb{A}_k^N$ the affine space of dimension $N \geq 1$ defined over k. Let $\mathfrak{m}$ be the defining ideal of the origin $0 \in A$ and μ a positive integer. Define the set $\mathfrak{I}_\mu$ of ideals as follows:*

$$\mathfrak{I}_\mu := \{\mathfrak{a} \text{ ideal on } A \mid \mathfrak{m}^\mu \subset \mathfrak{a} \subset \mathfrak{m}\}.$$

Then the following are equivalent:

(i) *There is a positive integer $L_{N,\mu}$ (depending only on N and μ) such that for every ideal $\mathfrak{a} \in \mathfrak{I}_\mu$, $\mathrm{lct}(0; A, \mathfrak{a})$ is computed by a prime divisor E over A with $k_E \leq L_{N,\mu}$.*

(ii) *There is a positive integer $L'_{N,\mu}$ (depending only on N and μ) such that for every ideal $\mathfrak{a} \in \mathfrak{I}_\mu$, $z_m(0; A, \mathfrak{a})$ computes* $\mathrm{lct}(0; A, \mathfrak{a})$ *(i.e.,* $\mathrm{lct}(0; A, \mathfrak{a}) = z_m(0; A, \mathfrak{a})$ *holds) for some $m \leq L'_{N,\mu}$.*

Proof. First we prove (i) $\Rightarrow$ (ii). By the assumption (i), for an ideal $\mathfrak{a} \in \mathfrak{I}_\mu$ there is a prime divisor E over A with $k_E \leq L_{N,\mu}$ such that E computes $\mathrm{lct}(0; A, \mathfrak{a})$. Let $\mathrm{val}_E \mathfrak{a} = m + 1$, then we have

$$\mathrm{lct}(0; A, \mathfrak{a}) = \frac{k_E + 1}{m + 1} \leq \frac{L_{N,\mu} + 1}{m + 1}. \tag{9}$$

Here, as $\mathrm{val}_E \mathfrak{a} = m + 1$, we have

$$C_A(\mathrm{val}_E) \subset \mathrm{Cont}^{\geq m+1}(\mathfrak{a}).$$

Then considering the codimensions in A_∞ of the both hand sides, we obtain

$$k_E + 1 \geq \mathrm{codim}(\mathrm{Cont}^{\geq m+1}(\mathfrak{a}), A_\infty).$$

Therefore,

$$\frac{k_E + 1}{m + 1} \geq z_m(0; A, \mathfrak{a}).$$

Here, as the left hand side is $\mathrm{lct}(0; A, \mathfrak{a})$ and the right hand side is not less than $\mathrm{lct}(0; A, \mathfrak{a})$, the equality holds:

$$\mathrm{lct}(0; A, \mathfrak{a}) = z_m(0; A, \mathfrak{a})$$

and by (9) the value m satisfies

$$m = \frac{k_E + 1}{\mathrm{lct}(0; A, \mathfrak{a})} - 1 \leq \frac{L_{N,\mu} + 1}{\mathrm{lct}(0; A, \mathfrak{a})} - 1 \leq \frac{L_{N,\mu} + 1}{\mathrm{lct}(0; A, \mathfrak{m}^\mu)} - 1,$$

where note that the last term does not depend on the choice of $\mathfrak{a}$.

Next we prove (ii) $\Rightarrow$ (i). Take an ideal $\mathfrak{a} \in \mathfrak{I}_\mu$. As we assume (ii), there exists $m \leq L'_{N,\mu}$ such that

$$z_m(0; A, \mathfrak{a}) = \mathrm{lct}(0; A, \mathfrak{a}) \tag{10}$$

Take an irreducible component

$$C_A(q \cdot \mathrm{val}_E) \subset \mathrm{Cont}^{\geq m+1}(\mathfrak{a}) \tag{11}$$

such that $\mathrm{codim}(C_A(q \cdot \mathrm{val}_E), A_\infty) = \mathrm{codim}(\mathrm{Cont}^{\geq m+1}(\mathfrak{a}), A_\infty)$. Then we have

$$\frac{q(k_E + 1)}{m + 1} = z_m(0; A, \mathfrak{a}).$$

On the other hand, by the inclusion (11), it follows $q \cdot \mathrm{val}_E \mathfrak{a} \geq m + 1$. Then,

$$\mathrm{lct}(0; A, \mathfrak{a}) \leq \frac{k_E + 1}{\mathrm{val}_E \mathfrak{a}} = \frac{q(k_E + 1)}{q \cdot \mathrm{val}_E \mathfrak{a}} \leq \frac{q(k_E + 1)}{m + 1} = \mathrm{lct}(0; A, \mathfrak{a}).$$

Hence, E computes $\operatorname{lct}(0; A, \mathfrak{a})$ and $k_E + 1 \leq (m+1) \cdot \operatorname{lct}(0; A, \mathfrak{a}) \leq (L'_{N,\mu} + 1)N$. Thus we obtain a uniform bound

$$k_E \leq (L'_{N,\mu} + 1)N - 1. \quad \square$$

4. PROOFS OF THE MAIN RESULTS AND A PROBLEM

In this section, we will prove the main theorems stated in the first section and pose a problem whose affirmative answer would give descent of the statements to the whole set of pairs $(\mathbb{A}^N_k, \mathfrak{a}^e)$ of positive characteristic.

LEMMA 4.1. *Let k be an algebraically closed field of characteristic $p > 0$, $(\mathbb{A}^N_k, \mathfrak{a}^e)$ a pair and E a toric prime divisor over $\mathbb{A}^N_k$ with the center at* 0. *Then, there are a prime divisor $\widetilde{E}$ over $\mathbb{A}^N_{\mathbb{C}}$ with the center at* 0 *and multiideal $\widetilde{\mathfrak{a}}^e$ such that*

$$k_E = k_{\widetilde{E}}, \quad \operatorname{val}_E \mathfrak{a}_i = \operatorname{val}_{\widetilde{E}} \widetilde{\mathfrak{a}}_i, \quad \textit{and} \quad \widetilde{\mathfrak{a}}_i (\operatorname{mod} p) = \mathfrak{a}_i \quad \textit{for all } i,$$

where $\widetilde{\mathfrak{a}}^e = \widetilde{\mathfrak{a}}_1^{e_1} \cdots \widetilde{\mathfrak{a}}_s^{e_s}$ and $\mathfrak{a}^e = \mathfrak{a}_1^{e_1} \cdots \mathfrak{a}_s^{e_s}$.

Proof. Let the toric prime divisor E over $\mathbb{A}^N_k$ correspond to the vector $w = (w_1, \ldots, w_N) \in \mathbb{Z}_{>0}$. Let $\widetilde{E}$ be the toric prime divisor over $\mathbb{A}^N_{\mathbb{C}}$ corresponding to w. Then,

$$k_E + 1 = \langle w, \mathbf{1} \rangle = k_{\widetilde{E}} + 1,$$

where $\mathbf{1} = (1, 1, \ldots, 1)$.

For our lemma, it is sufficient to construct a lifting $\widetilde{\mathfrak{a}}_i$ of $\mathfrak{a}_i$ $(i = 1, \ldots, s)$ such that

$$\operatorname{val}_E \mathfrak{a}_i = \operatorname{val}_{\widetilde{E}} \widetilde{\mathfrak{a}}_i.$$

First, we will show that for a polynomial $f \in k[X_1, \ldots, X_N]$ there exists $\tilde{f} \in \mathbb{C}[X_1, \ldots, X_N]$ such that

$$\tilde{f} (\operatorname{mod} p) = f \quad \text{and} \quad \operatorname{val}_E f = \operatorname{val}_{\widetilde{E}} \tilde{f}.$$

Express

$$f = \sum_{u \in \mathbb{Z}^N_{\geq 0}} \alpha_u X^u.$$

Let $d := \operatorname{val}_E f = \min_{\alpha_u \neq 0} \langle w, u \rangle$. By Proposition 2.4, there exists a lifting $\overline{f} = \sum_{u \in \mathbb{Z}_{\geq 0}} \widetilde{\alpha}_u X^u \in \mathbb{C}[X_1, \ldots, X_N]$ of f. Divide $\overline{f}$ into the sum of two polynomials

$$\overline{f} = \tilde{f}_{<d} + \tilde{f}_{\geq d},$$

where $\tilde{f}_{<d} = \sum_{\langle w,u \rangle < d} \widetilde{\alpha}_u X^u$ and $\tilde{f}_{\geq d} = \sum_{\langle w,u \rangle \geq d} \widetilde{\alpha}_u X^u$. Note that $\tilde{f}_{<d} (\operatorname{mod} p) = 0$, as $\overline{f} (\operatorname{mod} p) = f$ and $\operatorname{val}_E f = d$. Define $\tilde{f} := \tilde{f}_{\geq d}$. Then, we obtain that

$$\tilde{f} (\operatorname{mod} p) = f \text{ and}$$

$$\mathrm{val}_{\widetilde{E}}\tilde{f} = \min_{X^u \in \tilde{f}} \langle w, u \rangle = d = \mathrm{val}_E f.$$

Next we apply the discussion above to the ideals. For each generator f_{ij} of $\mathfrak{a}_i$ we obtain a lifting $\tilde{f}_{ij}$ to characteristic 0 by the above procedure. Here, note that we can take them in a common subring supporting the liftings. Let them generate an ideal $\widetilde{\mathfrak{a}}_i \subset \widetilde{R}_0$. Then we obtain that

$$\widetilde{\mathfrak{a}}_i (\mathrm{mod}\ p) = \mathfrak{a}_i.$$

If f_{i1} computes $\mathrm{val}_E \mathfrak{a}_i$ then $\tilde{f}_{i1}$ computes $\mathrm{val}_{\widetilde{E}} \widetilde{\mathfrak{a}}_i$ by the definition of $\widetilde{\mathfrak{a}}_i$ and we obtain

$$\mathrm{val}_E \mathfrak{a}_i = \mathrm{val}_E f_{i1} = \mathrm{val}_{\widetilde{E}} \tilde{f}_{i1} = \mathrm{val}_{\widetilde{E}} \widetilde{\mathfrak{a}}_i,$$

which completes the proof. $\square$

Proof of Theorem 1.8. Let k be an algebraically closed field of characteristic $p > 0$. We divide our discussion into two cases:

(a) $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e) \geq 0$ and

(b) $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e) = -\infty$.

In case (a), let $\delta := \mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}^e) \geq 0$. Since $(\mathbb{A}_k^N, \mathfrak{a}^e) \in \mathcal{M}_k$, for every $\epsilon > 0$ there exists a toric prime divisor E_ϵ over $\mathbb{A}_k^N$ with the center at $\{0\}$ such that

$$a(E_\epsilon; \mathbb{A}_k^N, \mathfrak{a}^e) - \delta < \epsilon.$$

Then, by Lemma 4.1, there exist a multiideal $\widetilde{\mathfrak{a}}_\epsilon$ in $\mathbb{C}[X_1, \ldots, X_N]$ and a prime divisor $\widetilde{E}_\epsilon$ over $\mathbb{A}_{\mathbb{C}}^N$ such that

$$a(E_\epsilon; \mathbb{A}_k^N, \mathfrak{a}^e) = a(\widetilde{E}_\epsilon; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e). \tag{12}$$

Here, as we assume that $(M_{N,e})$ holds for the base field $\mathbb{C}$, $(D_{N,e})$ also holds for $\mathbb{C}$ by Proposition 3.16. Therefore, there exists $m_\epsilon \in \mathbb{Z}_{\geq 0}^s$ such that $|m_\epsilon| \leq \ell'_{N,e}$ and

$$s_{m_\epsilon}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e) = \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e). \tag{13}$$

On the other hand, by Lemma 3.13, we have:

$$s_{m_\epsilon}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e) \geq s_{m_\epsilon}(0; \mathbb{A}_k^N, \mathfrak{a}^e). \tag{14}$$

By composing (12), (13) and (14), we have

$$\delta + \epsilon > a(E_\epsilon; \mathbb{A}_k^N, \mathfrak{a}^e) = a(\widetilde{E}_\epsilon; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e) \geq \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e)$$
$$= s_{m_\epsilon}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_\epsilon^e) \geq s_{m_\epsilon}(0; \mathbb{A}_k^N, \mathfrak{a}^e) \geq \delta.$$

Therefore we have a sequence $\{m_\epsilon\}_\epsilon$ such that

$$|m_\epsilon| < \ell'_{N,e} \text{ and } |s_{m_\epsilon}(0; \mathbb{A}_k^N, \mathfrak{a}^e) - \delta| < \epsilon.$$

Since the set $\{m \in \mathbb{Z}^s_{\geq 0} \mid |m| \leq \ell'_{N,e}\}$ is finite, we have that $m_\epsilon = m$ (constant) for sufficiently small ϵ, and

$$s_m(0; \mathbb{A}^N_k, \mathfrak{a}^e) = \delta.$$

Therefore, we obtain that this m satisfies $|m| \leq \ell'_{N,e}$ and computes $\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e)$. By Proposition 3.16, this proves $(M_{N,e})$ holds in $\mathcal{M}_k$ and also there exists a prime divisor over $\mathbb{A}^N_k$ that computes $\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e)$.

In case (b) $\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e) = -\infty$, by the definition of $\mathcal{M}_k$, there exists a toric prime divisor E over $\mathbb{A}^N_k$ with the center at $\{0\}$ such that $a(E; \mathbb{A}^N_k, \mathfrak{a}^e) < 0$.

Then, by Lemma 4.1, there exist a multiideal $\widetilde{\mathfrak{a}}$ of $\mathbb{C}[X_1, \dots, X_N]$ and a prime divisor $\widetilde{E}$ over $\mathbb{A}^N_{\mathbb{C}}$ such that

$$a(E; \mathbb{A}^N_k, \mathfrak{a}^e) = a(\widetilde{E}; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e) < 0,$$

which implies $\mathrm{mld}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e) = -\infty$. Since we assume that $(M_{N,e})$ holds over the base field $\mathbb{C}$, $(D_{N,e})$ also holds by Proposition 3.16 and therefore there exists m such that $|m| < \ell'_{N,e}$ and $s_m(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e) < 0$. Here, as is seen in Lemma 3.13, we obtain

$$s_m(0; \mathbb{A}^N_k, \mathfrak{a}^e) \leq s_m(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e) < 0.$$

Thus we obtain m such that $|m| < \ell'_{N,e}$ and computes $\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e) = -\infty$. So, in this case $(M_{N,e})$ holds for k. $\square$

Proof of Corollary 1.10. Let $\mathfrak{a}$ be a set of monomial ideals $\{\mathfrak{a}_1, \dots, \mathfrak{a}_s\}$ of $k[X_1, \dots, X_N]$. Define a set of monomial ideals $\widetilde{\mathfrak{a}} = \{\widetilde{\mathfrak{a}}_1, \dots, \widetilde{\mathfrak{a}}_s\}$ of $\mathbb{C}[X_1, \dots, X_N]$ such that $\widetilde{\mathfrak{a}}_i$ is generated by the same monomials as $\mathfrak{a}_i$ for every i. Then,

$$\widetilde{\mathfrak{a}}_i (\mathrm{mod}\ p) = \mathfrak{a}_i.$$

It is known that both $(\mathbb{A}^N_k, \mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_s)$ and $(\mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_1 \cdot \widetilde{\mathfrak{a}}_2 \cdots \widetilde{\mathfrak{a}}_s)$ have toric log resolutions [5]. As the Newton polygon of the ideal $\mathfrak{a}_1 \cdot \mathfrak{a}_2 \cdots \mathfrak{a}_s$ is the same as that of $\widetilde{\mathfrak{a}}_1 \cdot \widetilde{\mathfrak{a}}_2 \cdots \widetilde{\mathfrak{a}}_s$, we can take a common fan corresponding to log resolutions. In general, on a log resolution there is a prime divisor computing the minimal log discrepancy. In our case all divisors on the log resolutions are toric divisors and they satisfy

$$\mathrm{val}_E(\mathfrak{a}_i) = \mathrm{val}_{\widetilde{E}}(\widetilde{\mathfrak{a}}_i) \quad \text{and} \quad k_E = k_{\widetilde{E}},$$

where E and $\widetilde{E}$ the toric divisors over $\mathbb{A}^N_k$ and $\mathbb{A}^N_{\mathbb{C}}$, respectively, corresponding to a common vector $w \in \mathbb{Z}^N_{\geq 0}$. Then, we can see that there is a toric prime divisor computing $\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e)$ and

$$\mathrm{mld}(0; \mathbb{A}^N_k, \mathfrak{a}^e) = \mathrm{mld}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e). \tag{15}$$

Hence, by Theorem 1.8, Mustaţă-Nakamura's conjecture descends to the set of the pairs $(\mathbb{A}^N_k, \mathfrak{a}^e)$ with monomial ideals $\mathfrak{a}_i$. On the other hand, the conjecture is known to hold for the pairs $(\mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}^e)$ with monomial ideals $\widetilde{\mathfrak{a}}_i$ in characteristic 0. Therefore, for the whole set of pairs $(\mathbb{A}^N_k, \mathfrak{a}^e)$ with monomial ideals $\mathfrak{a}_i$ Mustaţă-Nakamura's conjecture holds. As the conjecture is equivalent to ACC conjecture in characteristic 0, the equality (15) gives ACC in positive characteristic.

Proof of Theorem 1.11. The statement is proved over the base field of characteristic 0 by Shibata (see the proof of Theorem A.3 in the appendix). Let k be an algebraically closed field of characteristic $p > 0$ and $\mathfrak{a} \subset k[X_0, \dots, X_N]$ a nonzero proper ideal and assume $\mathfrak{m}^\mu \subset \mathfrak{a} \subset \mathfrak{m}$, where $\mathfrak{m} = (X_0, \dots, X_N)$. Take a system of generators $\mathbf{f} = \{f_1, \dots, f_u\}$ of $\mathfrak{a}$. We may assume that $\mathbf{f}$ contains all monomials of degree μ in variables $X_1, \dots, X_N$. Let $c := \mathrm{lct}(0; \mathbb{A}^N_k, \mathfrak{a})$.

Now by the assumption that $(\mathbb{A}^N_k, \mathfrak{a}) \in \mathcal{L}_k$, for every $\epsilon > 0$ there exists a toric prime divisor E_ϵ over $\mathbb{A}^N_k$ with the center 0 such that

$$\left| \frac{k_{E_\epsilon} + 1}{\mathrm{val}_{E_\epsilon} \mathfrak{a}} - c \right| < \epsilon. \tag{16}$$

Then, by Lemma 4.1, we obtain a prime divisor $\widetilde{E}_\epsilon$ over $\mathbb{A}^N_{\mathbb{C}}$ and a lifting $\widetilde{\mathfrak{a}}_\epsilon$ of $\mathfrak{a}$ such that

$$k_{E_\epsilon} = k_{\widetilde{E}_\epsilon} \quad \text{and} \quad \mathrm{val}_{E_\epsilon}(\mathfrak{a}) = \mathrm{val}_{\widetilde{E}_\epsilon}(\widetilde{\mathfrak{a}}_\epsilon).$$

On the other hand, by the definition of $\widetilde{\mathfrak{a}}_\epsilon$ in Lemma 4.1 we obtain

$$\widetilde{\mathfrak{m}}^\mu \subset \widetilde{\mathfrak{a}}_\epsilon. \tag{17}$$

Now by (16), we obtain

$$c + \epsilon > \frac{k_{E_\epsilon} + 1}{\mathrm{val}_{E_\epsilon} \mathfrak{a}} = \frac{k_{\widetilde{E}_\epsilon} + 1}{\mathrm{val}_{\widetilde{E}_\epsilon} \widetilde{\mathfrak{a}}_\epsilon} \geq \mathrm{lct}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_\epsilon). \tag{18}$$

Here, because of (17), we can apply Proposition 1.4 to $(\mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_\epsilon)$ and by Lemma 3.21 we obtain that the last term of the above inequalities is expressed as follows:

$$\mathrm{lct}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_\epsilon) = z_{m_\epsilon}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_\epsilon) \tag{19}$$

for some $m_\epsilon \leq L'_{N,\mu}$ where $L'_{N,\mu}$ depends only on N and μ. By the same discussion as about s_m in Lemma 3.13, we obtain

$$z_{m_\epsilon}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}_\epsilon) \geq z_{m_\epsilon}(0; \mathbb{A}^N_k, \mathfrak{a}) \geq c. \tag{20}$$

By composing (16), (18), (19) and (20) we obtain

$$c + \epsilon \geq z_{m_\epsilon}(0; \mathbb{A}^N_k, \mathfrak{a}) \geq c. \tag{21}$$

As m_ϵ is a positive integer bounded by $L'_{N,\mu}$, we have $m_\epsilon = m$ (constant) for sufficiently small ϵ and the equality

$$z_m(0; \mathbb{A}_k^N, \mathfrak{a}) = c.$$

This shows the existence of m computing $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$ and also the uniform bound $L'_{N,\mu}$ of m. By Lemma 3.21 this yields the bound $L_{N,\mu}$ of k_E for some prime divisor E computing the log canonical threshold. □

Proof of Corollary 1.12. Take a pair $(\mathbb{A}_k^N, \mathfrak{a}) \in \mathcal{L}_k$ and assume that $\mathfrak{a}$ is $\mathfrak{m}$-primary. Then, there exists $\mu \in \mathbb{N}$ such that

$$\mathfrak{m}^\mu \subset \mathfrak{a}.$$

Hence, by Theorem 1.11, we obtain a prime divisor over $\mathbb{A}_k^N$ computing $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$. □

Now we consider a generalization of the main theorems to the descent to the whole set of pairs in positive characteristic. First we pose a problem which asks if Lemma 4.1 holds for every prime divisor E over $\mathbb{A}_k^N$ with the center 0.

Problem 4.2. Let k be an algebraically closed field of characteristic $p > 0$, $(\mathbb{A}_k^N, \mathfrak{a}^e)$ a pair and E a prime divisor over $\mathbb{A}_k^N$ with the center 0. Then, are there a prime divisor $\widetilde{E}$ over $\mathbb{A}_{\mathbb{C}}^N$ with the center 0 and multiideal $\widetilde{\mathfrak{a}}^e$ such that

$$k_E = k_{\widetilde{E}}, \quad \mathrm{val}_E \mathfrak{a}_i = \mathrm{val}_{\widetilde{E}} \widetilde{\mathfrak{a}}_i, \quad \text{and} \quad \widetilde{\mathfrak{a}}_i (\mathrm{mod}\ p) = \mathfrak{a}_i \quad \text{for all } i\ ?$$

Concerning with this problem, we obtain the following:

PROPOSITION 4.3. *If Problem 4.2 is affirmatively solved, then Mustaţă-Nakamura's conjecture and ACC conjecture over $\mathbb{C}$ descend to the whole set of pairs $(\mathbb{A}_k^N, \mathfrak{a}^e)$ defined over an algebraically closed field k of positive characteristic.*

Proof. About Mustaţă-Nakamura's conjecture, the proof is the same as the proof of Theorem 1.8 and we have only to replace the word "toric prime divisor" by "prime divisor". To show the descent of ACC conjecture, we assume that ACC conjecture holds over $\mathbb{C}$. Let

$$\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(1)}^{e_{(1)}}) < \mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(2)}^{e_{(2)}}) < \cdots < \mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(i)}^{e_{(i)}}) < \cdots \tag{22}$$

be an ascending chain with $e_{(i)} \subset J$ and multiideals $\mathfrak{a}_{(i)}$ of $k[X_1, \ldots, X_N]$ for every $i \in \mathbb{N}$, where J is a DCC set.

As we assume that the conjecture (A_N) holds over the base field $\mathbb{C}$, by [10, Theorem 4.6], the conjecture $(M_{N,e})$ holds over the base field $\mathbb{C}$ for every finite set $e \subset \mathbb{R}_{>0}$. Then, by the above discussion, for each $i \in \mathbb{N}$ there exists a prime divisor E_i computing $\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(i)}^{e_{(i)}})$. Therefore, if Problem 4.2 is affirmatively solved, there exist $\widetilde{\mathfrak{a}}_{(i)}$ and $\widetilde{E}_i$ over the base field $\mathbb{C}$ such that $\widetilde{\mathfrak{a}}_{(i)}(\bmod p) = \mathfrak{a}_{(i)}$ and

$$\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(i)}^{e(i)}) = a(E_i; \mathbb{A}_k^N, \mathfrak{a}_{(i)}^{e(i)}) = a(\widetilde{E}_i; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_{(i)}^{e(i)}) \geq \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_{(i)}^{e(i)}).$$

By Lemma 3.13, we obtain

$$\mathrm{mld}(0; \mathbb{A}_k^N, \mathfrak{a}_{(i)}^{e(i)}) = \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_{(i)}^{e(i)}).$$

Now, the ascending chain (22) coincides with the ascending chain on $\mathbb{A}_{\mathbb{C}}$:

$$\mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_1^{e(1)}) < \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_2^{e(2)}) < \cdots < \mathrm{mld}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}_i^{e(i)}) < \cdots .$$

By the assumption (A_N) over $\mathbb{C}$, this sequence stops at a finite stage, which yields (A_N) over k. □

PROPOSITION 4.4. *Assume that Problem 4.2 is affirmatively solved and moreover under the notation in Problem 4.2 assume that* $\widetilde{\mathfrak{m}}^\mu \subset \widetilde{\mathfrak{a}}$ *when a pair* $(\mathbb{A}_k^N, \mathfrak{a})$ *satisfies* $\mathfrak{m}^\mu \subset \mathfrak{a}$. *Then, for every pair* $(\mathbb{A}_k^N, \mathfrak{a})$ *with an* $\mathfrak{m}$*-primary ideal* $\mathfrak{a}$, *there exists a prime divisor over* $\mathbb{A}_k^N$ *computing* $\mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$.

Proof. We have only to replace the word "toric prime divisor" in the proof of Theorem 1.11 by "prime divisor" and the observe that the proof of Corollary 1.12 works. □

The following is provided by Mircea Mustaţă.

PROPOSITION 4.5 (M. Mustaţă). *Assume Problem 4.2 is affirmatively solved. Let* $\mathfrak{a}$ *be an ideal in* $k[X_1, \ldots, X_N]$, *where* k *is an algebraically closed field of characteristic* $p > 0$. *Then either there is a divisor* E *with center* 0 *that computes* $c = \mathrm{lct}(0; \mathbb{A}_k^N, \mathfrak{a})$ *or* c *lies in the set* T_{N-1} *of log canonical thresholds in characteristic 0, for ideals in* $k[X_1, \ldots, X_{N-1}]$. *In any case, we get that the log canonical threshold of every ideal in positive characteristic is a rational number.*

Proof. In order to prove the assertion, we use the fact that if c does not lie in T_{N-1}, then c is not an accumulation point of log canonical thresholds in T_N. Choose $\epsilon > 0$ such that there is no element of T_N in $(c, c+\epsilon)$. Take a prime divisor E over $\mathbb{A}_k^N$ with the center 0 such that

$$\frac{k_E + 1}{\mathrm{val}_E(\mathfrak{a})} < c + \epsilon,$$

and then by choosing a prime divisor $\widetilde{E}$ over an ideal $\widetilde{\mathfrak{a}}$ in $\mathbb{C}[X_1, \ldots, X_N]$ satisfying the conditions in the Problem 4.2. Then

$$c + \epsilon > \frac{k_E + 1}{\mathrm{val}_E(\mathfrak{a})} = \frac{k_{\widetilde{E}} + 1}{\mathrm{val}_{\widetilde{E}}(\widetilde{\mathfrak{a}})} \geq \mathrm{lct}(0; \mathbb{A}_{\mathbb{C}}^N, \widetilde{\mathfrak{a}}) \geq c.$$

Here, the last inequality follows from

$$z_m(0, \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}) \geq z_m(0, \mathbb{A}^N_k, \mathfrak{a}) \quad \text{for every} \quad m \in \mathbb{N},$$

which is proved in the same way as the inequality of s_m's in Lemma 3.13. By the choice of ϵ we obtain

$$\operatorname{lct}(0; \mathbb{A}_{\mathbb{C}}, \mathfrak{a}) = c.$$

This already shows that c is a rational number. Using again the fact that c is not in T_{N-1}, it follows that we have a prime divisor F over $\mathbb{A}^N_{\mathbb{C}}$ with center 0 that computes $\operatorname{lct}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}})$. In particular, there is $m \in \mathbb{Z}_{>0}$ such that $\operatorname{lct}(0; \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}})$ is computed by $z_m(0, \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}})$. Using the inequality $z_m(0, \mathbb{A}^N_{\mathbb{C}}, \widetilde{\mathfrak{a}}) \geq z_m(0, \mathbb{A}^N_k, \mathfrak{a})$ and the fact that the two log canonical thresholds are equal, we see that in fact $\operatorname{lct}(0, \mathbb{A}^N_k, \mathfrak{a})$ is computed by $z_m(0, \mathbb{A}^N_k, \mathfrak{a})$, and we obtain our assertion. □

APPENDIX

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A. PROOF OF PROPOSITION 1.4

In this section, we prove that there is a bound of minimal discrepancies of prime divisors computing log canonical thresholds of ideals which contain a fixed power of the maximal ideal. We use the theory of generic limit of ideals to show this statement. Throughout this section, we always assume that the base field k is an algebraically closed field of characteristic 0.

We recall the definition of the generic limit of ideals and some of its properties. The reader is referred to [2] for details.

Let $R = k[[X_1, \dots, X_N]]$ with the maximal ideal $\mathfrak{m}$ and for every field extension L/k let $R_L = L[[X_1, \dots, X_N]]$ and $\mathfrak{m}_L = \mathfrak{m}R_L$. For every d, let $\mathcal{H}_d$ be the Hilbert scheme parametrizing the ideals in R containing $\mathfrak{m}^d$. We have a natural surjective map $\tau_d : \mathcal{H}_d \to \mathcal{H}_{d-1}$ and by generic flatness, we can cover $\mathcal{H}_d$ by disjoint locally closed subsets such that the restriction of τ_d to each of these subsets is a morphism.

We now fix a positive integer s. Consider the product $(\mathcal{H}_d)^s$ and the map $t_d : (\mathcal{H}_d)^s \to (\mathcal{H}_{d-1})^s$ that is given by τ_d on each component.

We define a generic limit of the collection of s-tuples $\{(\mathfrak{a}_1^{(i)}, \ldots, \mathfrak{a}_s^{(i)})\}_{i\in I}$ of ideals in R indexed by an infinite set I, as follows. We can construct irreducible closed subsets $Z_d \subset (\mathcal{H}_d)^s$ such that

(1) t_d induces a dominant rational map $Z_d \dashrightarrow Z_{d-1}$,
(2) $I_d := \{i \in I | (\mathfrak{a}_1^{(i)} + \mathfrak{m}^d, \ldots, \mathfrak{a}_s^{(i)} + \mathfrak{m}^d) \in Z_d\}$ is infinite,
(3) the set of points in $(\mathcal{H}_d)^s$ indexed by I_d is dense in Z_d.

Let $K := \cup_{d\geq 1} k(Z_d)$. For each d, the morphism $\mathrm{Spec} K \to Z_d$ corresponds to an s-tuple $(\widetilde{\mathfrak{a}}_1^{(d)}, \ldots, \widetilde{\mathfrak{a}}_s^{(d)})$. Furthermore there exist ideals $\mathfrak{a}_j$ in R_K for $1 \leq j \leq s$ such that $\widetilde{\mathfrak{a}}_j^{(d)} = \mathfrak{a}_j + \mathfrak{m}_K^d$. We call the s-tuple $(\mathfrak{a}_1, \ldots, \mathfrak{a}_s)$ the generic limit of the collection of s-tuples $\{(\mathfrak{a}_1^{(i)}, \ldots, \mathfrak{a}_s^{(i)})\}_{i\in I}$.

Lemma A.1 (Lemma 3.1 and Lemma 3.2 in [2]). *With the above notation, suppose that* $\mathfrak{b} \subset \mathfrak{a}_j^{(i)} \subset \mathfrak{m}$ *for some ideal* $\mathfrak{b} \subset R$ *and every* i, j. *Then* $\mathfrak{b}R_K \subset \mathfrak{a}_j \subset \mathfrak{m}_K$ *for every* j.

We put $X = \mathrm{Spec}\, R$ and $\widetilde{X} = \mathrm{Spec} R_K$. We denote by x and $\tilde{x}$ the closed points of X and $\widetilde{X}$, respectively.

Proposition A.2 (Proposition 3.3 in [2], Proposition 3.1 in [13]). *With the above notation, suppose that* $\mathfrak{a}_j^{(i)}$ *are proper nonzero ideals. We write* $(\mathfrak{a}^{(i)})^e := \prod_{j=1}^s (\mathfrak{a}_j^{(i)})^{e_j}$ *and* $\mathfrak{a}^e := \prod_{j=1}^s \mathfrak{a}_j^{e_j}$ *for a positive exponent* $e = \{e_1, \ldots, e_s\}$. *Suppose that* $\mathfrak{a}_j \neq 0$ *for all* j. *Then the following hold:*

(i) *For every positive exponent* $e = \{e_1, \ldots, e_s\}$, *then* $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \mathfrak{a}^e)$ *is a limit point of the set* $\{\mathrm{lct}(x; X, (\mathfrak{a}^{(i)})^e) | i \geq 1, i \in I\}$.
(ii) *If* E *is a prime divisor over* $\widetilde{X}$ *computing* $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \mathfrak{a}^e)$ *for some positive exponent* $e = \{e_1, \ldots, e_s\}$ *and having center at the closed point* $\widetilde{x}$, *then for every* $d \gg 0$ *there is an infinite subset* $I'_d \subset I$ *depending on* E *and* e *and satisfying the following property: for every* $i \in I'_d$ *there is a divisor* E_i *over* X *that computes* $\mathrm{lct}(x; X, \prod_{j=1}^s (\mathfrak{a}_j^{(i)} + \mathfrak{m}^d)^{e_j})$, *which is equal to* $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \prod_{j=1}^s (\mathfrak{a}_j + \mathfrak{m}_K^d)^{e_j})$ *and such that* $\mathrm{val}_E(\mathfrak{m}_K) = \mathrm{val}_{E_i}(\mathfrak{m})$, $k_E = k_{E_i}$ *and* $\mathrm{val}_E(\mathfrak{a}_j + \mathfrak{m}_K^d) = \mathrm{val}_{E_i}(\mathfrak{a}_j^{(i)} + \mathfrak{m}^d)$ *for* $1 \leq j \leq s$.

Theorem A.3. *Let* $R = k[[X_1, \ldots, X_N]]$ *with the maximal ideal* $\mathfrak{m}$. *Let* $X = \mathrm{Spec}\, R$ *and* x *the closed point of* X.

Then, for a positive integer μ, *there is a positive integer* $L_{N,\mu}$ *depending only on* N *and* μ *such that for every* $\mathfrak{m}$*-primary ideal* $\mathfrak{b}$ *on* X *with* $\mathfrak{m}^\mu \subset \mathfrak{b}$, *there is some prime divisor* E *over* X *that computes* $\mathrm{lct}(x; X, \mathfrak{b})$ *and satisfies* $k_E \leq L_{N,\mu}$.

Proof. We argue by contradiction. If the conclusion of the theorem fails, then we can find a sequence of $\mathfrak{m}$-primary ideals $(\mathfrak{a}_i)_{i\geq 1}$ such that for every i, $\mathfrak{m}^\mu \subset \mathfrak{a}_i \subset \mathfrak{m}$ and

$$\lim_{i\to\infty}\mathrm{md}(\mathrm{lct}(x; X, \mathfrak{a}_i)) = \infty.$$

Let $\mathfrak{a} \subset K[[X_1 \dots, X_n]]$ be the generic limit of $(\mathfrak{a}_i)_{i\geq 1}$. Let $R_K = K[[X_1, \dots, X_N]]$ with the maximal ideal $\mathfrak{m}_K$. We put $\widetilde{X} = \mathrm{Spec} R_K$. Let $\widetilde{x}$ be the closed point of $\widetilde{X}$. Then by Lemma A.1, we have

$$\mathfrak{m}_K^\mu \subset \mathfrak{a} \subset \mathfrak{m}_K.$$

Since $\mathfrak{a}$ is an $\mathfrak{m}_K$-primary ideal, any divisor over $\widetilde{X}$ computing $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \mathfrak{a})$ has the center at the closed point $\widetilde{x}$. Let E be a divisor over $\widetilde{X}$ computing $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \mathfrak{a})$.

Then by Proposition A.2, for every $d \gg \mu$ there is an infinite subset $I'_d \subset \mathbb{N}$ with the following property: for every $i \in I'_d$ there is a divisor E_i over X that computes $\mathrm{lct}(x; X, \mathfrak{a}_i)$, which is equal to $\mathrm{lct}(\widetilde{x}; \widetilde{X}, \mathfrak{a})$ and we have

$$k_E = k_{E_i} \text{ for every } i \in I'_d.$$

This is a contradiction to the assumption that $\lim_{i\to\infty}\mathrm{md}(\mathrm{lct}(x; X, \mathfrak{a}_i)) = \infty$. □

The above theorem does not hold if we do not assume the inclusion $\mathfrak{m}^\mu \subset \mathfrak{a}$ of ideals for the fixed power μ.

Example A.4. Let $\mathfrak{a}_i = (x, y^i) \subset k[x, y]$ for $i \in \mathbb{N}$. Then by the main theorem in [5], we have

$$\mathrm{lct}(0; \mathbb{A}^2_k, \mathfrak{a}_i) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (1, 1) \in t \cdot \mathrm{Newt}(x, y^i)\},$$

where $\mathrm{Newt}(x, y^i) \subset \mathbb{R}^2_{\geq 0}$ is the Newton polygon corresponding to the ideal (x, y^i). Therefore

$$\mathrm{lct}(0; \mathbb{A}^2_k, \mathfrak{a}_i) = (i + 1)/i.$$

Let E_i be a prime divisor over $\mathbb{A}^2_k$ computing $\mathrm{lct}(0; \mathbb{A}^2_k, \mathfrak{a}_i)$. Then

$$(k_{E_i} + 1)/\mathrm{val}_{E_i}(\mathfrak{a}_i) = (i + 1)/i.$$

We can rewrite this as $i(k_{E_i} + 1) = (i + 1)\mathrm{val}_{E_i}(\mathfrak{a}_i)$. Note that k_{E_i} and $\mathrm{val}_{E_i}(\mathfrak{a}_i)$ are integers. Since i and $i+1$ are coprime, the integer $k_{E_i}+1$ is divisible by $i+1$. Therefore $k_{E_i}+1 \geq i+1$. This implies that $\lim_{i\to\infty}\mathrm{md}(\mathrm{lct}(0; \mathbb{A}^2_k, \mathfrak{a}_i)) = \infty$.

Remark A.5. In the theorem we can replace the condition $\mathfrak{m}^\mu \subset \mathfrak{b}$ by the following:

$$\ell(\mathcal{O}_X/\mathfrak{b}) \le \mu,$$

because from this inequality, the condition $\mathfrak{m}^\mu \subset \mathfrak{b}$ follows.

By the same reason, it is also possible to replace the condition $\mathfrak{m}^\mu \subset \mathfrak{b}$ by the following inequality involving the multiplicity $e(\mathfrak{b})$ of $\mathfrak{b}$:

$$e(\mathfrak{b}) \le \mu.$$

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