COFRONTALS

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In this paper, we introduce the notion of cofrontal mappings, as the dual objects to frontal mappings, and study their basic local and global properties. Cofrontals are very special mappings and far from generic nor stable except for the case of submersions. It is observed that any smooth mapping can be C^0-approximated by a possibly "unfair" cofrontal or a frontal. However, global "fair" cofrontals are very restrictive to exist. Then we give a method to construction "fair" cofrontals with fiber-dimension one and a target-local diffeomorphism classification of such cofrontals, under some finiteness condition.

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1. INTRODUCTION

In the previous papers (see [8–11]) we have introduced and studied the notion of frontal map-germs. A map-germ $f : (N, a) \to (M, b)$ from an n-dimensional manifold N to an m-dimensional manifold M is called a *frontal* if $n \leq m$ and there exists a smooth n-plane field $\widetilde{f}$ along f, *i.e.* which commutes

$$\begin{tikzcd} & \mathrm{Gr}(n, TM) \arrow[d, "\pi"] \\ (N, a) \arrow[ur, "\widetilde{f}"] \arrow[r, "f"'] & (M, b), \end{tikzcd}$$

and which satisfies $\mathrm{Im} T_x f \subseteq \widetilde{f}(x)$ for any $x \in (N, a)$. Here $\mathrm{Gr}(n, TM)$ means the Grassmannian bundle over M with fibers $\mathrm{Gr}(n, T_y M)$, the Grassmannians of n-dimensional subspaces in $T_y M, y \in M$. The condition on $\widetilde{f}$ is equivalent to that $\widetilde{f}$ is an integral mapping for the canonical distribution on the Grassmannian bundle.

In this paper, in a dual manner to frontals, we introduce the notion of cofrontals: A map-germ $f : (N, a) \to (M, b)$ is called a *cofrontal* if $n \geq m$ and

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there exists an integrable vector-subbundle $K = K_f$ of TN of corank m, which is regarded as a section

$$
\begin{tikzcd}
\mathrm{Gr}(n-m, TN) & \\
(N,a) \arrow[u, "K_f"] \arrow[r, "f"'] & (M,b),
\end{tikzcd}
$$

and which satisfies the condition $(K_f)_x \subseteq \mathrm{Ker}(T_x f)$ for any $x \in (N,a)$. We impose the integrability condition on K in addition. If f is fair (Definition 2.13), *i.e.* the singular locus of the confrontal f has no interior point nearby $a \in N$, then the integrability of the germ K follows automatically. Moreover in this case K is uniquely determined from the cofrontal f (Lemma 2.16). In some sense, frontals are mappings such that the images of differentials are well-behaved, and cofrontals are mappings such that the kernels of differentials are well-behaved.

A global mapping $f : N \to M$ is called a *frontal* (resp. a *cofrontal*) if all germs $f_a : (N,a) \to (M, f(a))$ of f at every $a \in N$ are frontal (resp. cofrontal). Moreover a cofrontal f is called *fair* if all germs f_a at f at every $a \in N$ are fair cofrontal germs (Definition 3.1).

Important examples of cofrontals are obtained as mappings which are constant along Seifert fibers ([2], cf. Example 3.2).

We see that frontals and cofrontals are not stable except for the trivial cases, immersions and submersions and far from generic classes in the space of all C^∞ mappings. Nevertheless, we see that they enjoy rather interesting properties to be studied. For example, we see that any smooth map is approximated by a frontal or a cofrontal in C^0-topology, at least if the source manifold is compact (Proposition 3.3). In this paper, we will describe such basic but interesting properties of cofrontals mainly.

If $f : N \to M$ is a fair cofrontal, then the kernel field of f exists uniquely and globally, and therefore the source manifold N has a strict restriction if a global fair cofrontal exists on N. Note that, for given manifolds N, M with $n < m$, if N is compact then there exists a fair frontal $N \to M$ (Remark 4.3).

It is known that the local structures of fair (proper) frontals are understood by map-germs between spaces with the same dimension ($n = m$), together with the process of "openings" ([10, 11]). On the other hand, the local structures of fair cofrontals turn to be reduced to the case $n = m$. In fact, as for the *source-local* problem, the classification of cofrontal singularities is reduced to the case $n = m$ completely (Proposition 2.4, Lemma 2.16).

Note that frontals were studied mainly in the case $m - n = 1$, *i.e.* the case of hypersurfaces, motivated by the study on wave-fronts ([1, 10]).

In this paper, as for cofrontals, we study the cases of relative dimension

$n - m = 1$. We provide a general *target-local* classification of fair cofrontals $N^{m+1} \to M^m$ with relative dimension 1 under a mild condition. In fact the target-local classification problem of cofrontal mappings is reduced to that of the right-left classification of multi-germs $(\mathbb{R}^m, S) \to (\mathbb{R}^m, 0)$ together with a right symmetry of the multi-germ (Theorem 5.9). It is interesting to apply the classification results of map-germs $(\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$, in particular in the case $m = 2$ (see [12, 19, 20, 22] for instance), to classifications of concrete classes of cofrontals.

In §2, we introduce the notion of cofrontal map-germs comparing with that of frontals and clarify their local characters. In §3, we introduce global cofrontals and show some approximation result of mappings by frontals and cofrontals. After given several notions and examples related to fair frontals in §4, we give a classification of cofrontals of fiber-dimension one under the condition of "reduction-finite" (Definition 2.5, Definition 5.5, Theorem 5.9) in § 5.

In this paper, all manifolds and mappings are assumed to be smooth, *i.e.* of class C^∞ unless otherwise stated.

2. COFRONTAL SINGULARITIES

Let N, M be smooth manifold of dimension n and m respectively, and $f : (N, a) \to (M, b)$ a smooth map-germ. Suppose $n \geq m$.

Definition 2.1 (Cofrontal map-germ, kernel field). The germ f is called a *cofrontal map-germ* or a *cofrontal* in short, if there exists a germ of smooth (C^∞) integrable subbundle $K \subset TN$, $K = (K_x)_{x \in (N,a)}$, of rank $n - m$ such that

$$K_x \subseteq \operatorname{Ker}(T_x f : T_x N \to T_{f(x)} M),$$

for any $x \in N$ nearby a. Here $T_x f : T_x N \to T_{f(x)} M$ is the differential of f at $x \in (N, a)$.

Then K is called a *kernel field* of the cofrontal f.

Note that the kernel field is regarded as a section $K : (N, a) \to \mathrm{Gr}(n - m, TN)$ satisfying $(T_x f)(K_x) = \{0\}$, $x \in (N, a)$.

Compare with the notion of frontals (cf. [7, 10, 11]). Here we recall the definition of frontals: Let $f : (N, a) \to (\mathbb{R}^m, b)$ be a map-germ. Suppose $n \leq m$. Then f is called a *frontal map-germ* or a *frontal* in short, if there exists a smooth family of n-planes $\widetilde{f}(t) \subseteq T_{f(t)}\mathbb{R}^m$ along f, $t \in (N, a)$, satisfying the condition $\mathrm{Image}(T_t f : T_t N \to T_{f(t)}\mathbb{R}^m) \subseteq \widetilde{f}(t)$ $(\subset T_{f(t)}\mathbb{R}^m)$, for any $t \in (N, a)$. The family $\widetilde{f}(t)$ is called a *Legendre lift* of the frontal f.

In some sense, cofrontals are dual objects to frontals.

Example 2.2. (1) Any *immersion* is a frontal. The Legendre lift is given by $\widetilde{f} := (T_t f(T_t N))_{t\in(N,a)}$. Any *submersion* is a cofrontal. The kernel field K is given by $K := (\mathrm{Ker}(T_x f))_{x\in(N,a)}$.

(2) Any map-germ $(N,a) \to (M,b)$ between same dimensional manifolds $(n = m)$ is a frontal and a cofrontal simultaneously. In fact, the Legendre lift is given by $\widetilde{f}(t) := T_{f(t)}M, t \in (N,a)$ and the kernel field K is given by the zero-section of TN.

(3) Any *constant map-germ* $(N,a) \to (M,b)$ is a frontal if $n \leq m$ and a cofrontal if $n \geq m$. In fact, we can take any family of n-planes along the germ as a Legendre lift and any subbundle $K \subset TN$ of rank $n - m$ as a kernel field.

Remark 2.3. As was mentioned, the differentials of cofrontals have a mild behavior. This reminds us Thom's a_f-condition: Let $f : N \to M$ be a smooth map, X, Y submanifolds in N, and $x \in X \cap \overline{Y}$. Then Y is a_f-regular over X at x if a sequence y_i of points in Y converges to x and

$$\mathrm{Ker}(T_{y_i}(f|_Y)) \to T \subseteq T_x N, (i \to \infty),$$

then $\mathrm{Ker}(T_x(f|_X)) \subseteq T$.

Let $f : N \to M$ is a cofrontal and take any fiber $X = f^{-1}(b), b \in M$. Then X is a submanifold of N and $Y = N \setminus X$ is a_f-regular over X.

Let $\mathcal{E}_{N,a} := \{h : (N,a) \to \mathbb{R}\}$ denote the $\mathbb{R}$-algebra of smooth function-germs on (N,a).

Recall that *Jacobi ideal* J_f of a map-germ $f : (N,a) \to (M,b)$ is defined as the ideal generated in $\mathcal{E}_{N,a}$ by all $\min\{n,m\}$-minor determinants of Jacobi matrix $J(f)$ of f. Note that J_f is independent of the choices of local coordinates on (N,a) and (M,b).

Proposition 2.4 (Criterion of cofrontality). *Let $f : (N,a) \to (M,b)$ be a map-germ with $n = \dim(N) \geq m = \dim(M)$. If f is a cofrontal, then there exists a germ of submersion $\pi : (N,a) \to (\overline{N},\overline{a})$ to an m-dimensional manifold $\overline{N}$ and a smooth map-germ $\overline{f} : (\overline{N},\overline{a}) \to (M,b)$ such that $f = \overline{f} \circ \pi$. Moreover the Jacobi ideal J_f of f is principal, i.e. it is generated by one element. In fact J_f is generated by $\lambda = \pi^*(\overline{\lambda})$ for the Jacobian determinant $\overline{\lambda}$ of $\overline{f}$.*

Conversely, if the Jacobi ideal J_f is principal and the singular locus

$$S(f) = \{x \in (N,a) \mid \mathrm{rank}(T_x f : T_x N \to T_{f(x)}M) < m\}$$

of f is nowhere dense in (N,a), then f is a cofrontal.

Definition 2.5 (Reductions of cofrontals). We call $\overline{f}$ a *reduction* of the cofrontal-germ f. A germ of cofrontal $f : (N,a) \to (M,b)$ is called *reduction-finite* if a reduction $\overline{f} : (\overline{N},\overline{a}) \to (M,b)$ of f is $\mathcal{K}$-finite (or finite briefly), *i.e.* the dimension of $Q_{\overline{f}} := \mathcal{E}_{\overline{N},\overline{a}}/\overline{f}^*(m_b)$ is finite, where $\overline{f}^* : \mathcal{E}_{M,b} \to \mathcal{E}_{\overline{N},\overline{a}}$ is the

$\mathbb{R}$-algebra homomorphism defined by $\overline{f}^*(h) = h \circ \overline{f}$, and $m_b \subset \mathcal{E}_{M,b}$ is the maximal ideal of function-germs vanishing at b (see [14] [5] [21] [1]).

Remark 2.6. In Lemma 2.3 of [11], it is shown that if $f : (N, a) \to (M, b), n \leq m$ is a frontal, then the Jacobi ideal J_f is principal and that conversely if J_f is principal and $S(f) = \{x \in (N, a) \mid \text{rank}(T_x f) < n\}$ is nowhere dense, then f is a frontal.

Remark 2.7. If $\overline{f} : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ is $\mathcal{K}$-finite, then the zero set of $\overline{f}$ is isolated and any nearby germ of $\overline{f}$ has the same property. The number of fibers of $\overline{f}$ is uniformly bounded by $\dim(Q_{\overline{f}})$ (Propositions 2.2, 2,4 of Ch.VII in [5], see also [3, 13]).

Proof of Proposition 2.4. Let f be a cofrontal and K be a kernel field of f. Since K is an integrable subbundle of TN of rank $n - m$, there exists a submersion $\pi : (N, a) \to (\mathbb{R}^m, 0)$ such that $K_x = \text{Ker}(\pi_* : T_x N \to T_{\pi(x)}\mathbb{R}^m)$ for any $x \in (N, a)$, *i.e.* π-fibers form the foliation induced by K. Take any curve $\gamma : (\mathbb{R}, 0) \to N$ in a fiber of π. Then $(f \circ \gamma)'(t) = (T_{\gamma(t)} f)(\gamma\,'(t)) = 0$. Therefore f is constant along the curve γ. Hence f is constant on π-fibers. Then there exists a map-germ $\overline{f} : (\mathbb{R}^m, 0) \to (M, b)$ such that $f = \overline{f} \circ \pi$. Take a smooth section $s : (\mathbb{R}^m, 0) \to (N, a)$. Then $\overline{f} = \overline{f} \circ \pi \circ s = f \circ s$. Therefore $\overline{f}$ is a smooth map-germ.

Take a system of local coordinates $x_1, \dots, x_m, x_{m+1}, \dots, x_n$ such that π is given by

$$\pi(x_1, \dots, x_m, x_{m+1}, \dots, x_n) = (x_1, \dots, x_m)$$

and therefore K_x is generated by $\partial/\partial x_{m+1}, \dots, \partial/\partial x_n$ in $T_x N$. Then f is expressed as

$$f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_m)).$$

Then J_f is generated by one element $\det(\partial f_i/\partial x_j)_{1 \leq i,j \leq m} = \pi^*(\det(\partial \overline{f}_i /\partial x_j)_{1 \leq i,j \leq m}) = \pi^*(\overline{\lambda})$ and therefore J_f is a principal ideal in $\mathcal{E}_{N,a}$.

Conversely, suppose J_f is a principal ideal generated by one element $\lambda \in J_f$ and $S(f)$ is nowhere dense. Denote by Γ the set of subsets $I \subseteq \{1, 2, \dots, n\}$ with $\#(I) = m$. For a map-germ $f : (N, a) \to (M, b), n \geq m$ and $I \in \Gamma$, we set $D_I = \det(\partial f_i/\partial x_j)_{1 \leq i \leq m, j \in I}$ for some coordinates $x_1, \dots, x_n$ of (N, a) and $y_1, \dots, y_m$ of (M, b) with $f_i = y_i \circ f$. For any $I \in \Gamma$, there exists $h_I \in \mathcal{E}_a$ such that $D_I = k_I \lambda$. Since $S(f)$ is nowhere dense, there exists $I_0 \in \Gamma$ such that $D_{I_0} \neq 0$. Since $\lambda \in J_f$, there exists $\ell_I \in \mathcal{E}_a$ for any $I \in \Gamma$ such that $\lambda = \sum_{I \in \Gamma} \ell_I D_I$. Then $(1 - \sum_{I \in \Gamma} \ell_I k_I)\lambda = 0$. If $k_I(a) = 0$ for any $I \in \Gamma$, then $1 - \sum_{I \in \Gamma} \ell_I k_I$ is invertible in $\mathcal{E}_a$, therefore $\lambda = 0$ and then we have $J_f = 0$. This contradicts to the assumption that $S(f)$ is nowhere dense. Hence there exists $I_0 \in \Gamma$ such that $(\ell_{I_0} k_{I_0})(a) \neq 0$. Then $k_{I_0}(a) \neq 0$. Therefore J_f is generated by D_{I_0}. Hence $D_I = h_I D_{I_0}$ for any $I \in \Gamma$ with $h_{I_0}(a) = 1$. Then the Plücker-Grassmann coordinates $(h_I)_{I \in \Gamma}$ give a smooth section $K : (\mathbb{R}^a, a) \to \mathrm{Gr}(n-m, TN) \cong \mathrm{Gr}(m, T^*\mathbb{R}^a)$, which is regarded as a subbundle $K \subseteq TN$ of rank $n-m$ and $K_x \subseteq \mathrm{Ker}(T_x f)$ for any $x \in (N, a)$. Moreover K_x coincides with $\mathrm{Ker}(T_x f)$ for $x \in (N \setminus S(f), a)$ and therefore K is integrable outside of $S(f)$. Since $S(f)$ is nowhere dense, K is integrable. Thus f is a cofrontal map-germ with the kernel field K. $\square$

COROLLARY 2.8. *Let $f : (N, a) \to (M, b)$ be a map-germ. Suppose f is analytic and $J_f \neq 0$. Then f is a frontal or a cofrontal if and only if J_f is a principal ideal.*

Proof. By Lemma 2.4 and Remark 2.6, if f is a frontal or a cofrontal, then J_f is principal. If J_f is principal, $J_f \neq 0$ and f is analytic, then $S(f)$ is nowhere dense. Thus by Lemma 2.4 and Remark 2.6, f is a frontal if $n \leq m$ or a cofrontal if $n \geq m$. $\square$

Example 2.9. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ be the map-germ given by $f(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2, 0)$. Then f is analytic and $J_f = 0$ is principal. However f is not a cofrontal. In fact, suppose f is a cofrontal and K a kernel field of f of rank 1. Let

$$\xi(x) = \xi_1(x)\partial/\partial x_1 + \xi_2(x)\partial/\partial x_2 + \xi_3(x)\partial/\partial x_3, \ \xi(0) \neq 0,$$

be a generator of K. Then $\xi_1(x)x_1 + \xi_2(x)x_2 + \xi_3(x)x_3$ is identically zero in a neighborhood of 0 in $\mathbb{R}^3$. In particular, we have $\xi_1(x_1, 0, 0)x_1 = 0$ and therefore $\xi_1(x_1, 0, 0) = 0$, so $\xi_1(0, 0, 0) = 0$. Similarly, we have also $\xi_2(0, 0, 0) = 0$ and $\xi_3(0, 0, 0) = 0$. This leads a contradiction.

Definition 2.10 (Jacobians of frontals and cofrontals). Let $f : (N, a) \to (M, b)$ be a frontal or a cofrontal. Then a generator $\lambda \in \mathcal{E}_a$ of J_f is called a *Jacobian* (or a *singularity identifier*) of the cofrontal f, which is uniquely determined from f up to multiplication of a unit in $\mathcal{E}_a$.

Remark 2.11. The singular locus $S(f) = \{x \in (N, a) \mid \mathrm{rank}(T_x f : T_x N \to T_{f(x)} M) < \min\{n, m\}\}$ of a frontal or a cofrontal f is given by the zero-locus of the Jacobian λ of f.

Remark 2.12. Let $f : (N, a) \to (M, b)$ be a cofrontal and K a kernel field of f. Set

$$K_x^{\perp} := \{\alpha \in T_x^* N \mid \alpha(v) = 0 \text{ for any } v \in K_x\}.$$

Then $K^{\perp}$ is a germ of subbundle of the cotangent bundle T^*N of rank m. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be a local frame of $K^{\perp}$.

Then there is a unique $\lambda \in \mathcal{E}_a$ such that

$$df_1 \wedge df_2 \wedge \cdots \wedge df_m = \lambda \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m.$$

Then λ generates J_f and therefore λ is a Jacobian of the cofrontal f.

Definition 2.13 (Fair frontals and cofrontals). A frontal or a cofrontal $f : (N, a) \to (M, b)$ is called *fair* if the singular locus $S(f)$ is nowhere dense in (N, a).

Remark 2.14. A cofrontal f is fair if and only if a reduction $\overline{f}$ (Definition 2.5) is fair. In fact if $f = \overline{f} \circ \pi$ for a submersion-germ $\pi : (N, a) \to (\mathbb{R}^m, 0)$, we have $S(f) = \pi^{-1}(S(\overline{f}))$, and therefore $S(f)$ is nowhere dense in (N, a) if and only if $S(\overline{f})$ is nowhere dense in $(\mathbb{R}^m, 0)$. If a cofrontal f is reduction-finite (Definition 2.5), then f is fair, since a reduction $\overline{f}$ is $\mathcal{K}$-finite so is necessarily fair.

Remark 2.15. In [10, 11], a frontal with nowhere dense singular locus was called *proper*. However in the global study the terminology "proper" is rather confusing, in particular for the study of cofrontals, since its usage is different from the ordinary meaning of properness (inverse images of any compact is compact). Therefore, in this paper, we use the terminology "fair" instead of "proper".

Lemma 2.16. *Let $f : (N, a) \to (M, b)$ be a fair cofrontal or* $\dim(N) = \dim(M)$. *Then the kernel filed K of f is uniquely determined and the reduction $\overline{f}$ of f (Definition 2.5) is uniquely determined up to right equivalence.*

Proof. On the regular locus $N \setminus S(f)$, there is the unique kernel field K defined by $K_x := \mathrm{Ker}(T_x f : T_x N \to T_{f(x)} M)$. Let f be a fair cofrontal. Then $N \setminus S(f)$ is dense in (N, a). Therefore the extension of K to (N, a) is unique if it exists. Let $n = m$. Then the unique kernel field K is defined by the zero-section of TN (Example 2.2 (2)). Then the submersion $\pi : (N, a) \to (\mathbb{R}^m, 0)$ induced by K is uniquely determined up to left equivalence. Let $\pi' : (N, a) \to (\mathbb{R}^m, 0)$ be induced by K and $\overline{f}$ and $\overline{f}'$ be both reductions of f with $f = \overline{f} \circ \pi = \overline{f}' \circ \pi'$. Then $\pi' = \sigma \circ \pi$ for some diffeomorphism-germ $\sigma : (\mathbb{R}^m, 0) \to (\mathbb{R}^m, 0)$ and $\overline{f} \circ \pi = (\overline{f}' \circ \sigma) \circ \pi$. Since π is a submersion, we have $\overline{f} = \overline{f}' \circ \sigma$. $\square$

Let $f : (N, a) \to (M, b)$ be a cofrontal (resp. a fair cofrontal) and $K : (N, a) \to \mathrm{Gr}(n - m, TM)$ be a kernel field of f. Recall that $K \subset TN$ is a germ of integrable subbundle of rank $n - m$.

Definition 2.17 (Adapted coordinates). A system $(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ of local coordinates of N centered at a is called *adapted* to a kernel field K of a cofrontal f, or simply, to f, if

$$K_x = \left\langle \left(\frac{\partial}{\partial x_{m+1}}\right)_x, \dots, \left(\frac{\partial}{\partial x_n}\right)_x \right\rangle_{\mathbb{R}} = \{v \in T_x N \mid dx_1(v) = 0, \dots, dx_m(v) = 0\},$$

for any $x \in (N, a)$.

Since a kernel field K of a cofrontal is assumed to be integrable, we have

LEMMA 2.18. *Any cofrontal* $f : (N, a) \to (M, b)$ *has an adapted system of local coordinates on* (N, a).

Remark 2.19. For an adapted system of coordinates $(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ of f, the Jacobian λ is given by the ordinary Jacobian $\frac{\partial(f_1,\dots,f_m)}{\partial(x_1,\dots,x_m)}$.

3. GLOBAL COFRONTALS

We will define the class of (co)frontal maps and fair (co)frontal maps, and we make clear the difference of these classes of mappings.

Definition 3.1 (Global cofrontal mappings). Let N, M be smooth manifolds of dimension n, m respectively.

Suppose $n \geq m$. A smooth mapping $f : N \to M$ is called a *cofrontal map* or a *cofrontal* briefly if the germ $f_a : (N, a) \to (M, f(a))$ at a is a cofrontal for any $a \in N$ (Definition 2.1). A cofrontal $f : N \to M$ is called *fair* if f_a is a fair cofrontal for any $a \in N$, *i.e.* if the singular locus $S(f) := \{x \in N \mid \mathrm{rank}(T_x f : T_x N \to T_{f(x)} M) < m\}$ is nowhere dense in N (Definition 2.13).

Suppose $n \leq m$. A smooth mapping $f : N \to M$ is called a *frontal map* or a *frontal* briefly if the germ $f_a : (N, a) \to (M, f(a))$ is a frontal for any $a \in N$. A frontal $f : N \to M$ is called *fair* if f_a is a fair frontal for any $a \in N$, *i.e.* if the singular locus $S(f) := \{x \in N \mid \mathrm{rank}(T_x f : T_x N \to T_{f(x)} M) < n\}$ is nowhere dense in N.

Example 3.2. (1) Any submersion is a cofrontal. Any immersion is a frontal.

(2) Any constant mapping $N \to M$ is a cofrontal of a frontal depending on $\dim(N) \geq \dim(M)$ or $\dim(N) \leq \dim(M)$.

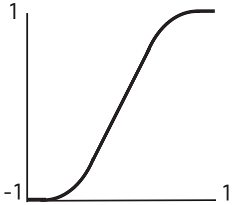
(3) Let $\mathcal{F}$ be a foliation of codimension m on a manifold N of dimension n. If a mapping $f : N^n \to M^m$ is constant on any leaf of $\mathcal{F}$, then f is a cofrontal.

(4) As a motivating example from symplectic geometry, consider a Lagrangian foliation $\mathcal{L}$ on a symplectic manifold N^{2n} and a system of functions $f_1, \dots, f_n$ on N. Then $f = (f_1, \dots, f_n) : N \to \mathbb{R}^n$ is a cofrontal if f is constant along each leaf of $\mathcal{L}$.

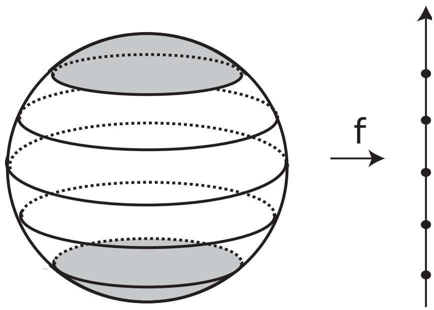
First we observe "unfair" (co)frontal maps are not so restrictive in topological or homotopical sense. In what follows, we suppose N is compact for simplicity.

Proposition 3.3 (C^0-approximation). *Any smooth (C^∞) map $f : N \to M$ is C^0-approximated by a frontal or a cofrontal $g : N \to M$, i.e., for any open neighborhood $\mathcal{U}$ of f, for C^0-topology on the space $C^\infty(N, M)$ of all C^∞ mappings, there exists a frontal or a cofrontal g which belongs to $\mathcal{U}$. Moreover any smooth map $f : N \to M$ is homotopic to a frontal or a cofrontal $g : N \to M$.*

Example 3.4. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $g : S^2 \to \mathbb{R}$ the height function, *i.e.* $g(x_1, x_2, x_3) = x_3$. Then g is never a cofrontal. Let $\varepsilon > 0$. Let $\varphi : [-1, 1] \to [-1, 1]$ be any smooth map satisfying that $\varphi(y) = -1(-1 \leq y - 1 + \varepsilon), \varphi(y) = 1(1 - \varepsilon \leq y \leq 1)$, and that φ is a diffeomorphism from $(-1+\varepsilon, 1-\varepsilon)$ to $(-1, 1)$. Then $f = \varphi \circ g$ is a cofrontal. See the figure: In the right picture, f restricted to the north (resp. south) gray part is constant.



graph of φ.



A cofrontal on the sphere.

Note that f can be taken to be arbitrarily near g in C^0-topology.

Similar construction can be applied to any proper Morse function $g : N \to \mathbb{R}$ and we have a cofrontal which is a C^0-approximation to g.

Proof of Proposition 3.3. Let $f : N \to M$ be a smooth mapping. Take any open neighborhood $\mathcal{U}$ of f in $C^\infty(N, M)$ for C^0-topology. Then, for any neighborhood $\mathcal{U}'$ of f for C^∞-topology, there exist a mapping $f' : N \to M$ which belongs to $\mathcal{U}'$ such that f' has a Thom stratification $(\mathcal{S}, \mathcal{T})$ satisfying the following conditions:

(1) $\mathcal{S}$ is a Whitney stratification of N and $\mathcal{T}$ is a Whitney stratification of M.

(2) For any $S \in \mathcal{S}$, there exists a $T \in \mathcal{T}$ such that $f'|_S : S \to T$ is a surjective submersion.

(3) The critical locus

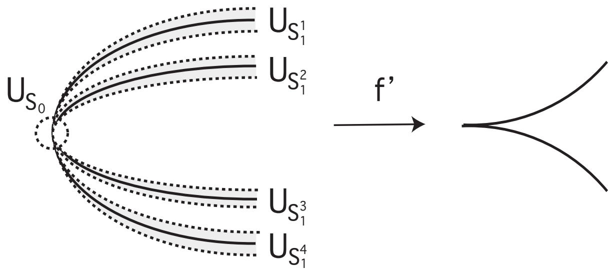
$$\Sigma(f') := \{x \in N \mid \operatorname{rank}(T_x f' : T_x N \to T_{f(x)} M) < m\}$$

is a union of strata of $\mathcal{S}$ and, for any stratum $S \in \mathcal{S}$ in $\Sigma(S)$, $f|_S : S \to M$ is an immersion.

(4) There exists a compatible tubular system $(\pi_S, \rho_S)_{S\in\mathcal{S}}$ for $\mathcal{S}$, *i.e.*, a normed normal bundle $E_S \to S$ to S in N, a positive smooth function $\varepsilon : S \to \mathbb{R}$ and a diffeomorphism $\phi_S : (E_S)_{<\varepsilon} \to N$ on on the image $U_S = \Phi_S((E_S)_{<\varepsilon})$, which is a tubular neighborhood of S in N. Here $(E_S)_{<\varepsilon} := \{v \in E_S \mid v \in T_xN, \|v\| < \varepsilon(x)\}$ and $\rho_S(v) = \|v\|^2$. The projection π_S is regarded as the projection from the tubular neighborhood U_S to S via ϕ_S, and ρ_S is the squared norm function on a normed normal bundle of S, which is regarded as a function on the tubular neighborhood. Then compatibility condition means that, for any $S, S' \in \mathcal{S}$ with $S' \subseteq \overline{S}$,

$$\pi_{S'} \circ \pi_S = \pi_{S'}, \quad \rho_{S'} \circ \pi_S = \rho_S,$$

in the intersection of a tubular neighborhood of S and that of S', and that for any $S, S' \in \mathcal{S}$ with $\dim(S) = \dim(S')$, the intersection $U_S \cap U_{S'} = \emptyset$.



This is obtained as a byproduct of the proof of the topological stability theorem due to Mather [15–17]. In fact, the argument goes as follows: First f is C^∞-approximated by a mapping $f'' \in \mathcal{U}'$ of finite singularity type (Proposition 7.1 of [17]). Then f'' has an unfolding (F, N', M', i, j) such that F is proper and infinitesimally stable (Proposition 7.2 of [17]). Here $i : N \to N'$ and $j : M \to M'$ are embeddings and

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ f\downarrow & & \downarrow F \\ M & \xrightarrow{j} & M' \end{array}$$

is a fibre-product. For the infinitesimally stable mapping F, the restriction of F to the singular locus $\Sigma(F)$ is proper and uniformly finite-to-one (Proposition 1.1 of [17]). Then F has a stratification $(\mathcal{S}', \mathcal{T}')$ satisfying the conditions (1)(2)(3) (Proposition 11.4 of [16], §3 of [17]). Since $\mathcal{S}'$ is obtained as the pull-back of $\mathcal{T}'$, $\mathcal{S}'$ is Thom regular over F by the condition (3). Therefore $(\mathcal{S}', \mathcal{T}')$ is a Thom stratification of F. Then, by perturbing j, a C^∞-approximation $f' : N \to P$ of f'', $f' \in \mathcal{U}'$ is obtained. Moreover, as the restriction of $(\mathcal{S}', \mathcal{T}')$, the required

Thom stratification $(\mathcal{S}, \mathcal{T})$ of f' is obtained. Then the condition (4) is satisfied by Proposition 6.3 of [16]. (See also [4, 15].)

Now, since $\mathcal{U}$ is open for C^∞-topology, we can take f' from $\mathcal{U}$. For $0 \le i \le n$, denote by $S^{(i)}$ is the i-skeleton of $\mathcal{S}$, *i.e.* the union of all strata of $\mathcal{S}$ of dimension $\le i$. Set $U^{(i)} = N \setminus S^i$, the union of all strata of $\mathcal{S}$ of dimension $> i$. We will modify f' first on $U^{(n-1)}$ and then $U^{(n-2)}$ and so on to get the approximation g.

Actually, we perform as follows: First we suppose $n > m$. Let τ be a sufficiently small positive real number and $\tau = \tau_\delta : [0,1] \to \mathbb{R}$ a smooth function such that $\tau(t) = 0(0 \le t < \delta), \tau(t) = 1(1 - \delta < t \le 1$. Define $f_{n-1} : U^{(n-1)} \to M$ by

$$f_{n-1}(x) := f\left(\phi_S\left(\tau\left(\frac{1}{\varepsilon(\pi_S(\phi_S^{-1}(x)))}\|\phi_S^{-1}(x)\|\right)\phi_S^{-1}(x)\right)\right),$$

for $x \in U_S \cap U^{(n-1)}$ with $\dim(S) = n-1$, and by $f_{n-1}(x) = f(x)$ otherwise. Note that, by the mapping f_{n-1}, the mapping f is modified along points near $S \in \mathcal{S}$ with $\dim(S) = n-1$ contracts to S and then mapped by f. The modified map f_{n-1} is a smooth map and a cofrontal. Also we have that f_{n-1} is homotopic to $f|_{U^{(n-1)}}$. Moreover note that f_{n-1} is not a C^∞-approximation but a C^0-approximation of f on $U^{(n-1)}$. Define $f_{n-2} : U^{(n-2)} \to M$ by setting $f_{n-2}(x)$ similar as above for $x \in U_S \cap U^{(n-2)}$ with $\dim(S) = n-2$, by $f_{n-2}(x) = f_{n-1}(x)$ otherwise. Then f_{n-2} is a smooth map, a cofrontal and a C^0 approximation of f on $U^{(n-2)}$. Iterating this procedure we have $f_0 : U^{(0)} \to M$ and finally $f_{-1} : U^{(-1)} = N \to M$, which is a smooth map, a cofrontal, belongs to $\mathcal{U}$ and is homotopic to f.

If $n = m$, then we have nothing to do.

Suppose $n < m$. Note that in this case $\Sigma(f) = N$. Then by the same procedure as above, we have a *frontal* f_i which is a C^0-approximation of $f|U^{(i)}$ and is homotopic to $f|U^{(i)}$ for $i = n-1, n-2, \dots, 0, -1$. Note that we may take as a Legendre lift of f_i any extension of the Legendre lift of $f|_S, \dim(S) = i$ over U_S. Thus we have a frontal f which belongs to $\mathcal{U}$ and is homotopic to f. $\square$

4. GLOBAL FAIR COFRONTALS

Contrary to the case of "unfair" cofrontals, the following lemmata show that the sauce space of a *fair* cofrontal must be very restrictive.

LEMMA 4.1. *Let $f : N \to M$ be a fair cofrontal. Then there exists a unique kernel field K of f, i.e. there exists a unique integrable subbundle $K \subseteq TN$ of rank $n - m$ such that $K_x \subseteq \mathrm{Ker}(T_x f : T_x N \to T_{f(x)} M)$ for any $x \in N$.*

Proof. By Lemma 2.16, the germ of kernel field is uniquely determined for each $x \in N$. By the local existence and *uniqueness*, we have the global existence of the kernel field of f. $\square$

LEMMA 4.2. *Let $f : N \to M$ be a fair cofrontal and K the kernel field of f. Let $\mathcal{F}$ be the foliation induced by the integrable subbundle K of TN of rank $n - m$. Then the closure of any leaf of $\mathcal{F}$ is nowhere dense in N.*

Proof. Let L be a leaf of $\mathcal{F}$. Then f restricted to L is constant. (See the proof of Proposition 2.4. Note that L is assumed to be connected by the definition of leaves.) Then f restricted to the closure $\overline{L}$ of L is constant just by the continuity of f. Assume $\overline{L}$ has an interior point. Then also $S(f)$ necessarily has an interior point. This leads us to a contradiction with the fairness. $\square$

Remark 4.3. Let N, M be smooth manifolds with $\dim(N) = n \leq m = \dim(M)$. Suppose that N is compact or both N, M are non-compact. Then there exists a proper fair frontal $f : N \to M$. In fact take any closed submanifold $N' \subseteq M$ of dimension n and its inclusion $i : N' \hookrightarrow M$. Take any proper smooth map $g : N \to N'$ whose singular locus $S(g)$ is nowhere dense, for instance, g is a topologically stable map. Then $f := i \circ g$ is a proper fair frontal map.

The following is clear.

LEMMA 4.4. *Let $g : L \to M$ be a cofrontal and $\pi : N \to L$ be a submersion. Then $g \circ \pi : N \to M$ is a cofrontal. $g \circ \pi$ is fair if and only if g is fair.*

Definition 4.5 (Reducible and irreducible cofrontals). Let $f : N \to M$ be a cofrontal with $\dim(N) = n > m = \dim(M)$. The frontal f is called *reducible* if there exists a submersion $\pi : N \to \widetilde{N}$ to an ℓ-dimensional manifold $\widetilde{N}$ and a cofrontal $g : \widetilde{N} \to M$ with $n > \ell \geq m$ such that $f = g \circ \pi$. A cofrontal is called *irreducible* if it is not reducible.

PROPOSITION 4.6. *Let $f : N \to M$ be a fair cofrontal with $n > m$ and K its kernel field. If the leaf space forms an m-dimensional manifold $\overline{N}$ and $\pi : N \to \overline{N}$ is a surjective smooth submersion such that* $\mathrm{Ker}(\pi_*) = K$, *then f is reducible. In fact, there exists a smooth map $g : \overline{N} \to M$ such that $f = g \circ \pi$.*

Proof. Since f is constant on each leaf of K, we have a map $g : \overline{N} \to M$ such that $f = g \circ \pi$. Take any leaf L of K and any point $x \in L$, then the reduction of the germ of f at x is given by the germ of g at $L \in \overline{N}$. Therefore g is smooth at L (see Proposition 2.4). Thus g is a smooth map. $\square$

Example 4.7 (Irreducible fair frontals). (1) Let the open Möbius band N is given as the quotient of $\mathbb{R}^2$ by the cyclic action generated by the transformation

$\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $\psi(x_1, x_2) = (x_1 + 1, -x_2)$. Then $(x_1, x_2) \mapsto x_2^2$ induces a well-defined map $f : N \to \mathbb{R}$ which is an irreducible fair cofrontal.

(2) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus. Let K be the Klein bottle defined as the quotient by the involution $\varphi : T \to T$, $\varphi([x_1, x_2]) := [x_1 + \frac{1}{2}, 1 - x_2]$. Define $f : K \to \mathbb{R}$ by $f([[x_1, x_2]]) := (x_2 - \frac{1}{2})^2$. Here $[x_1, x_2]$ (resp. $[[x_1, x_2]]$) be the point on T (resp. K) represented by $(x_1, x_2) \in \mathbb{R}^2$. Then f is well-defined smooth mapping which is an irreducible fair cofrontal.

Example 4.8 (Cofrontal of reduction-non-finite). Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth such that $\varphi(t) = t, (t < \frac{1}{3}, t\frac{2}{3} < t)$, $\frac{1}{3} < |\varphi(t)| < \frac{2}{3}$ and that $\varphi^{-1}(\frac{1}{2})$ is an infinite set having just one point $t = \frac{1}{2}$ as a non-isolated point. Then define $f : T^2 = \mathbb{R}^2/\mathbb{Z}^2 \to S^1 = \mathbb{R}/\mathbb{Z}$ by $f([t_1, t_2]) = [\varphi(t_2)]$ modulo $\mathbb{Z}$, where $t_2 \in [0, 1]$. Then f is a fair cofrontal such that f is not reduction-finite and the fiber $f^{-1}([\frac{1}{2}])$ has infinite many connected components.

5. CLASSIFICATION OF COFRONTALS OF FIBER-DIMENSION ONE

To give a target-local classification theorem of cofrontals, we start with an algebraic consideration. Let $\mathrm{Diff}(N, a)$ denote the group of diffeomorphisms $(N, a) \to (N, a)$.

Definition 5.1. Let $f : (N, a) \to (M, b)$ be a smooth map-germ. Then the *right symmetry group G_f* of f is defined by

$$G_f := \{\sigma \in \mathrm{Diff}(N, a) \mid f \circ \sigma = f\}.$$

LEMMA 5.2. *Let $f : (N, a) \to (M, b)$ and $g : (N', a') \to (M', b')$ be smooth map-germs. If f and g are right-left equivalent ($\mathcal{A}$-equivalent), then G_f and G_g are isomorphic as groups.*

Proof. Suppose $\tau \circ f = g \circ \sigma$ for diffeomorphism-germs $\sigma : (N, a) \to (N', a')$ and $\tau : (M, b) \to (M', b')$. Let $\varphi \in G_f$. Then

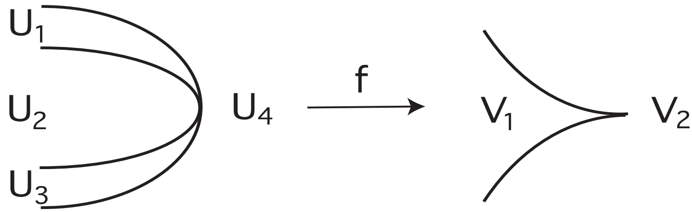
$$g \circ (\sigma \circ \varphi \circ \sigma^{-1}) = (g \circ \sigma) \circ \varphi \circ \sigma^{-1} = (\tau \circ f) \circ \varphi \circ \sigma^{-1} = \tau \circ (f \circ \varphi) \circ \sigma^{-1} \tau \circ f \circ \sigma^{-1} = g.$$

Therefore $\sigma \circ \varphi \circ \sigma^{-1} \in G_g$. The correspondence $G_f \to G_g$ defined by $\varphi \mapsto \sigma \circ \varphi \circ \sigma^{-1}$ induces a group isomorphism. $\square$

Example 5.3. (1) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a *fold* which is defined by $f(x_1, x_2) = (x_1, x_2^2)$. Then the right symmetry group $G_f \cong \mathbb{Z}/2\mathbb{Z}$.

(2) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a *cusp* which is defined by $f(x_1, x_2) = (x_1, x_2^3 + x_1 x_2)$ [22]. Then G_f is trivial, *i.e.* G_f consists of only the identity map-germ on $(\mathbb{R}^2, 0)$. This can be seen, for instance, as follows: There are

natural stratification $(\mathcal{S}, \mathcal{T})$ of f such that $\mathcal{S}$ has four open strata U_1, U_2, U_3 and U_4 , $\mathcal{T}$ has two open strata V_1 and V_2, each $U_i, i = 1, 2, 3$, is mapped to V_1, and U_4 to V_2 respectively by f, and that each restriction of f is injective.



Let $\sigma \in G_f$. Since σ preserves inverse images of f for each point in the target, σ must be the identity on U_4, and it permutes three points from U_1, U_2, U_3. However, by the continuity of f, σ maps each U_i to itself. Thus σ restricted to each fiber of f turns to be the identity, and therefore σ itself turns to be the identity.

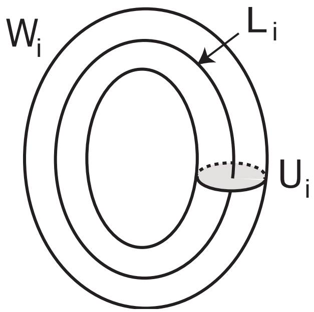
(3) Let $f : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be defined by $f(x_1, x_2) = (x_1^2, x_2^2)$. Then $G_f \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(4) Let $f : (\mathbb{R}^2, 0) = (\mathbb{C}, 0) \to (\mathbb{C}, 0) = (\mathbb{R}^2, 0)$ be defined by $f(z) = z^\ell, (z \in \mathbb{C})$. Then we have $G_f \cong \mathbb{Z}/\ell\mathbb{Z}$.

(5) Let G be a finite reflection group on $\mathbb{R}^n$ and $h_1, h_2, \dots, h_n$ be a system of generators of the invariant ring of G consisting of homogeneous polynomials (cf. Chevalley's theorem [6]). Then the right symmetry group G_h of $h = (h_1, \dots, h_n) : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is isomorphic to G.

Proposition 5.4 (Construction of cofrontals of fiber-dimension one). *Let $h : (\mathbb{R}^m, 0) \to (M, b)$ be a smooth map-germ and $\sigma \in G_h$. Let $h : U \to M$ and $\sigma : U \to U$ be representatives of h and σ respectively such that $h \circ \sigma = h$ on U. Set $N = ([0, 1] \times U)/\sim$ where $(0, \overline{x}) \sim (1, \sigma(\overline{x}))$. Then N is a $(m + 1)$-dimensional manifold and $f = f_{h,\sigma} : N \to M, f([t, \overline{x}]) = h(\overline{x})$ is well-defined and is a cofrontal.*

In general, let $h_1, \dots, h_s : (\mathbb{R}^m, 0) \to (M, b)$ be smooth map-germs and $\sigma_i \in G_{h_i}, (1 \le i \le s)$. Let $h_i : U_i \to M$ and $\sigma_i : U_i \to U_i$ be representatives of h_i and σ_i, $(1 \le i \le s)$ respectively, such that $h_i \circ \sigma_i = h_i$ on U_i. Set $N_i = ([0, 1] \times U_i)/\sim$ where $(0, \overline{x}) \sim (1, \sigma_i(\overline{x}))$. Take the disjoint union $N = \bigcup_{i=1}^s N_i$. which is an $(m+1)$-dimensional manifold. Define $f = f_{h_1,\dots,h_s;\sigma_1,\dots,\sigma_s} : N \to M$ by $f([t, \overline{x}]) = h_i(\overline{x})$ for $[t, \overline{x}] \in N_i$, $1 \le i \le s$. Then f is well-defined and f is a cofrontal.



Proof. Since $\sigma_i \in G_{h_i}$, f is well-defined and smooth. Moreover the t-direction defines well-defined subbundle $K \subset TN$ of ranks 1. Since f is constant along K, we see that f is a cofrontal. □

Definition 5.5 (Reduction-finite cofrontals.). A cofrontal $f : N \to M$ is called *reduction-finite* if any germ of cofrontal $f_a : (N, a) \to (M, f(a))$ is reduction-finite in the sense of Definition 2.5.

Remark 5.6. The cofrontal f in Proposition 5.4 is fair if and only if all $h_i, i = 1, \dots, s$ are fair. Moreover f is reduction-finite if and only if all $h_i, i = 1, \dots, s$ are $\mathcal{K}$-finite.

Definition 5.7 ([18]). Let $f : N \to M, f' : N' \to M'$ be smooth map-germs and $b \in M, b' \in M'$. Then the germ of f over b is right-left equivalent to the germ of f' over b', if there exists an open neighborhood U of b in M, an open neighborhood U' of b' in M', a diffeomorphism $\Phi : f^{-1}(U) \to f'^{-1}(U')$ and a diffeomorphism $\varphi : U \to U'$ such that the diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\Phi} & f'^{-1}(U') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow[\varphi]{} & U' \end{array}$$

commutes.

Remark 5.8. The right-left equivalence class of the germ of $f_{g,\sigma} : N \to M$ over $b \in M$ in Proposition 5.4 depends only on the right-left equivalence class of the germ g and the conjugacy class of σ in G_g. Similarly the right-left equivalence class of the germ of $f_{h_1,\dots,h_s;\sigma_1,\dots,\sigma_s} : N \to M$ over $b \in M$ the right-left equivalence class of the multi-germ $(g_1, \dots, g_s)$ from the disjoint union of s-copies of $(\mathbb{R}^m, 0)$ to (M, b), and the conjugacy classes of σ_i in G_{g_i}.

Theorem 5.9 (Classification theorem of cofrontals with one-dimensional fibers). *Let N be a compact smooth manifold of dimension $m + 1$, and M a* *smooth manifold of dimension m. Let $f : N \to M$ be any reduction-finite cofrontal and $b \in M$. Then the germ f over b is right-left equivalent to the germ $f_{h_1,\dots,h_s;\sigma_1,\dots,\sigma_s}$ over b for some non-negative integer s, $\mathcal{K}$-finite map-germs $h_i : (\mathbb{R}^m, 0) \to (M, b)$ and elements $\sigma_i \in G_{g_i}$ of finite order $(1 \le i \le s)$.*

LEMMA 5.10. *Let $h : (\mathbb{R}^m, S) \to (M, b)$ be a multi-germ with $S = \{x_1, \dots, x_s\}$. Suppose all germ $h_i = h_{x_i} : (\mathbb{R}^m, x_i) \to (\mathbb{R}^m, 0)$ are $\mathcal{K}$-finite. Let $\sigma_i \in G_{h_i}$. Then there exist open neighborhood V of b, open neighborhood U_i of x_i and representatives $h_i : U_i \to M$ of h_i and $\sigma_i : U_i \to U_i$ such that $h_i^{-1}(V) = U_i$ and $h_i\sigma_i = h_i$ on U_i for $i = 1, \dots, s$.*

Proof. Let $g : (\mathbb{R}^m, 0) \to (M, b)$ be a $\mathcal{K}$-finite map-germ. Then $m_0^k \subset f^*(m_b)$ for some positive integer k. Then there exist $\alpha > 0, C > 0$ such that $C\|x\|^\alpha \le \|g(x)\|$ on a neighborhood of $0 \in \mathbb{R}^m$. Therefore, for any representative $g : W \to M$ of g, and for any neighborhood W' of 0 with $W' \subseteq W$, there exists an open neighborhood V of b such that $g^{-1}(V) \subset W'$. For each h_i take such an open neighborhood V_i of b such that $h_i \circ \sigma_i = h_i$ holds on $h^{-1}(V_i)$. Then set $V = \cap_{i=1}^s V_i$ and $U_i = h_i^{-1}(V)$. Then $\sigma(U_i) = U_i$ and $h_i \circ \sigma_i = h_i$ on U_i. $\square$

LEMMA 5.11. *Let $f : N \to M$ be a cofrontal of reduction-finite, N compact and $b \in M$. Then the fiber $f^{-1}(b)$ over b consists of a finite number of disjoint circles in N. Each connected component has an open neighborhood consisting of leaves of a kernel field of f.*

Proof. First we remark that, since the cofrontal f is reduction-finite, f is fair and therefore there exists the unique global kernel field K of f. Take any $a \in N$. Let L be the leaf through a of the foliation $\mathcal{F}$ defined by K. Then the germ f_a has a $\mathcal{K}$-finite reduction. Then there exists an adapted open neighborhood W_a of form $U \times V, \dim(U) = m, \dim(V) = n-m$ such that $\{p\} \times V$ is contained in a leaf of $\mathcal{F}$ for any $p \in U$. If we take U sufficiently small, then $W_a \cap f^{-1}(b) = W_a \cap L$, since f is reduction-finite. Set $W = \cup_{a \in L} W_a$. Then W is an open set in N and $W \cap f^{-1}(b) = W \cap L$. Thus we have seen that L is a closed, therefore compact submanifold in N. In particular L is diffeomorphic to the circle S^1. Moreover L has an open neighborhood consists of leaves of $\mathcal{F}$. Then we have that the number of connected components is finite. $\square$

Proof of Theorem 5.9. Since N is compact and the cofrontal f is reduction-finite, $f^{-1}(b)$ consists of a finite number of disjoint circles $L_1, L_2, \dots L_s$ in N by Lemma 5.11. Each L_i has an open neighborhood W_i consisting of leaves of the foliation $\mathcal{F}$ of the kernel field K of f. By taking each W_i small enough, we have that $W_i \cap W_j = \emptyset$ for $i \ne j$. Now f is locally constant on each leaf of $\mathcal{F}$. (See the proof of Lemma 4.2.) Since each leaf is assumed to be connected, f is constant on each leaf of $\mathcal{F}$. Take a transversal U_i of dimension m to the leaves on W_i through a point on L_i. Then we have the Poincaré map $\sigma_i : U_i \to U_i$ by moving along leaves of $\mathcal{F}$. We have that $f \circ \sigma_i = \sigma_i$ on U_i. Set $h_i = f|_{U_i}$. Then we have that the germ of f over b is right-left equivalent to $f_{g_1,\dots,g_s;\sigma_1,\dots,\sigma_s}$ over b. Taking U_i sufficiently small, then the number of fibers of h_i is bounded (Remark 2.7). Then any σ_i-orbit on U_i has bounded period. Therefore σ_i must be of finite order. $\square$

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