Dedicated to Tzee-Char Kuo for his 81st birthday

GLOBAL DIRECTIONAL PROPERTIES OF SINGULAR SPACES

SATOSHI KOIKE and LAURENŢIU PĂUNESCU

In a previous series of papers [6–10] we have discussed several local directional properties in order to introduce local bi-Lipschitz invariants and to explore the local structure of sets satisfying the sequence selection property (denoted by (SSP)). As an application of them, we gave a classification of planar spirals in the Appendix of [8].

In this paper, in order to introduce global bi-Lipschitz invariants, we globalise the (SSP) structure preserving theorem, directional weak transversality theorem and directional transversality theorem, proved in [8]. In particular, we show a few global complex transversality results.

In addition, to study more global directional properties, we introduce the notion of geometric directional bundle of a singular space, and prove many fundamental properties of it. Furthermore, we introduce a notion of *$\mathcal{GD}$-idempotent* related to the geometric directional bundle, aiming at distinguishing various graphs of oscillations including the zigzag curves.

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1. INTRODUCTION

Directional properties at singular points of algebraic and analytic sets were first studied by H. Whitney [27, 28], in order to show that those sets are locally topologically trivial fibrations. Indeed, Whitney introduced the directional properties, now called Whitney, (a), (b) regularities on stratified sets, and used them to achieve that purpose. The local triviality was shown in Whitney's framework by René Thom [22, 23] as Thom's 1st Isotopy Lemma, and then generalised to the mapping case as Thom's 2nd Isotopy Lemma [4, 12, 22]. Thom established these results by using controlled tubular neighbourhood

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systems. (This mathematical field is called the *Whitney-Thom stratification theory.*) In [27] Whitney conjectured that a stronger triviality than topological one holds, formulating his fibering conjecture. Recently, the Whitney fibering conjecture was solved by A. Parusiński and the second author [19]. After the work of Whitney and Thom, many more regularity conditions were introduced on stratified sets, for details see D. Trotman [25]. Furthermore, several Lipschitz equisingular conditions similar to the above Whitney and Thom's topological ones were considered by T. Mostowski [14] and A. Parusiński [16–18].

In this decade, we have worked on directional properties related to the Lipschitz equisingularity problem. In particular, we introduced in [6] a directional property called sequence selection property, denoted by (SSP) for short, to show that the dimension of the common direction set of two subanalytic subsets is a local Lipschitz invariant. In addition, we generalised the above result to the case of a general real closed field, and further showed more properties on (SSP) in [7]. After the above works, we have worked on condition (SSP) in order to establish the geometry of sets satisfying (SSP). In fact, we proved in [8] several fundamental directional properties of sets satisfying (SSP) with respect to bi-Lipschitz homeomorphisms. For instance, we proved two types of (SSP) structure preserving theorems in [8]. In order to unify them, we defined the notion of directional homeomorphism in [9], and further proved three important directional properties with respect to the directional homeomorphisms. In the works concerning the Whitney-Thom stratification theory and its related regularity conditions, directional properties are used to show triviality of family of sets or mappings. On the other hand, we are using directional properties in our work to introduce bi-Lipschitz invariants.

Using local directional properties shown in [8], we gave a classification of planar spirals. The research on spirals has a very long history. The structure of important planar spirals has been clarified. Some local directional properties connected with the local bi-Lipschitz homeomorphisms can be useful to distinguish such spirals. On the other hand, lots of types of oscillations including zigzags appear in many areas of Natural Science and Social Science as the graphs of their examining functions or measuring functions, for instance, the density of carbon dioxide, the shift of exchange rate and so on. Local directional properties and local Lipschitz invariants are not enough to distinguish many of them. The purpose in this paper is to study global directional properties and introduce global Lipschitz invariants in order to distinguish more graphs of oscillations.

In §2 we first give the definitions of the direction set and condition (SSP) and recall two important directional properties concerning (SSP) and local bi-Lipschitz transformations. Then we mention the directional dimension theorem.

The generalised directional dimension theorems in two different directions [7,10] are also mentioned. At the end of this section, we give the list of planar spirals whose classification was shown in [8].

In §3 we discuss numerical properties concerning summation and multiplication over the sets of subsets of $\mathbb{R}^n$ satisfying condition (SSP). Then, in order to introduce global bi-Lipschitz invariants, we globalise the (SSP) structure preserving theorem, directional weak transversality theorem and directional transversality theorem, results which are proved in [8]. Subsequently, we show a few global complex transversality results. In particular, we prove that the directional complex-weak transversality (resp. the directional transversality) of a subset of $\mathbb{C}^n$ (resp. a complex analytic variety) to the canonical stratification of a given complex analytic variety is preserved under bi-Lipschitz homeomorphisms.

In §4, in order to study more global directional properties, we introduce the notion of geometric directional bundle of a singular space as a generalisation of tangent bundle of a smooth manifold. We discuss many fundamental properties of geometric directional bundle in this section.

In §5 we introduce the notion of *$\mathcal{GD}$-idempotent* for a subset of $\mathbb{R}^n$, notion related to the geometric directional bundle. Then we propose the stabilisation problem of the operation $\mathcal{GD}$. We show the stabilisation problem affirmatively in the case where $n = 2$, and give a characterisation of *$\mathcal{GD}$-idempotent* for a subset of $\mathbb{R}^2$. We use this characterisation, aiming at distinguishing various graphs of oscillations including the zigzag curves.

Throughout this paper, we use the following notations:

Let $\{a_m\}$, $\{b_m\}$ be sequences of points of $\mathbb{R}^n$ tending to the origin $0 \in \mathbb{R}^n$.

(1) If $\lim_{m\to\infty} \frac{\|a_m\|}{\|b_m\|} = 0$, then we write $\|a_m\| \ll \|b_m\|$.

(2) If there are a natural number $N \in \mathbb{N}$ and a real number $K > 0$ such that

$$\|a_m\| \leq K\|b_m\|, \quad \forall m \geq N,$$

then we write $\|a_m\| \precsim \|b_m\|$ (or $\|b_m\| \succsim \|a_m\|$). If $\|a_m\| \precsim \|b_m\|$ and $\|b_m\| \precsim \|a_m\|$, we write $\|a_m\| \approx \|b_m\|$.

2. DIRECTIONAL DIMENSION THEOREM AND PLANAR SPIRALS

In this section, we explain the directional dimension theorem treated in [6, 7, 10] and a classification of planar spirals given in [8].

2.1. **Direction set and condition (SSP)**

Let us recall the direction set.

Definition 2.1. Let A be a subset of $\mathbb{R}^n$, and let $p \in \mathbb{R}^n$ such that $p \in \overline{A}$, where $\overline{A}$ denotes the closure of A in $\mathbb{R}^n$. We define the *direction set* $D_p(A)$ of A at p by

$$D_p(A) := \{a \in S^{n-1} \mid \exists \{x_i\} \subset A \backslash \{p\},\ x_i \to p \in \mathbb{R}^n \text{ s.t. } \frac{x_i - p}{\|x_i - p\|} \to a,\ i \to \infty\}.$$

Here S^{n-1} denotes the unit sphere centred at $0 \in \mathbb{R}^n$.

For a subset $D_p(A) \subset S^{n-1}$, we denote by $LD_p(A)$ the half-cone of $D_p(A)$ with $0 \in \mathbb{R}^n$ as the vertex, and call it the *real tangent cone* of A at p:

$$LD_p(A) := \{ta \in \mathbb{R}^n \mid a \in D_p(A),\ t \geq 0\}.$$

For $p \in \overline{A}$, for simplicity, we put $L_pD(A) := p + LD_p(A)$, and call it the *geometric tangent cone* of A at p.

In the case where $p = 0 \in \mathbb{R}^n$, we write $D(A) := D_0(A)$ and $LD(A) := L_0D(A)$ for short.

We next recall the original notion of sequence selection property with respect to the direction set at a point.

Definition 2.2. Let A be a set-germ at $p \in \mathbb{R}^n$ such that $p \in \overline{A}$. We say that A satisfies *condition* (SSP) at p, if for any sequence of points $\{a_m\}$ of $\mathbb{R}^n$ tending to $p \in \mathbb{R}^n$, such that $\lim_{m\to\infty} \frac{a_m - p}{\|a_m - p\|} \in D_p(A)$, there is a sequence of points $\{b_m\} \subset A$ such that,

$$\|a_m - b_m\| \ll \|a_m - p\|,\ \|b_m - p\|,$$

i.e. $\lim_{m\to\infty} \frac{\|a_m - b_m\|}{\|a_m - p\|} = 0$.

Remark 2.3. Let $A \subset \mathbb{R}^n$ be a set-germ at $p \in \mathbb{R}^n$ such that $p \in \overline{A}$.

(1) The cone $L_pD(A)$ satisfies condition (SSP) at p.

(2) If A is subanalytic, then it satisfies condition (SSP) at p. See H. Hironaka [5] for subanalyticity.

In regard to the condition (SSP), we recall two important local directional properties concerning the behaviour under a local bi-Lipschitz homeomorphism.

Let $A \subset \mathbb{R}^n$ be a set-germ at $p \in \mathbb{R}^n$ such that $p \in \overline{A}$, and let $h : (\mathbb{R}^n, p) \to (\mathbb{R}^n, h(p))$ be a bi-Lipschitz homeomorphism. Then we have the following.

Theorem 2.4 ([6]). *Suppose that A satisfies condition (SSP) at p. Then we have*

$$D_{h(p)}(h(L_pD(A))) = D_{h(p)}(h(A)).$$

THEOREM 2.5 ([8]). *Suppose that A satisfies condition (SSP) at p. Then $h(A)$ satisfies condition (SSP) at $h(p)$ if and only if $h(L_pD(A))$ satisfies condition (SSP) at $h(p)$.*

2.2. **Directional dimension theorem**

Using directional properties, we first proved the following result.

THEOREM 2.6 ([6]). *Let A, $B \subset \mathbb{R}^n$ be subanalytic set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A)$, $h(B)$ are also subanalytic. Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

M. Oka constructed in [15] a complex polynomial family with isolated singularities which is μ-constant but not μ^*-constant. When we regard the Oka family as a real family, using the above theorem we can see that the real Oka family is not Lipschitz trivial as a family of zero-sets, nevertheless is topologically trivial as a family of functions.

Theorem 2.6 was generalised to the case of a general real closed field. Let R denote a real closed field with an o-minimal structure, and consider the topology on R given by the open intervals of R, analogous to that on $\mathbb{R}$. See [1] or [2] for o-minimal structure. We have the following directional dimension theorem.

THEOREM 2.7 ([7]). *Let A, $B \subset R^n$ be definable set-germs at $0 \in R^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (R^n, 0) \to (R^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that $h(A)$, $h(B)$ are also definable. Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

In the proof of Theorem 2.6 of [6], condition (SSP) took an important role. However it is not always a good condition over a general real closed field. Instead, we had to use the assumption of definability to show the above theorem.

On the other hand, J. Edison Sampaio gave in [20] a much shorter proof for Theorem 2.6 using the Arzela-Ascoli's theorem. In this respect, we generalise the theorem to the following form.

THEOREM 2.8 ([10]). *Let A, $B \subset \mathbb{R}^n$ be set-germs at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism.*

Suppose that A, B, $h(A)$, $h(B)$ *satisfy condition* (SSP) *at* $0 \in \mathbb{R}^n$. *Then we have the equality of dimensions,*

$$\dim(D(h(A)) \cap D(h(B))) = \dim(D(A) \cap D(B)).$$

Remark 2.9. The argument with the Arzela-Ascoli's theorem does not always work on a general real closed field. Therefore we don't have a simpler proof for Theorem 2.7.

2.3. **Classification of planar spirals**

In the appendix of [8], we gave a classification of planar spirals using directional properties. For the reader's convenience we recall it below.

We consider polar coordinates $(r, \theta), 0 < r, \theta < \infty$. Let $R : (0, \infty) \to (0, \infty)$ be a continuous function. We say that $S_0 : r = R(\theta)$ is a *spiral* at $0 \in \mathbb{R}^2$ if $0 \in \overline{S_0}$, R is strictly monotone and

$$\lim_{\theta \to \infty} R(\theta) = 0 \quad \text{or} \quad \lim_{\theta \to \infty} R(\theta) = \infty.$$

In the first case we write $R(\infty) = 0$ and note that the extension $R : (0, \infty] \to [0, \infty)$ is continuous and injective. In the second case we write $R(0) = 0$ and note that also the extension $R : [0, \infty) \to [0, \infty)$ is continuous and injective.

Let us introduce the homeomorphism germ induced by a spiral, defined in polar coordinates by:

$$h_{S_0} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0), \; h_{S_0}(r, \alpha) = (r, R^{-1}(r) + \alpha), \; 0 \le \alpha < 2\pi.$$

For $0 \le \alpha < 2\pi$ we put $L_\alpha := \{(r, \alpha) \mid 0 \le r < \infty\}$ and $S_\alpha := h_{S_0}(L_\alpha)$. Note that S_0 is just the spiral $r = R(\theta)$ together with $0 \in \mathbb{R}^2$.

If $0 \le \alpha < 2\pi$ we denote by R_α the rotation of $\mathbb{R}^2$ centred at the origin and of angle α. Then we have the following property.

Property 2.10. $S_\alpha = R_\alpha(S_0)$ *and* $D(h_{S_0}(L_\alpha)) = R_\alpha(D(S_0))$.

Here we recall a result concerning the direction set of the image of an intersection set.

Theorem 2.11 ([8]). *Let* $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *be a homeomorphism, and let* U, $V \subset \mathbb{R}^n$ *be set-germs at* $0 \in \mathbb{R}^n$ *such that* $0 \in \overline{U} \cap \overline{V}$. *Suppose that the following four conditions hold.*

(1) $D(U \cap V) = D(U) \cap D(V)$.
(2) $U \cap V$ *satisfies condition* (SSP) *at* $0 \in \mathbb{R}^n$.
(3) $h(U)$ *satisfies condition* (SSP) *at* $0 \in \mathbb{R}^n$.
(4) h *is bi-Lipschitz.*

Then we have $D(h(U \cap V)) = D(h(U)) \cap D(h(V))$.

We use the above theorem to classify the local structure of spirals. We apply it as $h = h_{S_0}$, $n = 2$, $U = L_\alpha$ and $V = L_\beta$ for $\alpha \neq \beta$, $\alpha, \beta \in [0, 2\pi)$.

Let us check the assumptions in Theorem 2.11. We have

$$D(L_\alpha \cap L_\beta) = D(L_\alpha) \cap D(L_\beta) = \emptyset,$$

and $L_\alpha \cap L_\beta = \{0\}$ satisfies condition (SSP) at $0 \in \mathbb{R}^2$. Therefore the first two assumptions are always satisfied.

We first consider the case when $\#(D(S_0)) > 1$, which is equivalent to the following condition: There are $\alpha \neq \beta, \alpha, \beta \in [0, 2\pi)$ such that $D(h_{S_0}(L_\alpha)) \cap D(h_{S_0}(L_\beta)) \neq \emptyset$.

On the other hand, since $D(h_{S_0}(L_\alpha \cap L_\beta)) = \emptyset$, the conclusion of Theorem 2.11 does not hold. We can therefore divide the case $\#(D(S_0)) > 1$ into the following three cases:

(1) Let us assume that h_{S_0} is a bi-Lipschitz homeomorphism, namely assumption (4) in Theorem 2.11 is satisfied. Then assumption (3) is not satisfied. Therefore S_0 does not satisfy condition (SSP) at $0 \in \mathbb{R}^2$.

(2) Let us assume that S_0 satisfies condition (SSP) at $0 \in \mathbb{R}^2$, namely assumption (3) is satisified. Then assumption (4) is not satisfied. Therefore h_{S_0} is not a bi-Lipschitz homeomorphism.

(3) S_0 does not satisfy condition (SSP) at $0 \in \mathbb{R}^2$, and h_{S_0} is not a bi-Lipschitz homeomorphism.

Finally, we have the remaining case when $\#D(S_0) = 1$. This condition is equivalent to the condition that we have $D(h_{S_0}(L_\alpha)) \cap D(h_{S_0}(L_\beta)) = \emptyset$ for $\forall \alpha \neq \beta, \alpha, \beta \in [0, 2\pi)$.

Let us remember several important planar spirals.

Example 2.12. Let (r, θ) be polar coordinates.

(1) An *equiangular spiral* or *logarithmic spiral* S_0 is defined by

$$r = ae^{b\theta} \quad (a > 0,\ b < 0;\ 0 < \theta \leq \infty).$$

Then the induced homeomorphism h_{S_0} is bi-Lipschitz (Example 3.3 in [6]). Note that we have to replace $\log(x^2 + y^2)$ with $\frac{1}{2}\log(x^2 + y^2)$ in the example.

In our daily life an equiangular spiral is seen in the growth with self-symmetry (D'Arcy Thompson [24]).

(2) A *hyperbolic spiral* S_0 is defined by

$$r = \frac{a}{\theta} \quad (a \neq 0;\ 0 < \theta \leq \infty).$$

Then S_0 satisfies condition (SSP) at $0 \in \mathbb{R}^2$ [8]. A direct proof of this fact is given by S. Miyake [13].

(3) The *Archimedean spiral* is defined by

$$r = a\theta + b \quad (a \neq 0,\ b \geq 0;\ 0 \leq \theta < \infty).$$

The Archimedean spiral has two arms, smoothly connected at $0 \in \mathbb{R}^2$.

(4) *Fermat's spiral* is defined by

$$r = \pm a\theta^{\frac{1}{2}} \quad (a > 0;\ 0 \leq \theta < \infty).$$

Fermat's spiral also has two arms, smoothly connected at $0 \in \mathbb{R}^2$.

Concerning the spirals in Example 2.12, we have the following table.

List of planar spirals

	$\#D(S_0) = 1$	$\#D(S_0) > 1$ ($\#D(S_0) = \infty$)
S_0 satisfies condition (SSP). S_0 induces a Lipschitz homeomorphism.	Archimedean spiral Fermat's spiral	
S_0 does not satisfy condition (SSP). S_0 induces a Lipschitz homeomorphism.		logarithmic spiral
S_0 satisfies condition (SSP). S_0 does not induce a Lipschitz homeomorphism.		hyperbolic spiral
S_0 does not satisfy condition (SSP). S_0 does not induce a Lipschitz homeomorphism.		We can construct an artificial example.

2.4. A similar classification of graphs of oscillations

Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a continuous function with graph G_f, and let $h : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a homeomorphism germ defined by

$$h(x, y) := (x, y - f(x)).$$

Then $h(G_f) = \{x\text{-axis}\}$. We consider G_f as an *oscillation* created by f.

Let us recall our important property.

THEOREM 2.13 ([6]). *Let $A \subset \mathbb{R}^n$ be a set-germ at $0 \in \mathbb{R}^n$ such that $0 \in \overline{A}$, and let $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ be a bi-Lipschitz homeomorphism. Suppose that A satisfies condition (SSP) at $0 \in \mathbb{R}^n$. Then we have*

$$D(h(LD(A))) = D(h(A)).$$

Set $G_f^+ := G_f \cap \{x \geq 0\}$. Then we have

$$D(h(G_f^+)) = D(\{\text{non-negative } x\text{-axis}\}) = \{(1,0)\}.$$

Let us apply Theorem 2.13 as $A = G_f^+$. If $\# D(h(LD(G_f^+))) > 1$, then the equality of the theorem does not hold. Then the homeomorphism h is not bi-Lipschitz or G_f^+ does not satisfy condition (SSP). Concerning G_f^+, we have a similar classification of graphs of oscillations as that of the planar spirals (see the list in the previous subsection).

Let $G_f^- := G_f \cap \{x \leq 0\}$. Then, concerning G_f^-, we have the same kind of classification as above.

This kind of classification is too rough for oscillations. Local directional properties and local Lipschitz invariants seem to be not enough to distinguish graphs of oscillations. In this paper, we shall explain an approach to study global directional properties and introduce global Lipschitz invariants in order to distinguish more graphs of oscillations.

3. SETS OF (SSP) SETS

In this section, we consider the following sets consisting of subsets of $\mathbb{R}^n$ satisfying condition (SSP):

$$\mathcal{A}_{SSP}(\mathbb{R}^n) := \{A \subset \mathbb{R}^n \mid A \text{ satisfies } (SSP) \text{ at any point of } \mathbb{R}^n\},$$

$$\mathcal{A}^0_{SSP}(\mathbb{R}^n) := \{A \subset \mathbb{R}^n \mid A \neq \emptyset \text{ and } A \text{ satisfies (SSP) at any point of } \mathbb{R}^n\}.$$

Clearly $\mathcal{A}^0_{SSP}(\mathbb{R}^n) \cup \{\emptyset\} = \mathcal{A}_{SSP}(\mathbb{R}^n)$ (the empty set $\emptyset$ satisfies condition (SSP) at any point of $\mathbb{R}^n$).

Concerning condition (SSP) for a subset of $\mathbb{R}^n$ we have the following properties.

PROPERTY 3.1. (1) *Let $A \subset \mathbb{R}^n$ such that $A \neq \emptyset$.*

(i) *If p is not a point of the closure of A, then A satisifies condition (SSP) at p.*

(ii) *If p is an isolated point of A, then A satisfies condition (SSP) at p.*

(iii) *If A satisfies condition (SSP) at any accumulation point of A, then A satisfies condition (SSP) at any point of $\mathbb{R}^n$.*

(2) *For any point $p \in \mathbb{R}^n$, $\{p\} \in \mathcal{A}^0_{SSP}(\mathbb{R}^n) \subset \mathcal{A}_{SSP}(\mathbb{R}^n)$.*

3.1. Set-theoretical properties

In [8–10] we gave several local set-theoretical properties on condition (SSP). Some of them can be globalised. We give such examples in this subsection.

Lemma 3.2 ([9]). *Let $A \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$ such that $p \in \overline{A}$. Then A satisfies condition (SSP) at $p \in \mathbb{R}^n$ if and only if so does $\overline{A}$.*

We have the following corollary of Lemma 3.2.

Corollary 3.3. *Let $A \subset \mathbb{R}^n$. Then $A \in \mathcal{A}_{SSP}(\mathbb{R}^n)$ if and only if $\overline{A} \in \mathcal{A}_{SSP}(\mathbb{R}^n)$.*

Lemma 3.4 ([8]). *Let $A \subset \mathbb{R}^n$ and $p \in \mathbb{R}^n$ such that $p \in \overline{A}$. If A is a finite union of sets, all of which satisfy condition (SSP) at $p \in \mathbb{R}^n$, then A satisfies condition (SSP) at $p \in \mathbb{R}^n$.*

The next proposition is an immediate consequence of Lemma 3.4.

Proposition 3.5. *Let $\{A_\lambda\}$ be a locally finite family of subsets of $\mathbb{R}^n$. If $A_\lambda \in \mathcal{A}_{SSP}(\mathbb{R}^n)$ $(\lambda \in \Lambda)$, then we have*

$$\bigcup_{\lambda \in \Lambda} A_\lambda \in \mathcal{A}_{SSP}(\mathbb{R}^n).$$

3.2. Numerical properties

Let $\mathcal{A}$ denote the set $\{a_m\}$ of positive real numbers tending to $0 \in \mathbb{R}$. In [10] we gave the following criterion for a sequence in $\mathcal{A}$ to satisfy the condition (SSP).

Lemma 3.6. *For $A \in \mathcal{A}$, $A = \{a_m\}$ numbered in decreasing order, satisfies condition (SSP) at $0 \in \mathbb{R}$ if and only if*

$$\frac{a_m}{a_{m+1}} \to 1 \text{ as } m \to \infty.$$

Let $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. Then we have a corollary of Lemma 3.6.

Corollary 3.7. *For $A \subset \mathbb{R}_+ \subset \mathbb{R}$ such that $0 \in \overline{A}$, the following conditions are equivalent.*

(1) *A satisfies condition (SSP) at $0 \in \mathbb{R}$.*

(2) *There exists a countable subset B of A such that B satisfies condition (SSP) at $0 \in \mathbb{R}$.*

(3) *There exists a decreasing sequence $\{b_m\} \in \mathcal{A}$ such that $\lim_{m\to\infty} \frac{b_m}{b_{m+1}} = 1$.*

Proof. By definition, it is obvious that condition (2) implies condition (1). By renumbering, the equivalence of conditions (2) and (3) follows from Lemma 3.6. Therefore we show that condition (1) implies condition (2).

Consider the sequence $\{\frac{1}{m}\} \in \mathcal{A}$. Since A satisfies condition (SSP) at $0 \in \mathbb{R}$, there exists a sequence $\{a_m\} \subset A$ such that $|a_m - \frac{1}{m}| \ll |\frac{1}{m}|$.

Let $\{c_m\}$ be arbitrary sequence in $\mathcal{A}$. By Lemma 3.6, $\{\frac{1}{m}\}$ satisfies condition (SSP) at $0 \in \mathbb{R}$. Therefore there exists a subsequence $\{\frac{1}{n(m)}\}$ of $\{\frac{1}{m}\}$ such that $|\frac{1}{n(m)} - c_m| \ll |\frac{1}{n(m)}|$. It follows that

$$|a_{n(m)} - c_m| \leq |a_{n(m)} - \frac{1}{n(m)}| + |\frac{1}{n(m)} - c_m| \ll \frac{1}{n(m)} \sim c_m.$$

Thus $B := \{a_{n(m)}\}$ satisfies condition (SSP) at $0 \in \mathbb{R}$. □

Remark 3.8. The equivalence of the first two statements follows as well from the separability of $\mathbb{R}$ and by Lemma 3.2.

Remark 3.9. In [3] we use a generalisation of condition (SSP), denoted by $\widetilde{(SSP)}$, to prove several similar results and show that there is no bi-Lipschitz homeomorphism of $\mathbb{R}^2$ that maps a spiral with a sub-exponential decay of winding radii to an unwound arc. This result is sharp as shows an example of a logarithmic spiral.

For $A,\ B \subset \mathbb{R}$, we define the summation $+$ as follows:

$$A + B = B + A := \{a + b \mid a \in A,\ b \in B\},$$

where $A + \emptyset = \emptyset + B := \emptyset$. Using Property 3.1 and Corollary 3.7, we can see the following properties on the summation.

PROPOSITION 3.10. (1) *If* $A,\ B \in \mathcal{A}_{SSP}(\mathbb{R})$, *then* $A + B \in \mathcal{A}_{SSP}(\mathbb{R})$. *The unit with respect to* $+$ *in* $\mathcal{A}_{SSP}(\mathbb{R})$ *is* $\{0\}$.

(2) *If* $A,\ B \in \mathcal{A}^0_{SSP}(\mathbb{R})$, *then* $A + B \in \mathcal{A}^0_{SSP}(\mathbb{R})$. *The unit with respect to* $+$ *in* $\mathcal{A}^0_{SSP}(\mathbb{R})$ *is* $\{0\}$.

(3) *Define* $\tau : \mathcal{A}_{SSP}(\mathbb{R}) \times \mathcal{A}^0_{SSP}(\mathbb{R}) \to \mathcal{A}^0_{SSP}(\mathbb{R})$ *by* $\tau(A, B) = A + B$. *Then* τ *is well-defined.*

For $A,\ B \subset \mathbb{R}$, we define also the multiplication as follows:

$$AB = BA := \{ab \mid a \in A,\ b \in B\},$$

where $A\emptyset = \emptyset B := \emptyset$. Then we have the following.

PROPOSITION 3.11. (1) *If* $A,\ B \in \mathcal{A}_{SSP}(\mathbb{R})$, *then* $AB \in \mathcal{A}_{SSP}(\mathbb{R})$. *The unit with respect to the multiplication in* $\mathcal{A}_{SSP}(\mathbb{R})$ *is* $\{1\}$.

(2) *If* $A,\ B \in \mathcal{A}^0_{SSP}(\mathbb{R})$, *then* $AB \in \mathcal{A}^0_{SSP}(\mathbb{R})$. *The unit with respect to the multiplication in* $\mathcal{A}^0_{SSP}(\mathbb{R})$ *is* $\{1\}$.

Remark 3.12. Note that $\mathcal{A}^0_{SSP}(\mathbb{R})$ does not form a commutative ring, because there does not exist an inverse in general with respect to $+$. Nevertheless in our context both the addition and the multiplication are associative and commutative operations.

3.3. (SSP) structure preserving theorem

We recall the notion of (SSP) map for a map-germ, and define the notion for global maps.

Definition 3.13. (1) Let $A \subset \mathbb{R}^m$ be a set-germ at $p \in \mathbb{R}^m$ such that $p \in \overline{A}$, and let $B \subset \mathbb{R}^n$ be a set-germ at $q \in \mathbb{R}^n$ such that $q \in \overline{B}$. We say that a map-germ $h : (A, p) \to (B, q)$, $q = h(p)$, is an (SSP) *map* at (p, q) if the graph of h satisfies condition (SSP) at $(p, q) \in \mathbb{R}^m \times \mathbb{R}^n$.

(2) We say that a map $h : \mathbb{R}^m \to \mathbb{R}^n$ is an (SSP) *map* if the graph of h satisfies condition (SSP) at any point of the closure of the graph of h (or at any $(p, q) \in \mathbb{R}^m \times \mathbb{R}^n$).

In [8] we proved the (SSP) structure preserving theorem in the local case.

THEOREM 3.14. *Let $p \in \mathbb{R}^n$, let $h : (\mathbb{R}^n, p) \to (\mathbb{R}^n, h(p))$ be an (SSP) bi-Lipschitz homeomorphism, and let $A \subset \mathbb{R}^n$ be a set-germ at $p \in \mathbb{R}^n$ such that $p \in \overline{A}$. Then A satisfies condition (SSP) at $p \in \mathbb{R}^n$ if and only if $h(A)$ satisfies condition (SSP) at $h(p) \in \mathbb{R}^n$.*

As a corollary of the (SSP) structure preserving theorem, we have the following global result.

COROLLARY 3.15. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an (SSP) bi-Lipschitz homeomorphism. Then h induces a one-to-one correspondence*

$$h_* : \mathcal{A}_{SSP}(\mathbb{R}^n) \to \mathcal{A}_{SSP}(\mathbb{R}^n)$$

defined by $h_*(A) := h(A)$, and similarly also a one-to-one correspondence

$$h^0_* : \mathcal{A}^0_{SSP}(\mathbb{R}^n) \to \mathcal{A}^0_{SSP}(\mathbb{R}^n).$$

defined by $h^0_*(A) := h(A)$.

3.4. Transversality results

We first recall the local notion of weak transversality, and define its global analogue.

Definition 3.16. (1) Let A, $B \subset \mathbb{R}^n$ be set-germs at $p \in \mathbb{R}^n$ such that $p \in \overline{A} \cap \overline{B}$. We say that A and B are *directionally weakly transverse* or *weakly transverse* simply *at* $p \in \mathbb{R}^n$ if $D_p(A) \cap D_p(B) = \emptyset$ (or $LD_p(A) \cap LD_p(B) = \{0\}$).

(2) Let A, $B \subset \mathbb{R}^n$. We say that A and B are *directionally weakly transverse* or *weakly transverse* simply if A and B are weakly transverse at any point $p \in \overline{A} \cap \overline{B}$.

In [8] we proved the preserving of local weak transversality result.

THEOREM 3.17. *Let $p \in \mathbb{R}^n$, let $A,\ B \subset \mathbb{R}^n$ be set-germs at $p \in \mathbb{R}^n$ such that $p \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, p) \to (\mathbb{R}^n, h(p))$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP) at $p \in \mathbb{R}^n$, and $h(A)$ or $h(B)$ satisfies condition (SSP) at $h(p) \in \mathbb{R}^n$. Then A and B are weakly transverse at $p \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are weakly transverse at $h(p) = q \in \mathbb{R}^n$.*

As a corollary of the above weak transversality theorem, we have the following global result.

COROLLARY 3.18. *Let $A,\ B \subset \mathbb{R}^n$, and let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz homeomorphism. Suppose that A or B satisfies condition (SSP) at any $p \in \overline{A} \cap \overline{B}$, and $h(A)$ or $h(B)$ satisfies condition (SSP) at any $q \in \overline{h(A)} \cap \overline{h(B)}$. Then A and B are weakly transverse if and only if $h(A)$ and $h(B)$ are weakly transverse.*

By corollaries 3.15 and 3.18, we have one more corollary.

COROLLARY 3.19. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an (SSP) bi-Lipschitz homeomorphism. Then, for any $A \in \mathcal{A}_{SSP}(\mathbb{R}^n)$ and $B \subset \mathbb{R}^n$, A and B are weakly transverse if and only if $h(A)$ and $h(B)$ are weakly transverse.*

We recall also the local notion of transversality, and similarly define the global notion.

Definition 3.20. (1) Let $A,\ B \subset \mathbb{R}^n$ be set-germs at $p \in \mathbb{R}^n$ such that $p \in \overline{A} \cap \overline{B}$. We say that A and B are *directionally transverse* or *transverse* simply *at* $p \in \mathbb{R}^n$ if

$$\dim LD_p(A) + \dim LD_p(B) - \dim(LD_p(A) \cap LD_p(B)) = n.$$

(2) Let $A,\ B \subset \mathbb{R}^n$. We say that A and B are *directionally transverse* or *transverse* simply if A and B are transverse at any point $p \in \overline{A} \cap \overline{B}$.

In [10] we proved the following preserving of local transversality result.

THEOREM 3.21. *Let $p \in \mathbb{R}^n$, let $A,\ B \subset \mathbb{R}^n$ be set-germs at $p \in \mathbb{R}^n$ such that $p \in \overline{A} \cap \overline{B}$, and let $h : (\mathbb{R}^n, p) \to (\mathbb{R}^n, h(p))$ be a bi-Lipschitz homeomorphism. Suppose that A and B satisfy condition (SSP) at $p \in \mathbb{R}^n$, and $h(A)$ and $h(B)$ satisfy condition (SSP) at $h(p) \in \mathbb{R}^n$. Then A and B are transverse at $p \in \mathbb{R}^n$ if and only if $h(A)$ and $h(B)$ are transverse at $h(p) \in \mathbb{R}^n$.*

As a corollary of the above local transversality theorem, we have the following global result.

COROLLARY 3.22. *Let $A, B \subset \mathbb{R}^n$, and let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a bi-Lipschitz homeomorphism. Suppose that A and B satisfy condition (SSP) at any $p \in \overline{A} \cap \overline{B}$, and $h(A)$ and $h(B)$ satisfy condition (SSP) at any $q \in \overline{h(A)} \cap \overline{h(B)}$. Then A and B are transverse if and only if $h(A)$ and $h(B)$ are transverse.*

We have one more corollary.

COROLLARY 3.23. *Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be an (SSP) bi-Lipschitz homeomorphism. Then, for any $A, B \in \mathcal{A}_{SSP}(\mathbb{R}^n)$, A and B are transverse if and only if $h(A)$ and $h(B)$ are transverse.*

3.5. **Complex transversality results**

Let us recall the complex tangent cone. Let $A \subset \mathbb{C}^n$ be a set-germ at $p \in \mathbb{C}^n$ such that $p \in \overline{A}$. The *complex tangent cone* of A at p is defined as follows:

$$LD^*_p(A) := \{v \in \mathbb{C}^n : \exists \{C_i\} \subset \mathbb{C},\ \exists \{v_i\} \subset A \setminus \{p\} \to p \in \mathbb{C}^n \text{ s.t. } \lim_{i \to \infty} c_i(v_i - p) = v\}.$$

Let $\mathbb{C}D_p(A) := \{cv \in \mathbb{C}^n \ : \ c \in \mathbb{C},\ v \in D_p(A)\}$ for $p \in \mathbb{C}^n$.

Concerning $\mathbb{C}D_p(A)$, we have the following property.

LEMMA 3.24 ([8]). *$LD^*_p(A) = \mathbb{C}D_p(A)$.*

By the above lemma, we can easily see the following.

LEMMA 3.25. *$LD^*_p(A) = LD^*_p(\overline{A})$.*

We recall one more lemma.

LEMMA 3.26 ([8]). *Let p be a point of an analytic subvariety $V \subset \mathbb{C}^n$, and let $A \subset \mathbb{C}^n$ such that $p \in \overline{A}$. Then we have*

$$LD^*_P(V) \cap LD^*_p(A) = \{0\} \quad \textit{if and only if} \quad LD_P(V) \cap LD_p(A) = \{0\}.$$

We next define the complex version of weak transversality similarly to the real case.

Definition 3.27. (1) Let $A, B \subset \mathbb{C}^n$ be set-germs at $p \in \mathbb{C}^n$ such that $p \in \overline{A} \cap \overline{B}$. We say that A and B are *directionally complex-weakly transverse* or *complex-weakly transverse* simply *at* $p \in \mathbb{C}^n$ if

$$LD^*_p(A) \cap LD^*_p(B) = \{0\}.$$

(2) Let $A, B \subset \mathbb{C}^n$. We say that A and B are *directionally complex-weakly transverse* or *complex-weakly transverse* simply if A and B are complex-weakly transverse at any point $p \in \overline{A} \cap \overline{B}$.

Now we recall Shiffman's theorem.

THEOREM 3.28 (B. Shiffman [21]). *Let U be open in $\mathbb{C}^n$, and let E be closed in U. Let A be a pure k-dimensional analytic set in $U \setminus E$, and let A' be the closure of A in U. If E has Hausdorff $(2k-1)$-measure zero, then A' is a pure k-dimensional analytic set in U.*

Using Shiffman's theorem and the above lemmas, we can show the following lemma.

LEMMA 3.29. *Let $V \subset \mathbb{C}^n$ be an analytic variety, let W be an analytic subvariety of V with $\dim W < \dim V$, and let $A \subset \mathbb{C}^n$. Let $p \in \mathbb{C}^n$ such that $p \in \overline{V \setminus W} \cap \overline{A}$. Then $V \setminus W$ and A are complex-weakly transverse at p if and only if $V \setminus W$ and A are weakly transverse at p.*

Proof. We first remark that by Theorem 3.28, $\overline{V \setminus W}$ is an analytic set in $\mathbb{C}^n$. We assume that $V \setminus W$ and A are complex-weakly transverse at p, namely

$$LD_p^*(V \setminus W) \cap LD_p^*(A) = \{0\}.$$

By Lemma 3.25, this condition is equivalent to

$$LD_p^*(\overline{V \setminus W}) \cap LD_p^*(A) = \{0\}.$$

Then it follows from Lemma 3.26 that the above condition is equivalent to

$$LD_p(\overline{V \setminus W}) \cap LD_p(A) = \{0\}.$$

Since $LD_p(\overline{V \setminus W}) = LD_p(V \setminus W)$, this condition is equivalent to

$$LD_p(V \setminus W) \cap LD_p(A) = \{0\},$$

namely $V \setminus W$ and A are weakly transverse at p. □

By definition, the next global result follows from the above local lemma.

PROPOSITION 3.30. *Let $V \subset \mathbb{C}^n$ be an analytic variety, let W be an analytic subvariety of V with $\dim W < \dim V$, and let $A \subset \mathbb{C}^n$. Then $V \setminus W$ and A are complex-weakly transverse if and only if $V \setminus W$ and A are weakly transverse.*

Recently, J. Edison Sampaio proved in [20] a very useful result on Lipschitz regularity. Here we recall it.

Definition 3.31. Let $x_0 \in X \subset \mathbb{R}^n$ be a subanalytic set. We say that X is *Lipschitz regular* at x_0 if there exists an open neighbourhood of x_0 in X which is bi-Lipschitz homeomorphic to an Euclidean ball.

THEOREM 3.32 (J. Edison Sampaio [20]). *Let $X \subset \mathbb{C}^n$ be an analytic variety. If X is Lipschitz regular at $x_0 \in X$, then x_0 is a smooth point of X.*

Remark 3.33. Let us consider the notion of *topological regularity* similarly to the Lipschitz one. It is known by Lê Dũng Tráng [11] that if $X \subset \mathbb{C}^n$ is a hypersurface and topologically regular at $x_0 \in X$, then x_0 is a smooth point of X. But this does not always hold in the non-hypersurface case. On the other hand, the above result in the Lipschitz case holds in the general case, not only in the hypersurface case.

Let $V \subset \mathbb{C}^n$ be an analytic variety. We set $V := \Sigma^0(V)$. If V is smooth, we stop here. But if V is not smooth, let us denote by $\Sigma^1(V)$ the set of non-smooth points of V. Note that $\Sigma^1(V)$ is an analytic subvariety of V with $\dim \Sigma^1(V) < \dim V$. Then if $\Sigma^1(V)$ exists and is smooth, we stop here. But if $\Sigma^1(V)$ exists and is not smooth, let us denote by $\Sigma^2(V)$ the set of non-smooth points of $\Sigma^1(V)$. Note that $\Sigma^2(V)$ is an analytic subvariety of $\Sigma^1(V)$ with $\dim \Sigma^2(V) < \dim \Sigma^1(V)$. We continue this procedure until $\Sigma^k(V)$ is smooth for some k, $0 \le k \le \dim V$. Note that we are assuming that $\Sigma^j(V)$ is not smooth for $0 \le j < k$.

By the above construction, we get a filtration of a complex analytic variety V by complex analytic subvarieties:

$$V = \Sigma^0(V) \supset \Sigma^1(V) \supset \cdots \supset \Sigma^k(V).$$

Then we set

$$S^j(V) := \Sigma^j(V) \setminus \Sigma^{j+1}(V) \ (0 \le j < k-1), \ S^k(V) := \Sigma^k(V).$$

Note that

$$V = \bigcup_{j=0}^{k} S^j(V)$$

is a disjoint union where $\dim S^j(V) < \dim S^{j-1}(V)$ for $1 \le j \le k$ and each $S^j(V)$ is smooth for $0 \le j \le k$. Therefore

$$\mathcal{S}_c(V) := \{S^0(V), S^1(V), \cdots, S^k(V)\}$$

gives a smooth stratification of V. We call $\mathcal{S}_c(V)$ the *canonical stratification* of V.

Let V, $W \subset \mathbb{C}^n$ be analytic varieties. Then V and W admit the canonical stratifications

$$\mathcal{S}_c(V) := \{S^0(V), S^1(V), \cdots, S^k(V)\}, \ \mathcal{S}_c(W) := \{S^0(W), S^1(W), \cdots, S^m(W)\}.$$

Then we have the following theorem.

Theorem 3.34. *Suppose that there exists a bi-Lipschitz homeomorphism* $h : \mathbb{C}^n \to \mathbb{C}^n$ *such that* $h(V) = W$. *Then* $k = m$, *and* $h(S^j(V)) = S^j(W)$ *for* $0 \le j \le k$, *in other words,* $h(\mathcal{S}_c(V)) = \mathcal{S}_c(W)$.

Proof. By Theorem 3.32 we can see that a smooth point of V corresponds to a smooth point of W by the bi-Lipschitz homeomorphism h. It follows that

$$h(S^0(V)) = S^0(W),\ h(\Sigma^1(V)) = \Sigma^1(W).$$

Thus we can show this theorem, using the downward induction on the dimension of $\Sigma^j(V)$ and $\Sigma^j(W)$. $\square$

Concerning the above canonical stratification of a complex analytic variety, by Proposition 3.30, Corollary 3.18 and Theorem 3.34, we have the following transversality theorem.

THEOREM 3.35. *Let $V,\ W \subset \mathbb{C}^n$ be complex analytic varieties, and let $A \subset \mathbb{C}^n$. Suppose that there exists a bi-Lipschitz homeopmorphism $h : \mathbb{C}^n \to \mathbb{C}^n$ such that $h(V) = W$. Then A is complex-weakly transverse to the canonical stratification $\mathcal{S}_c(V)$ of V if and only if $h(A)$ is complex-weakly transverse to the canonical stratification $\mathcal{S}_c(W)$ of W.*

In [8] we introduced the local notion of transversality for complex singular varieties. Let us recall it, and give the global notion.

Definition 3.36. (1) Let $V, W \subset \mathbb{C}^n$ be analytic varieties and $p \in V \cap W$. Then we say that V and W are *directionally transverse* or *transverse* simply *at* $p \in \mathbb{C}^n$ if the following equality holds:

$$\dim_{\mathbb{C}} LD^*_p(V) + \dim_{\mathbb{C}} LD^*_p(W) - n = \dim_{\mathbb{C}}(LD^*_p(V) \cap LD^*_p(W)).$$

(2) Let $V, W \subset \mathbb{C}^n$ be analytic varieties. Then we say that V and W are *directionally transverse* or *transverse* simply if V and W are transverse at any point $p \in V \cap W$.

Remark 3.37. (1) The above notion of transversality can be generalised for general sets $V,\ W \subset \mathbb{C}^n$.

(2) Let $V \subset \mathbb{C}^n$ be an analytic variety, and let V_1 be an analytic subvariety of V. Then, for $p \in V \setminus V_1$, $LD^*_p(V \setminus V_1) = LD^*_p(V)$.

In [8] we proved the following transversality theorem in the local case.

THEOREM 3.38. *Let $p \in \mathbb{C}^n$, let $V,\ W \subset \mathbb{C}^n$ be analytic varieties such that $p \in V \cap W$, and let $h : (\mathbb{C}^n, p) \to (\mathbb{C}^n, h(p))$ be a bi-Lipschitz homeomorphism. Suppose that $h(V),\ h(W) \subset \mathbb{C}^n$ are analytic varieties. Then V and W are transverse at $p \in \mathbb{C}^n$ if and only if $h(V)$ and $h(W)$ are transverse at $h(p) \in \mathbb{C}^n$.*

By Theorem 3.38 and Remark 3.37, we have a global transversality theorem.

THEOREM 3.39. *Let $V,\ W \subset \mathbb{C}^n$ be analytic varieties, let V_1 be an analytic subvariety of V, and let $h : \mathbb{C}^n \to \mathbb{C}^n$ be a bi-Lipschitz homeomorphism.*

Suppose that $h(V)$, $h(W)$, $h(V_1) \subset \mathbb{C}^n$ are analytic varieties. Then $V \setminus V_1$ and W are transverse if and only if $h(V \setminus V_1) = h(V) \setminus h(V_1)$ and $h(W)$ are transverse.

By Theorems 3.34 and 3.39, we have the following transversality theorem.

THEOREM 3.40. *Let U, V, $W \subset \mathbb{C}^n$ be complex analytic varieties. Suppose that there exists a bi-Lipschitz homeomorphism $h : \mathbb{C}^n \to \mathbb{C}^n$ such that $h(V) = W$, and that $h(U) \subset \mathbb{C}^n$ is an analytic variety. Then U is transverse to the canonical stratification $\mathcal{S}_c(V)$ of V if and only if $h(U)$ is transverse to the canonical stratification $\mathcal{S}_c(W)$ of W.*

4. GEOMETRIC DIRECTIONAL BUNDLES

In this section, we introduce the notion of geometric directional bundle, and describe several fundamental properties of it.

Definition 4.1. Let $A \subset \mathbb{R}^n$, and let $W \subset \mathbb{R}^n$ such that $\emptyset \neq W \subset \overline{A}$. We define the *direction set* $D_W(A)$ of A over W by

$$D_W(A) := \bigcup_{p \in W} (p, D_p(A)) \subseteq W \times S^{n-1} \subseteq \mathbb{R}^n \times S^{n-1}.$$

Remark 4.2. Note that in general the inclusion $\overline{D_A(A)} \subset \overline{D_{\overline{A}}(\overline{A})}$ is strict. Moreover $D_{\overline{A}}(A) = D_{\overline{A}}(\overline{A})$ and is not necessarily equal to $\overline{D_A(\overline{A})}$; take for instance $A = \{1/n, n \in \mathbb{N}\}$.

We call

$$\mathcal{GD}_W(A) := W \times S^{n-1} \cap \overline{D_A(A)},$$

the *geometric directional bundle* of A over W, where $\overline{D_A(A)}$ denotes the closure in $\mathbb{R}^n \times S^{n-1}$.

Let $\Pi : \mathbb{R}^n \times S^{n-1} \to S^{n-1}$ be the canonical projections defined by $\Pi(x, a) = a$. In case $W = \{p\}$ for $p \in \overline{A}$, we write

$$\mathcal{GD}_p(A) := \Pi(\mathcal{GD}_{\{p\}}(A)) \subseteq S^{n-1}.$$

This is the set of all possible limits of directional cones $D_q(A), q \in A, q \to p$.

For understanding directional properties it is also important to study the sets

$$\mathcal{GD}_W(\overline{A}) = (W \times S^{n-1}) \cap \overline{D_{\overline{A}}(A)},$$

which we call the *complete geometric directional bundle.*

We consider the half cone of $\mathcal{GD}_p(A)$ with $0 \in \mathbb{R}^n$ as the vertex

$$L\mathcal{GD}_p(A) := \{tv \in \mathbb{R}^n \mid v \in \mathcal{GD}_p(A),\ t \geq 0\}.$$

We call it the *real tangent bundle cone* of A at p.

Similarly we define the *geometric bundle cone* of A at p, to be its translation by p

$$L_p\mathcal{GD}(A) := p + L\mathcal{GD}_p(A).$$

Here we give some elementary properties of geometric directional bundle.

PROPERTY 4.3. *Let $W,\ A \subset \mathbb{R}^n$ such that $W \subset \overline{A}$.*
(1) *If $W \subset A$, then $D_W(A) \subset \mathcal{GD}_W(A)$.*
(2) *If $V \subset W$, then $\mathcal{GD}_V(A) \subset \mathcal{GD}_W(A)$.*
(3) $D_W(A) = D_W(\overline{A}) \subset \mathcal{GD}_W(\overline{A})$.
(4) $D_W(A) \cup \mathcal{GD}_W(A) \subset \mathcal{GD}_W(\overline{A})$.

Proof. We show only (3). The others follow similarly.

Since $D_p(A) = D_p(\overline{A})$ for $p \in W$, we have $D_W(A) = D_W(\overline{A})$. On the other hand, we have

$$D_W(\overline{A}) = \bigcup_{p\in W} (p, D_p(\overline{A})) \subset W \times S^{n-1} \cap \overline{\bigcup_{p\in \overline{A}} (p, D_p(\overline{A}))} = W \times S^{n-1} \cap \overline{D_{\overline{A}}(\overline{A})}$$
$$= \mathcal{GD}_W(\overline{A}). \quad \square$$

In Property 4.3 (4) the opposite inclusion does not always hold, in other words, the equality does not always hold. Actually, we have the following negative examples.

Example 4.4. (1) Let $Z \subset \mathbb{R}^2$ be an equiangular zigzag such that $0 \in Z$ (see Fig. EAZZ), and let $A := Z \setminus \{0\}$. Set $W := A$. Then we can see that

$$D_{\{0\}}(A) \subset \mathcal{GD}_W(\overline{A}),\ \text{but}\ D_{\{0\}}(A) \not\subset D_W(A) \cup \mathcal{GD}_W(A).$$

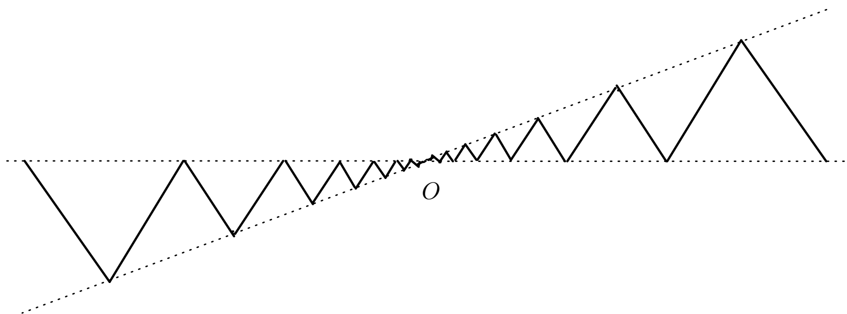
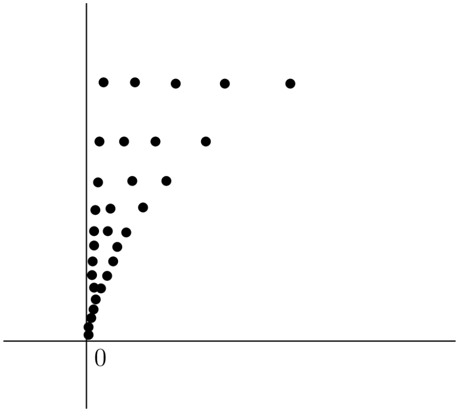


Fig. EAZZ.

(2) Let $P_{i,n} := (1/i, 1/n) \in \mathbb{R}^2$ for $i, n \in \mathbb{N}$. Set

$$A := \{P_{i,n} \in \mathbb{R}^2\ :\ i > n^2 > 0\}.$$

Then $0 \in \overline{A} \setminus A$ and $D_{P_{i,n}}(A) = \emptyset$ ($i > n^2$). It follows that $\mathcal{GD}_0(A) = \emptyset$ and $D_0(A) = \{(0,1)\}$. Therefore we have $D_0(A) \cup \mathcal{GD}_0(A) = \{(0,1)\}$. On the other hand, we have $\mathcal{GD}_0(\overline{A}) = \{(1,0),(0,1)\}$. Thus



Remark 4.5. $D_A(A)$ is not always closed even if $\overline{A} = A$. For instance, we can see this fact in the case of the above equiangular zigzag Z in Example 4.4 (1).

PROPERTY 4.6. *Let W, $A \subset \mathbb{R}^n$ suth that $W \subset \overline{A}$. Then the following properties hold.*

(1) $\mathcal{GD}_{\overline{A}}(A) = \overline{D_A(A)}$.

(2) $\mathcal{GD}_{\overline{A}}(\overline{A}) = \overline{D_{\overline{A}}(\overline{A})}$.

(3) $D_W(A) \cap \overline{D_A(A)} \subset \mathcal{GD}_W(A)$.

(4) $\mathcal{GD}_W(\overline{A}) = \mathcal{GD}_W(A) \cup (W \times S^{n-1} \cap \overline{D_{\overline{A}\setminus A}(\overline{A})})$.

Proof. Properties (1)–(3) are immediate consequences of the definitions of direction set and geometric directional bundle. We can see property (4) as follows; this property points out the limits of directional cones coming from $\overline{A} \setminus A$:

$$\begin{aligned}
\mathcal{GD}_W(\overline{A}) &= W \times S^{n-1} \cap \overline{D_{\overline{A}}(\overline{A})} \\
&= (W \times S^{n-1} \cap \overline{D_A(A)}) \cup (W \times S^{n-1} \cap \overline{D_{\overline{A}\setminus A}(\overline{A})}) \\
&= \mathcal{GD}_W(A) \cup (W \times S^{n-1} \cap \overline{D_{\overline{A}\setminus A}(\overline{A})}). \quad \square
\end{aligned}$$

Remark 4.7. As mentioned in Remark 4.5, $D_{\overline{A}}(\overline{A}) = \bigcup_{p\in\overline{A}} D_{\{p\}}(\overline{A})$ is not always closed, but

$$\bigcup_{p\in\overline{A}} \mathcal{GD}_{\{p\}}(\overline{A}) = (\overline{A} \times S^{n-1}) \cap \overline{D_{\overline{A}}(\overline{A})} = \overline{D_{\overline{A}}(\overline{A})}$$

is always closed.

At the end of this section, we give some fundamental properties of geometric directional bundle concerning the union.

Property 4.8. *Let* U_λ *(*$\lambda \in \Lambda$*),* $A \subset \mathbb{R}^n$ *such that* $U_\lambda \subset \overline{A}$ *for any* $\lambda \in \Lambda$*. Then we have*

$$\mathcal{GD}_{\bigcup_{\lambda\in\Lambda} U_\lambda}(\overline{A}) = \bigcup_{\lambda\in\Lambda} \mathcal{GD}_{U_\lambda}(\overline{A}).$$

Property 4.9. *Let* $U, A, B \subset \mathbb{R}^n$ *suth that* $U \subset \overline{A} \cap \overline{B}$*. Then we have*

$$\mathcal{GD}_U(\overline{A}) \cup \mathcal{GD}_U(\overline{B}) = \mathcal{GD}_U(\overline{A \cup B}).$$

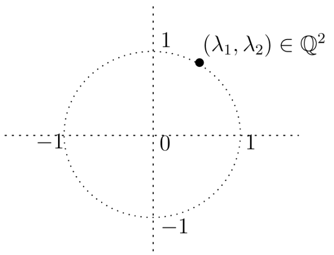
Property 4.10. *Let* U, A_λ *(*$\lambda \in \Lambda$*)* $\subset \mathbb{R}^n$ *suth that* $U \subset \overline{A_\lambda}$ *for any* $\lambda \in \Lambda$*. Then we have*

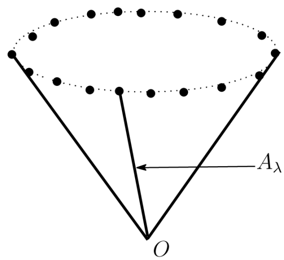
$$\overline{\bigcup_{\lambda\in\Lambda} \mathcal{GD}_{\overline{U}}(\overline{A_\lambda})} \subset \mathcal{GD}_{\overline{U}}(\overline{\bigcup_{\lambda\in\Lambda} A_\lambda}).$$

Remark 4.11. In Property 4.10 the equality does not always hold. Let $U = \{0\}$ where $0 \in \mathbb{R}^3$, and let

$$\Lambda := \{\lambda = (\lambda_1, \lambda_2, 1) \in \mathbb{Q}^2 \times \{1\} \mid \lambda_1^2 + \lambda_2^2 = 1\} \subset \mathbb{R}^3.$$

For $\lambda \in \Lambda$, A_λ denotes the half line passing through λ and with $0 \in \mathbb{R}^3$ as the starting point.





Let $C := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}$. Then we can see that

$$\overline{\bigcup_{\lambda\in\Lambda} \mathcal{GD}_{\{0\}}(\overline{A_\lambda})} = C \cap S^2 \quad \text{and} \quad \mathcal{GD}_{\{0\}}(\overline{\bigcup_{\lambda\in\Lambda} A_\lambda}) = D \cap S^2,$$

where $D := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \geq z^2\}$.

5. $\mathcal{GD}$-IDEMPOTENCE

Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. Then we simply write $\mathcal{GD}(A) := \mathcal{GD}_0(A)$. For $m = 2, 3, \cdots$, we define

$$\mathcal{GD}^m(A) := \mathcal{GD}(L\mathcal{GD}^{m-1}(A)).$$

We consider the notion of stabilisation concerning the operation $\mathcal{GD}$.

Question 1. Is the operator $\mathcal{GD}$ stabilised? Namely, for $A \subset \mathbb{R}^n$ with $0 \in \overline{A}$, does there exist a natural number $m \in \mathbb{N}$ such that

$$\mathcal{GD}^m(A) = \mathcal{GD}^{m+1}(A) = \mathcal{GD}^{m+2}(A) \cdots \ ?$$

We have a positive answer to the above question in the case where $n = 2$.

PROPOSITION 5.1. *Let $A \subset \mathbb{R}^2$ such that $0 \in \overline{A}$. Then we have*

$$\mathcal{GD}^3(A) = \mathcal{GD}^4(A) = \mathcal{GD}^5(A) = \cdots .$$

In other words, the operator $\mathcal{GD}$ is stabilised at degree 3.

Proof. By definition, $\mathcal{GD}(A) = \mathcal{GD}_0(A) = \Pi^{-1}(\{0\}) \cap \overline{B_A}$. Therefore $\mathcal{GD}(A)$ is closed in $\mathbb{R}^2$, and $\mathcal{GD}(A) \subset S^1$.

We first consider the case $\#(\mathcal{GD}(A)) = \infty$. Since $\mathcal{GD}(A)$ is compact, it has an accumulation point $p \in S^1$. Let ℓ_p be the half line passing through p and with $0 \in \mathbb{R}^2$ as the starting point, and let us denote by $S^1_+(p)$ the closed unit semicircle connecting p and $-p$ anti-clockwise. Then $D_q(L\mathcal{GD}(A)) \supset S^1_+(p)$ for any $q \in \ell_p \setminus \{0\}$, or $D_q(L\mathcal{GD}(A)) \supset S^1_+(-p)$ for any $q \in \ell_p \setminus \{0\}$. Therefore we have $\mathcal{GD}^2(A) \supset S^1_+(p)$ or $\mathcal{GD}^2(A) \supset S^1_+(-p)$. It follows that $\mathcal{GD}^3(A) = \mathcal{GD}(L\mathcal{GD}^2(A)) = S^1$. Thus we have

$$\mathcal{GD}^3(A) = \mathcal{GD}^4(A) = \mathcal{GD}^5(A) = \cdots .$$

We next consider the case $\#(\mathcal{GD}(A)) < \infty$. Let us express $\mathcal{GD}(A)$ as

$$\mathcal{GD}(A) = \{p_1, \cdots, p_s, -p_1, \cdots, -p_s, q_1, \cdots, q_t\},$$

where all the elements $p_i, -p_j, q_k$ in the right side are different. Then we have

$$\mathcal{GD}^2(A) = \mathcal{GD}(L\mathcal{GD}(A)) = \{p_1, \cdots, p_s, -p_1, \cdots, -p_s, q_1, \cdots, q_t, -q_1, \cdots, -q_t\}.$$

It follows that

$$\mathcal{GD}^2(A) = \mathcal{GD}^3(A) = \mathcal{GD}^4(A) = \cdots . \quad \square$$

Example 5.2. For $i, j \in \mathbb{N}$, let $a_{i,j} := (\frac{1}{i}, \frac{1}{ij}) \in \mathbb{R}^2$. If we fix $j \in \mathbb{N}$, then $a_{i,j}$ are points on the half life ℓ^j passing through $(1, \frac{1}{j})$ and with $(0,0)$ as the starting point. Set $A := \{a_{i,j} \in \mathbb{R}^2 | i, j \in \mathbb{N}\}$.

Let ℓ^0 be the positive x-axis with $(0,0)$, and let ℓ^∞ be the positive y-axis with $(0,0)$. Then we have

$$L\mathcal{GD}(A) = \ell^1 \cup \ell^2 \cup \cdots \cup \ell^0 \cup \ell^\infty.$$

Let p_j be the intersection of S^1 and ℓ^j for $j \in \mathbb{N}$. Then we have

$$\mathcal{GD}^2(A) = \mathcal{GD}(L\mathcal{GD}(A)) = S^1_+((1,0)) \cup \{-p_1, -p_2, \cdots\} \cup \{(0,-1)\}.$$

Therefore $\mathcal{GD}^3(A) = S^1$. It follows that $\mathcal{GD}^3(A) = \mathcal{GD}^4(A) = \mathcal{GD}^5(A) = \cdots$, but $\mathcal{GD}^2(A) \neq \mathcal{GD}^3(A)$.

Let us consider the following notion in order to distinguish subsets of $\mathbb{R}^n$.

Definition 5.3. Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. We say that $\mathcal{GD}$ is *idempotent for A* or A is called *$\mathcal{GD}$-idempotent*, if $\mathcal{GDGD}(A) = \mathcal{GD}(A)$.

We make an observation for the stabilisation of the operator $\mathcal{GD}$.

OBSERVATION 1. *Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that there exists a positive integer $m \in \mathbb{N}$ such that $\mathcal{GD}^m(A)$ has an interior point as a subspace of S^{n-1}. Then we have $L\mathcal{GD}^{m+1}(A) = \mathbb{R}^n$.*

Proof. Let P be the interior point of $\mathcal{GD}^m(A)$. Then there exists $\epsilon > 0$ such that $U_\epsilon(P) \subset \mathcal{GD}^m(A)$, where $U_\epsilon(P)$ is an ϵ-neighbourhood of P in S^{n-1}. Therefore we have

$$LU_\epsilon(P) \subset L\mathcal{GD}^m(A).$$

It follows that

$$\mathbb{R}^n = L\mathcal{GD}(LU_\epsilon(P)) \subset L\mathcal{GD}(\mathcal{GD}^m(A)) = L\mathcal{GD}^{m+1}(A) \subset \mathbb{R}^n.$$

Therefore we have $L\mathcal{GD}^{m+1}(A) = \mathbb{R}^n$. □

As a corollary of Observation 1, we have the following.

COROLLARY 5.4. *Let $A \subset \mathbb{R}^n$ such that $0 \in \overline{A}$. Suppose that there exists a positive integer $m \in \mathbb{N}$ such that $\mathcal{GD}^m(A)$ has an interior point as a subspace of S^{n-1}. Then for A, the operator $\mathcal{GD}$ is stabilised at degree m if and only if $L\mathcal{GD}^m(A) = \mathbb{R}^n$.*

In particular, if $D(A)$ has an interior point as a subspace of S^{n-1}, then A is $\mathcal{GD}$-idempotent if and only if $L\mathcal{GD}(A) = \mathbb{R}^n$.

Let us apply the notion of $\mathcal{GD}$-idempotent to the graphs of oscillations. As in subsection 2.4, let us denote by G the graph of a continuous function $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, and set $G^+ := G \cap \{x \geq 0\}$ and $G^- := G \cap \{x \leq 0\}$.

LEMMA 5.5. *If $\#D(G^+) > 1$, then $D(G^+)$ has an interior point as a subspace of S^1.*

Proof. Let $S^1_{\geq 0} := \{(\cos\theta, \sin\theta) | -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\} \subset S^1$. For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, set $A_\theta := (\cos\theta, \sin\theta) \in S^1_{\geq 0}$.

Since $\#D(G^+) > 1$, there exist $-\frac{\pi}{2} \leq \theta_1 < \theta_2 \leq \frac{\pi}{2}$ such that $A_{\theta_1}, A_{\theta_2} \in D(G^+)$. Let θ be any element of $S^1_{\geq}$ such that $\theta_1 < \theta < \theta_2$, and let ℓ_{A_θ} be the half line passing through A_θ and with $(0,0)$ as the starting point. Then, using the mean value theorem, we can show that $\ell_\theta \cap G^+$ contains a sequence of points converging to $(0,0)$. Therefore we have $A_\theta \in D(G^+)$. It follows that $D(G^+)$ has an interior point as a subspace of S^1. $\square$

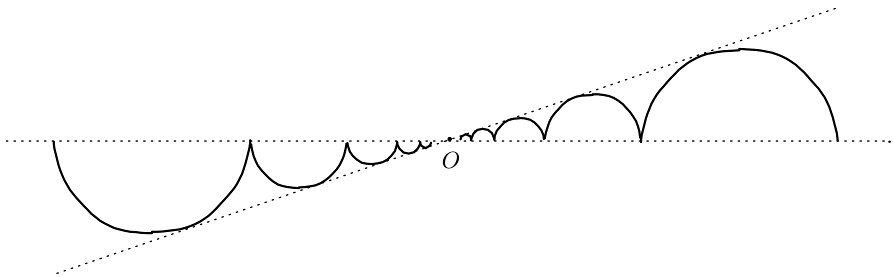
By Corollary 5.4 and Lemma 5.5, we have the following criterion for G^+ to be $\mathcal{GD}$-idempotent.

CRITERION 5.6. *Suppose that* $\#D(G^+) > 1$. *Then* G^+ *is* $\mathcal{GD}$*-idempotent if and only if* $L\mathcal{GD}(G^+) = \mathbb{R}^2$.

We have a similar criterion for the $\mathcal{GD}$-idempotent of G_- to that of G_+.

Example 5.7. By Criterion 5.6, we can see that G^+ of the equiangular zigzag G_f is not $\mathcal{GD}$-idempotent.

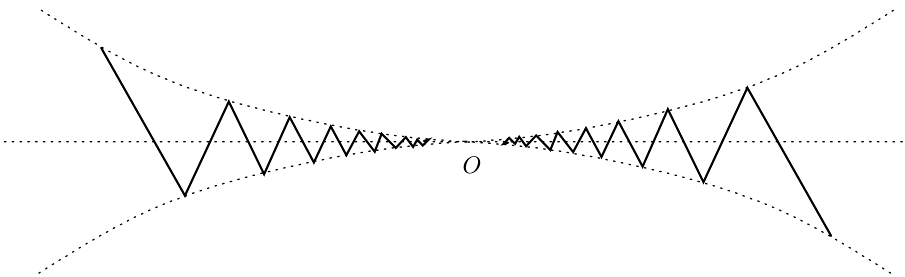
Example 5.8. The semicircle oscillation consists of infinitely many semicircles as follows:



By Criterion 5.6, we can see that G^+ of the semicircle oscillation is $\mathcal{GD}$-idempotent.

In the case where $\#D(G^+) = \#D(G^-) = 1$, we cannot apply the above criterion. We call such an oscillation *flat*. We give an example of flat zigzag.

Example 5.9 (Equiangular flat zigzag). Let A be the equiangular flat zigzag as follows.

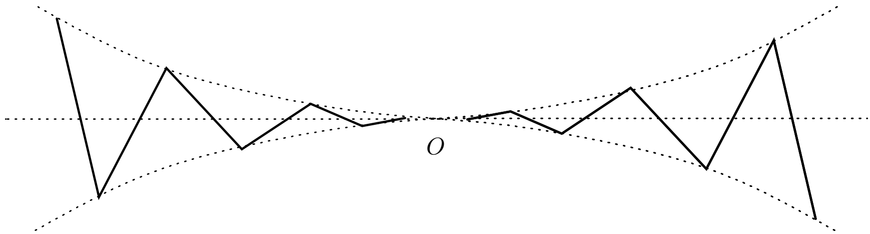


Then $\mathcal{GD}(G^+)$ consists of 5 points, but $\mathcal{GDGD}(G^+)$ consists of 6 points. Therefore $\mathcal{GDGD}(G^+) \neq \mathcal{GD}(G^+)$.

On the other hand, both of $D(A)$ and $\mathcal{GDGD}(A)$ consist of the same 6 points. Therefore we have $\mathcal{GDGD}(A) = \mathcal{GD}(A)$.

It follows that G^+ is not $\mathcal{GD}$-idempotent but A is $\mathcal{GD}$-idempotent like the equiangular zigzag (see Example 5.7 also).

Example 5.10 (Super quick flat zigzag). Let A be the super quick flat zigzag as follows.



Then we have $D(A) = \mathcal{GD}(A) = \{(\pm 1, 0)\}$. A satisifies condition (SSP) at $0 \in \mathbb{R}^2$. Since A is flat, it induces a bi-Lipschitz homeomorphism.

We can see also that G^+ and A are $\mathcal{GD}$-idempotent. Therefore the $\mathcal{GD}$-idempotence of G^+ distinguishes the super quick flat zigzag from the equiangular flat zigzag.

REFERENCES

[1] M. Coste, *An introduction to o-minimal geometry.* Dottorato di Ricerca in Matematica, Dip. Mat. Pisa. Instituti Editoriali e Poligrafici Internazionali, 2000.

[2] L. van den Dries, *Tame topology and o-minimal structures.* LMS Lecture Notes Series **248**, Cambridge Univ. Press, 1997.

[3] A. Fish and L. Paunescu, *Unwinding spirals I.* Methods Appl. Anal. (Special issue dedicated to the memory of John Mather) **25** (2018), *3*, 225–232.

[4] C.G. Gibson, K. Wirthmuller, A. du Plessis and E.J.N. Looijenga, *Topological stability of smooth mappings.* Lect. Notes in Math. **552**, Springer, Berlin, 1976.

[5] H. Hironaka, *Subanalytic sets.* Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Yasuo Akizuki, pp. 453–493, Kinokuniya, Tokyo, 1973.

[6] S. Koike and L. Paunescu, *The directional dimension of subanalytic sets is invariant under bi-Lipschitz homeomorphisms.* Ann. Inst. Fourier **59** (2009), 2448–2467.

[7] S. Koike, T.L. Loi, L. Paunescu and M. Shiota, *Directional properties of sets definable in o-minimal structures.* Ann. Inst. Fourier **63** (2013), 2017–2047.

[8] S. Koike and L. Paunescu, *On the geometry of sets satisfying the sequence selection property.* J. Math. Soc. Japan **67** (2015), 721–751.

[9] S. Koike and L. Paunescu, *(SSP) geometry with directional homeomorphisms.* J. Singularity **13** (2015), 169–178.

[10] S. Koike and L. Paunescu, *Applications of the sequence selection property to bi-Lipschitz geometry.* Eur. J. Math. **5** (2019), *4*, 1202–1211.

[11] L.D. Tráng, *Calcul du nombre de cycles évanouissants d'une hypersurface complexe.* Ann. Inst. Fourier **23** (1973), 261–270.

[12] J.N. Mather, *Notes on topological stability.* Bull. Amer. Math. Soc. (N.S.) **49** (2012), 475–506.

[13] S. Miyake, *Geometry of spirals.* Master's thesis, Hyogo University of Teacher Education, 2014 (in Japanese).

[14] T. Mostowski, *Lipschitz equisingularity.* Dissertationes Math. (Rozprawy Mat.) **243** (1985).

[15] M. Oka, *On the weak simultaneous resolution of a negligible truncation of the Newton boundary.* Contemp. Math. **90** (1989), 199–210.

[16] A. Parusiński, *Lipschitz stratification of real analytic sets.* Singularities (Warsaw, 1985), Banach Center Publ. **20** (1988), 323–333.

[17] A. Parusiński, *Lipschitz properties of semi-analytic sets.* Ann. Inst. Fourier **38** (1988), 189–213.

[18] A. Parusiński, *Regular projections for subanalytic sets.* C. R. Acad. Sci. Paris, Sér. I Math. **307** (1988), 343–347.

[19] A. Parusiński and L. Paunescu, *Arc-wise analytic stratification, Whitney fibering conjecture and Zariski equisingularity.* Adv. Math. **309** (2017), 254–305.

[20] J. Edison Sampaio, *Bi-Lipschitz homeomorphic subanalytic sets have bi-Lipschitz homeomorphic tangent cones.* Selecta Math. (N.S.) **22** (2016), 553–559.

[21] B. Shiffman, *On the removal of singularities of analytic sets.* Michigan Math. J. **15** (1968), 111–120.

[22] R. Thom, *Local topological properties of differentiable mappings.* Differential Analysis, Bombay Colloq., pp. 191–202, Oxford Univ. Press, London, 1964.

[23] R. Thom, *Ensembles et morphismes stratifiés.* Bull. Amer. Math. Soc. **75** (1969), 240–284.

[24] D'Arcy Thompson, *On growth and form.* Cambridge Univ. Press, 1917.

[25] D. Trotman, *Comparing regularity conditions on stratrifications.* Singularities, Part 2 (Arcata, Calf. 1981), 575–586, Proc. Sympo. Pure Math. **40**, Amer. Math. Soc. Providence, R.I., 1983.

[26] H. Whitney, *Elementary structure of real algebraic varieties.* Ann. of Math. **66** (1957), 545–556.

[27] H. Whitney, *Local topological properties of analytic varieties.* In: S.S. Cairns (Ed.), *Differential and Combinatorial Topology.* A Symposium in Honor of M. Morse, pp. 205–244, Princeton Univ. Press, 1965.

[28] H. Whitney, *Tangents to an analytic variety.* Ann. of Math. **81** (1965), 496–549.

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Hyogo University of Teacher Education,
Department of Mathematics,
Kato, Hyogo 673-1494, Japan
koike@hyogo-u.ac.jp

University of Sydney,
School of Mathematics and Statistics,
Sydney, NSW, 2006, Australia
laurentiu.paunescu@sydney.edu.au