AN OLD MAN'S MATHEMATICAL STORIES

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Laurentiu insisted that I should give a talk. His idea was that a residue value of an old man is that he may have some stories to tell the younger generations, whence the title.

I believe what he actually meant was that the only residue value an old man like me might still have is to tell stories which might be of some interest to you.

Most stories took place a very long time ago, they may not be 100% accurate.

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1. THE CHICAGO LECTURE SERIES "HOW TO DO RESEARCH"

This lecture series was organised by Professor Saunders Mac Lane in 1962, when I was half way through a Ph. D program. He was, of course, the first speaker, the others were: Albert, Calderón, Halmos, Kaplansky, Steenrod (visiting professor) and Zygmund.

The purpose was to advise us graduate students how to do research. Hence, only we were allowed to attend. Yet many junior staffs still sneaked into the room, hid behind us; some were caught, but none was actually thrown out.

Every speaker, including Mac Lane himself, began his lecture by saying "I don't know how to do research", and changed the title to "*How I Do Research*". (So much the better!) However, there were many common advices. Some impressive advices are as follows.

(i) Compute And Think. That is, begin with an idea, compute examples; try to understand the meaning; design more examples, compute again, then try to understand, etc..

Mac Lane showed us the amount of paper, some 2 feet thick, he spent computing the homology of the Eilenberg-Mac Lane spaces $K(\pi, n)$, case by case, dimension by dimension.

Steenrod, with his finger pointing at us, said: "I have computed more H_*, H^* than all of you put together." We happily accepted the humiliation, for we learned *Algebraic Topology* as a kind of general abstract non-sense, knowing only a handful of simple examples.

(ii) Every Theorem Is An Exercise. This method of learning was very much emphasised by Mac Lane, Calderón, Halmos and Steenrod. *A theorem is the tip of a pyramid, based on examples, counterexamples and the idea of proof.* In order to appreciate the structure of the pyramid one must try to find a proof. *A theorem sails through the reefs of counterexamples.*

Calderón told us when he was a student (in Argentina), he came across a theorem of Zygmund, he found a proof, hence did not bother to read Zygmund's paper. When he came to Chicago, one day he discussed this theorem with Zygmund, he was amazed to find that his proof was actually better than Zygmund's original one.

Of course, there are too many theorems in the world. Hence, in order to do research, one must ignore almost all theorems; a theorem is an exercise only if it is relevant to you.

(iii) Steenrod: Ask Questions Which Need Not Make Sense. We all laughed. But Steenrod was serious. This was what he told us: "Around 1940, *Algebraic Topology* amounted to computation of $H_*(X)$, $H^*(X)$, for all kinds of X.

I asked myself: *Can I axiomatise the theory*? This question did not make sense.

In 1945, Eilenberg-Mac Lane's *Category Theory* paper appeared. I was so excited. This paper had more impact on me than any other paper; it completely changed my way of thinking. Now, H_*, H^* are *functors*, cup product, Sq^i, etc. are *natural transformations*.

This was *precisely* the language I need for the axiomatisation."

This he and Eilenberg did in their book *Foundation of Algebraic Topology* (1952).

I was impressed and inspired by this advice of Steenrod.

Non-standard Analysis appeared in the 1960's. In order not to use (ϵ, δ) in *Calculus*, A. Robinson introduced galaxy, galaxy of galaxies, ..., of infinitesimals (logic symbols).

I asked myself: *Can I give geometric meaning to infinitesimals?*

This question now makes sense in the *Newton-Puiseux Infinitesimal Analysis* [13]; a fractional power series generates an irreducible curve-germ, a *geometric infinitesimal.* See (9.2).

I believe the geometric infinitesimals, being *self-similar* (as in *Fractal Geometry*), can be used to teach *Calculus*, and to build models ("pictures") in *Quantum Mechanics.*

Another question which did not make sense was: Are the classical discriminants

$$b^2 - 4ac, \quad 4p^3 - 27q^2, ...,$$

one for each degree of polynomial, *related* in some way?

Thanks to Koike and Parusiński [14], this question will make sense once we know the *canonical stratification* of $J^\omega(2,1)$; the discriminant varieties appear amid the strata (§7).

2. NO DEFINITION IS WRONG?

Steenrod gave a lecture; his paper appeared in *L'Enseignement Math.*, 153–178 (1962).

In 1956, Cartan and Eilenberg wrote the first book on *Homological Algebra.* At the very beginning of this book they define the *cohomology algebra* as the *infinite direct sum*

$$H^*(X) := H^0(X) \oplus \cdots \oplus H^p(X) \oplus \cdots ,$$

and call the cup product "*anti-commutative*":

$$u_p \cup v_q = (-1)^{pq} v_q \cup u_p \in H^{p+q}.$$

It is then awkward to compare

$$(u_0 + \cdots + u_p) \cup (v_0 + \cdots + v_q) \quad \text{with} \quad (v_0 + \cdots + v_q) \cup (u_0 + \cdots + u_p).$$

Similarly, although $Sq^i(u_p \cup v_q)$ has a beautiful expansion, there isn't one for

$$Sq^i[(u_0 + \cdots + u_p) \cup (v_0 + \cdots + v_q)].$$

Steenrod: "Adding cohomology classes of different dimensions has no geometric meaning. The definition is wrong, hence the whole book is wrong."

Steenrod's Correction: The *cohomology algebra* is, by definition, a ***graded*** algebra

$$H^*(X) := \{H^0(X), ..., H^p(X), ...\}.$$

A graded algebra is ***commutative*** if, ***by definition***,

$$u_p \cdot v_q = (-1)^{pq} v_q \cdot u_p,$$

and *connected* if H^0 is isomorphic to the ground field. Then everything goes smoothly.

Mac Lane adopted Steenrod's definition in his book *Homology* (1963). Unfortunately, even nowadays we still see meaningless expressions like $dx + dx \wedge dy$.

3. CHEVALLEY AND THOM: IN GENERAL POSITION

In 1954(?), when Thom (then a young man) was visiting IAS, Chevalley (Columbia) invited him to dinner. Like every good Frenchman, they had coffee after the meal.

Chevalley: "The notion ***in general position*** in *Algebraic Geometry* can be ***transplanted*** to *Differential Topology*."

Thom was inspired, leading to his *Transversality Theorem*, and then the *Stability Theory* of mappings. A branch of modern mathematics thus grew out of a cup of coffee.

A short conversation with a big master can keep us busy for a very long time.

This notion is also used in the development of *Stratification Theory*. See §8.

4. THOM *VS.* GROTHENDIECK (PRIDE AND PREJUDICE)

In 1964(?) Shih WeiShu overheard a conversation in the tea room of IHES.

Thom: Does your theory help understand algebraic varieties in $\mathbb{R}^n$?

Grothendieck: No.

Thom: Then, what is your theory good for?

The conversation was of course in French. Shih WeiShu told me in Chinese in 1965. The above is my translation. Some years later, the book *Structural Stability and Morphogenesis* appeared [34]. On p. 35, Thom wrote: "in the case of any natural phenomenon governed by an algebraic equation it is of paramount importance to know whether this equation has solutions, ***real*** roots, and precisely this question is suppressed when complex scalars are used ...." He then gave a number of examples.

I used to have a copy of Whitney's book *Complex Analytic Varieties* (1972). I was amazed to find in the *Preface*: "This (book) is a prerequisite for a full study of the real case."

Thom and Whitney indicated to us a new frontier: ***Real Algebraic Geometry***.

How to do research? ***Have Pride and Prejudice!***

The first classical result of this kind is Sturm's Theorem. This is an algorithm to compute the number of real roots of a given $p(x) \in \mathbb{R}[x]$ in a given interval $[a, b]$.

Another one is Harnack's Theorem [5] which gives the maximum possible number of connected components of a curve C_n in $\mathbb{R}^2$, defined by a polynomial of degree n.

In his renowned 1979 M. Sc thesis [24,25], S. McCallum generalised Sturm's theorem to $\mathbb{R}^2$. He created an algorithm to determine the topological structure of C_n in a given compact rectangle D (some components can be isolated points). More precisely, he constructed a simplicial pair $(\mathcal{S}, \mathcal{R})$ homeomorphic to $(D, C_n \cap D)$.

Nowadays, *Real Algebraic Geometry* is an active field of research. See, *e.g.*, [1,3].

5. EQUIVALENCE OF SINGULARITIES (*EQUI-SINGULARITIES*)

Regardez les singularités, il n'y a que ça qui compte.

G. Julia.

The late Professor Roger Richardson had a very interesting observation. He pointed out to me that Zariski and Whitney were colleagues at Harvard. (Whitney moved to IAS later.) They actually worked on the same research project, namely, to understand *singular algebraic varieties*; but their approaches were very different. Roger was quite right.

Zariski followed the traditional approach: blow-up, blow-up, ..., till all singular points disappear, to achieve a parametrisation. His student Hironaka finally succeeded.

I see Whitney as an "*environmentalist*", who would not break anything, let alone blow things up. He kept singular points as they are, only to classify them: *equi-singularities*. His idea was to define $P \sim P'$ on a given variety if they have the "*same local environment*".

Whitney's Project: Find a definition for "same local environment".

(Two professors, one tried to prove a theorem, the other devoted himself to finding a definition. Between theorem and definition, which one is more difficult?)

Let us explain Whitney's idea on two typical singular varieties in $\mathbb{R}^3$:

$$V_a : \ y^2 = x^2(x + z^2) \quad \text{and} \quad V_b : \ y^2 = z^2x^3 + x^5. \tag{5.1}$$

A section of V_a by $z = c$, $c \neq 0$, is an α-shaped curve, crossing itself at 0; that with $c = 0$ is the cusp $y^2 = x^3$; that with $y = 0$ is $x = -z^2$ plus the z-axis. Take smooth points P_1, P_2 on V_a, and P_3, P_4 on the z-axis, not 0. The equivalence relation " $\sim$ " must be such that

$$P_1 \sim P_2 \not\sim P_3 \sim P_4 \not\sim 0.$$

As for V_b, each section by $z = c$, $c \neq 0$, is analytically equivalent to $y^2 = x^3$; that by $z = 0$ is a sharper cusp $y^2 = x^5$. The origin 0 should have the worst local environment.

Question: When should we call two Taylor series "equivalent"?

Once an equivalence relation is chosen, we can do the following. Take *any* variety V, defined, say, by $f(x) = 0$. Consider the Taylor expansions f_P, $f_{P'}$ of f at P, P' resp.. We say P, P' have *"the same local environment"* on V if f_P and $f_{P'}$ are equivalent.

The equivalence relation should be so good that it induces a *finite* decomposition

$$V = M_1 \cup \cdots \cup M_N, \quad M_i \text{ being the equivalence classes,}$$

where M_i are manifolds, patching up in a nice way.

That is, we are searching for a "universal stratification" of varieties. The power series rings, with the equivalence relation, are "universal objects".

(In *Algebraic Topology*, $K(\pi, n)$ are *universal objects* for cohomology operations. We do not know whether or not "*universal object*" has been formally defined in *Category Theory*.)

6. ON THE ROAD TO UNIVERSAL STRATIFICATION

There are hurdles on the road, due to Whitney, Koike, Paunescu, Kobayashi, etc.

***The Whitney Hurdle* [37, 38].** Consider

(6.1) $$W_t(x, y) = xy(x - y)(x - ty) \colon (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}, 0), \quad 1 < t < \infty.$$

Take $t \neq t'$. It is intuitively clear (and not hard to prove) that there exists

$$h : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0), \quad W_{t'} \circ h = W_t,$$

where h is a homeomorphism sending

$$x = 0 \text{ to } x = 0; \ y = 0 \text{ to } y = 0; \ x = y \text{ to } x = y; \text{ and } x = ty \text{ to } x = t'y.$$

Whitney: Such h ***can never*** be a C^1-diffeomorphism.

For a proof, suppose h is C^1. Then the *linear approximation* at 0 must be

$$dh|_0 = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \text{ a contradiction, proof complete.}$$

This hurdle was "*blown away*" in [19] as follows. Let

$$\beta : (\mathscr{M}^2, C) \longrightarrow (\mathbb{R}^2, 0) \tag{6.2}$$

be the blow-up of $\mathbb{R}^2$ at 0, where $\mathscr{M}^2$ is the Möbius band, C the centre circle.

A (real) *analytic isomorphism* Ψ and a homeomorphism h were found such that

$$\begin{array}{ccc}
(\mathscr{M}^2, C) & \xrightarrow{\quad\Psi\quad} & (\mathscr{M}^2, C) \\
\downarrow \beta & & \downarrow \beta \\
(\mathbb{R}^2, 0) & \xrightarrow{\quad h \quad} & (\mathbb{R}^2, 0) \\
& {\searrow}^{W_t} \quad {}^{W_{t'}}{\swarrow} & \\
& (\mathbb{R}, 0) &
\end{array}$$

is commutative. That is, as observed on $\mathscr{M}^2$, the Whitney hurdle is no longer there.

It is, therefore, tempting to propose the following definition. We say

$$f,\ g :\ (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}, 0) \tag{6.3}$$

are *blow-analytically equivalent*, $f \sim_{b.a.} g$, if there exist $\tilde{\beta}_1$, $\tilde{\beta}_2$, each being a *finite composition of blow-ups*, an analytic isomorphism Ψ, and a homeomorphism h such that the diagram

$$\begin{array}{ccc}
(\mathscr{M}_1, E_1) & \xrightarrow{\quad\Psi\quad} & (\mathscr{M}_2, E_2) \\
\downarrow \tilde{\beta}_1 & & \downarrow \tilde{\beta}_2 \\
(\mathbb{R}^n, 0) & \xrightarrow{\quad h \quad} & (\mathbb{R}^n, 0) \\
& {\searrow}^{f} \quad {}^{g}{\swarrow} & \\
& (\mathbb{R}, 0) &
\end{array}$$

is commutative, where E_i are the exceptional divisors (like the centre circle C in $\mathscr{M}^2$).

Unfortunately, we don't know how to prove $\sim_{b.a.}$ is an equivalence relation. This involves an open question in *Algebraic Geometry*. Namely, let

$$\tilde{\beta}_i : \mathscr{M}_i \longrightarrow \mathscr{M}, \quad i = 1,\ 2, \quad \text{(finite compositions of blow-ups)}$$

be given. Can we find

$$\tilde{\beta}_i' : \tilde{\mathscr{M}} \longrightarrow \mathscr{M}_i, \quad i = 1,\ 2,$$

such that

$$\tilde{\beta}_1 \circ \tilde{\beta}_1' = \tilde{\beta}_2 \circ \tilde{\beta}_2' \ ?$$

That is, in the category where morphisms are blow-downs, do "*pullbacks*" exist?

In order to avoid this hurdle we used *"modifications"* instead of blow-ups in [20].

The next hurdle is much more serious. Koike, in [12], found that the Briançon-Spader family is blow-analytically trivial, but tangential arcs are carried to transverse arcs. This result shows that the definition has to be substantially strengthened. (See [6].)

Another renowned hurdle is the Paunescu blow-analytic homeomorphism [30]

$$\mathcal{P} : (\mathbb{R}^3, 0) \longrightarrow (\mathbb{R}^3, 0), \quad (x, y, z) \mapsto (x, y, z - xy^5(x^4 + y^6)^{-1}), \tag{6.4}$$

(arc-analytic components) carrying a cusp to a smooth curve: $(t^3, t^2, 0) \to (t^3, t^2, t)$.

Hurdles also exist in $\mathbb{R}^2$. Kobayashi, in [10, 11], constructed

$$\tilde{\beta}_i : \mathscr{M} \longrightarrow \mathbb{R}^2, \quad i = 1,\ 2,$$

such that $\tilde{\beta}_2 \circ \tilde{\beta}_1^{-1}$, like $\mathcal{P}$, carries a cusp to a straight line. See also Valle [36].

Accordingly, Laurentiu, mimicking Steenrod, once said: "the definition of blow-analyticity is wrong, hence the whole theory is wrong."

Can we still find the universal stratification? I think "Yes". But hard works lie ahead.

Philosophers say: Every observation is made through a pair of glasses (telescopes, microscopes, naked eyes, etc.). Of course we all know wrong glasses give misleading images.

In mathematics, coordinate systems are glasses. When teaching *Calculus*, we all use

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}, \quad (x, y) \mapsto x^2 y(x^2 + y^2)^{-1}, \tag{6.5}$$

(Cartesian glasses) to show students that a good looking function need not be C^1.

However, if we wear the *polar* glasses, we actually see a beautiful *analytic* function:

$$\text{Cylinder} \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} : \quad (r, \theta) \mapsto (r\cos\theta, r\sin\theta) \mapsto r\cos^2\theta \sin\theta.$$

With the *Möbius* glasses (6.2), $f \circ \beta$ is also *analytic.* What we proved in [19] is that if we wear the Möbius glasses, the Whitney hurdle is not there anymore.

We now believe if we wear the *Newton-Puiseux glasses*, or perhaps the *Fukui glasses* (9.4), the other hurdles may also disappear. Laurentiu and I have taken the first step in [21].

7. BEYOND A THEOREM OF KOIKE-PARUSIŃSKI

A significant theorem of Koike–Parusiński [14, 15] asserts that in the case $n = 2$, if $f \sim_{b.a.} g$ then there exists a commutative diagram

$$\begin{array}{ccccccccccc}
\mathscr{M}_N & \xrightarrow{\beta_N} & \mathscr{M}_{N-1} & \xrightarrow{\beta_{N-1}} & \cdots & \xrightarrow{\beta_2} & \mathscr{M}_1 & \xrightarrow{\beta_1} & (\mathbb{R}^2, 0) & & \\
\downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle h_{N-1}} & & & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & \searrow^{f} & \\
 & & & & & & & & & & (\mathbb{R}, 0) \\
 & & & & & & & & & \nearrow_{g} & \\
\mathscr{M}'_N & \xrightarrow{\beta'_N} & \mathscr{M}'_{N-1} & \xrightarrow{\beta'_{N-1}} & \cdots & \xrightarrow{\beta'_2} & \mathscr{M}'_1 & \xrightarrow{\beta'_1} & (\mathbb{R}^2, 0) & &
\end{array}$$

where h_i are homeomorphisms, β_i, β'_i are blow-ups, and Ψ is an analytic isomorphism.

It follows that $\sim_{b.a}$ is an equivalence relation in $\mathbb{R}\{x, y\}$. Moreover, by Theorem 5.1 [14], there is no hurdle. Hence the definition of $\sim_{b.a.}$ is a *"correct"* one in $\mathbb{R}\{x, y\}$.

As a result, we can already propose two research projects in $\mathbb{R}\{x, y\}$.

The first one is to find the equivalence classes. This amounts to finding the strata of the *(blow-analytic) canonical stratification* of $J^\omega(2, 1)$ [32]. (*Compute and Think!*)

Each stratum is a *generalised* discriminant variety. Are they related?

Here is a Steenrod type question: Are there operations on the strata to enhance the structure of $J^\omega(2, 1)$, whence connecting the discriminants? (In *Algebraic Topology*, Cartan used cohomology operations, such as Sq^i, to enhance the structure of the Eilenberg–Mac Lane algebra $H^*(\pi, n)$. Can something similar also happen here?)

The second project is to develop the *universal unfolding theory* in $\mathbb{R}\{x, y\}$.

When Thom introduced the notion of *universal unfolding* in his book [34], he was too much ahead of his time. There was no language to *properly* express his ideas.

We now have the language to develop the theory in $\mathbb{R}\{x, y\}$. Following Thom, let

$$f(x, y) = H_2(x, y) + \cdots + H_m(x, y) + \cdots \in \mathbb{R}\{x, y\}$$

be given, 0 being an isolated singularity. Let m be the smallest integer such that

$$f(x, y) \sim_{b.a.} H_2(x, y) + \cdots + H_m(x, y) \quad \text{(blow-analytic truncation).}$$

The set of blow-analytic equivalence classes of

$$H_2(x,y)+\cdots+H_m(x,y)+\textstyle\sum_{i+j\le m} a_{ij}x^iy^j, \quad a_{ij} \text{ indeterminates,}$$

is *the (blow-analytic) universal unfolding* of $f(x,y)$.

Project. Find the universal unfoldings in $\mathbb{R}\{x,y\}$. (*Compute and Think!*)

The seven elementary catastrophes arise from strata of co-dim ≤ 4; they appear in space-time $\mathbb{R}^4$. The strata of co-dim ≥ 5 will produce catastrophe models in $\mathbb{R}^N$, $N \ge 5$.

Of course, the *ultimate* goal is to find the *canonical stratification* of the general jet space $J^\omega(n,p)$, and compute the *universal unfoldings*. But the "correct" equivalence relation is, as yet, to be defined. (Perhaps use the Newton-Puiseux/Fukui glasses?)

In mathematics, we come across definitions and theorems. For a student, theorems are much harder than definitions; for us, however, definitions are much harder than theorems.

8. FROM WHITNEY-THOM TO PARUSIŃSKI-PAUNESCU

Whitney, blocked by his hurdle, took a different approach. He defined his *(a,b)-regularity conditions* in [37,38]. Thom wrote two papers [32,33], with his *Isotopy Theorems*. These are the foundation papers of the *Whitney-Thom Stratification Theory* [8,27,35].

Whitney's definition works well on V_a. But V_b is (a,b)-regular over the *entire* z-axis [17]; that is, all points on the z-axis have the "same local environment". Following Steenrod, we say:*"the definition is wrong, hence the whole theory is wrong."* (*Pride and Prejudice!*)

(Moving to w-regularity would not help: $y^4 = z^2x^5 + x^7$ is w-regular over the entire z-axis. Koike has pointed out that $y^2 = x^2(x^2+z^2)$ is also w-regular over the z-axis.)

In a remarkable paper [29], Parusiński and Paunescu defined what we call the *PaPa regularity condition*. This condition asks for the existence of an *arc-analytic foliation* along each stratum. Their fundamental theorem asserts that *every algebraic/analytic variety admits a PaPa regular stratification.* Neither V_a nor V_b are PaPa regular over the z-axis at 0.

The new theory is definitely better than the classical theory of Whitney-Thom.

As in every theory, there should be a *criterion* for the foundation definition. We suggest a way to find one, inspired by a simple example of Thom ([34], p. 30). Consider

$$f(x,y) = x^2 + H_3(x,y) + H_4(x,y) + \cdots \in \mathbb{R}\{x,y\}.$$

If $H_3 = y^3$, then terms of degree ≥ 4 can be topologically (also analytically) deleted:

$$x^2 + y^3 + [H_4 + \cdots] \sim_{top} x^2 + y^3.$$

If $H_3 = 2xy^2$, then terms of degree ≥ 4 *cannot* be ignored:

$$x^2 + 2xy^2 + [y^4 \pm y^{100}] = (x + y^2)^2 \pm y^{100}.$$

But if $H_3 = 2xy^2 + \epsilon y^3$, $\epsilon \neq 0$, then, again,

$$x^2 + (2xy^2 + \epsilon y^3) + [H_4 + \cdots] \sim_{top} x^2 + (2xy^2 + \epsilon y^3) \sim_{top} x^2 + \epsilon y^3.$$

Observation. The (0-dimensional) projective varieties $H_2 = 0$, $H_3 = 0$ in $\mathbb{R}P^1$ are *"in general position"* in the first and the third cases, but not so in the second case.

It was proved that if $H_p = 0$, $H_q = 0$ are *"in general position"* in $\mathbb{R}P^{n-1}$, then

$$V := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid H_p(x) + H_q(x) + t[H_{q+1}(x) + \cdots] = 0\} \tag{8.1}$$

is Whitney *(a,b)-regular* over the t-axis [16, 18]. (This led to the "Ratio Test" in [17].)

Question. Suppose $H_p = 0$, $H_q = 0$ are "*in general position*" in $\mathbb{R}P^{n-1}$. Is the above V PaPa-regular over the entire t-axis?

I am sure the answer is "Yes", and will lead to a criterion for the PaPa regularity.

A Steenrod type question: Is the PaPa-condition "*ultimate*", i.e., cannot be "improved" anymore? (In [22], a Grassmann blow-up *improves* the Trotman t^s-condition. We do not know how to define t^s-regularity for $s < 0$. Is the PaPa condition also the answer?)

9. THE CURVE SELECTION LEMMA AND I

The *Curve Selection Lemma* plays a vital rôle in the development of *Modern Mathematics.*

It is also the most powerful tool in *Singularity Theory.* To prove an analytic germ $f(x)$ has property P, it *suffices* to show that $f(\lambda(t))$ has property P for *every analytic arc* λ.

By an *analytic arc* at $0 \in \mathbb{R}^n$ we mean

$$\lambda : [0, \epsilon) \longrightarrow \mathbb{R}^n, \quad \lambda(t) = (a_1(t), ..., a_n(t)), \ \lambda(0) = 0, \tag{9.1}$$

where $a_i(t)$ are *convergent power series.*

The discovery of this lemma, as I see it, might have begun with a min-max problem.

Let $p(x, y) \in \mathbb{R}\{x, y\}$ be given. Suppose $p(at, bt)$ is minimum at $t = 0$, for all given *constants* a, b. Then, is it true that $p(x, y)$ is minimum at $(0, 0)$?

Lagrange (1736–1813) said "Yes". Later, Peano (1858–1936) found a counterexample:

$$p(x, y) = (y - x^2)(y - 3x^2),$$

where $p(x, y) < 0$ between the two parabolas. For more details see [9].

Let us modify the question. Suppose $p(a(t), b(t))$ is minimum at $t = 0$ for every *analytic arc* $(a(t), b(t))$ at 0. Is it true that $p(0, 0)$ is a minimum of $p(x, y)$?

Peano's counterexample is no longer one, since $p(t, 2t^2) < 0$. In fact, now the answer is actually "Yes". This is an easy consequence of

The Curve Selection Lemma (Bruhat-Cartan-Wallace, 1957-8, [26]). *Suppose*

$$S = \{x \in \mathbb{R}^n \,|\, p_1(x) \geq 0, ..., p_k(x) \geq 0, p_{k+1}(x) > 0, ..., p_{k+s}(x) > 0\}$$

contains points arbitrarily close to 0*, where* p_i *are analytic. Then there exists* λ *at* 0*,*

$$0 \neq \lambda(t) \in S \text{ for } t > 0.$$

If Peano had gone one step farther, namely considered analytic arcs $(a(t), b(t))$, then, being Peano, he might have been led to discover the Lemma. (Missed it by two parentheses?)

Let me tell you how I learned this lemma from Bochnak and Lojasiewicz.

In 1969, I was in Manchester, I knew the converse of the so-called Kuiper-Kuo Theorem [16] must be true. I struggled to find a proof, but failed. There was a hurdle.

In 1970, Bochnak and Lojasiewicz came to attend the *Liverpool Singularities Semester* (March-August). In June, Bochnak gave a seminar, proving what I could not prove [4].

When I finished my lecture in Manchester, rushed to Liverpool, the seminar was already over, the room was empty. But Bochnak wrote small, the entire lecture was still on the six blackboards. I sat there and read. I was the only audience, and there was no speaker anymore. But this is one of the most inspiring lectures I have ever attended.

What Bochnak wrote on the first couple of blackboards was known to me, and I saw they were approaching the same hurdle. Then he wrote: "by the *Curve Selection Lemma*, we can *choose* an analytic arc along which ...." Ah, in this way they got around the hurdle!

As I had circled around this lemma so hard for so long without knowing it, once I learned it, I fully appreciate its meaning and power. I have used it all the time ever since.

It completely changed my way of thinking; *e.g.*, in order to study $f(x,y)$, I now study

$$f_{\mathbb{F}}:\mathbb{F}\longrightarrow\mathbb{F},\quad \xi\mapsto f(\xi,y),$$

where $\mathbb{F}$ is the field of fractional power series in y. (Eliminating t from (9.1) gives

$$x=\xi(y):=c_1y^{n_1/N}+c_2y^{n_2/N}+\cdots\in\mathbb{F},\quad N\geq 1,\ n_1<n_2<\cdots.)$$

This $f_{\mathbb{F}}$ is a function of *one variable* in the *Newton-Puiseux Infinitesimal Analysis*, where definitions and theorems are found as extensions of those in *Calculus/Complex Variables.*

Fractional power series are "*infinitesimals*". Examples of *real* infinitesimals at $0\in\mathbb{R}$:

(9.2) $$x=0\ll x=y^{5/2}\ <\ x=2y^{5/2}\ll x=y^2,\quad y>0.$$

In order to express my appreciation to them, I introduced the name *"Bochnak-Lojasiewicz arc"* in my paper [18]. (Thence Bochnak called me his best friend.)

Let me tell you a few more stories to end my talk.

Koike told me that M. Suzuki has an interesting principle: in many cases, a topological property in the *complex* case corresponds to a blow-analytic property in the *real* case.

The hurdle (6.1) is blow-analytically trivial, the complex case is topologically trivial.

There are several examples of this kind which I learned from Koike, *e.g.*, a theorem of Oka in the complex case [28] led Suzuki to find his theorem in [31].

The renowned Zariski Conjecture led Suzuki to conjecture that

(9.3) $$f,\,g\in\mathbb{R}\{x_1,...,x_n\},\ f\sim_{b.a}g\implies m(f)=m(g)\quad\text{(same multiplicity)}.$$

He proved some special cases. Then, Fukui, in [7], gave an ingenious proof of the general case. The key idea of Fukui is to consider the induced mapping, *the Fukui glasses*:

(9.4) $$f_M:M^n\longrightarrow M,\quad \lambda(t)\mapsto f(\lambda(t)),$$

where M is the maximal ideal of $\mathbb{R}\{t\}$. The following is clearly true:

$$m(f)=\min\{O_t(p(t))\,|\,p(t)\in Im(f_M)\}.$$

An analytic arc at $0 \in \mathbb{R}^n$ *lifts* via $\tilde{\beta}_1$ to one in $\mathscr{M}_1$ at E_1, where Ψ carries it to one in $\mathscr{M}_2$ at E_2, then blow-down by $\tilde{\beta}_2$. Whence $Im(f_M) = Im(g_M)$, (9.3) follows.

The Fukui glasses may be more useful than the Newton-Puiseux glasses; as we see in his proof, $y^{1/N}$ is magnified to t, the Puiseux denominator N is not a hurdle.

In 1988, Kurdyka explained to me the notion of *arc-analyticity*, which grew out of his *arc-symmetry theory* [23]. A blow-analytic germ is obviously arc-analytic. He pointed out to me that a kind of converse must be true. (For meromorphic functions, such as (6.4), (6.5), the converse is true by Hironaka's Theorem.)

Then Bierstone-Milman [2] proved the converse is indeed true under some mild conditions. Hence blow-analyticity and arc-analyticity are, more or less, equivalent notions.

Arc-analyticity, Blow-analyticity, the Curve Selection Lemma, and the Newton-Puiseux Infinitesimal Analysis are closely related.

I spent many years trying in vain to prove the existence of blow-analytic stratifications. This attempt is perhaps no longer relevant. The PaPa stratification (arc-analytic foliations) should be sufficient for all purposes of stratification.

My attempt, though failed, led to an interesting problem: A proper analytic mapping

$$f : \mathscr{M}_1 \longrightarrow \mathscr{M}_2 \quad \text{(real/complex manifolds)}$$

is given, with critical points $\mathcal{C}_f$, and critical values $\mathcal{V}_f$. Can f be "*desingularised*"?

That is to say, we would like to find

$$\tilde{\beta}_i : \tilde{\mathscr{M}}_i \longrightarrow \mathscr{M}_i, \quad i = 1, 2, \quad \text{(compositions of blow-ups)}$$

to desingularise $\mathcal{C}_f$, $\mathcal{V}_f$ such that the following is true. The exceptional divisor E_i of $\tilde{\beta}_i$ is a normal crossing family of hypersurfaces, whence providing a stratification of $\tilde{\mathscr{M}}_i$, the strata being called "*canonical*", $i = 1, 2$. There exists an analytic mapping Φ such that

$$\begin{array}{ccc} (\tilde{\mathscr{M}}_1, E_1) & \xrightarrow{\quad \Phi \quad} & (\tilde{\mathscr{M}}_2, E_2) \\ \Big\downarrow \tilde{\beta}_1 & & \Big\downarrow \tilde{\beta}_2 \\ (\mathscr{M}_1, \mathcal{C}_f) & \xrightarrow{\quad f \quad} & (\mathscr{M}_2, \mathcal{V}_f) \end{array}$$

is commutative; and Φ is a *stratified submersion*, i.e., each canonical stratum of

$\tilde{\mathscr{M}}_1$ is mapped *submersively* onto one of $\tilde{\mathscr{M}}_2$. (If $\dim \mathscr{M}_2 = 1$, this is Hironaka's Theorem.)

H. King also came across this problem in his research on *Real Algebraic Goemetry*; he explained to me a proof when $\dim \mathscr{M}_2 = 2$.

10. THE LOCAL-TO-GLOBAL DIALECTICS

We are all familiar with the notion of being "local" and "global", although this is not formally defined. Big theorems in *Global Analysis* assert a passage from "local" to "global" (Gauss-Bonnet, Hadamard, Liebmann, Morse, etc.). Examples also appear elsewhere:

(1) If $f : (a, b) \to \mathbb{R}$ is differentiable, $f'(x) \geq 0$, then f is increasing;
(2) A finitely generated *abelian* group is a product of cyclic groups;
(3) A closed, connected, *locally convex* set in $\mathbb{R}^n$ is (globally) convex.

Question. Can we define (perhaps in *Category Theory*) "*local-to-global type theorem*"?

11. EILENBERG-MAC LANE'S CATEGORY THEORY PAPER

Mac Lane was very proud of their *Category Theory.* He told us when the paper, some 63 pages long, was submitted for publication, the referee, G. Mackey (Harvard), held it up for a very long time. So he and Eilenberg invited Mackey to eat lunch and asked him why. Mackey's reply: "*I have never seen such a long paper so trivial.*"

12. THE STONE AGE OF MATHEMATICS

Soon after World War II ended in 1945, Professor Marshall H. Stone was appointed Chairman of the Mathematics Department, the University of Chicago.

In the American system, the chairman has very big power. Some chairmen just do their administrative duties. There are also some who feel insecure; if they want to be number one in the department, they can cause a lot of damage.

Professor Stone was not like that. Besides, he had a very good sense in identifying good mathematics and good mathematicians. He was able to hire such top mathematicians like Chern, Mac Lane, Weil, Zygmund, etc. (He almost succeeded in getting Whitney.) He made the Chicago Mathematics Department one of the two best in the world.

Professor Stone was very much respected and admired. The era of his chairmanship has since been called *"The Stone Age of Mathematics at the University of Chicago"*.

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