COMPUTING LOGARITHMIC VECTOR FIELDS AND BRUCE-ROBERTS MILNOR NUMBERS *VIA* LOCAL COHOMOLOGY CLASSES

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An effective method is proposed for computing logarithmic vector fields along a hypersurface with an isolated singularity. The resulting algorithm outputs a set of generators of the module, over a local ring, of logarithmic vector fields. The outputs have nice expressions to study complex analytic structures of logarithmic vector fields. A method for computing Bruce-Robert Milnor numbers is described as an application. The keys of our approach are the use of local cohomology and the concept of polar variety. Computation results of Bruce-Roberts Milnor numbers are also given for some functions on non-quasi homogeneous hypersurface with an isolated singularity.

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1. INTRODUCTION

We present an effective method for computing logarithmic vector fields associated with an isolated hypersurface singularity. As an application of the proposed method, we consider Bruce-Roberts Milnor numbers in the context of symbolic computation. We describe in particular an algorithm for computing Bruce-Roberts Milnor numbers.

Logarithmic vector fields along a hypersurface were introduced by K. Saito [20]. After that, logarithmic vector fields have been studied and utilized by many people in various problems. Notably, H. Hauser and G. Müller [7, 8] showed that two germs of hypersurfaces with an isolated singular point are biholomorphic if and only if the corresponding Lie algebra of logarithmic vector fields are isomorphic. Accordingly, computing logarithmic vector fields is of considerable importance in complex analysis of singularities.

In our previous paper [11], by utilizing results of [21], we considered a special class of semi-quasihomogeneous singularities and have obtained a computation method of logarithmic vector fields. In the present paper, we extend the result mentioned above to the case of isolated hypersurface singularities. We show that, by adopting the technique used in [12,15], the use of parametric local cohomology systems [14] allows us to construct an effective algorithm for computing logarithmic vector fields.

Bruce-Roberts Milnor number was introduced in 1988 by J. W. Bruce and R. M. Roberts [2] as a generalization of the Milnor number, a multiplicity of an isolated critical point of a holomorphic function germ. This number is defined for a critical point of a holomorphic function on a singular variety in terms of logarithmic vector fields. Recently, Bruce-Robert Milnor numbers are investigated by several authors [1,4,5,18,19]. However, many problems related with Bruce-Roberts Milnor numbers remain unsolved.

We consider, in the present paper, Bruce-Roberts Milnor numbers in the context of symbolic computation. We describe, as an application of logarithmic vector fields, an algorithm for computing Bruce-Roberts Milnor numbers.

This paper is organized as follows. Section 2 briefly reviews some basic facts on logarithmic vector fields associated with an isolated hypersurface singularity and algebraic cohomology classes. Section 3 introduces a concept of generic coordinates system for a holomorphic defining function of a hypersurface. The use of generic coordinates system is a key of our resulting algorithms. Section 4 presents an effective method for computing a set of generators of module, over a local ring, of logarithmic vector fields associated with an isolated hypersurface singularity. Section 5 describes an algorithm for computing Bruce-Roberts Milnor numbers. Some examples of computation for functions on non quasi-homogeneous hypersurfaces are also given.

2. LOGARITHMIC VECTOR FIELDS AND ALGEBRAIC LOCAL COHOMOLOGY CLASSES

Here, we briefly review the relation between logarithmic vector fields and algebraic local cohomology classes. The details are in [11,21].

Let $S = \{x \in X | f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin O in $\mathbb{C}^n$, where X is an open neighborhood of the origin O and f is a holomorphic function. Let $\mathcal{O}_X$ be the sheaf on X of holomorphic functions, $\mathcal{O}_{X,O}$ the stalk at the origin of the sheaf $\mathcal{O}_X$.

Throughout this paper, we use the notation x as the abbreviation of n variables $x_1, \ldots, x_n$. The set of natural numbers $\mathbb{N}$ includes zero. K is the field of rational numbers $\mathbb{Q}$ or the field of complex numbers $\mathbb{C}$.

2.1. **Logarithmic vector fields**

The definition of logarithmic vector fields is the following.

Definition 1 (K. Saito [20]). A holomorphic vector field

$$v = a_1(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n},$$

$a_i(x) \in \mathcal{O}_X, i = 1, \ldots, n$, is **logarithmic** along S if $v(f)$ belongs to the ideal $\langle f \rangle$ generated by f in $\mathcal{O}_X$.

Let $\mathcal{D}\mathrm{er}_X(-\log S)$ denote the sheaf of logarithmic vector fields along S and $\mathcal{D}\mathrm{er}_{X,O}(-\log S)$ the stalk at O of $\mathcal{D}\mathrm{er}_X(-\log S)$.

A logarithmic vector field v generated over $\mathcal{O}_{X,O}$ by

$$f\left(\frac{\partial}{\partial x_1}\right), \ldots, f\left(\frac{\partial}{\partial x_n}\right) \text{ and } \left(\frac{\partial f}{\partial x_j}\right)\left(\frac{\partial}{\partial x_i}\right) - \left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial}{\partial x_j}\right), \quad 1 \leq i < j \leq n,$$

is called *trivial.*

PROPOSITION 2 (S. Tajima [21]). *Let* $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ *be a regular sequence. Let* $v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$ *be a logarithmic vector field along* S*. Then, the following conditions are equivalent.*

(i) v *is trivial.*

(ii) $a(x) \in \langle f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}\rangle$.

2.2. **Algebraic local cohomology**

All local cohomology classes, in this paper, are algebraic local cohomology classes that belong to the set defined by

$$H^n_{[O]}(\mathcal{O}_X) = \lim_{k\to\infty} \mathrm{Ext}^n_{\mathcal{O}_X}(\mathcal{O}_{X,O}/\langle x_1, x_2, \ldots, x_n\rangle^k, \mathcal{O}_X)$$

where $\langle x_1, x_2, \ldots, x_n\rangle$ is the maximal ideal generated by $x_1, \ldots, x_n$. We represent an algebraic local cohomology class as a polynomial $\sum c_\lambda \xi^\lambda$ where $c_\lambda \in K$, $\lambda \in \mathbb{N}^n$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. For each $i \in \{1, 2, \ldots, n\}$, ξ_i corresponds to x_i. The multiplication by x^α is defined as

$$x^\alpha * \xi^\lambda = \begin{cases} \xi^{\lambda-\alpha}, & \lambda_i \geq \alpha_i, i = 1, \ldots, n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and $\lambda - \alpha = (\lambda_1 - \alpha_1, \ldots, \lambda_n - \alpha_n) \in \mathbb{N}^n$.

Let $\mathbb{T}^n$ be the monoid of terms in $K[\xi_1,\ldots,\xi_n]$ and let fix a global term order $\prec$ on $\mathbb{T}^n$. For a given algebraic local cohomology class of the form

$$\eta = c_\lambda \xi^\lambda + \sum_{\xi^{\lambda'} \prec \xi^\lambda} c_{\lambda'} \xi^{\lambda'}, \quad c_\lambda \neq 0$$

we call ξ^λ the *head term*, c_λ the *head coefficient* and $\xi^{\lambda'}$ the *lower terms*. We write the head term as $\mathrm{ht}(\eta)$, the set of terms of η as $\mathrm{Term}(\eta) = \{\xi^\kappa | \eta = \sum_{\kappa\in\mathbb{N}^n} c_\kappa \xi^\kappa, c_\kappa \neq 0, c_\kappa \in K\}$ and the set of lower terms of η as $\mathrm{LL}(\eta) = \{\lambda^\kappa \in \mathrm{Term}(\eta) | \xi^\kappa \neq \mathrm{ht}(\eta)\}$. For instance, let $\eta = \xi_1^3\xi_2 + 3\xi_1\xi_2^2 - 2\xi_1$ and $\prec$ be the total degree lexicographic term order with $\xi_2 \prec \xi_1$. Then, $\mathrm{ht}(\eta) = \xi_1^3\xi_2$, $\mathrm{Term}(\eta) = \{\xi_1, \xi_1\xi_2^2, \xi_1^3\xi_2\} \subset \mathbb{T}^2$ and $\mathrm{LL}(\eta) = \{\xi_1, \xi_1\xi_2^2\}$.

Let H be a finite subset of $H^n_{[O]}(\mathcal{O}_X)$. We write the set of head terms of H as $\mathrm{ht}(H)$, the set of lower terms of H as $\mathrm{LL}(H) = \bigcup_{\eta\in H} \mathrm{LL}(\eta)$ and the set of terms of H as $\mathrm{Term}(H) = \bigcup_{\eta\in H} \mathrm{Term}(\eta)$. Moreover, we write the set of monomial elements of H as $\mathrm{ML}(H)$, the set of linear combination elements of H as $\mathrm{SL}(H)$, i.e., $H = \mathrm{ML}(H) \cup \mathrm{SL}(H)$. For instance, let $H = \{\xi_1\xi_2^2, \xi_1\xi_2, \xi_1^3 + 2\xi_1\xi_2 + \xi_2, \xi_2^2 - 4\xi_2, \xi_1\}$, then $\mathrm{ML}(H) = \{\xi_1\xi_2^2, \xi_1\xi_2, \xi_1\}$, $\mathrm{SL}(H) = \{\xi_1^3 + 2\xi_1\xi_2 + \xi_2, \xi_2^2 - 4\xi_2\}$, $\mathrm{LL}(H) = \{\xi_1\xi_2, \xi_2\}$ and $\mathrm{Term}(H) = \{\xi_1, \xi_2, \xi_1\xi_2, \xi_2^2, \xi_1\xi_2^2, \xi_1^3\}$.

Let ξ^λ be a term and Ψ a set of terms in $\mathbb{T}^n$ where $\lambda \in \mathbb{N}^n$. We call $\xi^\lambda\xi_i$ a neighbor of ξ^λ for each $i = 1, 2, \ldots, n$. We write the *neighbors* of Ψ as $\mathrm{Neighbor}(\Psi)$, i.e., $\mathrm{Neighbor}(\Psi) = \{\xi^\kappa \cdot \xi_i | \xi^\kappa \in \Psi\}$.

Definition 3 (inverse orders). Let $\prec$ be a local or global term order. Then, the *inverse order* $\prec^{-1}$ of $\prec$ is defined by $\xi^\alpha \prec \xi^\beta \iff \xi^\beta \prec^{-1} \xi^\alpha$ where $\alpha, \beta \in \mathbb{N}^n$.

Definition 4 (minimal bases). A basis $\{\xi^{\alpha_1}, \xi^{\alpha_2}, \ldots, \xi^{\alpha_l}\}$ for a monomial ideal is said to be *minimal* if no ξ^{α_i} in the basis divides other ξ^{α_j} for $i \neq j$, where $\alpha_1, \alpha_2, \ldots, \alpha_l \in \mathbb{N}^n$.

Notation 5 ([14]). Let $\prec$ be a global term order on $\mathbb{T}^n$ and H a finite subset of $H^n_{[O]}(\mathcal{O}_X)$ such that for all $\eta \in H$, the head coefficient of η is 1, $\mathrm{ht}(\eta) \notin \mathrm{ht}(H\backslash\{\eta\})$ and $\mathrm{ht}(\eta) \notin \mathrm{LL}(H)$ w.r.t. $\prec$. Let $\eta \in H$ represent $\xi^\lambda + \sum_{\xi^{\lambda'} \prec \xi^\lambda} c_{(\lambda,\lambda')}\xi^{\lambda'}$ where $c_{(\lambda,\lambda')} \in K$ and $\lambda, \lambda' \in \mathbb{N}^n$. Then, the transfer SB_H is defined by the following:

$$\begin{cases} \mathrm{SB}_H(\xi^\alpha) = x^\alpha - \displaystyle\sum_{\xi^\kappa\in \mathrm{ht}(\mathrm{SL}(H))} c_{(\kappa,\lambda)} x^\kappa, & \text{in } K[x] \text{ if } \xi^\lambda \in \mathrm{LL}(H), \\ \mathrm{SB}_H(\xi^\alpha) = x^\alpha, & \text{in } K[x] \text{ if } \xi^\lambda \notin \mathrm{LL}(H), \end{cases}$$

where $\alpha, \kappa \in \mathbb{N}^n$.

Let Ψ be a set of terms in $\mathbb{T}^n$. Then, the set $\mathrm{SB}_H(\Psi)$ is also defined by $\mathrm{SB}_H(\Psi) = \{\mathrm{SB}_H(\xi^\lambda) | \xi^\lambda \in \Psi\}$.

Set

$$H_{\Gamma(f)} = \left\{ \eta \in H^n_{[O]}(\mathcal{O}_X) \,\middle|\, f * \eta = \frac{\partial f}{\partial x_2} * \eta = \frac{\partial f}{\partial x_3} * \eta = \cdots = \frac{\partial f}{\partial x_n} * \eta = 0 \right\}$$

and assume that the sequence $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ is a regular sequence [10]. Then, by Grothendieck local duality theorem [6], $H_{\Gamma(f)}$ is a finite dimensional vector space.

Set

$$H_{\Phi(f)} = \left\{ \frac{\partial f}{\partial x_1} * \eta \,\middle|\, \eta \in H_{\Gamma(f)} \right\}$$

and

$$Ann_{\mathcal{O}_{X,O}}(H_{\Phi(f)}) = \{a(x) \in \mathcal{O}_{X,O} \,|\, a(x) * \eta = 0, \forall \eta \in H_\Phi(f)\}.$$

In [12, 14, 22], computation methods of a basis of the vector spaces $H_{\Gamma(f)}$ and $H_{\Phi(f)}$ have been introduced.

Theorem 6 ([14]). *Using the same notation as in above, let H be a basis of the vector space $H_{\Phi(f)}$ such that for all $\eta \in H$, the head coefficient of η is 1 and $\mathrm{ht}(\eta) \notin \mathrm{LL}(H)$ w.r.t. a global term order $\prec$. Let Ψ be the minimal basis of the ideal generated by $\mathrm{Neighbor}(\mathrm{ht}(H)) \setminus \mathrm{ht}(H)$.*

Then, $\mathrm{SB}_H(\Psi)$ is the reduced standard basis of the ideal $Ann_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. the local term order $\prec^{-1}$ in $\mathcal{O}_{X,O}$.

By using a basis of the vector space $H_{\Phi(f)}$ (set of algebraic local cohomology classes), we are able to obtain a reduced standard bases of $Ann_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$.

The following properties will be used to construct an algorithm for computing logarithmic vector fields along S.

Theorem 7 (S. Tajima [21]). *Let $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ be a regular sequence. Let $a(x) \in \mathcal{O}_{X,O}$. Then, the following conditions are equivalent.*

(i) $a(x) \in Ann_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$.

(ii) *There exists a logarithmic vector field v along S such that*

$$v = a(x)\frac{\partial}{\partial x_1} + a_2(x)\frac{\partial}{\partial x_2} + \cdots + a_n(x)\frac{\partial}{\partial x_n}$$

where $a_2(x), \ldots, a_n(x) \in \mathcal{O}_{X,O}$.

Let $\mu(f)$ denote the Milnor number of f at the origin defined by

$$\mu(f) = \dim\left(\mathcal{O}_{X,O}/\left\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle\right).$$

Let $\tau(f)$ denote the Tjurina number of f at the origin defined by

$$\tau(f) = \dim\left(\mathcal{O}_{X,O}/\left\langle f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right\rangle\right).$$

A result of [21] together with a result of D. T. Lê [10] and B. Teissier [24] yields the following.

Theorem 8 (S.Tajima [21]). *Let $f, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}$ be a regular sequence and $H_{\Phi(f)} = \left\{ \frac{\partial f}{\partial x_1} * \eta \,\middle|\, \eta \in H_{\Gamma(f)} \right\}$. Then,*

$$\dim_{\mathbb{C}}\left(H_{\Phi(f)}\right) = \mu(f) - \tau(f) + \mu(f|_{x_1=0})$$

where $f|_{x_1=0}$ is the restriction of f on the hyperplane $x_1 = 0$.

Let $p' = (p'_1, p'_2, \ldots, p'_n)$ be a non-zero vector and let $[p']$ be the corresponding point in the complex projective space $\mathbb{P}^{n-1}$. We identify the hyperplane

$$H_{p'} = \{(x_1, \ldots, x_n) \in \mathbb{C}^n \mid p'_1x_1 + p'_2x_2 + \cdots + p'_nx_n = 0\}$$

with the point $[p']$ in $\mathbb{P}^{n-1}$. Then, the number $\mu^{(n-1)}(f)$ is defined as

$$\mu^{(n-1)}(f) = \min_{[p']\in\mathbb{P}^{n-1}} \mu(f|_{H_{p'}})$$

where $f|_{H_{p'}}$ is the restriction of f on $H_{p'}$ and $\mu(f|_{H_{p'}})$ is the Milnor number of $f|_{H_{p'}}$ at the origin O.

Theorem 9 (B. Teissier [25]). *Let $f(x_1, \ldots, x_n)$ be a holomorphic function with an isolated singularity at the origin O. Let $U = \{[p'] \in \mathbb{P}^{n-1} \mid \mu(f|_{H_{p'}}) = \mu^{(n-1)}(f)\}$. Then, U is a Zariski open dense subset of $\mathbb{P}^{n-1}$.*

Algorithms for computing $\mu^{(n-1)}(f)$ are given in [15, 16]. Accordingly, $\mu(f), \tau(f)$ and $\mu^{(n-1)}(f)$ are computable.

3. GENERIC COORDINATES OF f

Let $(x'_1, x'_2, \ldots, x'_n)$ be a coordinate system that is defined by

$$\begin{aligned}
x'_1 &= x_1 + p_2x_2 + \cdots + p_nx_n,\\
x'_2 &= x_2,\\
&\vdots\\
x'_n &= x_n,
\end{aligned}$$

where $p_2, \ldots, p_n$ are indeterminates.

Set

$$f_p(x'_1,\ldots,x'_n)=f(x'_1-p_2x'_2-\cdots-p_nx'_n,x'_2,\ldots,x'_n),$$

$$H_{\Gamma(f_p)}=\left\{\eta\in H^n_{[O]}(\mathcal{O}_X)\,\middle|\, f_p*\eta=\frac{\partial f_p}{\partial x'_2}*\eta=\frac{\partial f_p}{\partial x'_3}*\eta=\cdots=\frac{\partial f_p}{\partial x'_n}*\eta=0\right\}$$

and

$$H_{\Phi(f_p)}=\left\{\frac{\partial f_p}{\partial x'_1}*\eta\,\middle|\,\eta\in H_{\Gamma(f_p)}\right\}$$

where $p=(p_2,\ldots,p_n)\in\mathbb{C}^{n-1}$.

We introduce a number $\varphi(f)$ as

$$\varphi(f)=\min_{p\in\mathbb{C}^{n-1}}\left(\dim_{\mathbb{C}}(H_{\Phi(f_p)})\right).$$

The following corollary immediately follows from Theorem 9 and Theorem 8.

Corollary 10. *Let $f_p(x'_1,\ldots,x'_n)=f(x'_1-p_2x'_2-\cdots-p_nx'_n,x'_2,\ldots,x'_n)$ where $p=(p_2,\ldots,p_n)\in\mathbb{C}^{n-1}$. Let $U_\Phi=\{p\in\mathbb{C}^{n-1}\mid \dim_{\mathbb{C}}(H_{\Phi(f_p)})=\varphi(f)\}$. Then, U_Φ is a Zariski open dense subset of $\mathbb{C}^{n-1}$. Moreover, $\varphi(f)=\mu(f)-\tau(f)+\mu^{(n-1)}(f)$.*

As there exist algorithms for computing $\mu^{(n-1)}(f)$, the number $\varphi(f)$ is computable. We say that the coordinate $(x'_1,x'_2,\ldots,x'_n)$ is *generic*, if $\dim_{\mathbb{C}}(H_{\Phi(f_p)})=\varphi(f)$.

Example 11. Let us consider $f(x_1,x_2)=x_1^3+x_1x_2^9+x_2^{14}\in\mathbb{C}[x_1,x_2]$ (E_{25} singularity) that defines an isolated singularity at the origin O in $\mathbb{C}^2$. In this case, $\mu(f)=25$, $\tau(f)=22$ and $\mu^{(1)}(f)=2$ and $\varphi(f)=25-22+2=5$. Note that (x_1,x_2) is not generic for f because $\dim_{\mathbb{C}}(H_{\Phi(f)})=16$.

Let us consider a coordinate (x'_1,x'_2) that is defined by

$$\begin{aligned}x'_1&=x_1+p_2x_2,\\x'_2&=x_2,\end{aligned}$$

where p_2 is an indeterminate. Set $f_p(x'_1,x'_2)=f(x'_1-p_2x'_2,x'_2)$ and regard p_2 as a parameter. Then, by computing parametric local cohomology system (see [14]) w.r.t. $\langle f_p,\frac{\partial f_p}{\partial x'_2}\rangle$, we have the following

1. if $p_2=0$, then $\dim_{\mathbb{C}}(H_{\Phi(f_p)})=16$,
2. if $p_2\neq 0$, then $\dim_{\mathbb{C}}(H_{\Phi(f_p)})=5$.

Notice that $p_2\neq 0$ is a Zariski open subset of $\mathbb{C}$. If $p_2\neq 0$, then

$$\begin{aligned}x'_1&=x_1+p_2x_2,\\x'_2&=x_2,\end{aligned}$$

is a generic coordinate for $f_p(x'_1,x'_2)$.

4. COMPUTATION METHOD OF LOGARITHMIC VECTOR FIELDS

Here we give an algorithm for computing a basis of the module $\mathcal{D}\mathrm{er}_{X,O}(-\log S)$ where $S = \{x \in X | f(x) = 0\}$ is a hypersurface with an isolated singularity at the origin O. We generalize the algorithm [11] for semi-quasihomogeneous hypersurfaces, to hypersurfaces with an isolated singularity. In this section, we assume that $(x_1, x_2, \ldots, x_n)$ is generic for $f(x)$, for simplicity.

The following lemma will be utilized to construct the main algorithm.

Lemma 12. *Let $G = \{q_1, \ldots, q_r\}$ be the reduced standard basis of the annihilating ideal $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. a local term order $\prec$. Then, for each $j \in \{1, 2, \ldots, r\}$, there exists a polynomial*

$$g_j \in \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle q_j \frac{\partial f}{\partial x_1}\rangle$$

such that $g_j(O) \neq 0$ where

$$\begin{aligned}&\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle q_j \frac{\partial f}{\partial x_1}\rangle \\ &= \{g \in K[x_1, \ldots, x_n] | g(q_j \frac{\partial f}{\partial x_1}) \in \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle\}\end{aligned}$$

is the ideal quotient in the global ring $K[x_1, \ldots, x_n]$.

Proof. By the Grothendieck local duality theorem, $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Gamma(f)}) = \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle$. Let $q(x) \in \mathcal{O}_{X,O}$. Then,

$$\begin{aligned} & q(x) \in \mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)}) \\ & \iff \text{for all } \eta \in H_{\Gamma(f)},\ q(x)\left(\frac{\partial f}{\partial x_1} * \eta\right) = \left(q(x)\frac{\partial f}{\partial x_1}\right) * \eta = 0, \\ & \iff q(x) \in \left\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f \right\rangle : \left\langle \frac{\partial f}{\partial x_1} \right\rangle. \end{aligned}$$

Thus, $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)}) = \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle \frac{\partial f}{\partial x_1}\rangle$.

Let $I = \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle \frac{\partial f}{\partial x_1}\rangle$ in $K[x_1, \ldots, x_n]$ (global ring). Then, as $q_j \in \mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ in $\mathcal{O}_{X,O}$, by Lemma 1 of [13], there exists $g \in I : \langle q_j \rangle$ such that $g(O) \neq 0$ in $K[x_1, \ldots, x_n]$. Therefore,

$$\begin{aligned} g \in I : \langle q_j \rangle &= \left(\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle \frac{\partial f}{\partial x_1}\rangle\right) : \langle q_j \rangle \\ &= \langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \ldots, \frac{\partial f}{\partial x_n}, f\rangle : \langle q_j \frac{\partial f}{\partial x_1}\rangle. \quad \square \end{aligned}$$

We are now ready to give the main theorem.

THEOREM 13. *Using the same notation as in Lemma 12, let M_j be the reduced Gröbner basis of the module of syzygies of* $g_j q_j \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}, f$ *w.r.t. a position over term order, for each* $1 \leq j \leq r$. *Then, there exists* $(1, c_{j2}, c_{j3}, \ldots, c_{jn}, c_{jn+1}) \in M_j$, *namely,*

$$v_j = q_j \frac{\partial}{\partial x_1} + \frac{c_{j2}}{g_j}\frac{\partial}{\partial x_2} + \frac{c_{j3}}{g_j}\frac{\partial}{\partial x_3} + \cdots + \frac{c_{jn}}{g_j}\frac{\partial}{\partial x_n}$$

is a logarithmic vector field along S. Moreover, $v_1, \ldots, v_r$ and trivial vector fields generate $\mathcal{D}er_{X,O}(-\log S)$.

Proof. From Lemma 12, for each $j \in \{1, \ldots, r\}$, $g_j(q_j \frac{\partial f}{\partial x_1})$ can be written as

$$g_j(q_j \frac{\partial f}{\partial x_1}) = d_2 \frac{\partial f}{\partial x_2} + \cdots + d_n \frac{\partial f}{\partial x_n} + d_{n+1} f$$

where $d_2, \ldots, d_n, d_{n+1} \in K[x_1, \ldots, x_n]$. As $(1, -d_2, -d_3, \ldots, -d_n, -d_{n+1}) \in \langle M_j \rangle$ and M_j is the reduced Gröbner basis w.r.t. the position over term order, therefore, there exists $(1, c_{j2}, c_{j3}, \ldots, c_{jn}, c_{jn+1}) \in M_j$ where $c_{j2}, c_{j3}, \ldots, c_{jn}, c_{jn+1} \in K[x_1, \ldots, x_n]$.

The condition $g_j(O) \neq 0$ implies that $\frac{d_j}{g_j}$ are well-defined as elements of $\mathcal{O}_{X,O}$. Hence,

$$q_j \frac{\partial f}{\partial x_1} = (\tfrac{d_2}{g_j}) \cdot \frac{\partial f}{\partial x_2} + \cdots + (\tfrac{d_n}{g_j}) \cdot \frac{\partial f}{\partial x_n} + (\tfrac{d_{n+1}}{g_j}) \cdot f$$

and this implies

$$q_j \frac{\partial f}{\partial x_1} - (\tfrac{d_2}{g_j}) \cdot \frac{\partial f}{\partial x_2} - \cdots - (\tfrac{d_n}{g_j}) \cdot \frac{\partial f}{\partial x_n} \in \langle f \rangle$$

in $\mathcal{O}_{X,O}$. Therefore,

$$v_j = q_j \frac{\partial}{\partial x_1} + (\tfrac{-d_2}{g_j})\frac{\partial}{\partial x_2} + \cdots + (\tfrac{-d_n}{g_j})\frac{\partial}{\partial x_n}$$

is a logarithmic vector field along S. By Theorem 7 and Proposition 2, $v_1, \ldots, v_r$ and trivial vector fields generate $\mathcal{D}er_{X,O}(-\log S)$. □

Remark. The algorithm for computing $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ consists of only linear algebra. (See [14, 22].) In Theorem 13, ideal quotients and syzygies in the "global" ring $K[x_1, \ldots, x_n]$ are utilized. Thus, almost all computation of $\mathcal{D}er_{X,O}(-\log S)$ is performed in the "global" ring.

Theorem 13 leads us to construct Algorithm 1.

Algorithm 1 has been implemented in the computer algebra system Risa/Asir.

Let us compare the output of Algorithm 1 with that of the function `syz` of the computer algebra system SINGULAR [3].

Let $f = x_1^2 x_2 + x_2^3 + x_2^4$ and $S = \{x \in X | f(x) = 0\}$. Then, Algorithm 1 outputs $\{v_1, v_2, v_3\}$ as a basis of $\mathcal{D}er_{X,O}(-\log S)$.

Algorithm 1

Input: $f(x_1,\dots,x_n)$: a polynomial with an isolated singularity at the origin O where $(x_1,\dots,x_n)$ is generic for f.
Output: $[V_N, V_T]$: $V_N \cup V_T$ is a basis of $\mathcal{D}er_{X,O}(-\log S)$. Each $v \in V_N$ forms $v = d_1\frac{\partial}{\partial x_1} + d_2\frac{\partial}{\partial x_2} + \cdots + d_n\frac{\partial}{\partial x_n}$ where $d_1 \neq 0$ and $d_1,\dots,d_n \in \mathcal{O}_X$. Each $v' \in V_T$ forms $d'_2\frac{\partial}{\partial x_2} + \cdots + d'_n\frac{\partial}{\partial x_n}$ where $d'_2,\dots,d'_n \in \mathcal{O}_X$.
BEGIN
$V_N \leftarrow \emptyset$;
$\{q_1,\dots,q_r\} \leftarrow$ Compute the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. a local term order $\prec$;
for each $1 \le i \le r$ **do**
$G_i \leftarrow$ Compute a Gröbner basis of $\langle \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}, \dots, \frac{\partial f}{\partial x_n}, f\rangle : \langle q_i\frac{\partial f}{\partial x_1}\rangle$ in $K[x_1,\dots,x_n]$;
$g \leftarrow$ Select a polynomial g from G s.t. $g(O) \neq 0$;
Syz $\leftarrow$ Compute the reduced Gröbner basis of a module of syzygies of $(gq\frac{\partial f}{\partial x_1}), \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n}, f$ w.r.t. a position over term module order in $K[x_1,\dots,x_n]^{n+1}$;
$(1, d_2.\dots, d_n, d_{n+1}) \leftarrow$ Select a vector whose form is $(1, d_2,\dots,d_n,d_{n+1})$, from Syz;
$V_N \leftarrow V_N \cup \{gq\frac{\partial}{\partial x_1} + d_2\frac{\partial}{\partial x_2} + \cdots + d_n\frac{\partial}{\partial x_n}\}$;
end-for
$\mathrm{Syz}_0 \leftarrow \{(0, d'_2,\dots,d'_n) | (0, d'_2,\dots,d'_n, d'_{n+1}) \in \mathrm{Syz}\}$;
$V_T \leftarrow \{d'_2\frac{\partial}{\partial x_2} + \cdots + d'_n\frac{\partial}{\partial x_n} | (0, d'_2,\dots,d'_n) \in \mathrm{Syz}_0\}$;
return $[V_N, V_T]$;
END

$$
\begin{aligned}
v_1 &= (3x_2+2)x_1\frac{\partial}{\partial x_1} + (2x_2+2x_2^2)\frac{\partial}{\partial x_2},\\
v_2 &= (3x_2+2)x_2^2\frac{\partial}{\partial x_1} + (-2x_1x_2)\frac{\partial}{\partial x_2},\\
v_3 &= (x_1^2x_2+x_2^4+x_2^3)\frac{\partial}{\partial x_2}
\end{aligned}
$$

where $\{x_1, x_2^2\}$ is the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ w.r.t. the local total degree lexicographic term order with (x_1,x_2).

It is known that a basis of $\mathcal{D}er_{X,O}(-\log S)$ can be obtained by computing syzygies of $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$ w.r.t. a local term order. By using the functions `syz` (for computing syzygies) and `std` (for computing standard bases) of the computer algebra system Singular [3], a basis of $\mathcal{D}er_{X,O}(-\log S)$ can be obtained. The following $\{v'_1, v'_2\}$ is a standard basis of $\mathcal{D}er_{X,O}(-\log S)$.

$$
v'_1 = (3x_1x_2+2x_1)\frac{\partial}{\partial x_1} + (2x_2+2x_2^2)\frac{\partial}{\partial x_2},
$$

$$v_2' = (x_1^2 + 3x_2^2 - 3x_1^2x_2 + 3x_2^3)\frac{\partial}{\partial x_1} + (-2x_1x_2 - 2x_1x_2^2)\frac{\partial}{\partial x_2}.$$

Let us examine two holomorphic functions $3x_1x_2 + 2x_1$, $x_1^2 + 3x_2^2 - 3x_1^2x_2 + 3x_2^3$, the coefficients of $\frac{\partial}{\partial x_1}$ of the output above. It is true that $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)}) = \langle 3x_1x_2 + 2x_1, x_1^2 + 3x_2^2 - 3x_1^2x_2 + 3x_2^3\rangle$, however, the fact $\langle 3x_1x_2+2x_1, x_1^2+3x_2^2-3x_1^2x_2+3x_2^3\rangle = \langle x_1, x_2^2\rangle$ cannot be read off directly from the output of std(syz($\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$)). For instance, consider the factorization $3x_1x_2+2x_1 = (3x_2+2)x_1$. Since the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$ is $\{x_1, x_2^2\}$, the factor x_1 is essential, the other factor $3x_2 + 2$ is of less importance. The function std does not tell such information. Next, $x_1^2 + 3x_2^2 - 3x_1^2x_2 + 3x_2^3$ can be reduced by $3x_1x_2 + 2x_1$ into $3x_2^2 + 3x_2^3$ which is equal to $3(x_2 + 1)x_2^2$. This fact is not given in the output of std(syz($\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$)).

In general, the function std does not output the reduced standard basis of the module of syzygies. In contrast, Algorithm 1 presented above always outputs the reduced standard basis of the module, over a local ring, of syzygies.

If we compute the reduced Gröbner basis of a module of syzygies of $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, f$, w.r.t. a position over term order (with the degree lexicographic term order with $x_1 > x_2$) in $K[x_1, x_2]^3$, then we have the following

$$v_1'' = (3x_1^2+x_2^2)\frac{\partial}{\partial x_1}+2x_1x_2\frac{\partial}{\partial x_2}, \qquad v_2'' = (3x_1x_2+2x_1)\frac{\partial}{\partial x_1}+(2x_2^2+2x_2)\frac{\partial}{\partial x_2},$$

$$v_3'' = (3x_2^3 + 2x_2^2)\frac{\partial}{\partial x_1} + (-2x_1x_2^2)\frac{\partial}{\partial x_2}, \qquad v_4'' = (x_1^2x_2 + x_2^4 + x_2^3)\frac{\partial}{\partial x_2}.$$

The output, the reduced Gröbner basis, consists of four logarithmic vector fields. Since four is bigger than two and three, it contains some redundancy that comes from global properties or some other issues. Since we are interested in local complex analytic properties of logarithmic vector fields, a presence of such unnecessary information is not favorable. Note also that since $\{v_1'', v_2'', v_3'', v_4''\}$ is a reduced Gröbner basis, $3x_1^2+x_2^2$ cannot be reduced by $\{3x_1x_2+2x_1, 3x_2^3+2x_2^2\}$ in $K[x_1, x_2]$. This means that such a direct use of syzygy computation in polynomial ring is not appropriate for our purpose.

Algorithm 1 is designed to analyze complex analytic structures and properties of the modules, over a local ring, of logarithmic vector fields. As the output tells us in particular the reduced standard basis of $\mathrm{Ann}_{\mathcal{O}_{X,O}}(H_{\Phi(f)})$, we are able to analyze the structure of $\mathcal{D}er_{X,O}(-\log S)$. The other existing algorithms do not have such a significant feature.

The output of Algorithm 1 is $[V_N, V_T]$. Thus, we are able to analyze how the logarithmic vector fields of V_N affect the structure of $\mathcal{D}er_{X,O}(-\log S)$, and how the trivial vector fields affect the structure of $\mathcal{D}er_{X,O}(-\log S)$. This is also an advantage of Algorithm 1.

Example 14. Let $f = x^2z + y^3 + z^4 + yz^2 \in \mathbb{C}[x, y, z]$ and $S = \{x_0 \in X|\psi(x_0) = 0\}$. Our implementation outputs the following as a basis of $\mathcal{D}er_{X,O}(-\log S)$.

$V_N = \{(-27y + 4)x\frac{\partial}{\partial x} + (-24y^2 + 4y - 4z^2)\frac{\partial}{\partial y} + (-18yz + 4z)\frac{\partial}{\partial z},$
$(-1)(3y^2 + z^2)\frac{\partial}{\partial x} + (2xz)\frac{\partial}{\partial y},$
$(-27y + 4)yz\frac{\partial}{\partial x} + (-2xy + 18xz^2)\frac{\partial}{\partial y} + (-6xz)\frac{\partial}{\partial z},$
$(-27y + 4)y^3\frac{\partial}{\partial x} + (18xy^2z - 2xzy - 6xz^3)\frac{\partial}{\partial y} + (2xz^2)\frac{\partial}{\partial z}\},$
$V_T = \{(y^3 - yz^2 - 3z^4)\frac{\partial}{\partial y} + (3y^2z + z^3)\frac{\partial}{\partial z},$
$(x^2 + 2yz + 4z^3)\frac{\partial}{\partial y} + (-3y^2 - z^2)\frac{\partial}{\partial z},$
$(x^2z + y^3 + z^4 + yz^2)\frac{\partial}{\partial z}\}.$

5. BRUCE-ROBERTS MILNOR NUMBERS

Bruce-Roberts Milnor number is a generalization of the Milnor number. Recently in [18], Nuño-Ballesteros, Oréfice and Tomazella gave a closed formula of Bruce-Roberts Milnor numbers for the case h and f are weighted homogeneous polynomial functions w.r.t. a same weight vector. To the best of our knowledge, no other closed formulas are known even for the case of semi-quasihomogeneous isolated hypersurface singularities. This is a motivation of the present and the following sections.

Let $S = \{x \in X|f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin O in $\mathbb{C}^n$ where f is a holomorphic function. Let h be a holomorphic function. Let

$$J_{BR}(h, S) = \langle v(h)|v \in \mathcal{D}\mathrm{er}_{X,O}(-\log S)\rangle.$$

The definition of Bruce-Roberts Milnor numbers is the following.

Definition 15. The Bruce-Roberts Milnor number of h w.r.t. S, is given by

$$\mu_{BR}(h, S) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/J_{BR}(h, S)).$$

Let us consider a set of local cohomology associated with $J_{BR}(h, S)$, *i.e.*,

$$H_{BR(h,S)} = \{\eta \in H^n_{[O]}(\mathcal{O}_X)|g * \eta = 0, \forall g \in J_{BR}(h, S)\}.$$

Let $[V_N, V_T]$ denote the output of Algorithm 1. Since the ideal $J_{BR(h,S)}$ in the local ring $\mathcal{O}_{X,O}$ is generated by $\{v(h) \mid v \in V_N \cup V_T\}$, we have

$$H_{BR(h,S)} = \{\eta \mid v(h) * \eta = 0, v \in V_N \cup V_T\}.$$

Therefore, a basis of the vector space $H_{BR(h,S)}$ can be computed by the algorithm described in [14].

The following theorem is the direct consequence of Theorem 31 of [14].

THEOREM 16. *Using the same notation as in above, let H be a basis of the vector space $H_{BR(h,S)}$ such that for all $\eta \in H$, the head coefficient of η is 1, $ht(\eta) \notin ht(H \backslash \{\eta\})$ and $ht(\eta) \notin \mathrm{LL}(H)$ w.r.t. a global term order $\prec$. Let Ψ be the minimal basis of the ideal generated by* $\mathrm{Neighbor}(\mathrm{ht}(H)) \backslash \mathrm{ht}(H)$. *(Some of notations are from Section 2.2.) Then,*

(1) $\mu_{BR}(h,S) = \dim_{\mathbb{C}}(H_{BR(h,S)}) = |H|$.
(2) $\mathrm{SB}_H(\Psi)$ *is the reduced standard basis of the ideal $J_{BR}(h,S)$ w.r.t. the local term order $\prec^{-1}$ in $\mathcal{O}_{X,O}$.*

Notation 17. Let $g = \sum c_\kappa x_1^{\kappa_1} x_2^{\kappa_2} \cdots x_n^{\kappa_n} \in K[x_1, \ldots, x_n]$ where $c_\kappa \in K$. Let $\prec$ be a global term order on $\mathbb{T}^n$ and H a finite subset of $K[\xi_1, \ldots, \xi_n]$.

(1) For all $i \in \{1, 2, \ldots, n\}$, a map $\mathcal{CV}$ is defined as changing variables x_i into ξ_i. Thus, $\mathcal{CV}(g) = \sum c_\kappa \xi_1^{\kappa_1} \xi_2^{\kappa_2} \cdots \xi_n^{\kappa_n}$ in $K[\xi_1, \ldots, \xi_n]$.
(2) For a term $x^\alpha \in K[x_1, \ldots, x_n]$, the transfer Mem_H is defined by the following:

$$\begin{cases} \mathrm{Mem}_H(\xi^\alpha) = \mathrm{SB}_H(\xi^\alpha), \quad \text{in } K[x_1, \ldots, x_n] & \text{if} \quad \xi^\alpha \in \mathrm{Term}(H), \\ \mathrm{Mem}_H(\xi^\alpha) = 0, & \text{if} \quad \xi^\alpha \notin \mathrm{Term}(H) \end{cases}$$

where the notation $\mathrm{Term}(H)$ is from Section 2.2.

COROLLARY 18. *Using the same notation as in Theorem 16, let $g = \sum c_\kappa x^\kappa$ be a polynomial in $K[x_1, \ldots, x_n]$ where $c_\kappa \in K$ and $\kappa \in \mathbb{N}^n$. Then, $g \in J_{BR}(h,S)$ in $\mathcal{O}_{X,O}$ if and only if $\sum c_\kappa \mathrm{Mem}_H(\mathcal{CV}(x^\kappa)) = 0$.*

Proof. If $\xi^\alpha \notin \mathrm{SL}(H) \cup \mathrm{ML}(H)$, it is obvious that for all $\eta \in H$, $x^\alpha * \eta = 0$ because there exists a degree of x_i such that the degree is bigger than a degree of ξ_i of all terms of η. This means $x^\alpha \in J_{BR}(h,S)$. The rest of corollary follows from Theorem 16. $\square$

We are able to obtain Bruce-Roberts Milnor numbers, reduced standard bases of $J_{BR}(h,S)$ and solve ideal membership problems of $J_{BR}(h,S)$ by utilizing basis local cohomology classes of the vector space $H_{BR(h,S)}$.

Example 19. Let us consider $f = x_1^2 x_2 + x_2^3 + x_1^2 x_2^3$, $h = x_1^2 + x_2^2$ and $S = \{x \in X | f(x) = 0\}$. A basis of $\mathcal{D}er_{X,O}(-\log S)$ is $\{v_1 = (-x_1^2 + 4)x\frac{\partial}{\partial x_1} + (2x_2^2 + 4x_2)\frac{\partial}{\partial x_2}, v_2 = (-x_1^2 + 4)x_2^2\frac{\partial}{\partial x_1} + (-2x_2^3 - 6x_2^2 - 4x_2)x_1\frac{\partial}{\partial x_2}, v_3 = (-x_1^2 x_2^2 - x_1^2 x_2 - x_2^3)\frac{\partial}{\partial x_2}\}$. Thus, $J_{BR}(h,S) = \langle v_1(h), v_2(h), v_3(h) \rangle$.

Let us fix the term order $\prec$ as the total degree lexicographic term order with $\xi_2 \prec \xi_1$ where ξ_1, ξ_2 correspond to x_1, x_2. Then, a basis of the vector space $H_{BR(h,S)}$ is

$$H = \{1, \xi_1, \xi_2, \xi_1\xi_2, \xi_1^2 - \xi_2^2, \xi_1^2\xi_2 - \xi_2^3 + \frac{1}{2}\xi_2^2, \xi_1^3 - \xi_1\xi_2^2, \xi_1^4 - \xi_1^2\xi_2^2 + \xi_2^4 - \frac{1}{2}\xi_2^3 + \frac{1}{2}\xi_2^2\}$$

that can be obtained by the computation method of [14].

As $|H| = 8$, now we know $\mu_{BR}(h, S) = 8$.

The minimal basis of Neighbor(ht(H))$\setminus$ht(H) is $\{\xi_1^5, \xi^3\xi_2, \xi_2^2\}$. Since $\xi_1^5, \xi_1^3\xi_2 \notin \mathrm{LL}(H)$ and $\xi_2^2 \in \mathrm{LL}(H)$, $\mathrm{SB}_H(\xi_1^5) = x_1^5$, $\mathrm{SB}_H(\xi_1^3\xi_2) = x_1^3x_2$ and $\mathrm{SB}_H(\xi_2^2) = x_2^2 - x_1^2 - \frac{1}{2}x_1^2x_2 - \frac{1}{2}x_1^4$. Therefore, the reduced standard basis of $J_{BR}(h, S)$ w.r.t. the local term order $\prec^{-1}$ is

$$\{x_1^5, x_1^3x_2, x_2^2 - x_1^2 - \tfrac{1}{2}x_1^2x_2 - \tfrac{1}{2}x_1^4\}.$$

In the rest of this section, we give examples of Bruce-Roberts Milnor numbers as the application of logarithmic vector fields.

Let $\mathrm{Triv} = \{f\frac{\partial}{\partial x_i} \mid 1 \leq i \leq n\} \cup \{(\frac{\partial f}{\partial x_j})\frac{\partial}{\partial x_i} - (\frac{\partial f}{\partial x_i})\frac{\partial}{\partial x_j} \mid 1 \leq i < j \leq n\}$ a set of generators of trivial vector fields. As Algorithm 1 outputs the set V_N, we are able to consider the colength of the following two ideals.

$$J_T(h, S) = \langle v(h) | v \in \mathrm{Triv}\rangle, \quad J_N(h, S) = \langle v(h) | v \in V_N\rangle.$$

The two colengths

$$\mu_T(h, S) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/J_T(h, S)), \quad \mu_N(h, S) = \dim_{\mathbb{C}}(\mathcal{O}_{X,O}/J_N(h, S)),$$

can also be computed in the same manner as the case of Bruce-Roberts Milnor numbers, if the two ideals $J_N(h, S), J_T(h, S)$ are zero-dimensional.

Let $p' = (p'_1, p'_2, \ldots, p'_n)$ be a non-zero vector and $h_{p'} = p'_1x_1 + p'_2x_2 + \cdots + p'_nx_n$. Then, we define $\mu_{BR}(\ell_{gen}, S)$ as

$$\mu_{BR}(\ell_{gen}, S) = \min_{p' \in \mathbb{P}^{n-1}} \left(\mu_{BR}(h_{p'}, S)\right).$$

For example, let us consider $f = x_1^2x_2 + x_2^3 + x_2^4$ and $S = \{x \in X | f(x) = 0\}$. As we described in Section 4, a basis of $\mathcal{D}er_{X,O}(-\log S)$ is

$$\{v_1 = (3x_2 + 2)x_1\tfrac{\partial}{\partial x_1} + (2x_2 + 2x_2^2)\tfrac{\partial}{\partial x_2},$$
$$v_2 = (3x_2 + 2)x_2^2\tfrac{\partial}{\partial x_1} + (-2x_1x_2)\tfrac{\partial}{\partial x_2}, v_3 = (x_1^2x_2 + x_2^4 + x_2^3)\tfrac{\partial}{\partial x_2}\}.$$

Thus, computing parametric local cohomology classes [14] associated with the ideal $\langle v_1(h_{p'}), v_2(h_{p'}), v_3(h_{p'})\rangle$, we have the following, where $h_{p'} = p'_1x_1 + p'_2x_2$.

- If $p_1'^2 + p_2'^2 \neq 0$, then $\mu_{BR}(h_{p'}, S) = 2$.
- If $p_1'^2 + p_2'^2 = 0$ (and $(p'_1, p'_2) \neq (0, 0)$), then $\mu_{BR}(h_{p'}, S) = 3$.

Therefore, $\mu_{BR}(\ell_{gen}, S) = 2$.

As the example shows, $\mu_{BR}(\ell_{gen}, S)$ can also be computed by utilizing the algorithm for computing parametric local cohomology classes [14].

Since $\mu_{BR}(\ell_{gen}, S) = \varphi(f)$ holds, we have the following result.

Proposition 20. *Let $S = \{x \in X \mid f(x) = 0\}$ be a hypersurface with an isolated singularity at the origin. Then,*

$$\mu_{BR}(\ell_{gen}, S) = \mu(f) - \tau(f) + \mu^{(n-1)}(f)$$

holds.

Note that $\mu^{(n-1)}(f)$ is related with the local Euler obstruction of S at the origin [16, 23].

Table 1

List of some invariants

	h	f	$\mu(f)$	$\tau(f)$	$\mu_T(h,S)$	$\mu_N(h,S)$	$\mu_{BR}(\ell_{gen},S)$	$\mu_{BR}(h,S)$
1	$f_0 + xy^{11}$	f_0	24	24	93	69	2	69
2	$f_0 + xy^{10}$	f_0	24	24	90	66	2	66
3	$f_0 + xy^9$	f_0	24	24	87	63	2	63
4	f_0	$f_0 + xy^{10}$	24	22	90	68	4	68
5	$f_0 + xy^{11}$	$f_0 + xy^{10}$	24	22	90	68	4	68
6	$f_0 + 2xy^{10}$	$f_0 + xy^{10}$	24	22	90	68	4	68
7	$f_0 + xy^9$	$f_0 + xy^{10}$	24	22	87	65	4	65
8	$f_0 + 2xy^{11}$	$f_0 + xy^{11}$	24	23	93	70	3	70
9	$f_0 + 2xy^{11}$	$f_0 + xy^{10}$	24	22	90	68	4	68
10	$f_0 + 2xy^{11}$	$f_0 + xy^9$	24	21	87	66	5	66
11	$x^3 + y^3$	f_0	24	24	36	12	2	12
12	$x^3 + y^3$	$f_0 + xy^{11}$	24	23	36	13	3	13
13	$x^3 + y^3$	$f_0 + xy^{10}$	24	22	36	14	4	14
14	$x^3 + y^3$	$f_0 + xy^9$	24	21	36	15	5	15

Table 2

List of some invariants

	h	f	$\mu(f)$	$\tau(f)$	$\mu_T(h,S)$	$\mu_N(h,S)$	$\mu_{BR}(\ell_{gen},S)$	$\mu_{BR}(h,S)$
1	$f_0 + y^{18}$	$f_0 + y^{16}$	28	25	103	78	5	78
2	$f_0 + y^{17}$	$f_0 + y^{16}$	28	25	103	78	5	78
3	$f_0 + 2y^{16}$	$f_0 + y^{16}$	28	25	103	78	5	78
4	f_0	$f_0 + y^{18}$	28	27	109	82	3	82
5	f_0	$f_0 + y^{17}$	28	26	106	80	4	80
6	f_0	$f_0 + x^{16}$	28	25	103	78	5	78
7	$x^4 + y^4$	$f_0 + x^{16}$	28	25	48	23	5	23
8	$x^5 + y^5$	$f_0 + x^{16}$	28	25	58	33	5	33
9	$x^6 + y^6$	$f_0 + x^{16}$	28	25	70	45	5	45
10	$x^3 + y^3$	f_0	28	28	40	12	2	12
11	$x^3 + y^3$	$f_0 + y^{18}$	28	27	40	13	3	13
12	$x^3 + y^3$	$f_0 + y^{17}$	28	26	40	14	4	14
13	$x^3 + y^3$	$f_0 + x^{16}$	28	25	40	15	5	15

We have implemented algorithms for computing $\mu_T(h,S)$, $\mu_N(h,S)$, $\mu_{BR}(\ell_{gen},S)$, $\mu_{BR}(h,S)$ in the computer algebra system Risa/Asir [17]. We give some examples of computation.

Example 21. Let us consider $f_0 = x^3 + y^{13}$ (E_{24} singularity) and $S = \{x_0 \in X | f(x_0) = 0\}$. Then, our implementation returns $\mu_T(h,S)$, $\mu_N(h,S)$, $\mu_{BR}(\ell_{gen},S)$, $\mu_{BR}(h,S)$ as Table 1.

Example 22. In Table 2, $f_0 = x^3 + x^2y^5 + y^{15}$ (J_{28} singularity) and $S = \{x_0 \in X | \psi(x_0) = 0\}$. Then, our implementation returns $\mu_T(f,S)$, $\mu_N(f,S)$, $\mu_{BR}(\ell_{gen},S)$, $\mu_{BR}(f,S)$.

Next, we give cases of three variables.

Example 23. In Table 3, $f_0 = x^2y + z^4 + y^5$ (V_{18}^{*1} singularity) and $S = \{x \in X | \psi(x) = 0\}$.

In Table 4, $f_0 = x^2y+z^4+y^4z$ (V_{19}^{*1} singularity) and $S = \{x_0 \in X | \psi(x_0) = 0\}$. Our implementation returns $\mu_T(f,S)$, $\mu_N(f,S)$, $\mu_{BR}(\ell_{gen},S)$, $\mu_{BR}(f,S)$.

In these tables, "$-$" means that $J_N(f,S)$ is not zero-dimensional, namely, trivial vector fields are necessary to decide the structure of $\mathcal{D}er_{X,O}(-\log S)$ and Bruce-Roberts Milnor numbers.

Table 3
List of some invariants

	h	f	$\mu(f)$	$\tau(f)$	$\mu_T(h,S)$	$\mu_N(h,S)$	$\mu_{BR}(\ell_{gen},S)$	$\mu_{BR}(h,S)$
1	$x^4+y^3+z^4$	f_0	18	18	79	$-$	5	61
2	$x^4+y^3+z^4$	$f_0+y^4z^2$	18	17	79	68	6	62
3	$x^4+y^3+z^4$	f_0+y^4z	18	16	79	$-$	7	63
4	$x^4+y^3+z^4$	$f_0+y^3z^2$	18	16	79	69	7	63
5	$x^4+y^4+z^4$	f_0	18	18	98	$-$	5	80
6	$x^4+y^4+z^4$	$f_0+y^4z^2$	18	17	98	91	6	81
7	$x^4+y^4+z^4$	f_0+y^4z	18	16	98	$-$	7	82
8	$x^4+y^4+z^4$	$f_0+y^3z^2$	18	16	98	92	7	82

Table 4
List of some invariants

	h	f	$\mu(f)$	$\tau(f)$	$\mu_T(h,S)$	$\mu_N(h,S)$	$\mu_{BR}(\ell_{gen},S)$	$\mu_{BR}(h,S)$
1	$x^3+y^3+z^3$	f_0	19	19	58	$-$	5	39
2	$x^3+y^3+z^3$	$f_0+y^3z^3$	19	18	58	$-$	6	40
3	$x^3+y^3+z^3$	$f_0+y^3z^2$	19	17	58	$-$	7	41
4	$x^3+y^3+z^3$	$f_0+y^2z^3$	19	17	58	$-$	7	41

Notice that all these examples computed above satisfy $\mu_{BR}(h,S) = \mu_T(h, S) - \tau(f)$.

Note added in proof. The anonymous referee informed us that K. Kourlikouros studies Bruce-Roberts Milnor numbers of a germ of holomorphic function with respect to a hypersurface with an isolated singularity and gives in particular in [9], among others, more general formula that generalize the result presented in Proposition 20. Note that his approach and arguments are different from ours and the paper [9] is recently posted on arXiv after the submission of our present paper.

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