Dedicated to Professor Tzee-Char Kuo on his 81st birthday

A SIGNATURE INVARIANT FOR STABLE MAPS OF 3-MANIFOLDS INTO SURFACES

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Based on the signature formula for stable maps of closed oriented 4–manifolds into 3–manifolds, the author defined a Vassiliev type invariant of order one for stable maps of closed oriented 3–manifolds into surfaces. In this paper, we give an intrinsic formula for the invariant in terms of a certain linking form associated with a stable map for the 3–manifold. As a corollary, we get a signature formula for 4–manifolds with boundary in terms of their singular fibers of stable maps.

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1. INTRODUCTION

In this paper, we consider differentiable maps of class C^∞ between C^∞ manifolds. For such maps, Vassiliev [23] introduced a class of invariants which are now called *Vassiliev type invariants*: they are invariants for certain generic maps and behave nicely under generic deformations. For such an invariant, we have the notion of its *order*: for example, if the invariant is defined for C^∞ stable maps and behaves nicely under generic deformations that pass through unstable loci "transversely", then it is of order one.

For maps of 3–manifolds into surfaces, $\mathbf{Z}_2$–valued Vassiliev type invariants of order one have been thoroughly studied by Minoru Yamamoto in [25] by using singular fibers [17]. Here, a *singular fiber* is a map germ along the full inverse image of a singular value. More precisely, he formulated the notion of a semi-local invariant of order one for maps of a closed orientable 3–manifold into $\mathbf{R}^2$, and completely classified them: in fact, the space of such invariants is of dimension eight over $\mathbf{Z}_2$ and he explicitly identified the generating invariants. Unfortunately, he obtained only $\mathbf{Z}_2$–valued invariants and the orientations of the source 3–manifolds did not play an essential role.

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On the other hand, in the final chapter of [17], a $\mathbf{Z}$–valued Vassiliev type invariant of order one for maps of an *oriented* 3–manifold into $\mathbf{R}^2$ has been introduced. (In fact, it was Mikio Furuta that proposed the invariant.) Such an invariant had not appeared in Minoru Yamamoto's work [25]. For a C^∞ stable map $f : M \to \mathbf{R}^2$ of a closed oriented 3–manifold M, this integer valued invariant, denoted by $\sigma(f)$, is defined by using a generic extension $F : V \to \mathbf{R}^2 \times [0, \infty)$ of f, where V is a compact oriented 4–manifold with $\partial V = M$, $F|_{\partial V} = f : \partial V \to \mathbf{R}^2 \times \{0\}$, and $F|_{\mathrm{Int}\, V} : \mathrm{Int}\, V \to \mathbf{R}^2 \times (0, \infty)$ is a proper C^∞ stable map. More precisely, $\sigma(f)$ is defined as the difference between the signature of V and the algebraic number of singular fibers of type III^8 (for notations of singular fiber types, the reader is referred to [17]). Then, we can show that this invariant does not depend on particular choices of V nor F. In fact, this well-definedness is a consequence of the signature formula for C^∞ stable maps of closed oriented 4–manifolds into $\mathbf{R}^3$: for such a stable map, the signature of the source closed oriented 4–manifold coincides with the algebraic number of singular fibers of type III^8 [18, 19].

In this paper, we consider the invariant $\sigma(f) \in \mathbf{Z}$ for C^∞ stable maps $f : M \to N$ of closed oriented 3–manifolds into surfaces, and show that $\sigma(f)$ coincides with the signature of a certain linking form associated with f defined for the source 3–manifold M (Theorem 4.3). This enables us to compute $\sigma(f)$ without using dimension four. Furthermore, as a consequence, we obtain a signature formula for generic maps of compact oriented 4–manifolds with boundary into 3-manifolds with boundary (Corollary 4.7).

The paper is organized as follows. In Section 2, we first review certain characterizations of C^∞ stable maps of 3– and 4–dimensional manifolds. We also recall the notion of singular fibers, and then recall the signature formula for C^∞ stable maps of closed oriented 4–manifolds into 3–manifolds. In Section 3, we recall the definition of the invariant σ and give its basic properties. In Section 4, we state and prove our main theorem. Our idea is to construct a generic extension F as above "canonically" in a certain sense. Finally, in Section 5, we give explicit examples. We also give an explicit path in the mapping space which passes through a certain unstable locus transversely exactly once and for which the value of the invariant σ changes by ± 1. As an application of such an explicit path, we will construct a new explicit C^∞ stable map of $\mathbf{C}P^2$ into $\mathbf{R}^3$.

Throughout the paper, manifolds and maps are differentiable of class C^∞ unless otherwise indicated. For a topological space X, id_X denotes the identity map of X. All (co)homology groups are with integer coefficients unless otherwise indicated. The symbol "$\cong$" means an appropriate isomorphism between algebraic objects or a diffeomorphism between smooth manifolds.

2. PRELIMINARIES

Throughout the paper, the following notion of a C^∞ stable map (or stable map, for short) plays an important role.

Definition 2.1. Let $f : M \to N$ be a smooth map between smooth manifolds. Such a map is C^∞ *stable* (or *stable*, for short) if there exists a neighborhood U_f of f in the space $C^\infty(M, N)$ of all smooth maps of M into N endowed with the Whitney C^∞–topology such that for every $g \in U_f$, there exist diffeomorphisms $\Phi : M \to M$ and $\varphi : N \to N$ that make the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\Phi} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ N & \xrightarrow{\varphi} & N. \end{array}$$

About stable maps, the reader is referred to [5] for more details.

For stable maps in low dimensions, the following is known. (For details, the reader is referred to [9,17], for example.) In the following, we suppose that the source manifolds are closed, i.e. they are compact and have no boundary. Furthermore, for a smooth map f, we denote by $S(f)$ the set of singular points of f.

Remark 2.2. (1) A function $f : M \to \mathbf{R}$ on a closed manifold of dimension $n \geq 1$ is stable if and only if it has only non-degenerate critical points and the critical values are all distinct, i.e. f is a *Morse function*.

(2) Let M be a closed 3–manifold and N a surface. A map $f : M \to N$ is stable if and only if it has only fold and cusp singularities, $f|_{S(f) \smallsetminus C(f)}$ is an immersion with normal crossings, and $f^{-1}(f(x)) \cap S(f) = \{x\}$ for each $x \in C(f)$, where $C(f)$ is the set of cusp singular points. Here, a *fold* (or *cusp*) singularity is a singular point of f around which f is locally of the form $(x, y, z) \mapsto (x, y^2 \pm z^2)$ (resp. $(x, y^3 + xy - z^2)$) with respect to certain local coordinates. Note that in this case, $S(f)$ is a closed non-singular 1–dimensional submanifold of M and $C(f)$ is a finite set of points.

(3) Let M be a closed 4–manifold and N a 3–manifold. A map $f : M \to N$ is stable if and only if it has only fold, cusp and swallowtail singularities, and $f|_{S(f)} : S(f) \to N$ has only the multi-germs as described in Fig. 1. Here, a *fold* (resp., *cusp* or *swallowtail*) singularity is a singular point of f around which f is locally of the form $(x, y, z, w) \mapsto (x, y, z^2 \pm w^2)$ (resp. $(x, y, z^3 + yz - w^2)$ or $(x, y, z^4 + xz^2 + yz \pm w^2)$) with respect to certain local coordinates. Note that in this case, $S(f)$ is a closed non-singular 2–dimensional submanifold of M.

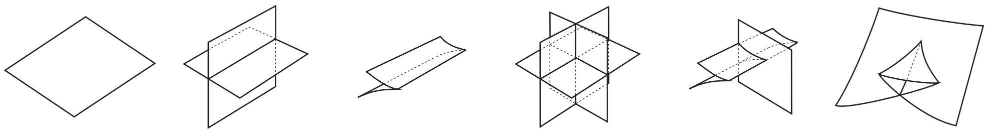

Fig. 1. List of local forms of $f(S(f))$ for stable maps f of 4–manifolds into 3–manifolds.

(4) In all the above situations, it is known that the set of all stable maps forms an open dense subset of the mapping space. Therefore, any smooth map can be approximated by a stable map. Furthermore, it is also known that if a smooth map g satisfies the above conditions on a neighborhood of a closed subset C of M, then such an approximating stable map f can be chosen in such a way that $f|_C = g|_C$.

In this paper, the notion of fibers also plays an important role. (For details, the reader is referred to [17].)

Definition 2.3. Let $f : M \to N$ be a smooth map. For a point $y \in N$, the *fiber of f over y* is the map germ $f : (M, f^{-1}(y)) \to (N, y)$ along the whole pre-image $f^{-1}(y)$. When $y \in f(S(f))$, we say that it is a *singular fiber*.

Let us consider two maps $f_i : M_i \to N_i$ and take $y_i \in N_i$, $i = 0, 1$. The fibers of f_0 and f_1 over y_0 and y_1, respectively, are said to be *equivalent* if there exist diffeomorphism germs Ψ and ψ that make the following diagram commutative:

$$\begin{array}{ccc} (M_0, f_0^{-1}(y_0)) & \xrightarrow{\Psi} & (M_1, f_1^{-1}(y_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (N_0, y_0) & \xrightarrow{\psi} & (N_1, y_1). \end{array}$$

Note that in [17], singular fibers of stable maps of closed orientable 3–manifolds into surfaces and those of closed orientable 4–manifolds into 3–manifolds have been completely classified.

Furthermore, in [18, 19], a signature formula as described below has been obtained. Let M be a closed oriented 4–manifold, N a 3–manifold, and $f : M \to N$ a stable map. Then, in the classification list of singular fibers of such maps, we have a class of singular fibers of type III^8, which appears discretely; in other words, there are always finitely many such singular fibers for a given f. (Refer to [17] about the notations for singular fibers. See also Fig. 2.) Moreover, using the given orientation of M, we can define a sign, positive or negative, for each III^8–type singular fiber. (Note that this does not depend on a local orientation of N.) We denote by $||\mathrm{III}^8(f)||$ the number of III^8–type singular fibers of f counted with signs. Then, we have

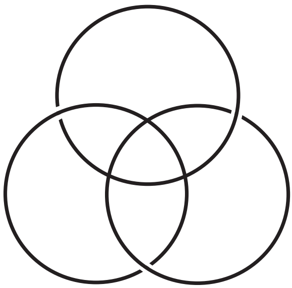

Fig. 2. III^8–type singular fiber.

$$\mathrm{sign}(M) = ||\mathrm{III}^8(f)||, \tag{2.1}$$

where $\mathrm{sign}(M)$ is the signature of the *intersection form*

$$Q_M : H_2(M; \mathbf{Q}) \times H_2(M; \mathbf{Q}) \to \mathbf{Q}$$

of M, and is called the *signature* of the 4–manifold M.

3. INVARIANT σ

In this section, we recall the definition of an invariant of stable maps of closed oriented 3–manifolds into surfaces and give some of its basic properties.

Let $f : M \to N$ be a stable map of a closed oriented 3–manifold M into a (possibly non-compact) surface N without boundary. It is known that there always exists a compact oriented 4–manifold V such that $\partial V = M$ as oriented manifolds (see [14, 20]). Let P be a (possibly non-compact) 3–manifold with $\partial P = N$.

Definition 3.1. A smooth map $F : V \to P$ is called a *generic null-cobordism* of f if it satisfies the following.

(1) $F^{-1}(\partial P) = \partial V = M$.
(2) For small collar neighborhoods $C(\partial V) = M \times [0, \varepsilon)$ and $C(\partial P) = N \times [0, \varepsilon)$ of ∂V in V and ∂P in P, respectively, with $0 < \varepsilon << 1$, we have that $F|_{C(\partial V)} : C(\partial V) \to C(\partial P)$ can be identified with $f \times \mathrm{id}_{[0,\varepsilon)} : M \times [0, \varepsilon) \to N \times [0, \varepsilon)$.
(3) The restriction $F|_{\mathrm{Int}\, V} : \mathrm{Int}\, V \to \mathrm{Int}\, P$ is a proper stable map.

LEMMA 3.2. *For a stable map $f : M \to N$ of a closed oriented 3–manifold M into a surface N, there always exists a generic null-cobordism $F : V \to P$ for some V and P.*

Proof. Since the oriented bordism group $\Omega_3(N)$ vanishes (see [2]), there exists a continuous map $V \to N$ extending f for some compact oriented 4–manifold V with $\partial V = M$. Then, for $P = N \times [0, \infty)$, we can construct a smooth map $F_1 : V \to P$ that satisfies ((1)) and ((2)) above (with F being replaced by F_1). This map F_1 can be approximated by a smooth map F such that $F|_{C(\partial V)} = F_1|_{C(\partial V)}$ and $F|_{\mathrm{Int}\, V} : \mathrm{Int}\, V \to N \times (0, \infty)$ is a proper stable map as required. □

Definition 3.3. Let $f : M \to N$ be a stable map of a closed oriented 3–manifold M into a surface N. We define the integer $\sigma(f)$ by

$$\sigma(f) = \mathrm{sign}(V) - ||\mathrm{III}^8(F)||,$$

where $F : V \to P$ is a generic null-cobordism of f, $\mathrm{sign}(V)$ is the signature of the compact oriented 4–manifold V with boundary, and $||\mathrm{III}^8(F)||$ is the number of singular fibers of type III^8 of F counted with signs.

LEMMA 3.4. *The integer $\sigma(f) \in \mathbf{Z}$ is well defined; i.e. it does not depend on the choice of a generic null-cobordism F as above.*

Proof. Suppose that we have another generic null-cobordism $F' : V' \to P'$ of f as above. Then $(-F) \cup F' : (-V) \cup V' \to P \cup P'$ is a stable map of the closed oriented 4–manifold $(-V) \cup V'$ into the 3–manifold $P \cup P'$ without boundary. Here, $-V$ denotes the 4–manifold V with the reversed orientation, $(-V) \cup V'$ is the closed oriented 4–manifold obtained by attaching $-V$ and V' along their boundary 3–manifold M orientation reversingly, $P \cup P'$ is the 3–manifold obtained by attaching P and P' along N, and $-F : -V \to P$ is identical to F (see Fig. 3). Then, by the signature formula (2.1) for singular fibers of stable maps of closed oriented 4–manifolds into 3–manifolds [18, 19], we have

$$\mathrm{sign}((-V) \cup V') = ||\mathrm{III}^8((-F) \cup F')||.$$

Now, by Novikov additivity (for example, see [4, 24]), we have

$$\mathrm{sign}((-V) \cup V') = -\,\mathrm{sign}(V) + \mathrm{sign}(V').$$

Furthermore, we also have

$$||\mathrm{III}^8((-F) \cup F')|| = -||\mathrm{III}^8(F)|| + ||\mathrm{III}^8(F')||,$$

since reversing the orientation of the source 4–manifold changes the signs of all III^8–type singular fibers. Hence, we have

$$\mathrm{sign}(V) - ||\mathrm{III}^8(F)|| = \mathrm{sign}(V') - ||\mathrm{III}^8(F')||.$$

This completes the proof. □

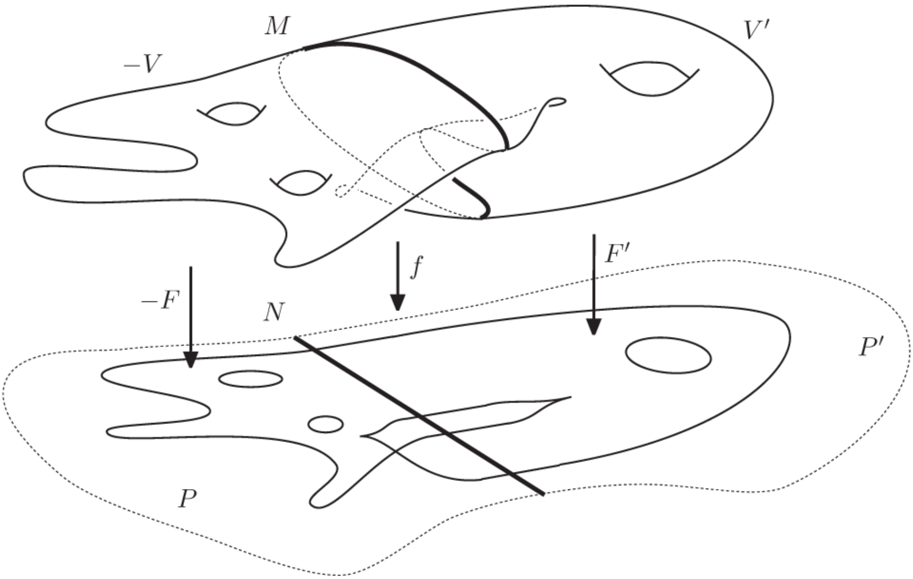

Fig. 3. Stable map $(-F) \cup F' : (-V) \cup V' \to P \cup P'$

Note that the invariant σ is originally defined in [17, §16.2].[1]

The following proposition is straightforward and the proof is left to the reader.

PROPOSITION 3.5. (1) *The value σ is invariant under* $\mathrm{R}^+\mathrm{L}$*–equivalence: i.e., for two stable maps $f_i : M_i \to N_i$, $i = 0, 1$, of closed oriented 3–manifolds into surfaces, if there exist diffeomorphisms $\Phi : M_0 \to M_1$ and $\varphi : N_0 \to N_1$ such that Φ is orientation preserving and $f_0 = \varphi^{-1} \circ f_1 \circ \Phi$, then we have $\sigma(f_0) = \sigma(f_1)$.*

(2) *For a stable map $f : M \to N$ of a closed oriented 3–manifold into a surface, we have $\sigma(-f) = -\sigma(f)$, where $-f : -M \to N$ is given by the map f with $-M$ being the 3–manifold M with the orientation reversed.*

(3) *The invariant σ is additive under disjoint union.*

Definition 3.6. Let $f_i : M_i \to N_i$, $i = 0, 1$, be stable maps of closed oriented 3–manifolds into (unoriented) surfaces. A *generic cobordism* between f_0 and f_1 is a smooth map $F : V \to P$ of a compact oriented 4–manifold V with $\partial V = (-M_0) \cup M_1$ into a 3–manifold P with $\partial P = N_0 \cup N_1$ which satisfies the following conditions, where $-M_0$ denotes the 3–manifold M_0 with the orientation reversed.

(1) $F^{-1}(N_i) = M_i$, $i = 0, 1$.

[1] The author is indebted to Mikio Furuta for the idea of construction of the invariant σ.

(2) For a small collar neighborhood $C(M_0) = M_0 \times [0, \varepsilon)$ of M_0 in V (resp. $C(M_1) = M_1 \times (1-\varepsilon, 1]$ of M_1 in V), and a small collar neighborhood $C(N_0) = N_0 \times [0, \varepsilon)$ of N_0 in P (resp. $C(N_1) = N_1 \times (1-\varepsilon, 1]$ of N_1 in P), $0 < \varepsilon << 1$, we have that $F|_{C(M_0)} = f_0 \times \mathrm{id}_{[0,\varepsilon)}$ and $F|_{C(M_1)} = f_1 \times \mathrm{id}_{(1-\varepsilon,1]}$.

(3) The map $F|_{\mathrm{Int}\, V} : \mathrm{Int}\, V \to \mathrm{Int}\, P$ is a proper stable map.

LEMMA 3.7. *Let $f_i : M_i \to N_i$, $i = 0, 1$, be stable maps of closed oriented 3–manifolds into surfaces. If there exists a generic cobordism $F : V \to P$ between f_0 and f_1, then we have*

$$\sigma(f_1) - \sigma(f_0) = \mathrm{sign}(V) - ||\mathrm{III}^8(F)||.$$

Proof. Let $F_0 : V_0 \to P_0$ be a generic null-cobordism of f_0. Then the map $F \cup F_0 : V \cup V_0 \to P \cup P_0$ gives a generic null-cobordism of f_1, where $V \cup V_0$ is the compact oriented 4–manifold obtained by gluing V and V_0 along M_0 orientation reversingly, and $P \cup P_0$ is the 3–manifold obtained by gluing P and P_0 along $N_0 = \partial P_0$. Note that $\partial(V \cup V_0) = M_1$ and $\partial(P \cup P_0) = N_1$. Then, we have

$$\sigma(f_1) = \mathrm{sign}(V_0) + \mathrm{sign}(V) - (||\mathrm{III}^8(F)|| + ||\mathrm{III}^8(F_0)||)$$

and

$$\sigma(f_0) = \mathrm{sign}(V_0) - ||\mathrm{III}^8(F_0)||.$$

Therefore, we get the desired result. (We could also show the lemma by regarding F as a generic null-cobordism for $(-f_0) \cup f$ and by using Proposition 3.5.) □

LEMMA 3.8. *The invariant σ defines a Vassiliev type invariant of order one for stable maps in $C^\infty(M, N)$.*

Proof. Let $f_t : M \to N$, $t \in [0, 1]$, be a generic smooth 1–parameter family of smooth maps connecting stable maps f_0 and f_1 such that it intersects codimension 1 unstable strata of $C^\infty(M, N)$ "transversely" at finitely many parameter values and that it does not intersect unstable strata of codimension ≥ 2 (for more details, see [12, 13], for example). Then, the map $F : M \times [0, 1] \to N \times [0, 1]$ defined by $F(x, t) = (f_t(x), t)$, $(x, t) \in M \times [0, 1]$, gives a generic cobordism with vanishing signature between f_0 and f_1. Therefore, if f_t does not pass through a codimension 1 unstable stratum corresponding to a III^8–type singular fiber of F, then we have $\sigma(f_0) = \sigma(f_1)$. In general, by Lemma 3.7, $\sigma(f_1) - \sigma(f_0)$ coincides with the intersection number of the path f_t in $C^\infty(M, N)$ with the co-oriented codimension 1 strata C_{III^8} corresponding to III^8–type singular fibers in such a way that the co-orientation is defined by using the sign of the singular fibers (see Fig. 4). In such a sense, σ defines a

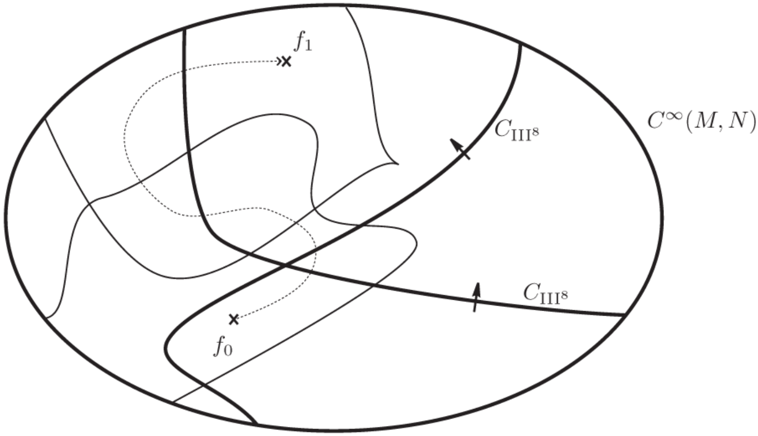

Fig. 4. Intersection of a generic path f_t and the codimension 1 strata C_{III^8}, where C_{III^8} corresponds to the thick curves. In this example, we have $\sigma(f_1) - \sigma(f_0) = 2$.

Vassiliev type invariant of order one, since the difference $\sigma(f_1)-\sigma(f_0)$ does not depend on the choice of the generic path connecting f_0 and f_1. □

4. MAIN THEOREM

In this section, we state and prove the main theorem of this paper, i.e. an intrinsic formula for the invariant σ. Let $f : M \to N$ be a stable map of a closed oriented 3–manifold M into a surface.

Definition 4.1. Let us consider the following equivalence relation for M. Two points in M are equivalent if they lie in the same connected component of the pre-image of a point in N by f. Let W_f denote the quotient space of M with respect to this equivalence relation and $q_f : M \to W_f$ the quotient map. Then, we see easily that there exists a unique continuous map $\overline{f} : W_f \to N$ such that $f = \overline{f} \circ q_f$. The commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ & {\scriptstyle q_f}\searrow \quad \nearrow{\scriptstyle \overline{f}} & \\ & W_f & \end{array}$$

is called the *Stein factorization* of f and the space W_f is called the *quotient space* or the *Reeb space* of f. Note that W_f can be regarded as the space of connected components of f–fibers. The continuous map $\overline{f}$ is sometimes called the *Reeb map* of f.

It is known that W_f is a compact polyhedron and its local structures have been completely determined (see [7, 9]). More precisely, every point of W_f has a regular neighborhood as depicted in Fig. 5.

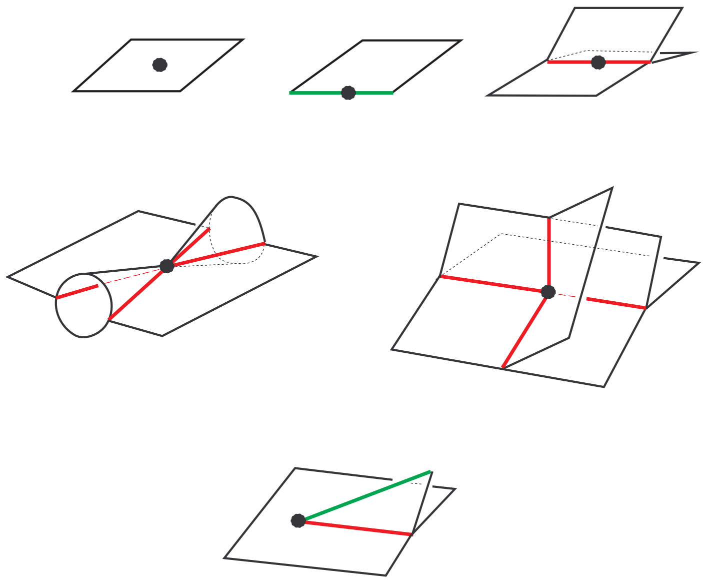

Fig. 5. Local structures of the quotient space W_f. The green lines indicate the q_f–image of definite fold points and the red ones that of indefinite fold points. The Reeb map $\overline{f}$ corresponds to the planar projection in the vertical direction.

Set $\Sigma_f = q_f(S(f))$, which is a 1–dimensional sub-polyhedron of W_f. Denote by $N(\Sigma_f)$ the regular neighborhood of Σ_f in W_f and by $\mathcal{R}_f$ the closure of $W_f \smallsetminus N(\Sigma_f)$. Then, each component R of $\mathcal{R}_f$ is a compact surface with boundary and a natural smooth structure is given so that the Reeb map $\overline{f}$ is an immersion.

Remark 4.2. A component R of $\mathcal{R}_f$ may have no boundary. This occurs if and only if f restricted to the component M_0 of M containing $q_f^{-1}(R)$ is a submersion. In such a case, R is a closed connected surface and the map $\bar{f} : R \to f(M_0)(\subset N)$ is a covering map.

Take a point p_R in the interior of each *orientable* component R of $\mathcal{R}_f$. Note that $q_f^{-1}(p_R)$ is diffeomorphic to the circle S^1. Furthermore, $q_f^{-1}(p_R)$ has a natural framing given by $q_f^{-1}(p'_R)$, where $p'_R \in \operatorname{Int} R$ is a point close to p_R. (This kind of a framing has already appeared in [16].) Let $\mathcal{L}_f$ denote the union of all $q_f^{-1}(p_R)$ over all orientable components R of $\mathcal{R}_f$. By the above observations, we see that $\mathcal{L}_f$ is a framed link in the source 3–manifold M. Let

$i : \mathcal{L}_f \to M$ denote the inclusion map and consider the induced homomorphism $i_* : H_1(\mathcal{L}_f) \to H_1(M)$. Set $K_f = \operatorname{Ker} i_*$, which is a free abelian group of finite rank. For two elements α and $\beta \in K_f$, we define their *linking number* $\operatorname{lk}(\alpha, \beta) \in \mathbf{Z}$ as follows. Take representatives a and b of α and β, respectively, which are framed and oriented links in M with integer multiplicities. By using their framings and by giving them appropriate orientations, we can slightly modify the components of a and b inside M so that they have multiplicity 1 and that they are all disjoint. (For example, if a component of a has multiplicity -3, then we take two distinct points p'_R and p''_R close to p_R and different from p_R, and consider the three component link consisting of the pre-images of the three points p_R, p'_R and p''_R by q_f, and with reversed orientations, where R is the component corresponding to that component of a.) As α is an element of $K_f = \operatorname{Ker} i_*$, there exists a 2–chain A in M with $\partial A = a$. We may assume that A intersects b "transversely" at finitely many points. Then, the linking number $\operatorname{lk}(\alpha, \beta) \in \mathbf{Z}$ is defined to be the sum of all the signs ($= \pm 1$) over all the intersection points of A with b.

We can show that this is well-defined as follows. Let A' be another 2–chain as above. Then, $A - A'$ forms a 2–cycle of M. As b is null-homologous in M, the intersection number of $A - A'$ with b must vanish. Hence the intersection number of A with b and that of A' with b coincide.

By a standard argument, we can also show that $\operatorname{lk}(\alpha, \beta) = \operatorname{lk}(\beta, \alpha)$. Thus, the linking number defines a symmetric bilinear form

$$\operatorname{lk} : K_f \times K_f \to \mathbf{Z}$$

over the integers.

Our main result of the present paper is the following.

THEOREM 4.3. *Let $f : M \to N$ be a stable map of a closed oriented 3–manifold into a surface. Then the invariant $\sigma(f)$ coincides with the signature of the linking form* $\operatorname{lk} : K_f \times K_f \to \mathbf{Z}$.

Proof. We shall construct a compact oriented 4–manifold V with $\partial V = M$ and a generic null-cobordism $F : V \to N \times [0, \infty)$ of $f : M \to N \times \{0\}$ such that F has no singular fiber of type III^8 as follows.

Let R be a component of $\mathcal{R}_f$, which might be non-orientable. Recall that by $\bar{f} : W_f \to N$, the compact surface R is immersed into N. Furthermore, the map $q_f : q_f^{-1}(R) \to R$ is the projection of a smooth S^1–bundle over R. Note that we may assume that the structure group of such an S^1–bundle is reduced to the group of rotations $O(2)$, since, as is well known, the group of diffeomorphisms of S^1 has $O(2)$ as a deformation retract.

Set $M_1 = q_f^{-1}(N(\Sigma_f))$ and $M_2 = q_f^{-1}(\mathcal{R}_f)$. Note that $M = M_1 \cup M_2$ attached along the boundaries, and that $M_1 \cap M_2 = \partial M_1 = \partial M_2$ is the total space of an S^1–bundle over $\partial\mathcal{R}_f$. Consider the map $f \times \mathrm{id}_{[0,2]}$ restricted to $V_1 = (M_1 \times [0,2]) \cup (M_2 \times [0,1])$, which we denote by $F_1 : V_1 \to N \times [0,\infty)$. The corresponding quotient map $q_{F_1} : V_1 \to W_{F_1}$ is an S^1–bundle over $\widetilde{\mathcal{R}_f} = q_{F_1}((\partial M_2 \times [1,2]) \cup (M_2 \times \{1\})) \subset W_{F_1}$, where $\widetilde{\mathcal{R}_f}$ can be naturally identified with $\mathcal{R}_f \cup (\partial\mathcal{R}_f \times [1,2])$ attached along $\partial\mathcal{R}_f = \partial\mathcal{R}_f \times \{1\}$. Note also that V_1 (after the corners being smoothed) is diffeomorphic to $M \times [0,1]$, which we orient in such a way that $\partial V_1 \cong (M \times \{0\}) \cup (-M \times \{1\})$.

Let E be the D^2–bundle associated with the above S^1–bundle over $\widetilde{\mathcal{R}_f}$. Note that, although E might not be orientable as a D^2–bundle, E can be oriented as a 4–manifold in such a way that the induced orientation on $q_{F_1}^{-1}(\widetilde{\mathcal{R}_f})$ matches that of $M \times \{1\}$. We denote by V_2 the compact oriented 4–manifold obtained by attaching E to V_1 along the S^1–bundle over $\widetilde{\mathcal{R}_f}$. By using the standard map $D^2 \to [1, 1+\varepsilon]$, for a sufficiently small $\varepsilon > 0$, defined by $(x,y) \mapsto 1 + \varepsilon(1 - (x^2 + y^2))$ on each fiber, we can extend the map $q_{F_1} : V_1 \to W_{F_1}$ to $q_2 : V_2 \to W_2$, where D^2 is the unit 2–disk in $\mathbf{R}^2$, and W_2 is the union of W_{F_1} and $\widetilde{\mathcal{R}_f} \times [1, 1+\varepsilon]$ attached along $\widetilde{\mathcal{R}_f} = \widetilde{\mathcal{R}_f} \times \{1\}$. Note that W_2 can be naturally embedded into $W_f \times [0,2]$ (see Fig. 6, which schematically depicts the structure of W_2). Then, by post-composing the map $\bar{f} \times \mathrm{id}_{[0,2]} : W_f \times [0,2] \to N \times [0,\infty)$ restricted to W_2, we get a smooth map $F_2 : V_2 \to N \times [0,\infty)$.

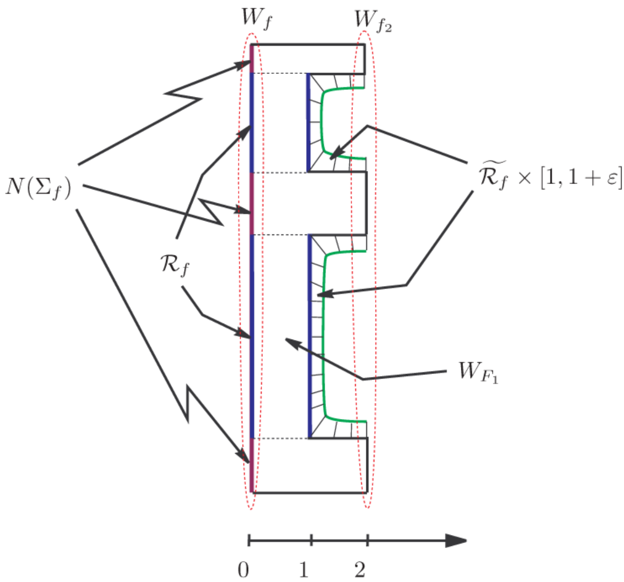

Fig. 6. Polyhedron W_2. The green lines depict the image of the newly introduced definite fold points.

By construction, we may assume that F_2 gives a generic cobordism between a smooth map $f_2 : M_2 \to N \times \{2\}$ and f, where $M_2 = F_2^{-1}(N \times \{2\})$ is a closed oriented 3–manifold. (By modifying F_2 near M_2 if necessary, we may assume that f_2 is stable.)

Lemma 4.4. *The signature* $\mathrm{sign}(V_2)$ *of the compact oriented* 4*–manifold* V_2 *coincides with the signature of the linking form* $\mathrm{lk} : K_f \times K_f \to \mathbf{Z}$.

Proof. Set $E' = E \cap V_1$, which is an S^1–bundle over $\widetilde{\mathcal{R}_f}$. Let us consider the Meyer–Vietoris exact sequence associated with the pair (V_1, E):

$$H_2(M\times[0,1])\oplus H_2(E)\xrightarrow{\ \iota\ }H_2(V_2)\xrightarrow{\ \rho\ }H_1(E')\xrightarrow{\ \tau\ }H_1(M\times[0,1])\oplus H_1(E).$$

(Recall that $V_1 \cong M \times [0, 1]$.)

Let us first consider the case where each component of $\mathcal{R}_f$ is orientable and has non-empty boundary. In this case, $\mathcal{R}_f$ is homotopy equivalent to a 1–dimensional complex, and E and E' are trivial bundles. Thus, we get the exact sequence

$$H_2(M\times[0,1])\xrightarrow{\ \iota\ }H_2(V_2)\xrightarrow{\ \rho\ }H_1(S^1\times\mathcal{R}_f)\xrightarrow{\ \tau\ }H_1(M\times[0,1])\oplus H_1(D^2\times\mathcal{R}_f).$$

As can be easily observed, $\mathrm{Ker}\,\tau$ can be naturally identified with K_f, which is free abelian of finite rank. Therefore, we get the split short exact sequence

$$0\longrightarrow \mathrm{Im}\,\iota\longrightarrow H_2(V_2)\longrightarrow K_f\longrightarrow 0.$$

Thus, we can choose a basis of the free part of $H_2(V_2)$ consisting of a basis B_1 of the free part of $\mathrm{Im}\,\iota$ together with a set B_2 of elements corresponding to a basis of K_f. Let us take an arbitrary member α of the basis B_1 of the free part of $\mathrm{Im}\,\iota$. Then, we can find a 2–cycle a representing α in $M \times [0, 1] \cong V_1$. This implies that its intersection number with all the members of the above basis $B_1 \cup B_2$ vanishes, since we can displace the 2–cycle a in the $[0, 1]$–direction. Therefore, the intersection matrix of V_2 with respect to the above basis coincides with the direct sum of a 0–matrix corresponding to B_1 and the intersection matrix corresponding to B_2.

Furthermore, by the construction of the linking form lk for K_f, we see that the intersection form on the free abelian group generated by B_2 coincides with the linking form. This can be seen by observing that the intersection number of two 2–chains in $M \times [0, 1]$ bounding 1–cycles in $M \times \{1\}$ representing elements of K_f coincides with their linking number in M.

Hence, the signature of V_2 coincides with the signature of the linking form lk.

If a component R of $\mathcal{R}_f$ is non-orientable and has non-empty boundary, then an S^1–fiber of $q_f^{-1}(R)$ represents a homology class in $H_1(q_f^{-1}(R))$ which is of order two. Therefore, by an argument using the Meyer–Vietoris exact sequence as above, we see that such a component does not contribute to the free part of $H_2(V)$. Thus, we can ignore such a component R when computing the signature of V_2.

Finally, suppose that a component R of $\mathcal{R}_f$ has no boundary. By the additivity of the signature and that of the linking form, we may assume that M is connected. Then, by Remark 4.2, $q_f^{-1}(R)$ coincides with M and is an S^1–bundle over a closed surface R. Hence, V_2 is diffeomorphic to the associated D^2–bundle over R. If R is non-orientable, then we see easily that the signature of V_2 is equal to zero, since its second homology group vanishes. If R is orientable, let e denote the Euler number of the S^1–bundle. In this case, $H_2(V_2)$ is infinite cyclic and is generated by the class represented by the zero-section of the D^2–bundle. As its self-intersection number coincides with e, we see that the signature of V_2 is equal to 1, 0 or -1 if e is positive, zero or negative, respectively. On the other hand, $\mathcal{L}_f$ is a framed link consisting of a framed S^1–fiber, and $|e|$ times the corresponding homology class generates K_f. Furthermore, we can show that the self-linking number of that generator is equal to e as follows. Let D be a small 2–disk in R and set R° to be the closure of $R \setminus D$. Then, M is diffeomorphic to the 3–manifold obtained by attaching $S^1 \times D$ and $S^1 \times R^\circ$ along their boundaries by a diffeomorphism $S^1 \times \partial D \to S^1 \times \partial R^\circ$ which preserves the fibers $S^1 \times \{*\}$ and which sends $\{*\} \times \partial D$ to a simple closed curve in $S^1 \times \partial R^\circ$ winding once in the direction of $\{*\} \times \partial R^\circ$ and e times in the direction of $S^1 \times \{*\}$ homologically. Therefore, the above-mentioned generator of K_f bounds a 2–chain consisting of $\{*\} \times D$ and $\{*\} \times R^\circ$. This 2–chain intersects the $|e|$ parallel copies of an S^1–fiber of $S^1 \times D$ at $|e|$ points. Taking into account the orientations, we see that the relevant linking number is equal to e. Hence, we have the desired result in this case as well.

This completes the proof. □

In view of Lemma 3.7, for the proof of the theorem, it suffices to show that $\sigma(f_2)$ vanishes, since F_2 has no singular fiber of III^8–type. Note that the quotient space W_{f_2} is homeomorphic to $N(\Sigma_f)$ (see Fig. 6).

By deforming f_2 by a generic homotopy in the space $C^\infty(M_2, N)$, we can realize the move of the quotient space in the Stein factorization as depicted in Fig. 7 (1). This corresponds to the reverse deformation of the elimination of a pair of cusps [8]. Note that for this move, one of the two overlapping sheets is split into two, and that we can choose either of them for the move. This kind of moves give a generic cobordism $F_3 : M_2 \times [2, 3] \to N \times [2, 3]$ between f_2 and a stable map $f_3 : M_2 \to N$.

Furthermore, by constructing an appropriate cobordism, we can realize the move as depicted in Fig. 7 (2) as well. This corresponds to attaching a

3–handle to $M_2 \times [2,3]$. Let us explain the details as follows.

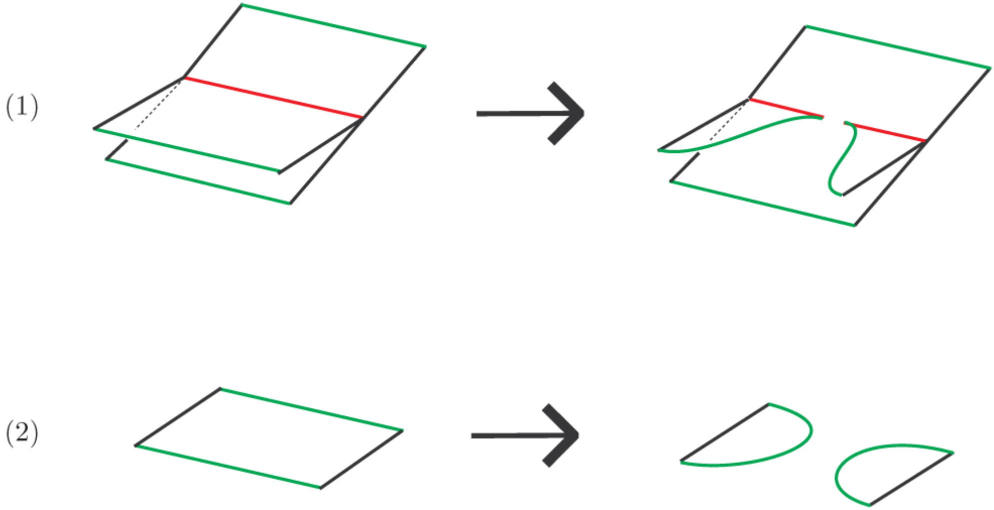

Fig. 7. Moves for the quotient space. The red lines depict the image of indefinite fold points and the green ones that of definite fold points.

Suppose that the quotient space W_{f_3} of the stable map $f_3 : M_2 \to N$ contains a sub-polyhedron T as in the left hand side picture of Fig. 7 (2). Then, $q_{f_3}^{-1}(T)$ is diffeomorphic to $S^2 \times [-1,1]$. Let $F_4' : V_4' \to N \times [3,4]$ be a smooth map of a compact oriented 4–manifold with boundary whose quotient space $W_{F_4'}$ is as depicted in Fig. 8, where F_4' restricted to $(F_4')^{-1}(N \times \{3\})$ coincides with f_3 restricted to $q_{f_3}^{-1}(T)$, and the vertical direction corresponds to that of $[3,4]$. Note that the green sheet on the boundary of $W_{F_4'}$ corresponds to the definite fold image, and that outside of that sheet $q_{F_4'}$ is a smooth S^1–bundle. Note also that V_4' is diffeomorphic to $D^3 \times [-1,1]$.

Now, consider $F_3 : M_2 \times [2,3] \to N \times [2,3]$ and we attach the map F_4' along $q_{f_3}^{-1}(T) \times \{3\}$. Then, the resulting map, after a suitable modification, gives a generic cobordism F_4 between f_2 and a stable map $f_4 : M_4 \to N \times \{4\}$ of a closed oriented 3–manifold in such a way that W_{f_4} is homeomorphic to the polyhedron obtained from W_{f_3} by applying the move as in Fig. 7 (2).

By repeating this kind of a procedure finitely many times, we get a quotient space which is a finite disjoint union of small connected polyhedrons, each of which is homeomorphic to one of the polyhedrons as depicted in Fig. 9. Each such polyhedron will be said to be *elementary*.

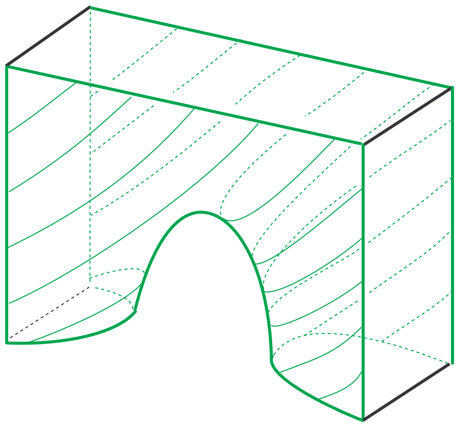

Fig. 8. Quotient space $W_{F_4'}$ of F_4'. The green sheet in front and on the back corresponds to the definite fold image.

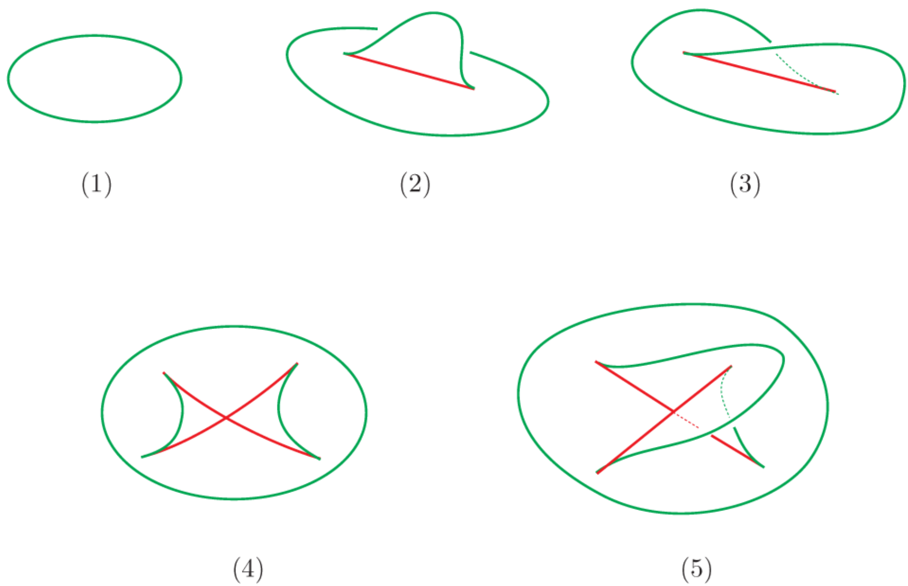

Fig. 9. Elementary polyhedrons.

Then, by generic homotopies of the Reeb maps, we may assume that the images of these elementary polyhedrons are disjoint and that on each elementary polyhedron, the Reeb map corresponds to the projection to the horizontal planes in Fig. 9.

As to an elementary polyhedron as in Fig. 9 (2), by a death deformation, we can turn it into that of Fig. 9 (1). As to (3), by a swallow-tail deformation, we can turn it into that of (1) as well. As to (4) and (5), by a swallow-tail deformation, we get (2), and then we get (1). For these deformations of the quotient space, the reader is referred to [10].

Note that in the deformations used in the above constructions, Reidemeister III type deformation for singular value curves never occurs so that the associated smooth map into $N \times [0, \infty)$ does not have any singular fiber of type III^8. Furthermore, the corresponding 4–manifold is constructed by attaching several 3–handles to $M_2 \times [2, 3]$.

Finally, we can attach a 4–disk corresponding to each elementary polyhedron of the form as depicted in Fig. 9 (1). The resulting map $F_5 : V_5 \to N \times [2, \infty)$ gives a generic null-cobordism of $f_2 : M_2 \to N \times \{2\}$.

By construction, the 4–manifold V_5 is diffeomorphic to the union of $M_2 \times [2, 3]$ and a 4–dimensional handlebody consisting of several 0– and 1–handles. Therefore, V_5 has vanishing signature. This implies that $\sigma(f_2) = 0$.

This completes the proof. □

In order to give a corollary to Theorem 4.3, let us first prove a lemma which simplifies the computation of the signature of the linking form, as follows.

For a stable map $f : M \to N$ of a closed oriented 3–manifold into a surface, we denote by $\mathcal{R}_f^1$ the union of those components of $\mathcal{R}_f$ such that the closure of the corresponding component of $W_f \smallsetminus \Sigma_f$ does not have the image of any definite fold point. Let $\mathcal{L}_f^1$ be the (framed) sublink of $\mathcal{L}_f$ corresponding to $\mathcal{R}_f^1$. Let $i^1 : \mathcal{L}_f^1 \to M$ be the inclusion map and set $K_f^1 = \mathrm{Ker}(i_*^1 : H_1(\mathcal{L}_f^1) \to H_1(M))$, which is a subgroup of K_f.

The following lemma helps to simplify the computation of the σ–invariant.

Lemma 4.5. *Let $f : M \to N$ be a stable map of a closed oriented 3–manifold into a surface. Then, the signature of the linking form* $\mathrm{lk} : K_f \times K_f \to \mathbf{Z}$ *coincides with that of the linking form* $\mathrm{lk}|_{K_f^1 \times K_f^1} : K_f^1 \times K_f^1 \to \mathbf{Z}$.

Proof. Set $\mathcal{R}_f^0 = \mathcal{R}_f \smallsetminus \mathcal{R}_f^1$ and let R be a component of $\mathcal{R}_f^0$. Let $\widetilde{R}$ be the closure of the component of $W_f \smallsetminus \Sigma_f$ containing R. Then, for a point $p_R \in \mathrm{Int}\, R$, there is an arc α embedded in $\widetilde{R}$ which connects p_R and the image of a definite fold point $d \in \Sigma_f$ and which intersects Σ_f transversely at d, where $\alpha \cap \partial\widetilde{R} = \{d\}$. Then, $q_f^{-1}(\alpha)$ is a 2–disk bounded by the regular fiber component $q_f^{-1}(p_R)$ and is consistent with the framing of the fiber component. Therefore, the homology class represented by $q_f^{-1}(p_R)$ belongs to K_f. This means that we have $K_f = K_f^1 \oplus H_1(\mathcal{L}_f^0)$, where $\mathcal{L}_f^0 = \mathcal{L}_f \cap q_f^{-1}(\mathcal{R}_f^0)$.

Furthermore, since the 2–disks are disjoint and do not intersect the components of $\mathcal{L}_f^1$, we see that the linking form lk on K_f is isomorphic to the direct sum of the form $\mathrm{lk}|_{K_f^1 \times K_f^1}$ and the zero form on $H_1(\mathcal{L}_f^0) \times H_1(\mathcal{L}_f^0)$. Hence, the result follows. □

Then, we have the following corollary immediately.

COROLLARY 4.6. *For a stable map $f : M \to N$ of a closed oriented 3–manifold into a surface, we always have*

$$|\sigma(f)| \leq \sharp\pi_0(\mathcal{R}_f^1),$$

where $\sharp\pi_0(\mathcal{R}_f^1)$ is the number of components of $\mathcal{R}_f^1$.

As a corollary to Definition 3.3, we have the following.

COROLLARY 4.7. *Let V be a compact oriented 4–manifold possibly with boundary, Q be a 3–manifold possibly with boundary, and $F : V \to Q$ be a smooth map with the following properties.*

(1) $F^{-1}(\partial Q) = \partial V$.

(2) *$f = F|_{\partial V} : \partial V \to \partial Q$ is a stable map.*

(3) *For small collar neighborhoods $C(\partial V) = \partial V \times [0, \varepsilon)$ and $C(\partial Q) = \partial Q \times [0, \varepsilon)$ of ∂V in V and of ∂Q in Q, respectively, $0 < \varepsilon << 1$, we have that $F|_{C(\partial V)} = f \times \mathrm{id}_{[0,\varepsilon)}$.*

(4) *The map $F|_{\mathrm{Int}\, V} : \mathrm{Int}\, V \to \mathrm{Int}\, Q$ is a proper stable map.*

Then, we have

$$\mathrm{sign}(V) = ||\mathrm{III}^8(F)|| + \sigma(f).$$

Compare Corollary 4.7 with the formula (2.1), which is valid only for closed oriented 4–manifolds. When the 4–manifold has nonempty boundary, then we need the correction term, which is given by $\sigma(f)$, where f is the stable map on the boundary 3–manifold, and $\sigma(f)$ can be computed by Theorem 4.3.

5. EXAMPLE

In this section, let us apply Theorem 4.3 to an explicit example.

Example 5.1. Let $h : S^3 \to S^2$ be a (positive) Hopf fibration and $pr : S^2 \to \mathbf{R}^2$ the restriction of the natural orthogonal projection $\mathbf{R}^3 \to \mathbf{R}^2$ to the unit 2–sphere S^2. Then, the composition $g = pr \circ h : S^3 \to \mathbf{R}^2$ is not stable, since its singular point set is of dimension 2. However, we can construct a stable map $f : S^3 \to \mathbf{R}^2$ which is a slight perturbation of g and whose quotient space W_f is a polyhedron as depicted in Fig. 10. Such a stable map is constructed, for example, in [11, Example 2.5].

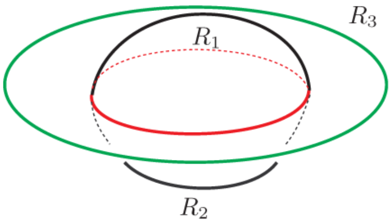

Fig. 10. Quotient space of the stable map $f : S^3 \to \mathbf{R}^2$.

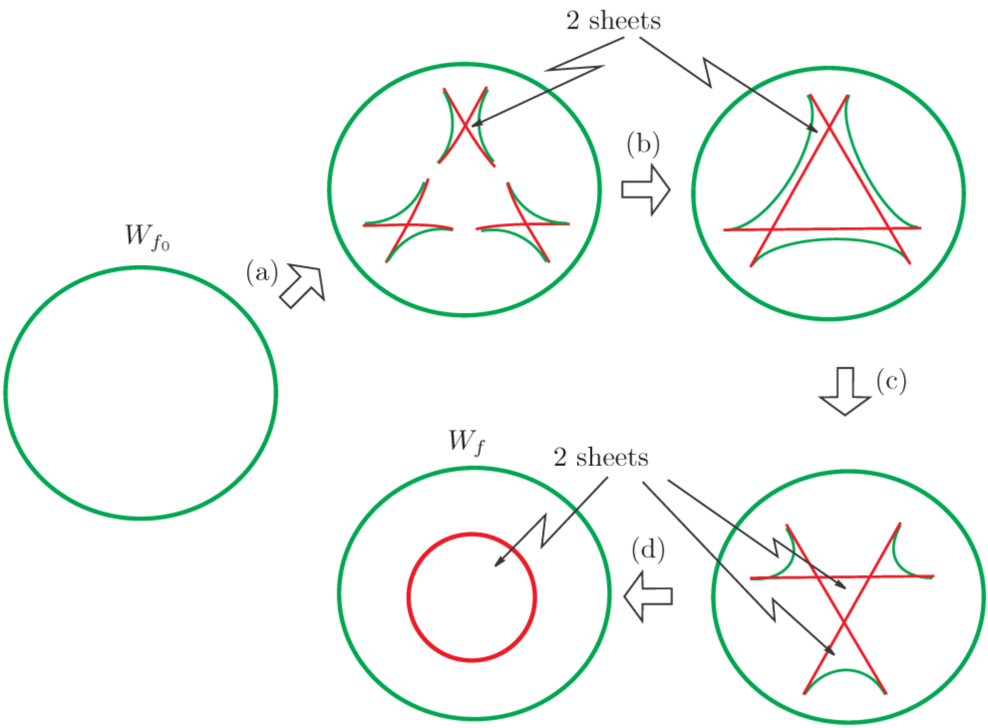

Fig. 11. Change in quotient space for a generic path connecting f_0 and f.

In fact, we can construct such a path for which the corresponding quotient space changes as depicted in Fig. 11.

Then, for f, $\mathcal{R}_f$ consists of three components R_1, R_2 and R_3. Since $H_1(S^3)$ vanishes, we have $K_f \cong \mathbf{Z}^3$. For the disk components R_1 and R_2, we see that $q_f^{-1}(p_{R_1}) \cup q_f^{-1}(p_{R_2})$ is a positive Hopf link and that the framing of each of the two components is given by $+1$. Furthermore, for the annulus component R_3, we see that $q_f^{-1}(p_{R_3})$ is a split unknot component and its framing is given by zero. Therefore, the linking matrix with respect to the natural basis of K_f is given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(Note that this conforms to the argument given in the proof of Lemma 4.5: the component $q_f^{-1}(p_{R_3})$ has no contribution to the signature.) Then, a straightforward calculation shows that it has signature $+1$, and hence that $\sigma(f) = 1$.

Now, let us give an explicit path in the mapping space $C^\infty(S^3, \mathbf{R}^2)$ connecting a standard map to f. The standard map that we consider here is the special generic map $f_0 : S^3 \to \mathbf{R}^2$ given by the restriction of the natural orthogonal projection $\mathbf{R}^4 \to \mathbf{R}^2$ to the unit 3–sphere S^3. (For the terminology "special generic map", the reader is referred to [1].)

In Fig. 11, the modification (a) corresponds to a path passing through the "birth transition" three times and the "swallow-tail transition" three times (for example, see [10, 15]). In other words, as a path in $C^\infty(S^3, \mathbf{R}^2)$, it passes through the strata of unstable maps six times in total. In Fig. 11, the modification (b) corresponds to a path passing through the "cusp elimination transition" (see [8]) three times. The modification (c) is most important: it corresponds to a path passing through the III^8–transition exactly once, i.e. it passes through a stratum of C_{III^8} transversely at one point in the positive direction. Finally, the modification (d) corresponds to a path passing through the "swallow-tail transition" three times.

These transitions can be verified by using known results [8, 15], except for the III^8–transition. In fact, in order that we can apply a III^8–transition at (c), we need to apply the cusp eliminations in (b) beforehand appropriately as follows.

After the modification (b), the behavior of the map restricted to the inverse image of a neighborhood of the central triangle in the quotient space depends on the so-called "gleam" attached to the triangle region. (For details of the notion of a gleam, the reader is referred to [3, 21, 22].) A gleam is an element of $(1/2)\mathbf{Z}$, the set of half integers, and is attached to each component of $\mathcal{R}_f^1$. This corresponds to a certain Euler class of the S^1–bundle over the corresponding component with respect to a certain "bi-section" over the boundary determined by the map.

In our explicit case, for each of the three cusp elimination transitions in (b), we need to choose a path in S^3 connecting the two relevant cusp points (see [8, 15]). Such a path goes through a 1–parameter family of regular S^1–fibers: then, we have the freedom to choose how many times (or in which direction) the path winds around regular S^1–fiber direction. This allows us to change the gleam of the central region by an arbitrary integer. On the other hand, we can easily check that the neighborhood of the triangular region in the quotient space is homeomorphic to that appearing in a III^8–transition. Hence, the difference that should be adjusted must be an integer. Therefore, by appropriately choosing the paths for cusp eliminations, we can arrange so that the resulting gleam coincides with the one appropriate for a III^8–transition to occur. In this way, the above construction is realized.

By Lemma 4.5 together with Theorem 4.3, we see immediately that $\sigma(f_0) = 0$, which can also be derived by directly using the definition. Therefore, we can deduce that $\sigma(f) = 1$, which conforms to the above calculation.

Remark 5.2. To be more precise, we note that the final stable map appearing in Fig. 11, say f', may not exactly coincide with f; however, we can show that f' is $\mathrm{R}^+\mathrm{L}$–equivalent to f as follows. We can observe that f' induces a fibered 2–component link with annulus fiber, by virtue of the structure of $W_{f'}$. Then, since the ambient space must be homeomorphic to S^3, we see that the monodromy must be the Dehn twist along the center circle of the annulus. Therefore, taking into account the orientation, we can construct an $\mathrm{R}^+\mathrm{L}$–equivalence between f' and f.

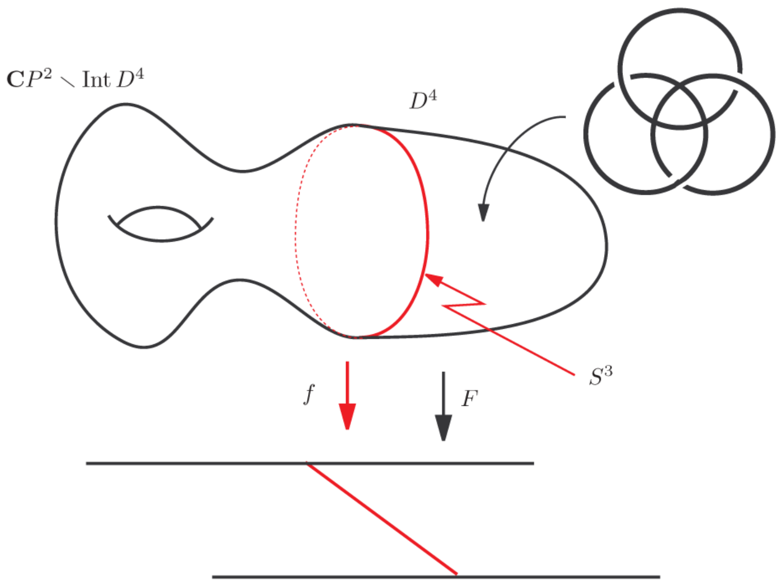

Fig. 12. An explicit stable map $\mathbf{C}P^2 \to \mathbf{R}^2 \times \mathbf{R}$.

As a byproduct of the above construction of an explicit path, we can construct a stable map $F : \mathbf{C}P^2 \to \mathbf{R}^2 \times \mathbf{R}$ with the following properties (see Fig. 12).

(1) We have $M = F^{-1}(\mathbf{R}^2 \times \{0\}) \cong S^3$ and the stable map f is R$^+$L–equivalent to the map $F|_M : M \to \mathbf{R}^2$.

(2) We have $V_+ = F^{-1}(\mathbf{R}^2 \times [0, \infty)) \cong D^4$.

(3) The map $F|_{V_+} : V_+ \to \mathbf{R}^2 \times [0, \infty)$ gives a generic null-cobordism for $-f$ and has exactly one (positive) III^8–type singular fiber, where $-f$ is the map f but with the orientation of the source 3–manifold reversed.

(4) We have $V_- = F^{-1}(\mathbf{R}^2 \times (-\infty, 0]) \cong \mathbf{C}P^2 \setminus \mathrm{Int}\, D^4$.

(5) The map $F|_{V_-} : V_- \to \mathbf{R}^2 \times (-\infty, 0]$ gives a generic null-cobordism for $f : M \to \mathbf{R}^2$, and has no singular fiber of type III^8.

As far as the author knows, this gives a new explicit example of a stable map $\mathbf{C}P^2 \to \mathbf{R}^3$. Note that Kobayashi [6] has already constructed some explicit stable maps of $\mathbf{C}P^2 \to \mathbf{R}^3$.

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