ALGEBRAIC AND AFFINE G-STRUCTURES ON NASH G-MANIFOLDS

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Necessary and sufficient conditions are given so that in a Nash G-manifold an algebraic or an affine Nash G-manifold structure exists.

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1. INTRODUCTION

A central problem in real geometry consists in finding an algebraic structure on a differentiable object. On this subject A. Tognoli [20] proved that any closed (*i.e.* compact without boundary) smooth (*i.e.* C^∞) manifold admits an algebraic structure: this means that it is diffeomorphic to a non singular real algebraic variety. Remark that there exist non-compact manifolds which are not diffeomorphic to real algebraic varieties, but M. Shiota [18] proved that any affine smooth Nash manifold is Nash diffeomorphic to a non singular real algebraic variety. In equivariant setting K.H. Dovermann, M. Masuda and T. Petrie [6] conjectured that, if G is a compact Lie group, then any closed smooth G-manifold is C^∞ G-diffeomorphic to a non singular real algebraic G-variety of some representation space of G (the so-called "Equivariant Nash Conjecture").

Now, following these facts, in this paper we study a problem that naturally comes out: to consider the Equivariant Nash Conjecture in the C^r Nash category ($1 \leq r \leq \infty$). Thus we formulate this question: Let M be a closed C^r Nash manifold and let G be a compact affine Nash group acting on M. Is MC^r Nash G-diffeomorphic to a non singular real algebraic G-variety of some Nash representation space of G? If it occurs we say that M has an algebraic G-variety structure.

In this paper, we solve completely this problem when the G action is free. Precisely, we prove:

Let G be a compact affine Nash group acting freely on a closed C^r Nash manifold M. Then:

i) If $r < \infty M$ has an algebraic G-variety structure;

ii) If $r = \infty M$ has an algebraic G-variety structure if and only if some precise conditions are satisfied (Theorem 5.3).

If the G action is not free we give an answer to the problem if M has a suitable global K-slice S, K being a closed subgroup of G. Indeed in Theorem 5.4 we prove that M has an algebraic G-variety structure if and only S has an affine Nash K-manifold structure (see below for this notion).

These results follow from a detailed study on the existence of affine G-structures on Nash G-manifolds. In fact, in the more general setting of affine Nash G-structures, the question we pose is the following:

Let M be a C^r Nash manifold ($1 \leq r \leq \infty$) and let G be a compact affine Nash group acting on M. We ask whether there exist a Nash representation of G, with space V, an invariant C^r Nash submanifold L of V and a C^r Nash G-diffeomorphism $M \to L$. If it occurs we say that M has an affine Nash G-manifold structure. Thus the G-variety structures are particular affine structures but, for free actions, we prove that if M is closed and has an affine G-structure then it also has an algebraic one (Theorem 5.2).

It may be of interest recalling that in the equivariant smooth category a smooth G-manifold M (G compact Lie group) is equivariantly embeddable in some representation space of G if and only if G has a finite number of orbit types in M [1].

In the smooth Nash setting the analogous result is not true, as the following example shows. Let S^1 be the standard circle. It is known that there exists in it a non affine Nash group structure [12]. Let call S^1, endowed with this structure, G. Then consider $M = G$ as a smooth Nash G-manifold where the action of G is the group multiplication. Thus, G has only one orbit type in M, but M is not Nash embeddable in any number space.

The key point of this example is, of course, that there exist non affine smooth Nash manifolds. But, what can we say if we restrict our attention to affine smooth Nash manifolds or, in general, to C^r Nash manifolds with $r < \infty$? Here we give some answers to this question for any action, not necessarily free. So, we find conditions which characterize large classes of affine C^r Nash manifolds ($1 \leq r \leq \infty$), with G action, in order to have an affine Nash G-manifold structure (Theorems 4.1, 4.4, 4.6). In particular, if the G action is free and M is compact and smooth the previous conditions are the same as in the algebraic case.

In the last section, we consider G actions with G non-compact. We must remark that G must be linear in order to equivariantly embed a G-manifold in a representation space of G (note that a compact affine Nash group is automatically linear) [7,10,14]. In this case, first we give a Nash G embedding theorem for homogeneous Nash G-manifolds (Theorem 6.5), then we characterize the affine C^r Nash G-structures ($r \le \infty$) by means of the existence of global slices (Theorem 6.6).

2. PRELIMINARY REMARKS

We briefly recall some definitions and facts [3,18,19]. The objects considered through the paper are of class C^r, $1 \le r \le \infty$.

Let U, V be semialgebraic open sets of $\mathbf{R}^n$ and $\mathbf{R}^m$, respectively. A C^r map $f : U \to V$ is said to be a C^r Nash map if its graph is semialgebraic. We know that if $r = \infty$ it is analytic. A C^r Nash manifold is a C^r manifold with a finite system of charts such that the coordinate changes are given by C^r Nash maps. A C^r Nash map $F : M \to N$ between C^r Nash manifolds is a C^r map which, in local coordinates, is given by C^r Nash maps. If there exists a C^r Nash embedding of M into some $\mathbf{R}^m$, we say that M is affine. If $r < \infty M$ is always affine, but there exist non affine smooth (*i.e.* C^∞) Nash manifolds. If M is a C^r Nash submanifold of $\mathbf{R}^m$ then it is semialgebraic in $\mathbf{R}^m$ and if $r = \infty$ it is analytic. Conversely, a semialgebraic C^r submanifold of $\mathbf{R}^m$ is a C^r Nash manifold.

A C^∞ Nash manifold G endowed with a group structure such that the group operations are C^∞ Nash maps is said to be a C^∞ Nash group. We consider only such groups and we call them simply Nash groups. They are affine if the Nash manifolds are affine. A Nash subgroup of G will be a subgroup of G which is a C^∞ Nash regular submanifold of G. Of course, it is a Nash group and it is closed. One defines as usual the notion of a C^r Nash action of a Nash group on a C^r Nash manifold.

We consider also C^r Nash fibre bundles: they are bundles such that base space, total space, fibre and projection are of class C^r Nash.

3. AN APPROXIMATION THEOREM

In this section, we construct a Nash bundle whose cross-sections are in bijective correspondence with suitable equivariant maps. It allows to obtain an approximation theorem of differentiable equivariant maps by Nash equivariant maps and it will be used later in order to obtain affine Nash G-structures. Let us begin with some definitions.

Definition 3.1. Let G be a Nash group and $1 \le r \le \infty$.

1. A C^r Nash G-manifold is a C^r Nash manifold with a given C^r Nash action of G on it.

2. A C^r Nash G-map between C^r Nash G-manifolds is an equivariant C^r Nash map.
3. A (linear) Nash representation of G is a smooth Nash homomorphism $G \to GL(n)$; this means a homomorphism of groups which is a smooth Nash map.
4. A C^r Nash G-manifold M is said to be an affine C^r Nash G-manifold or to have an affine Nash G-manifold structure if it is C^r Nash G-diffeomorphic to a G-invariant C^r Nash submanifold L of some representation space V of G. The space V equipped with the linear action of G via the given representation is said to be a linear G-space. If L is a non singular real algebraic variety, M is said to have an algebraic G-variety structure.

Given a C^r Nash G-manifold M, first of all we deal with the orbit space M/G.

THEOREM 3.2. *Let G be a compact Nash group and M an affine C^r Nash manifold on which G acts freely* $(1 \leq r \leq \infty)$ *(Recall that if $r < \infty M$ is automatically affine). Then:*

i) *The canonical map π: $M \to M/G$ is a C^r Nash principal bundle;*
ii) *Given a C^r Nash manifold L, a map $h : M/G \to L$ is of class C^r or C^r Nash if and only if $h \circ \pi$ is.*

Sketch of proof. It is known that M/G is a C^r manifold. Because in a Nash setting the implicit function theorem and the rank theorem hold true, we can adapt to the Nash case the proof given by Dieudonné in [4, 16.10.3] to construct differentiable charts for M/G. Note that M is semialgebraic, because it is affine. So, by [18, Proposition I.3.9] and by the proof of 16.10.3 in [4] we can construct a finite system $\{U_x, \lambda_x\}$ of C^r Nash charts for M and a finite system $\{\pi(U_x), \gamma_x\}$ of C^r Nash charts for M/G such that $\lambda_x(U_x) = V_x \times W_x$ where $V_x(W_x)$ is a semialgebraic open set of $\mathbf{R}^n$ $(\mathbf{R}^{m-n})$ $(m = \dim M)$, and $\gamma_x \circ \pi \circ \lambda_x^{-1} =$ canonical projection $V_x \times W_x \to W_x$. It follows that π is a C^r Nash map.

To prove that the bundle is Nash locally trivial, we remark that π is a submersion and hence for any point $y \in M/G$ there exist an open semialgebraic set U, $y \in U$, and a C^r Nash cross-section $\sigma : U \to M$. Therefore the map $\phi : U \times G \to \pi^{-1}(U)$, $\phi(u, g) = g\sigma(u)$, is a C^r Nash diffeomorphism.

ii) The assertion follows from the Nash local triviality of π. □

Now let M, L be affine C^r Nash manifolds $(r \leq \infty)$ with G action, where G is a compact Nash group. Let us suppose the action on M free. Following the topological case, we construct a Nash bundle whose cross-sections are in one-to-one correspondence with G-maps $M \to L$. This is done by using the twisted product $L \times_G M$. We recall that it is the orbit space of the action of G on $L \times M$ given by $(g, (l, m)) = (lg^{-1}, gm)$ and it is the total space of the bundle associated with the C^r Nash principal bundle $\pi : M \to M/G$ (3.2) and with L as fibre. It follows that the bundle $p : L \times_G M \to M/G$, $p([l,m]) = \pi(m)([l,m] = \text{orbit of } (l,m))$, is of class C^r Nash and the total space is a Nash manifold.

PROPOSITION 3.3. *Let G be a compact Nash group acting on the affine C^r Nash manifolds M and $L (r \leq \infty)$. Let us suppose that G acts freely on M. Then the cross-sections of class C^r or C^r Nash of the Nash bundle $p : L \times_G M \to M/G$ are in one-to-one correspondence with the G-maps $f : M \to L$ of the same class.*

Proof. The claim is true in the topological setting [2, II Theorem 2.6]. Then we must to prove that, given a G-map $f : M \to L$ of class C^r or C^r Nash, the associated continuous cross-section s is of the same class, and conversely. To do this, let $F : M \to L \times M$ be the map $F(m) = (f(m), m)$, which is of the same class of f, and consider the following commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & L \times M \\ \downarrow & & \downarrow \\ M/G & \longrightarrow & L \times_G M \end{array}$$

where the upper horizontal arrow is F, the horizontal arrow below is s and the vertical arrows are the canonical projections. Since G acts freely on $L \times M$, by 3.2 ii) we have that s is of the same class of f.

Conversely, let $s : M/G \to L \times_G M$ be a cross-section of class C^r or C^r Nash. By [2, II Lemma 2.5] we known that $L \times M$ is homeomorphic to the pull-back of π and p and, locally, this homeomorphism, say ϕ, is given by $\phi(l, g, u) = (lg, g, u)$, with inverse given by $\phi^{-1}(l, g, u) = (lg^{-1}, g, u)(u \in U \subset M/G)$. Since these maps are of class C^r Nash, $L \times M$ and the pull-back are C^r Nash diffeomorphic. Because $p(s(\pi(m)) = \pi(m)$, by the universal property of pull-backs there exists an equivariant map $\vartheta' : M \to$ pull-back such that $\vartheta'(m) = (s(\pi(m)), m)$ (ϑ' is of the class of s). Consider now ϕ^{-1}: pull-back $\to L \times M$ and the equivariant map $\vartheta = \phi^{-1} \circ \vartheta' : M \to L \times M$, $m \mapsto \phi^{-1}(s(\pi(m)), m)$. Composing ϑ with the first projection $q : L \times M \to L$ we get an equivariant map $f = q \circ \phi^{-1} \circ \vartheta' : M \to L$, associated with s, such that $s(\pi(m)) = [f(m), m]$ and which is of the same class of s. □

Using the bundle $p : L \times_G M \to M/G$ we obtain the following approximation theorem.

THEOREM 3.4. *Let G be a compact Nash group that acts on the affine C^r Nash manifolds M and $L (r \leq \infty)$. Let us suppose M compact and G acting* *freely on it. Then the C^r Nash G-maps $M \to L$ are dense in the space of the $C^t(1 \le t \le r)$ G-maps endowed with the compact-open topology if we have:*

i) $r < \infty$;

ii) $r = \infty$ *and the Nash manifolds* M/G *and* $L \times_G M$ *are affine.*

Proof. Let $f : M \to L$ be a C^t G-map. It gives rise to a C^t cross-section s of the C^r Nash bundle p (3.3). By 3.2 M/G and $L \times_G M$ are C^r Nash manifolds that are affine if $r < \infty$ or by hypothesis if $r = \infty$. Therefore there exist C^r Nash diffeomorphisms $\alpha : M/G \to S$, $\beta : L \times_G M \to T$, where S and T are analytic Nash manifolds contained respectively in $\mathbf{R}^p$ and in $\mathbf{R}^q$ [18, III 1.1; III 1.3]. Now C^1 approximate the map $\beta \circ s \circ \alpha^{-1} : S \to T$ by a polynomial $\lambda : \mathbf{R}^p \to \mathbf{R}^q$ in such a way $\lambda(S)$ is contained in an analytic Nash tubular neighbourhood of T in $\mathbf{R}^q$ with Nash retraction γ. So we obtain the C^r Nash map $\sigma = \beta^{-1} \circ \gamma \circ \lambda \circ \alpha : M/G \to L \times_G M$ which approximates s. If $p \circ \sigma$ is close enough to $p \circ s$, it is a C^r Nash diffeomorphism of M/G onto itself. Then the C^r Nash map $S = \sigma \circ (p \circ \sigma)^{-1} : M/G \to L \times_G M$ is a cross-section of p which C^1 approximates s. Looking now at the proof of 3.3 we have that the C^t G-map f, associated with s, is $f(m) = q(\phi^{-1}(s(\pi(m)), m)$ (see notation in 3.3) and that F, associated with S, is $F(m) = q(\phi^{-1}(S(\pi(m)), m)$. Thus if s and S are C^1 close, the same is true for f and F. □

4. AFFINE G-STRUCTURES WITH G COMPACT

In this section, we study the problem of the existence of an affine Nash G-structure on a C^r Nash G-manifold ($r \le \infty$) when G is compact affine.

THEOREM 4.1. *Let G be a compact affine Nash group acting freely on a closed C^r Nash manifold $M(r < \infty)$. Then M has an affine Nash G-manifold structure given by an analytic Nash G-manifold.*

Proof. By [15] M is C^r G-diffeomorphic to a smooth G-manifold N. By [8] N is C^∞ G-diffeomorphic to an analytic Nash G-submanifold L of a Nash representation space V of G. Then there exists a C^r G-diffeomorphism $f : M \to L$. Therefore, by 3.4 we can find a C^r Nash G-diffeomorphism $F : M \to L$. □

Remark that the previous theorem gives an equivariant version of a result by M. Shiota which asserts that a C^r Nash manifold ($0 < r < \infty$) is C^r Nash diffeomorphic to an analytic Nash manifold [18, Theorem III.1.3.].

If M is smooth some conditions, that we prove to be necessary and sufficient, must be fulfilled in order that M has an affine Nash G-manifold structure. Before, we need some ingredients. Let G be a compact affine Nash group and K a closed subgroup of G. Then:

THEOREM 4.2. a) *G is Nash isomorphic to a closed subgroup of an orthogonal group and K is a Nash subgroup of G.*

b) *The homogeneous space G/K is a smooth Nash G-manifold which has an algebraic G-variety structure.*

c) *A smooth representation of G is of class Nash.*

d) *If M is an affine C^r Nash K-manifold, the twisted product $G \times_K M$ is a C^r Nash G-manifold.*

Proof. See [7, Corollaries 1.5, 1.7; Theorems 2.3, 2.5; Lemma 2.6]. □

We must also recall a result by R. S. Palais on the extension of a representation.

THEOREM 4.3. *Let G be a Lie group which admits a faithful continuous linear representation in a finite dimensional real vector space. Let K be a compact subgroup of G and U a linear K-space. Then there is a linear G-space V which, considered as a linear K-space by restriction, contains U as an invariant linear subspace.*

Proof. See [14, Theorem 3.1]. □

Now we prove the result we were speaking about.

THEOREM 4.4. *Let M be a compact affine smooth Nash manifold and G a compact affine Nash group acting freely on M. Then M has an affine Nash G-manifold structure if and only if the following conditions hold true:*

a) *There exists a Nash representation of G, with space $\mathbf{R}^m$, such that the Nash manifold $\mathbf{R}^m \times_G M$ is affine;*
b) *There exists a smooth G-embedding $f : M \to \mathbf{R}^m$.*

Proof. Let us suppose the given conditions are satisfied. Remark that the Nash manifolds M/G and $\{O\} \times_G M$ ($O \in \mathbf{R}^m$) are Nash diffeomorphic and the second manifold is a Nash submanifold of $\mathbf{R}^m \times_G M$. Therefore M/G is affine. Then we can apply 3.4 and approximate f by a Nash G-embedding $M \to \mathbf{R}^m$. So, M has an affine Nash G-manifold structure.

Conversely, assume that M has an affine Nash G-manifold structure. Therefore there exist a Nash representation α of G, with space $\mathbf{R}^m$, an invariant smooth Nash submanifold L of $\mathbf{R}^m$ and a smooth Nash G-diffeomorphism $M \to L$.

We prove a). The Nash G-manifold $\mathbf{R}^m \times L$ is an invariant submanifold of $\mathbf{R}^{2m}$, which is the representation space of $\alpha \oplus \alpha$. Now, the algebra of $\mathbf{R}[\mathbf{R}^{2m}]^G$ of G-invariant polynomials on $\mathbf{R}^{2m}$ is finitely generated, say by $p_1, \ldots, p_s$ [21]. Therefore consider the map $p : \mathbf{R}^{2m} \to \mathbf{R}^s$ given by $p(x) = (p_1(x), \ldots, p_s(x))$. It is a proper polynomial map which separates the orbits and induces the map

$q : \mathbf{R}^{2m}/G \to \mathbf{R}^s$ such that $q \circ \pi = p$, where $\pi : \mathbf{R}^{2m} \to \mathbf{R}^{2m}/G$ is the canonical projection. Then q is a closed topological embedding and, in particular, $q|\mathbf{R}^m \times_G L$ is a regular Nash map [16], 3.2 ii). Thus $\mathbf{R}^m \times_G L$ is embeddable in $\mathbf{R}^s$. Now it suffices remarking that $\mathbf{R}^m \times_G M$ and $\mathbf{R}^m \times_G L$ are Nash diffeomorphic.

Since condition b) is satisfied by the definition of an affine Nash G-manifold structure, the claim is proved. □

We can generalize slightly Theorem 4.4. In order to do this, let M be again a compact affine smooth Nash manifold on which a compact affine Nash group G acts. Assume that M has only one orbit type G/H. Let N be the normalizer of H in G and $K = N/H$. This is a compact affine Nash group by 4.2 a), b), 3.2. Let $M^H = \{x \in M; hx = x \text{ for all } h \in H\}$ be the fixed point set of H on M. It is a compact smooth Nash submanifold of M [9, Proposition 2.25] and hence it is affine. The group K acts freely on M^H and $M^H \to M^H/K$ is a principal Nash K-bundle (3.2) i). Then we have:

THEOREM 4.5. *Let M, G, K be as above. Then M^H has an affine Nash K-manifold structure if and only if the following conditions are satisfied:*

a) *There is a Nash representation of K, with space $\mathbf{R}^m$, such that the Nash manifold $\mathbf{R}^m \times_K M^H$ is affine;*
b) *There exists a smooth K-embedding $M^H \to \mathbf{R}^m$.*

Proof. See proof of 4.4. □

About non free actions we have the following result.

THEOREM 4.6. *Let M be a C^r Nash manifold ($r \leq \infty$), G a compact affine Nash group and K a closed subgroup of G acting on M. Then the following conditions are equivalent:*

i) *M has an affine C^r Nash K-manifold structure;*
ii) *The twisted product $G \times_K M$ has an affine C^r Nash G-manifold structure.*

Proof. i) ⇒ ii) First note that K is a Nash subgroup of G by 4.2 a). By hypothesis there exists a C^r Nash K-embedding $f : M \to \mathbf{R}^m$ in the space of a Nash representation of K. By 4.2 a) we can suppose G a matrix group; then by 4.3 there is a continuous representation $\lambda : G \to GL(p)$, with space $\mathbf{R}^p$, such that $\mathbf{R}^p$, considered as a K-space by restriction, contains the K-space $\mathbf{R}^m$ as an invariant linear subspace. By 4.2 c) λ is a Nash homomorphism and hence it is a Nash representation of K with space $\mathbf{R}^p$. In this way, we obtain a C^r Nash K-embedding $f : M \to \mathbf{R}^p$. Now, by 4.2 d) $G \times_K M$ is a C^r Nash G-manifold; therefore let us consider the C^r G-map $\alpha : G \times_K M \to G/K \times \mathbf{R}^p$, $\alpha([g,t]) = (gK, gf(t))$. We claim that it is a C^r Nash embedding. To prove this, first consider the G-map $\beta : G \times_K M \to G \times_K \mathbf{R}^p$, $\beta([g,t]) = [g, f(t)]$.

It is a bijection onto its image. Moreover, consider the following commutative diagram:

$$\begin{array}{ccc} G\times M & \longrightarrow & G\times \mathbf{R}^p \\ \psi\downarrow & & q\downarrow \\ G\times_K M & \longrightarrow & G\times_K \mathbf{R}^p \end{array} \quad , \quad \begin{array}{ccc} (g,t) & \longrightarrow & (g,f(t)) \\ \downarrow & & \downarrow \\ [g,t] & \longrightarrow & [g,f(t)] \end{array} .$$

The vertical arrows are the canonical projections and are C^r Nash maps by 3.2 i); the arrow below is β. Since $\beta\circ\psi$ is a C^r Nash map, so is β, by 3.2 ii). Consider now the C^r Nash map $G\times f(M)\to G\times M, (g,y)\mapsto (g,f^{-1}(y))$, $y\in f(M)$. The map $\beta^{-1}: G\times_K f(M)\to G\times_K M$, is: $[g,y]\mapsto [g,f^{-1}(y)]$. Therefore a diagram similar to the previous one shows that β^{-1} is a C^r Nash map. Thus β is a C^r Nash embedding. Consider now the G-map $\gamma: G\times_K\mathbf{R}^p\to G/K\times\mathbf{R}^p$, $\gamma([g,x])=(gK,gx)$. It is a smooth Nash diffeomorphism. In fact it is a bijection and the map $\gamma\circ q$, $\gamma\circ q(g,x)=(gK,gx)$, is smooth and Nash and hence γ is of the same class. The inverse map γ^{-1} is such that $\gamma^{-1}(gK,x)=[g,g^{-1}x]$. To prove that it is smooth and Nash, consider the following commutative diagram:

$$\begin{array}{ccc} G\times \mathbf{R}^p & \longrightarrow & G\times \mathbf{R}^p \\ \downarrow & & \downarrow \\ G/K\times \mathbf{R}^p & \longrightarrow & G\times_K \mathbf{R}^p \end{array} \quad , \quad \begin{array}{ccc} (g,x) & \longrightarrow & (g,g^{-1}x) \\ \downarrow & & \downarrow \\ (gK,x) & \longrightarrow & [g,g^{-1}x] \end{array}$$

where the vertical arrows are the canonical projections and the arrow below is γ^{-1}. Then the claim follows from 3.2 ii). So $\alpha=\gamma\circ\beta$ is of class C^r Nash. Now, by 4.2 b) there exists a smooth Nash G-embedding of G/K into the representation space $\mathbf{R}^s$ of a Nash representation ϑ of G; therefore, if we consider the Nash representation $\lambda\oplus\vartheta: G\to GL(p+s)$, the manifold $G\times_K M$ is C^r Nash G-diffeomorphic to a Nash G-submanifold of $\mathbf{R}^{p+s}$.

ii) $\Rightarrow$ i) First remark that the canonical K-embedding $M\to G\times_K M$ is a C^r Nash map because it is the composition of the Nash maps $M\to G\times M\to G\times_K M$, $t\mapsto(e,t)\mapsto[e,t]$ ($e=$ identity element of G). Now, by hypothesis there exists a C^r Nash G-embedding $\varphi: G\times_K M\to V$ into the space of a Nash representation θ of G. Therefore the restriction $\varphi|M$ is a C^r Nash K-embedding into the space of the representation $\theta|K$. $\square$

5. ALGEBRAIC G-STRUCTURES

We begin by recalling the notion of a global slice. Let G be a Lie group, K a closed subgroup of G and M a C^h G-manifold, $1\le h\le\infty$. A K-invariant

C^h submanifold S of M is said to be a global C^h K-slice in M if the map $G \times_K S \to M, [g, s] \to gs$, is a C^h G-diffeomorphism.

We shall use the averaging operator A. Here we recall some basic facts about it. Let G be a compact Lie group and V, W representation spaces of G. Denote by $C^r(W, V)$ the set of all the C^r maps $W \to V$, $0 \leq r \leq \infty$. Let f be such a map and x any point of W. Denote the Haar measure on G by $\mathrm{d}g$. Then

$$A(f)(x) = \int_G g^{-1} f(gx) \mathrm{d}g.$$

LEMMA 5.1. 1. *$A(f)$ is equivariant and $A(f) = f$ if f is equivariant.*

2. *The operator A induces a map $C^r(W, V) \to C^r(W, V)$, $f \mapsto A(f)$, that is continuous with respect to the Whitney C^r topology.*
3. *If f is a polynomial, then so is $A(f)$.*

Proof. See [6, Lemma 4.1]. □

We have seen in 4.2 b) that the homogeneous Nash G-manifold G/K has an algebraic G-variety structure. In general, if we look for conditions in order to find an algebraic G-variety structure on a Nash G-manifold, next theorems answer the question for free actions.

THEOREM 5.2. *Let M be a closed C^r Nash manifold ($r \leq \infty$) and G a compact affine Nash group acting freely on M. Then M has an algebraic G-variety structure if and only if it has an affine Nash G-manifold structure.*

Proof. We have only to prove "if". Consequently, assume that M has an affine Nash G-manifold structure. First suppose $r = \infty$. Therefore there exists a smooth Nash G-embedding $f : M \to W$ of M into the space W of a Nash representation of G. By [5] there is a smooth G-embedding $h : M \to V$ of M into the space V of a representation σ of G and the image $h(M)$ is a non-singular real algebraic G-variety. Note that by 4.2 c) σ is of class Nash. Now C^1 approximate the smooth G-map $\alpha = h \circ f^{-1}|f(M) : f(M) \to h(M)$ by a polynomial $q : W \to V$. By 5.1 the G-map $A(q) C^1$ approximates $A(\alpha) = \alpha$ and is again a polynomial. In this way, using a Nash G-tubular neighbourhood of $h(M)$ in V, we get a smooth Nash G-map $\beta : f(M) \to h(M)$. If the approximation is close enough this map is a diffeomorphism and then the Nash G-diffeomorphism $\beta \circ f : M \to h(M)$ realizes an algebraic G-variety structure on M.

Now let $r < \infty$. By [15] there exists a C^r G-diffeomorphism $f : M \to L$, where L is a smooth G-manifold; by [5] there is a smooth G-diffeomorphism $h : L \to N$, where N is a non-singular real algebraic G-variety in the space of a Nash representation of G. Then consider the C^r G-diffeomorphism $h \circ f : M \to N$. Considering N as a C^r Nash G-manifold and using 3.4, we can obtain a C^r

Nash G-diffeomorphism $M \to N$ and then an algebraic G-variety structure on M. □

Collecting some results found above, we obtain:

THEOREM 5.3. *Let M, G be as in 5.2. Then:*

i) *If $r = \infty M$ has an algebraic G-variety structure if and only if the following condition holds true:*
M is affine and there exists a Nash representation of G, with space $\mathbf{R}^m$, such that the Nash manifold $\mathbf{R}^m \times_G M$ is affine and, moreover, there is a smooth G-embedding of M into $\mathbf{R}^m$.

ii) *If $r < \infty M$ has an algebraic G-variety structure.*

Proof. i) It follows from 4.4 and 5.2. ii) It follows from 4.1 and 5.2. □

If the G action is not free we obtain the following result.

THEOREM 5.4. *Let M be a closed C^r Nash manifold ($r \leq \infty$), G a compact affine Nash group acting on M and K a closed subgroup of G. Assume that there exists in M a closed global C^r Nash K-slice S on which K acts freely. Then M has an algebraic G-variety structure if and only if S has an affine C^r Nash K-manifold structure.*

Proof. Suppose that S has an affine Nash K-structure. By 5.2 it has an algebraic K-variety structure. This means that it is C^r Nash K-diffeomorphic to a non singular real algebraic K-variety T. Now, by [17, Corollary 1.4] the Nash G-manifold $G \times_K T$, considered as a smooth G-manifold, has the structure of a non singular real algebraic G-variety N. It follows that M and N are C^r G-diffeomorphic. We want to prove that they are C^r Nash G-diffeomorphic. From what we have just said, there exist a representation σ of G, with space W, and a C^r G-embedding $f : M \to W$ such that $f(M) = N$. Note that by 4.2 c) σ is of class Nash. Since G is compact, it is linear by 4.2 a). Therefore we can use Theorem 6.6 (see below): from the hypothesis made on S it follows that M has the structure of an affine C^r Nash G-manifold. So there is a C^r Nash G-embedding $h : M \to V$ of M into the space V of a Nash representation of G. Consider now the C^r G-diffeomorphism $f \circ h^{-1}$. Using the averaging operator A and a Nash G-tubular neighbourhood of N in W, we can obtain a Nash G-diffeomorphism $\beta : h(M) \to N$. Therefore the Nash G-map $\beta \circ h : M \to N$ gives an algebraic G-variety structure on M.

Conversely, suppose that M has an algebraic G-variety structure. So there is a Nash G-embedding $\varrho : M \to U$ of M into the space of a Nash representation ϑ of G such that $\varrho(M)$ is a non singular real algebraic G-variety. Therefore $\varrho|S$ and $\vartheta|K$ are what we want. □

6. AFFINE G-STRUCTURES WITH G NON-COMPACT

As we have already said (see Introduction), in order to equivariantly embed a G-manifold M into a linear G-space G must be linear, $G \subset GL(n)$. Moreover note that a compact subgroup K of a linear Nash group G is a Nash subgroup of G. In fact K is an algebraic group in the vector space $L(n)$ of all $n \times n$ matrices [13, Theorem 5 p. 133], and hence it is a Nash subgroup of $L(n)$; so it is a Nash subgroup of G. Now remark that the proof of 4.3 given by Palais works also in a Nash setting. Therefore, repeating that proof, we get the following result:

THEOREM 6.1. *Let $G \subset GL(n)$ be a linear Nash group, K a compact subgroup of G and $K \to GL(m)$ a Nash representation of K, with space $\mathbf{R}^m$. Then there is a Nash representation $G \to GL(p)$ of G, with space $\mathbf{R}^p$ which, considered as a K-space by restriction, contains $\mathbf{R}^m$ as an invariant subspace.*

THEOREM 6.2. *Let $G \subset GL(n)$ be a linear Nash group and K a compact subgroup of G. Then there exist a Nash representation of G, with space W, and a point $w \in W$ such that isotropy group G_w of w is K.*

Proof. K is a Nash subgroup of G. Since it is compact and affine, by [2, Theorem 0.3.5] and 4.2 c) we can suppose it is a Nash subgroup of $O(n)$. By [2, Theorem 0.5.2] there exist a continuous representation $\phi : O(n) \to O(m)$, for some m, with space U, and a point $u \in U$ such that $O(n)_u = K$. By 4.2 c) ϕ is a smooth Nash map. By 6.1 there exists a Nash representation $\lambda : GL(n) \to GL(p)$, with space $\mathbf{R}^p$ which, considered as an $O(n)$-space by restriction, contains U as an invariant subspace. Then $K = O(n) \cap GL(n)_u$. Consider now the Nash action $GL(n) \times S(n) \to S(n)$ of $GL(n)$ on the space $S(n)$ of the $n \times n$ symmetric matrices given by $(a, b) \mapsto aba^t$, where a^t is the transpose of the matrix $a \in GL(n)$. If $e \in S(n)$ is the identity matrix, the isotropy group $GL(n)_e$ of e is $O(n)$. If we identify $S(n)$ with $\mathbf{R}^q (q = \dim S(n))$ choosing an order of matrix elements, the previous action gives rise to a Nash representation $\sigma : GL(n) \to GL(q)$. So, in conclusion, we have the Nash representation $\alpha = \lambda \oplus \sigma : GL(n) \to GL(p+q)$, with space $W = \mathbf{R}^p \times \mathbf{R}^q$, and, if $w = (u, e) \in W$, it is $GL(n)_w = GL(n)_e \cap GL(n)_u = O(n) \cap GL(n)_u = K$. Finally, we have the Nash representation $\varrho = \alpha | G : G \to GL(p+q)$ of G and it is $G_w = K$. $\square$

In order to prove the Theorem 6.5 below, let us recall the notions of a proper G-space and of a Cartan G-space, which is a bit more general than that of a proper G-space. A locally compact space X on which a Lie group G acts is said to be a proper G-space, and G is said to act properly on X, if the set $G_A = \{g \in G; gA \cap A \neq \emptyset\}$ is a compact subset of G for every compact subset A of X. This is equivalent to the fact that the map $G \times X \to X \times X$,

$(g,x) \mapsto (gx,x)$ is proper. X is said instead to be a Cartan G-space if each point $x \in X$ has a compact neighbourhood A such that G_A is a compact subset of G. The relation between these two notions is given by the following proposition:

PROPOSITION 6.3. *If X is a locally compact G-space, the following conditions are equivalent:*

1. *X is a Cartan G-space and X/G is Hausdorff;*
2. *X is a proper G-space.*

Proof. See [14, Theorem 1.2.9]. □

Let us consider again the Nash action of $GL(n)$ on $S(n)$ given by $(a,b) \mapsto aba^t$ (see proof of 6.2). We have:

PROPOSITION 6.4. *Let $e \in S(n)$ be the identity matrix. Then the $GL(n)$-orbit of e in $S(n)$ is an open set A and $GL(n)$ acts properly on A.*

Proof. See [11, Lemma 2.1]. □

Identify now again, as in the proof of 6.2, the space $S(n)$ with $\mathbf{R}^q(q = \dim S(n))$ and let $v \in \mathbf{R}^q$ be the point which corresponds to the identity e and A_v the open set which corresponds to the open set $A \subset S(n)$ of 6.4. Therefore $GL(n)$ acts properly on A_v which is the orbit of v. We obtain the following equivariant embedding theorem for homogeneous smooth Nash G-manifolds:

THEOREM 6.5. *Let $G \subset GL(n)$ be a Nash linear group and K a compact subgroup of G. Then the homogeneous space G/K is a smooth Nash G-manifold which has an affine Nash G-manifold structure.*

Proof. First of all recall that K is a Nash subgroup of G; second, G/K is a smooth Nash manifold by 3.2 i); moreover, it is easy to see that the natural action of G on G/K, $(g', gK) \mapsto g'gK$ is of class Nash. Now, from 6.2 and its proof we have the following Nash representations of $GL(n)$, and hence of $G : \lambda : GL(n) \to GL(p), \sigma : GL(n) \to GL(q), \alpha : GL(n) \to GL(p+q)$, with spaces, respectively, $\mathbf{R}^p, \mathbf{R}^q, W = \mathbf{R}^p \times \mathbf{R}^q$ and a point $w = (u,v) \in W$ such that $G_w = K$; moreover $GL(n)$ acts properly on the orbit A_v of v in $\mathbf{R}^q$. Since G is closed in $GL(n)$, G also acts properly on A_v. Because a product of a proper G-space and a G-space is again a proper G-space, G acts properly on $\mathbf{R}^p \times A_v$. Therefore consider the Nash map $F : G \to \mathbf{R}^{p+q}$, $g \mapsto \varrho(g)w$, $\varrho = \alpha|G$, and the induced G-map $f : G/K \to \mathbf{R}^{p+q}$, $gK \mapsto \varrho(g)w$. If $\pi : G \to G/K$ is the canonical projection, it is $f \circ \pi = F$ and thus f is of class Nash by 3.2 i). Moreover remark that $f(G/K)$ is the orbit $G(w)$ of w. But the open set A_v of $\mathbf{R}^q$ contains the orbit $G(v)$ of v and hence the orbit $G(w)$ is contained in the proper G-space $\mathbf{R}^p \times A_v$, open in $\mathbf{R}^p \times \mathbf{R}^q$. By 6.4 $\mathbf{R}^p \times A_v$ is a Cartan G-space and therefore, by [14, Proposition 1.1.5], f is a topological embedding into $\mathbf{R}^p \times \mathbf{R}^q$; by [2, VI 1.2] f is a smooth immersion; then, by the inverse function theorem in a Nash setting, f is a Nash G-diffeomorphism onto its image. □

Remark now that the smooth Nash G-manifold G/K, which is affine, has the global Nash slice given by $\{eK\}$ (e = identity element). A link between affine Nash G-structures and global slices appears in a more general situation, as the following result shows.

THEOREM 6.6. *Let M be a C^r Nash manifold ($r \leq \infty$), G a linear Nash group acting on M and K a compact subgroup of G. Assume that there exists in M a global C^r Nash K-slice S. Then M has an affine C^r Nash G-manifold structure if and only if S has an affine C^r Nash K-manifold structure.*

Proof. Suppose that S has an affine C^r Nash K-manifold structure, that is that there exists a C^r Nash K-embedding $q : S \to \mathbf{R}^m$ of S into the space of a Nash representation of K. By 6.1 there is a Nash representation $\lambda : G \to GL(p)$ of G, with space $\mathbf{R}^p$ which, considered as a K-space by restriction, contains $\mathbf{R}^m$ as an invariant subspace. Therefore we have the Nash K-embedding $q : S \to \mathbf{R}^p$. Now, by hypothesis M is C^r Nash G-diffeomorphic to $G \times_K S$. Consider the C^r G-map $\alpha : G \times_K S \to G/K \times \mathbf{R}^p$, $\alpha([g, s]) = (gK, gq(s))$. We claim that it is a Nash embedding. At this point we are in the same situation of the proof of i) $\Rightarrow$ ii) in 4.6. Repeating that proof and using at the end the embedding Theorem 6.5 instead of 4.2 b), we equivariantly embed $G \times_K S$ into a Nash representation space of G.

Conversely, let us suppose that there exists a C^r Nash G-embedding $\varrho : M \to V$ of M into the space of a Nash representation ϑ of G. Therefore the restrictions $\varrho|S$ and $\vartheta|K$ give what we want. □

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