

PROPERTIES OF THE SPACE OF SECTIONS OF SOME BANACH BUNDLES

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One shows for Banach bundles in a certain class that having a second countable locally compact Hausdorff base space and separable fibers implies the separability of the Banach space of all the sections that vanish at infinity. In the reverse direction, it is proved that for a Banach bundle with locally compact Hausdorff base space the separability of the space of all the sections that vanish at infinity implies that the base space is second countable. If the base space is compact and the space of all the sections of the bundle is generated by a weakly compact subset then the base space is an Eberlein compact.

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1. INTRODUCTION

It is well known that a compact Hausdorff space T is second countable if and only if the Banach space $C(T)$ consisting of all the scalar valued functions on T is separable. Similarly, the compact Hausdorff T is an Eberlein compact if and only if $C(T)$ is generated by a weakly compact subset. We generalize to some extent these results to Banach bundles with compact and locally compact Hausdorff base spaces. To establish the separability of the space of sections we restrict the discussion to locally uniform Banach bundles, a class of Banach bundles that was introduced in [5]; its definition will be given below. Namely, we prove that the space of all the sections that vanish at infinity is separable provided that the Banach bundle is locally uniform, the base space is a second countable locally compact Hausdorff space, and all the fiber spaces of the bundle are separable. In the other direction, we prove that if the Banach space of all the sections of the Banach bundle $\xi := (\mathcal{E}, p, T)$ that vanish at infinity on the locally compact Hausdorff space T is separable then T is second countable; of course, each fiber is a separable Banach space. We also show that if the base space is compact Hausdorff and the space of all the sections of ξ is weakly compact generated (WCG) then T is Eberlein compact. For other separability results one should consult [4, 215 ff] and the references there.

We discuss here Banach bundles for which the norm is upper semi-continuous that is (H) Banach bundles in the terminology of [3]. We shall always suppose that the base space of a Banach bundle is Hausdorff. For a Banach bundle $\xi := (\mathcal{E}, p, T)$ we denote by $\Gamma(\xi)$ the space of all the continuous sections of ξ ; if the base space is compact, $\Gamma(\xi)$ is a Banach space when one endows it with the sup norm. Similarly, if T is locally compact then $\Gamma_0(\xi)$ stands for the space of all the sections of ξ that vanish at infinity; we endow it with the sup norm to make it a Banach space.

We shall denote the closed unit ball of the Banach space Y by Y_1 . Let X be a Banach space with its family of bounded closed subsets endowed with the Hausdorff metric, T a Hausdorff topological space and $t \rightarrow X(t)$ a map such that $X(t)$ is a closed subspace of X for each $t \in T$ and the map $t \rightarrow X(t)_1$ is continuous. Set $\mathcal{E} := \cup_{t \in T} (\{t\} \times X(t))$ and define $p : \mathcal{E} \rightarrow T$ by $p((t, x)) := t$. Given an open subset V of T and an open subset O of X we denote $\mathcal{U}(V, O) := \{(t, x) \mid t \in V, x \in O \cap X(t)\}$. It is shown in [5, Section 5] that the family of all the sets $\{\mathcal{U}(V, O)\}$ is the base of a topology on \mathcal{E} such that (\mathcal{E}, p, T) is a Banach bundle for which the norm is continuous on the bundle space \mathcal{E} . A Banach bundle isomorphic to a Banach bundle as detailed above is called a uniform Banach bundle. A Banach bundle $\xi := (\mathcal{E}, p, T)$ is called a locally uniform Banach bundle if there is a family $\{T_i\}$ of closed subsets of T such that $\{\text{Int}(T_i)\}$ is an open cover of T and every restriction $\xi \mid T_i$ is a uniform Banach bundle.

An Eberlein compact is a compact Hausdorff space that is homeomorphic to a weakly compact subset of a Banach space. A Banach space X is called weakly compact generated (WCG) if it contains a weakly compact subset whose linear span is dense in X . A thorough discussion of Eberlein compact spaces and of WCG Banach spaces can be found in [7].

2. RESULTS

We begin by discussing conditions on the Banach bundle that ensure the separability of the space of sections.

PROPOSITION 1. *Let $\xi := (\mathcal{E}, p, T)$ be a locally uniform Banach bundle with T compact and metrizable and $p^{-1}(t)$ separable for each $t \in T$. Then $\Gamma(\xi)$ is separable.*

The proof uses a lemma that undoubtedly is known but a proof is included for lack of a reference. The proof presented here relies on a rather advanced result but the lemma can be given a direct elementary proof.

LEMMA 2. *Let X be a Banach space and Y a closed subspace of X . If Y and X/Y are separable then X is separable.*

Proof. Let $\theta : X \rightarrow X/Y$ be the quotient map. By [1] there is a continuous map $\psi : X/Y \rightarrow X$ such that $\theta(\psi(z)) = z$ for every $z \in X/Y$. It is immediately seen that the map $x \rightarrow (x - \psi(\theta(x)), \theta(x))$ is a homeomorphism of X onto the separable space $Y \times X/Y$; the inverse map is $(y, z) \rightarrow y + \psi(z)$. \square

Proof of Proposition 1. We begin with a particular case: let ξ be a uniform Banach bundle. Thus we may consider the following setting: each fiber $p^{-1}(t)$ is a subspace of some Banach space X such that the map $t \rightarrow p^{-1}(t)$ from T to the family of all the closed subsets of X has the property that $t \rightarrow p^{-1}(t)_1$ is continuous when the collection of all the closed bounded subsets of X is endowed with the Hausdorff metric. We claim that the closed subspace \tilde{X} of X generated by $\cup\{p^{-1}(t) \mid t \in T\}$ is separable. Indeed, with $\{t_n\}$ a countable dense subset of T and $\{x_m^n\}_{m=1}^\infty$ a dense subset of $p^{-1}(t_n)_1$, it is easily seen that the set of all linear combinations with rational coefficients of $\{x_m^n \mid n, m \in \mathbb{N}\}$ is dense in \tilde{X} . From now on to simplify the notation, we shall assume that $\tilde{X} = X$. In this situation, the Banach space $\mathcal{C}(T, X) = \mathcal{C}(T) \otimes X$ is separable hence its subspace $\Gamma(\xi)$ is separable too.

We consider now the general case. Thus, by the compactness of T , there is a family $\{T_i\}_{i=1}^n$ of closed subsets of T such that $\{\text{Int}(T_i)\}_{i=1}^n$ is a cover of T , $\text{Int}(T_i) \neq \emptyset$ for each i , and the restriction of ξ to each T_i is a uniform Banach bundle. We shall prove the conclusion by induction. The step $n = 1$ was treated above and we shall suppose now that if the base space can be covered by $n - 1$ closed sets as before-mentioned then the conclusion of the theorem is valid. Set $S := \cup_{i=1}^{n-1} T_i$. The restriction map $\varphi \rightarrow \varphi \mid S$ maps $\Gamma(\xi)$ onto the separable space $\Gamma(\xi \mid S)$ by [3, p. 15]. The kernel of this map is $\{\varphi \in \Gamma(\xi) \mid \varphi \mid S \equiv 0\}$ which can be considered as a closed subspace of the separable space $\Gamma(\xi \mid T_n)$. Thus $\Gamma(\xi)$ is separable by Lemma 2. \square

In the above proposition, we required the bundle to be locally uniform. If one discards this hypothesis it may happen that $\Gamma(\xi)$ is not separable even if the base space is metrizable and all the fibers are separable Banach spaces. Such a situation occurs in [6, Example 5.1]. We consider now a setting more general than in Proposition 1.

THEOREM 3. *Let $\xi := (\mathcal{E}, p, T)$ be a locally uniform Banach bundle. If T is a second countable locally compact space and each fiber $p^{-1}(t)$, $t \in T$, is separable then $\Gamma_0(\xi)$ is separable.*

Proof. There is a sequence of compact subsets $\{T_n\}_{n=1}^\infty$ of T such that $\text{Int}T_1 \neq \emptyset$, $T_n \subseteq \text{Int}(T_{n+1})$, $n = 1, 2, \dots$, and $T = \cup_{n=1}^\infty T_n$, see [2, The-

orem XI.7.2]. Let f_n be a real valued continuous function on T such that $f_n | T_n \equiv 1$, $0 \leq f_n \leq 1$, and $f_n(t) = 0$ if $t \in T \setminus \text{Int}(T_{n+1})$, $n = 1, 2, \dots$. For each n the Banach space $\Gamma(\xi | T_n)$ is separable by Proposition 1 and we choose a dense countable set $\{\varphi_m^n\}_{m=1}^\infty$ in this space. By using again [3, p. 15] we extend each φ_m^n to a section of ξ which, for simplicity, we shall denote also by φ_m^n , with the requirement that

$$\|\varphi_m^n(t)\| \leq \|\varphi_m^n | T_n\|$$

for each $t \in T$. Denote now $\psi_m^n := f_n \varphi_m^{n+1}$. We claim that $\{\psi_m^n\}_{m,n=1}^\infty$ is dense in $\Gamma_0(\xi)$.

To prove the claim let $\varphi \in \Gamma_0(\xi)$, $\epsilon > 0$ and $n \in \mathbb{N}$ such that

$$\{t \in T \mid \|\varphi(t)\| \geq \epsilon\} \subseteq T_n.$$

Choose m such that $\|\varphi(t) - \varphi_m^{n+1}(t)\| < \epsilon$ for every $t \in T_{n+1}$. For this choice of m we have

$$\|\varphi(t) - \psi_m^n(t)\| = \|\varphi(t) - \varphi_m^{n+1}(t)\| < \epsilon, \quad t \in T_n,$$

and

$$\|\varphi_m^{n+1}(t)\| \leq \epsilon + \|\varphi(t)\| < 2\epsilon, \quad t \in \text{Int}(T_{n+1}) \setminus T_n,$$

hence

$$\begin{aligned} \|\varphi(t) - \psi_m^n(t)\| &= \|\varphi(t) - f_n(t)\varphi_m^{n+1}(t)\| \leq \|\varphi(t)\| + |f_n(t)|\|\varphi_m^{n+1}(t)\| < 3\epsilon, \\ & \quad t \in \text{Int}(T_{n+1}) \setminus T_n, \end{aligned}$$

and $\|\varphi(t) - \psi_m^n(t)\| = \|\varphi(t)\| < \epsilon$ if $t \in T \setminus \text{Int}(T_{n+1})$. We got $\|\varphi - \psi_m^n\| \leq 3\epsilon$ and this ends the proof. \square

Question 4. Let $\xi := (\mathcal{E}, p, T)$ be a Banach bundle with T an Eberlein compact. Suppose moreover that each fiber $p^{-1}(t)$ is WCG. Is $\Gamma(\xi)$ WCG? What if ξ is locally uniform?

We now discuss how certain properties of the space of sections are reflected in properties of the base space.

THEOREM 5. *Let $\xi := (\mathcal{E}, p, T)$ be a Banach bundle with T compact Hausdorff. If $\Gamma(\xi)$ is separable (WCG) then T is second countable (Eberlein compact, respectively) and each fiber space of ξ is separable (WCG, respectively).*

Proof. We begin by showing that each point t of the base space has a compact neighbourhood U_t such that the Banach space $C(U_t)$ is isomorphic with a subspace of $\Gamma(\xi | U_t)$. By [3, p. 15] there is a section φ_t of $\Gamma(\xi)$ such that $\|\varphi_t(t)\| = 1$. From the upper semicontinuity on \mathcal{E} of the norm it follows that $\{s \in T \mid \|\varphi_t(t) - \varphi_t(s)\| < 1/2\}$ is a neighbourhood V_t of t . For each $s \in V_t$ we have $\|\varphi_t(s)\| > 1 - 1/2 = 1/2$. Let U_t be a compact neighbourhood

of t contained in V_t . The map $f \rightarrow f \cdot \varphi_t | U_t$ is an isomorphism of $C(U_t)$ into $\Gamma(\xi | U_t)$. Indeed,

$$\frac{1}{2} \|f\| \leq \|f \cdot \varphi_t | U_t\| \leq \|f\| \|\varphi_t\|.$$

We shall denote by Φ_t this map of $C(U_t)$ into $\Gamma(\xi | U_t)$

Now, if $\Gamma(\xi)$ is separable each $\Gamma(\xi | U_t)$ is separable hence each $C(U_t)$ is separable. Thus each point $t \in T$ has a second countable neighbourhood and we infer that T is second countable.

If $\Gamma(\xi)$ is WCG then each $\Gamma(\xi | U_t)$ is WCG by being a quotient of $\Gamma(\xi)$. Therefore all the norm closed balls of $\Phi_t(C(U_t))^*$, $t \in T$, are Eberlein compacts in their weak star topology by [7, Theorem 4.9]. Hence the image B_t by $(\Phi_t^*)^{-1}$ of the norm closed unit ball of $C(U_t)^*$ is an Eberlein compact in its weak star topology since it is a weak star closed subset of a norm closed ball centered at the origin of $\Phi_t(C(U_t))^*$. Since U_t is homeomorphic with a weak star closed subset of B_t we infer that U_t is an Eberlein compact. It turns out that T is a finite union of Eberlein compacts. The disjoint union S of these finitely many Eberlein compacts is obviously an Eberlein compact. There is a natural continuous map of S onto T hence T is an Eberlein compact by [7, Theorem 2.15]. \square

For locally compact base spaces we have the following result.

THEOREM 6. *If $\xi := (\mathcal{E}, p, T)$ is a Banach bundle with a locally compact base space T and $\Gamma_0(\xi)$ is separable then T is second countable and each fiber space of ξ is separable.*

Proof. Let $\{\varphi_n\}_{n=1}^\infty$ be a dense countable subset of $\Gamma_0(\xi)$. For each n the set $\{t \in T \mid \|\varphi_n(t)\| > 0\}$ is σ -compact and T is the union of all these sets. As in the proof of Theorem 3 there is an increasing sequence of compact subsets $\{T_m\}$ such that $T_m \subset \text{Int}(T_{m+1})$ and $T = \bigcup_{m=1}^\infty T_m$. Each T_m is second countable by Theorem 5 hence T is second countable. The second assertion is obvious. \square

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