MEAN OF A FIRST-PASSAGE TIME FOR A TWO-DIMENSIONAL DIFFUSION PROCESS

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Let $X(t)$ denote the wear of a machine and $Y(t)$ the number of items it produces per unit time. A system of stochastic differential equations is considered for the vector process $(X(t), Y(t))$. The aim is to find the average time it takes the two-dimensional diffusion process to hit a certain boundary for the first time. The appropriate Kolmogorov backward equation is solved explicitly by making use of the method of similarity solutions.

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1. INTRODUCTION

Let $X(t)$ denote the wear of a machine at time t. Because wear cannot decrease, many authors have used a gamma process, which is a pure-jump increasing Lévy process, as a model for $X(t)$; see, for instance, [5]. Others have used one-dimensional diffusion processes, in particular Wiener processes; see [7] and the references therein. Ghamlouch *el al.* [2] have proposed a model based on a jump-diffusion process. Depending on the infinitesimal parameters of the diffusion processes, these processes can yield satisfactory results. However, any one-dimensional diffusion (or jump-diffusion) process will both increase and decrease in any time interval.

In order to obtain a more realistic model, we will consider (see [6]) two-dimensional (degenerate) diffusion processes $\{(X(t), Y(t)), t \geq 0\}$ defined by the system of stochastic differential equations

$$\mathrm{d}X(t) = Y(t)\,\mathrm{d}t, \tag{1}$$

$$\mathrm{d}Y(t) = f[Y(t)]\,\mathrm{d}t + \{v[Y(t)]\}^{1/2}\,\mathrm{d}W(t), \tag{2}$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion and $\{Y(t), t \geq 0\}$ is a diffusion process that always remains positive, so that wear always increases, as required. $Y(t)$ could be, for instance, the operating speed of the machine. The function $f(\cdot)$ in Eq. (2) is the infinitesimal mean of the process $\{Y(t), t \geq 0\}$, while $v(\cdot) > 0$ is its infinitesimal variance.

In this note, $Y(t)$ will be taken as the number of items produced by the machine per unit time (at time t), and we will assume that

$$\text{(3)} \qquad \mathrm{d}Y(t) = -Y(t)\,\mathrm{d}t + Y(t)\,\mathrm{d}W(t).$$

Hence, the process $\{Y(t), t \geq 0\}$ is a geometric Brownian motion with infinitesimal mean $-y$ and infinitesimal variance y^2. As is well known, if $Y(0) > 0$, the process will indeed always remain positive. Notice also that $Y(t)$ tends to decrease (because the infinitesimal mean is negative), which is realistic.

Next, let

$$\text{(4)} \qquad T(x,y) := \inf\left\{t \geq 0 \colon \frac{X(t)}{Y(t)} = k \mid X(0) = x, Y(0) = y\right\},$$

where k, x and y are all positive, and $x/y \leq k$. That is, the first-passage time T denotes the first time that the ratio $X(t)/Y(t)$ becomes too large, so that the useful life of the machine is over or, at least, the machine should be repaired. We see that this could be due to the fact that the machine is too old, or that its production per unit time has decreased too much.

We are interested in computing the expected value of the random variable T. In reliability theory, perhaps the most important quantity is indeed the expected lifetime of the machine or, in general, the device of interest. For instance, a manufacturer is interested in knowing this expected lifetime, in order to estimate the cost of the warranty it offers.

Let $\rho(t; x, y)$ denote the probability density function of the random variable $T(x,y)$. This function satisfies the Kolmogorov backward equation

$$\text{(5)} \qquad \frac{1}{2} y^2 \rho_{yy}(x,y) - y\,\rho_y(x,y) + y\,\rho_x(x,y) = \rho_t(x,y).$$

It follows that the moment-generating function of T (or the Laplace transform of its probability density function)

$$\text{(6)} \qquad M(x,y;a) := E\left[e^{-aT(x,y)}\right],$$

where a is a positive constant, satisfies the partial differential equation

$$\text{(7)} \qquad \frac{1}{2} y^2 M_{yy}(x,y;a) - y\,M_y(x,y;a) + y\,M_x(x,y;a) = a\,M(x,y;a)$$

and is such that (because $T(x,y) = 0$ if $x/y = k$)

$$\text{(8)} \qquad M(x,y;a) = 1 \quad \text{if } x/y = k.$$

Next, since

$$\text{(9)} \qquad E\left[e^{-aT(x,y)}\right] = 1 - a\,E[T(x,y)] + \frac{a^2}{2} E[T^2(x,y)] - \ldots,$$

Eq. (7) implies that

(10) $$m(x,y) := E[T(x,y)]$$

is a solution of

(11) $$\frac{1}{2}y^2 m_{yy}(x,y) - y m_y(x,y) + y m_x(x,y) = -1,$$

subject to the boundary condition

(12) $$m(x,y) = 0 \quad \text{if } x/y = k.$$

In the next section, we will solve the boundary-value problem (11), (12) explicitly by making use of the method of similarity solutions.

2. MEAN VALUE OF $T(x, y)$

As mentioned in the previous section, the function $M(x,y;a) := E[e^{-aT(x,y)}]$ satisfies Eq. (7), subject to the boundary condition (8). Using the results in [4], we could derive an exact and explicit expression for M. From this expression, we could theoretically obtain the function $m(x,y)$ by computing

(13) $$m(x,y) = -\lim_{a \downarrow 0} \frac{\partial M(x,y;a)}{\partial a}.$$

However, because M is expressed in terms of special functions (Whittaker functions and confluent hypergeometric functions, to be precise), making use of the above formula is actually very difficult. It is much easier to solve directly (11), (12) instead.

Lefebvre [4] (see also [3]) showed that $M(x,y;a)$ can actually be written as follows:

(14) $$M(x,y;a) = N(z;a),$$

where $z := x/y$. That is, he used the method of similarity solutions to solve the partial differential equation (7). He also proved that the solution, subject to the appropriate boundary conditions, was unique.

We will proceed in the same way here. That is, we can assume that

(15) $$m(x,y) = n(z),$$

with the similarity variable z being defined above. Then, we find that Eq. (11) is transformed into the ordinary differential equation

(16) $$\frac{1}{2}z^2 n''(z) + (2z+1)n'(z) = -1.$$

The boundary condition (12) becomes

(17) $$n(k) = 0.$$

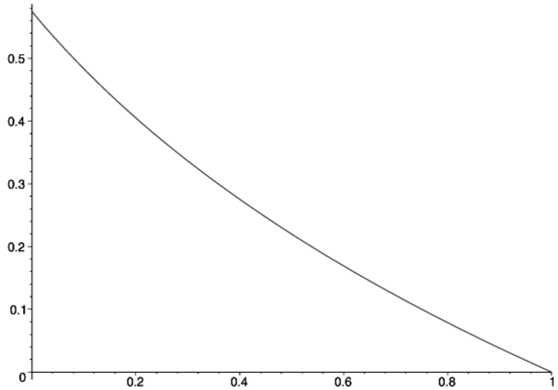

Fig. 1 – Expected value of the random variable $T(x, y)$, as a function of x/y, when $k = 1$.

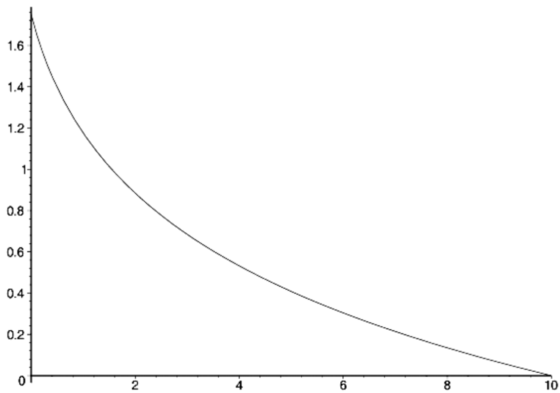

Fig. 2 – Expected value of the random variable $T(x, y)$, as a function of x/y, when $k = 10$.

Next, making use of the mathematical software *Maple*, we find that the solution of (16), (17) may be written as

$$n(z) = -\frac{1}{3}\int_z^k \frac{c_1 e^{2/u} - 2u^3 + 2u^2 - 4u + 8e^{2/u}\,\mathrm{Ei}(1, 2/u)}{u^4}\,\mathrm{d}u, \tag{18}$$

in which c_1 is an arbitrary constant and Ei is the exponential integral defined by (see [1, p. 228])

$$\mathrm{Ei}(1, z) = \int_1^\infty \frac{e^{-zt}}{t}\,\mathrm{d}t. \tag{19}$$

Now, we can show that $\int_z^k \frac{e^{2/u}}{u^4}\,\mathrm{d}u$ diverges as z decreases to zero. It follows that we must choose the constant $c_1 = 0$ in (18). Hence, we can state the following proposition.

PROPOSITION 2.1. *The mean value of the first-passage time $T(x, y)$ defined in (4) is given by*

$$m(x, y) = \frac{2}{3} \int_{x/y}^{k} \frac{u^3 - u^2 + 2u - 4e^{2/u}\,\mathrm{Ei}(1, 2/u)}{u^4}\, \mathrm{d}u. \tag{20}$$

To conclude this section, we present in Fig. 1 and Fig. 2 the function $m(x, y)$ when we choose the constant $k = 1$ and $k = 10$, respectively. We see that it is strictly decreasing, as it should be.

3. AN EXTENSION

In Section 2, we assumed that the wear $X(t)$ of the machine at time t depended on the variable $Y(t)$. We mentioned that $Y(t)$ could be the operating speed of the machine. In practice, there can be many variables that influence $X(t)$. For example, in addition to the operating speed, temperature is an important variable. Moreover, in some applications the variations of $X(t)$ could also depend on the value of $X(t)$ itself, and not only on the variables $Y_1(t)$, $Y_2(t)$, ... We could also accept that there are sometimes small decreases in the values taken by $X(t)$ because of measurement errors, etc. Therefore, we now consider the system of stochastic differential equations

$$\mathrm{d}X(t) = \frac{c}{\prod_{i=1}^{n}[Y_i(t)]^{a_i}}\, \mathrm{d}t + \mu_0\, X(t)\, \mathrm{d}t + \{\sigma_0^2 X^2(t)\}^{1/2}\, \mathrm{d}B_0(t), \tag{21}$$

$$\mathrm{d}Y_i(t) = \mu_i\, Y_i(t)\, \mathrm{d}t + \{\sigma_i^2 Y_i^2(t)\}^{1/2}\, \mathrm{d}B_i(t), \tag{22}$$

where $c \neq 0$, $a_i \neq 0$, $\sigma_i > 0$ and the processes $\{B_i(t), \geq 0\}$ are independent Brownian motions, for $i = 0, 1, \ldots, n$.

We define

$$T(x, y_1, \ldots, y_n) = \inf\left\{t \geq 0 : X(t) \prod_{i=1}^{n}[Y_i(t)]^{a_i} = k_1 \text{ or } k_2\right\}, \tag{23}$$

given that $X(0) = x$ and $Y_i(0) = y_i$ for $i = 1, \ldots, n$. We assume that $k_1 \leq x \prod_{i=1}^{n} y_i^{a_i} \leq k_2$.

The moment-generating function

$$\Phi(x, y_1, \ldots, y_n; \gamma) := E\left[e^{-\gamma T(x, y_1, \ldots, y_n)}\right] \tag{24}$$

satisfies the Kolmogorov backward equation

(25)

$$\frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 y_i^2\, \Phi_{y_i y_i} + \sum_{i=1}^{n} \mu_i\, y_i\, \Phi_{y_i} + \frac{1}{2}\sigma_0^2 x^2\, \Phi_{xx} + \left(\frac{c}{\prod_{i=1}^{n} y_i^{a_i}} + \mu_0\, x\right) \Phi_x = \gamma\, \Phi,$$

subject to the boundary conditions

$$\Phi(x, y_1, \ldots, y_n; \gamma) = 1 \quad \text{if } x \prod_{i=1}^{n} y_i^{a_i} = k_1 \text{ or } k_2. \tag{26}$$

Now, based on the above boundary conditions, we look for a solution of the form

$$\Phi(x, y_1, \ldots, y_n; \gamma) = \Psi(z), \tag{27}$$

where

$$z := x \prod_{i=1}^{n} y_i^{a_i}. \tag{28}$$

Under this assumption, we compute

(29)

$$\frac{\partial \Phi}{\partial x} = \prod_{i=1}^{n} y_i^{a_i}\, \Psi'(z), \; \frac{\partial^2 \Phi}{\partial x^2} = \left(\prod_{i=1}^{n} y_i^{a_i} \right)^2 \Psi''(z), \; \frac{\partial \Phi}{\partial y_i} = x\, a_i\, y_i^{a_i - 1} \prod_{j \neq i} y_j^{a_j}\, \Psi'(z)$$

and

$$\frac{\partial^2 \Phi}{\partial y_i^2} = x^2 \left(a_i\, y_i^{a_i - 1} \prod_{j \neq i} y_j^{a_j} \right)^2 \Psi''(z) + x\, a_i\, (a_i - 1)\, y_i^{a_i - 2} \prod_{j \neq i} y_j^{a_j}\, \Psi'(z). \tag{30}$$

Hence, we find that the partial differential equation (25) is transformed into the ordinary differential equation

$$\begin{aligned} &\frac{1}{2} \sum_{i=1}^{n} \sigma_i^2\, a_i^2\, z^2\, \Psi''(z) + \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2\, a_i\, (a_i - 1)\, z\, \Psi'(z) \\ &+ \sum_{i=1}^{n} \mu_i\, a_i\, z\, \Psi'(z) + \frac{1}{2} \sigma_0^2\, z^2\, \Psi''(z) + (c + \mu_0\, z)\; \Psi'(z) = \gamma\, \Psi(z). \end{aligned} \tag{31}$$

That is, we must solve

$$\begin{aligned} &\frac{1}{2} \left\{ \sigma_0^2 + \sum_{i=1}^{n} \sigma_i^2\, a_i^2 \right\} z^2\, \Psi''(z) \\ &+ \left\{ c + \left[\mu_0 + \sum_{i=1}^{n} \left(\frac{1}{2} \sigma_i^2\, a_i\, (a_i - 1) + \mu_i\, a_i \right) \right] z \right\} \Psi'(z) = \gamma\, \Psi(z), \end{aligned} \tag{32}$$

subject to

$$\Psi(k_1) = \Psi(k_2) = 1. \tag{33}$$

Let

$$\alpha^2 := \frac{1}{2}\left\{\sigma_0^2 + \sum_{i=1}^{n} \sigma_i^2\, a_i^2\right\} \tag{34}$$

and

$$\beta := \mu_0 + \sum_{i=1}^{n}\left(\frac{1}{2}\sigma_i^2\, a_i\,(a_i - 1) + \mu_i\, a_i\right). \tag{35}$$

Making once again use of the mathematical software *Maple*, we can find the general solution of

$$\alpha^2\, z^2\, \Psi''(z) + (c + \beta\, z)\, \Psi'(z) = \gamma\, \Psi(z). \tag{36}$$

We can therefore state the following proposition.

PROPOSITION 3.1. *The function* $\Phi(x, y_1, \ldots, y_n; \gamma) = \Psi(z)$ *is given by*

$$\begin{aligned}\Psi(z) = z^{\frac{\alpha^2 - \Delta - \beta}{2\alpha^2}} \Bigg\{ & c_1\, M\left(\frac{\beta + \Delta - \alpha^2}{2\alpha^2}, \frac{\alpha^2 + \Delta}{\alpha^2}, \frac{c}{\alpha^2\, z}\right) \\ & + c_2\, U\left(\frac{\beta + \Delta - \alpha^2}{2\alpha^2}, \frac{\alpha^2 + \Delta}{\alpha^2}, \frac{c}{\alpha^2\, z}\right)\Bigg\},\end{aligned} \tag{37}$$

in which

$$\Delta := \sqrt{\alpha^4 + (4\gamma - 2\beta)\,\alpha^2 + \beta^2}, \tag{38}$$

$M(\cdot, \cdot, \cdot)$ *and* $U(\cdot, \cdot, \cdot)$ *are confluent hypergeometric functions ([1, p. 504]), and the constants* c_1 *and* c_2 *are uniquely determined from the boundary conditions* $\Psi(k_1) = \Psi(k_2) = 1$.

4. CONCLUDING REMARKS

An exact and explicit expression has been obtained for the expected value of a random variable that represents the useful life of a machine. In the model that was considered, the variable $X(t)$ that denotes the wear of the machine at time t was defined in such a way that it was always increasing with time. To obtain this behaviour, we had to consider a system of two stochastic differential equations in which the derivative of $X(t)$ with respect to time was a deterministic function of another variable, $Y(t)$. The process $\{Y(t), t \geq 0\}$ was assumed to be a geometric Brownian motion, which always remains positive. We could have used another diffusion process that cannot become negative. We could also have used a Gaussian process, such has a Wiener process or an Ornstein-Uhlenbeck process, but with a reflecting boundary at the origin.

If we had worked in discrete time instead, then we could have considered a one-dimensional Markov chain model for the stochastic process $\{X_n, n = 0, 1, \ldots\}$ such that X_n increases with n. To obtain the expected value of the random variable that corresponds to $T(x, y)$, we would have to solve a difference equation, subject to the appropriate boundary condition.

Finally, a related problem for an $(n + 1)$-dimensional diffusion process was considered and solved explicitly. This problem generalizes the one in [4]. We could next compute the expected value of the first-passage time variable defined in this generalization.

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