RANDOM WALKS ON INFINITE TREES

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Some conditions for recurrence or transience of a random walk, with the transition probabilities matrix P, on an infinite tree T are obtained. Then defining a biexcessive function associated with the random walk as a function $q(x)$ on (T, P) such that $-\Delta q(x)$ is an excessive function, a representation for positive (with respect to a specific order) biexcessive functions and some of its consequences are given.

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1. INTRODUCTION

In an infinite tree T, a random walk is defined by the matrix $P = \{p(x,y)\}$ of transition probabilities. Fixing a state e in (T, P) if we define $|x|$ as the length of the shortest path from e to x, then for any state x, $|x| = n$, there is only one neighbour state $x^{\sim}$ such that $|x^{\sim}| = n-1$. We show here that some restrictions on the reverse probabilities $p(x, x^{\sim})$ determine the recurrence or the transience of the random walk. For a real-valued function $f(x)$ on T, the Laplacian is

$$\Delta f(x) = Pf(x) - f(x) = \sum_{y \sim x} p(x,y) f(y) - f(x).$$

Write $f \succ g$ if $f \geq g$ and $-\Delta f \geq -\Delta g$. Recall that the function $f(x)$ is excessive if $Pf(x) \leq f(x)$. Then defining a biexcessive function on T as a function $u(x)$ such that $-\Delta u(x)$ is excessive, we obtain a unique representation for biexcessive functions $u \succ 0$. Some properties of such biexcessive functions with respect to random walks on homogeneous trees are derived at the end.

2. PRELIMINARIES

An infinite graph considered here consists of a countable infinite number of vertices and a countable number of edges; two vertices x, y are said to be neighbours, noted $x \sim y$, if and only if there is an edge connecting x, y; if $x \sim x$, then there is a self-loop at x; the graph X is said to be connected if for any two vertices x, y there is a path $\{x = x_0, x_1, x_2, ., x_n = y\}$ connecting $x, y, x_i \sim x_{i+1}, 0 \leq i \leq n-1$; the graph is said to be locally finite if every vertex has only a finite number neighbours; a vertex is said to be a terminal vertex if it has only one neighbour; the graph is a tree if there is no closed path in it, that is there is no path with distinct vertices of the form $\{x = x_1, x_2, ., x_n = x\}$ for $n > 3$; a tree is a homogeneous tree of order q if every vertex has only q neighbours; a transition matrix $t = \{t(x, y)\}$ on X is defined by the conditions $t(x, y) \geq 0$ for any pair of vertices x, y and $t(x, y) > 0$ if and only if $x \sim y$; the transition values $t(x, y)$ and $t(y, x)$ need not be the same.

An infinite network (X, t) is an infinite graph that is a locally finite, connected graph without self-loops, having an associated transition matrix $t = \{t(x, y)\}$. If $u(x)$ is a real-valued function on an infinite network (X, t), then the Laplacian $\Delta u(x) = \sum_{y \sim x} t(x, y)[u(y) - u(x)]$; the function $u(x)$ is said to be superharmonic at the vertex x if $\Delta u(x) \leq 0$ and is harmonic at the vertex x if $\Delta u(x) = 0$. A non-negative superharmonic function $p(x)$ on X is termed a potential if for any harmonic function $h(x)$ such that $0 \leq h(x) \leq p(x)$ we necessarily have $h = 0$. If there exists a potential $p(x) > 0$ on X, then we say (see [1]) that (X, t) is a hyperbolic network, otherwise it is called a parabolic network. If $s(x)$ is a non-negative superharmonic function on the infinite network, then we have the unique Riesz decomposition $s(x) = p(x) + h(x)$ where $p(x)$ is a potential and $h(x)$ is a non-negative harmonic function. From this, using a form of the minimum principle (that is, if a superharmonic function attains its minimum at a vertex in X then it is a constant) we conclude that in a parabolic network every non-negative superharmonic function is constant.

The term random walk here signifies an infinite network (X, P) where X is the state space and $P = \{p(x, y)\}$ is the probability transition matrix whose entries $p(x, y)$ denote the transition probabilities from state x to state y with $\sum_{y \sim x} p(x, y) = 1$ for every x in X. Write the matrix $P^n = P.P^{n-1}$ inductively with $P^0 = I$. In this framework, a function $f(x)$ is said to be excessive (Woess [3]) if

$$Pf(x) = \sum_y p(x, y) f(y) \leq f(x).$$

Since $Pf(x) - f(x) = \Delta_P f(x)$ where Δ_P is the Laplacian of the network (X, P), then a real-valued function is excessive if and only if it is superharmonic. Now define for any two states x, y the Green kernel $G(x, y) = \sum_{n=0}^{\infty} p^n(x, y)$ where $p^n(x, y)$ is the entry in the matrix P^n. Note that $G(x, y)$ is finite for every pair x, y or ∞ for any pair since the matrix is irreducible (that is, the network

(X, P) is connected). Since $p^n(x, y)$ can be considered as the probability of the walker starting at the state space y reaches the state space x in n steps, when $G(x, y)$ is finite, it is considered as the expected number of visits the walker makes to x starting from y in a reversible random walk (recall that a random walk is said to be reversible if there exists a function $\varphi(x) > 0$ such that $\varphi(x)p(x, y) = \varphi(y)p(y, x)$ for any two states x, y). When it is the case that $G(x, y)$ is finite, the random walk (X, P) is called a transient walk; when $G = \infty$, the random walk is called a recurrent walk. Thus in a recurrent walk, the walker starting at y visits x infinitely often. Remarking that for a fixed state e, the function $G_e(x) = G(x, e)$ is excessive, we are led to the conclusion that the random walk (X, P) is transient if and only if it is a hyperbolic network.

3. RANDOM WALKS ON INFINITE TREES

In this section are given some conditions on the reverse transition probabilities of random walks on trees which when satisfied will show whether the walks are recurrent or transient.

Let (T, P) be a random walk where T is an infinite tree and $P = \{p(x, y)\}$ is the matrix of transition probabilities. For a fixed state e in T and any state $x \in T$, let $|x| = d(e, x)$ denote the length of the shortest path connecting e to x. Remark that for any vertex $x, |x| = n \geq 1$, all its neighbours $\{z_i\}$ are such that $|z_i| = n + 1$ except one $x^{\sim}$ for which $|x^{\sim}| = n - 1$.

THEOREM 3.1. *Let (T, P) be a random walk on an infinite tree with a state e fixed. If $p(x, x^{\sim}) \geq \frac{\alpha}{\alpha+1}$ for some $\alpha > 1$ and every $x, |x| \geq m$ for some integer m, then the random walk is recurrent.*

Proof. Let $s(x) = \alpha^n$ if $|x| = n$. For $n \geq m$, when $|x| = n$,

$$\begin{aligned} \Delta s(x) &= p(x, x^{\sim})[\alpha^{(n-1)} - \alpha^n] + [1 - p(x, x^{\sim})][\alpha^{(n+1)} - \alpha^n] \\ &= \alpha^{(n-1)}(\alpha - 1)[-p(x, x^{\sim})(\alpha + 1) + \alpha] \\ &\leq \alpha^{(n-1)}(\alpha - 1)[-\alpha + \alpha] = 0. \end{aligned}$$

Hence $s(x)$ is superharmonic at every vertex $x, |x| \geq m$, and tends to infinity at the vertex at infinity (that is, when $|x| \to \infty$). Hence by [2, Theorem 3.2], (T, P) is parabolic (recurrent). $\square$

Examples of recurrent networks:

(1) Let $X = \{..., -2, -1, 0, 1, 2, ...\}$ be a random walk determined by the matrix $P = \{p(x, y)\}$ where $p(n, n + 1) = p(-n, -n - 1) = p$ for all $n \geq 0$ and the value of any of the other transition probabilities is q where $p + q = 1$. Suppose $q \geq \frac{\alpha}{\alpha+1}, \alpha > 1$. Then the random walk (X, P) is recurrent.

We know that the standard random walk with $p = q = \frac{1}{2}$ is recurrent. In fact, if (T, P) is a random walk on an infinite tree T and if $p(x, x^{\sim}) \geq \frac{1}{2}$ for every state x, then the walk is recurrent. Consider the function $s(x) = n$, if $|x| = n$. Note that for $x, |x| \geq 1, \Delta s(x) = p(x, x^{\sim})[(n-1)-n]+[1-p(x, x^{\sim})][(n+1)-n] \leq 0$. Thus $s(x) > 0$ is superharmonic at every state $x, |x| \geq 1$, and tends to infinity when $|x| \to \infty$. Hence the random walk is recurrent.

(2) Let $X = \{0, 1, 2, ...\}$ be a random walk determined by $p(0, 1) = 1$; and if $n \geq 1$, $p(n, n-1) = q_n$, and $p(n, n+1) = 1 - q_n$. Suppose $q_n \geq \frac{\alpha}{\alpha+1}, \alpha \geq 1$. Then (X, P) is recurrent.

Theorem 3.2. *Let (T, P) be a random walk on an infinite tree with fixed vertex e. If $p(x, x^{\sim}) \leq \frac{1}{\alpha+1}$ for $|x| \geq 1, \alpha > 1$, then the random walk is transient.*

Proof. Consider the function $g(x) = \alpha^{-n}, |x| = n \geq 0$. Then for $|x| = n \geq 1$,

$$\begin{aligned} \Delta g(x) &= p(x, x^{\sim})[\alpha^{-n+1} - \alpha^{-n}] + [1 - p(x, x^{\sim})][\alpha^{-n-1} - \alpha^{-n}] \\ &= (\alpha - 1)\alpha^{-n-1}[(\alpha + 1)p(x, x^{\sim}) - 1] \leq 0. \end{aligned}$$

Hence $g(x)$ is superharmonic at every state $x, |x| \geq 1$. When $x = e, \Delta g(e) = (\alpha^{-1} - 1) < 0$. Hence $g(x) > 0$ is a non-harmonic superharmonic function on T, so that (T, P) represents a transient random walk. □

Corollary 3.3. *If (T, P) represents the random walk on a homogeneous tree T of order $(q + 1)$, with $p(x, y) = \frac{1}{q+1}$ for all pairs, then the random walk is transient if $q \geq 2$.*

Proof. Here $q \geq 2$ and $p(x, x^{\sim}) = \frac{1}{q+1} \leq \frac{1}{3}$. Hence taking $\alpha = 2$ in the above theorem, we conclude that the random walk (T, P) is transient. □

4. BIPOTENTIALS AND BIHARMONIC POTENTIALS ON INFINITE NETWORKS

A function u in the context of a random walk on an infinite network is called a biexcessive function if $-\Delta u(x)$ is an excessive function. We study in this section the properties of such functions on infinite homogeneous trees, among other related developments in infinite networks.

To start, let us introduce on the space of real-valued functions on an infinite network (X, t) an order $\succ$ to say that $f \succ g$ if there exists a non-negative superharmonic function s such that $f = g + s$. Consequently $f \succ g$ if and only if $f \geq g$ and $-\Delta f \geq -\Delta g$. In particular, $u \succ 0$ denotes that $u \geq 0$ is superharmonic.

Definition 4.1. (i) A real-valued function u on X is called bisuperharmonic (or biexcessive in the context of random walks) if $-\Delta u = v$ is superharmonic (respectively, excessive) on X.

(ii) A potential Q on X is called a bipotential if $-\Delta Q = p$ is also a potential.

(iii) A potential B on X is called a biharmonic potential if $-\Delta B = h$ is harmonic on X.

THEOREM 4.2. *Let $u \succ 0$ be a bisuperharmonic function on an infinite network (X, t). Then u is the unique sum of a bipotential, a biharmonic potential and a nonnegative harmonic function on X.*

Proof. If the network is parabolic, then the theorem is a trivial statement. Take the network as hyperbolic. Since $u \succ 0$, it is a superharmonic function; since u is bisuperharmonic also, then $-\Delta u = s$ is superharmonic and $s \geq 0$. Hence s is the unique sum of a potential p and a harmonic function $h \geq 0$. Now $u \geq 0$ being superharmonic, by using the Green function $G_y(x) = G(x, y)$ we have a representation for u as follows:

$$\begin{aligned} u(x) &= \sum_y G_y(x)[-\Delta u(y)] + (\text{a harmonic function } v \geq 0) \\ &= \sum_y G_y(x) s(y) + v(x) \\ &= \sum_y G_y(x)[p(y) + h(y)] + v(x) \\ &= Q(x) + B(x) + v(x), \end{aligned}$$

where $Q(x)$ is a potential, $-\Delta Q = p$; and $B(x)$ is a potential,$-\Delta B = h$. Thus $u(x)$ is the sum of a bipotential $Q(x)$, a biharmonic potential $B(x)$ and a nonnegative harmonic function $v(x)$. The uniqueness of the decomposition of u is proved using the Laplacian operator. □

It is possible that there does not exist any positive bipotential on (X, t) even if positive potentials exist on X (as in $\mathbb{R}^3$ or $\mathbb{R}^4$ in the continuous case). However if there exists a positive biharmonic potential on X, then there exist bipotentials on X.

PROPOSITION 4.3. *Let $B(x)$ be a positive biharmonic potential on X. Then there exist bipotentials on X.*

Proof. Let $-\Delta B(x) = h(x)$ where $h > 0$ is harmonic. Take a non-empty finite set A. Then $p(x) = R_h^A(x) = \inf\{s(x) : s > 0$ is superharmonic on $X, s \geq$

h on $A\}$ is a potential, $R_h^A(x) \leq h(x)$. (See [1, Theorem 3.1]). Now,

$$\begin{aligned} B(x) &= \sum_y G_y(x)[-\Delta B(y)] \\ &= \sum_y G_y(x)h(y) \\ &\geq \sum_y G_y(x)p(y) = Q(x). \end{aligned}$$

Then $Q(x)$ is a potential and $-\Delta Q(x) = p(x)$ is also a potential. Hence $Q(x)$ is a bipotential on X. □

COROLLARY 4.4. *There are positive bipotentials on X if and only if there exists a non-harmonic bisuperharmonic function $u \succ 0$ on X.*

Proof. Suppose $u \succ 0$ is bisuperharmonic but not harmonic. Write using Theorem 4.2, $u = Q + B + v$. Since u is not harmonic, both Q and B cannot be 0. If $Q \neq 0$, the corollary is valid; if $Q = 0$, then by the above proposition, there exist bipotentials on X. On the other hand, every positive bipotential is a bisuperharmonic function $Q \succ 0$ that is not harmonic. □

To give an example of a network on which bipotentials exist but no positive biharmonic potentials exist (as in $\mathbb{R}^n, n \geq 5$, in the continuous case) we shall turn to the standard random walk (T, P) on a homogeneous tree T of order $(q+1), q \geq 2$, and $p(x,y) = \frac{1}{q+1}$ for any two neighbouring states. Fix a state e in T and, as before, write $|x| = d(e,x)$. Note that there are $(q+1)q^{n-1}$ states with $|x| = n \geq 1$. When $|x| = n \geq 1$, the state x has q neighbours $\{z_i\}, |z_i| = n+1$ and one neighbour $x^\sim, |x^\sim| = n-1$. We have now the following lemma stating a mean-value property for harmonic functions on T.

LEMMA 4.5. *Let $h(x)$ be a harmonic function with respect to the random walk on (T, P) where T is the standard homogeneous tree of order $(q+1)$, and $p(x,y) = \frac{1}{(q+1)}$ for any pair of neighbouring states. Then $h(e) = \frac{1}{(q+1)q^{n-1}} \sum_{|x|=n} h(x)$.*

Proof. By induction; assume the equality up to n. Now for $|x| = n$,

$$h(x) = \frac{1}{(q+1)} h(x^\sim) + \sum_i \frac{1}{(q+1)} h(z_i).$$

Consequently, $\sum_{|x|=n} h(x) = \frac{q}{(q+1)} \sum_{|x|=n-1} h(x) + \frac{1}{(q+1)} \sum_{|x|=n+1} h(x)$, since each $x^\sim$ has q neighbours $y_i, |y_i| = n$. Now use the induction hypothesis to get

$$(q+1)q^{n-1}h(e) = \frac{q}{(q+1)}[(q+1)q^{n-2}h(e)] + \frac{1}{(q+1)} \sum_{|x|=n+1} h(x).$$

Hence $\frac{1}{(q+1)}\sum_{|x|=n+1} h(x) = [(q+1)q^{n-1} - q^{n-1}]h(e) = q^n h(e)$.
That is, $h(e) = \frac{1}{(q+1)q^n}\sum_{|x|=n+1} h(x)$. □

THEOREM 4.6. *For a random walk on the homogeneous tree (T, P) of order $(q+1), q \geq 2$, with constant transition probabilities $\frac{1}{(q+1)}$ for the neighbouring states, there are bipotentials on T but no positive biharmonic potentials.*

Proof. i) If $p(x) = q^{-n}, |x| = n$, it is easy to check that $\Delta p(x) = 0$ for every $x \neq e$ and $\Delta p(e) = (q^{-1} - 1) < 0$ so that $p(x) > 0$ is superharmonic on T. Since $p(x) \to 0$ when $|x| \to \infty$, then $p(x)$ is a potential on (T, P) which is harmonic at every state $x \neq e$. Consequently, if $G(x, y)$ is the Green kernel of (T, P), then $G(x, e) = \frac{q}{(q-1)}p(x) = \frac{1}{q^{n-1}(q-1)}$ if $|x| = n$. Then

$$\begin{aligned}
\sum_x G(x,e)p(x) &= G(e,e)p(e) + \sum_n [\sum_{|x|=n} G(x,e)p(x)] \\
&= G(e,e) + \sum_n [\sum_{|x|=n} \frac{1}{q^{n-1}(q-1)} \times \frac{1}{q^n}] \\
&= G(e,e) + \sum_n (q+1)q^{n-1} \times \frac{1}{q^{n-1}(q-1)} \times \frac{1}{q^n} \\
&= G(e,e) + \sum_n \frac{q+1}{q-1} \times \frac{1}{q^n} \\
&< \infty \text{ since } q \geq 2.
\end{aligned}$$

Consequently, $Q(y) = \sum_x G(x, y)p(x)$ is a potential such that $-\Delta Q = p$; that is Q is a bipotential function on (T, P).

ii) Now to show there is no positive biharmonic potential on (T, P): Suppose $h > 0$ is harmonic on (T, P). If there exists a potential $B(x)$ such that $-\Delta B(x) = h(x)$, then

$$\begin{aligned}
B(e) &= \sum_x G(x,e)h(x) \\
&= G(e,e)h(e) + \sum_{n\geq 1} [\sum_{|x|=n} G(x,e)h(x)] \\
&= h(e)G(e,e) + \sum_{n\geq 1} [\sum_{|x|=n} \frac{1}{q^{n-1}(q-1)} h(x)] \\
&= h(e)G(e,e) + \sum_{n\geq 1} [\frac{1}{q^{n-1}(q-1)} \sum_{|x|=n} h(x)] \\
&= h(e)G(e,e) + \sum_{n\geq 1} [\frac{1}{q^{n-1}(q-1)} (q+1)q^{n-1}h(e)]
\end{aligned}$$

$$= \quad h(e)G(e,e) + \sum_{n\geq 1} \frac{q+1}{q-1} h(e) = \infty,$$

a contradiction. Hence there is no positive biharmonic potential on (T, P). $\square$

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