PRERADICALS IN MODULES AND MORITA CONTEXTS

A.I. KASHU

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Let $(R, {}_RU_S, {}_SV_R, S)$ be a Morita context. We study the connection between the classes of preradicals $\mathbb{PR}(R)$ and $\mathbb{PR}(S)$ of the categories of modules R-Mod and S-Mod. Two mappings between $\mathbb{PR}(R)$ and $\mathbb{PR}(S)$ are constructed from the functors $Hom_R(U,\text{-})$ and $Hom_S(V,\text{-})$. We investigate several properties of these mappings, in particular their behavior relative to the order, intersection and heredity. Similar methods were used earlier for preradicals in an adjoint situation.

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1. INTRODUCTION. PRELIMINARY DEFINITIONS AND FACTS

The purpose of this article is the elucidation of relations between the preradicals of two module categories R-Mod and S-Mod in the case when there is given an arbitrary Morita context $(R, {}_RU_S, {}_SV_R, S)$. Then the functors $H^U = Hom_R(U,\text{-})$ and $H^V = Hom_S(V,\text{-})$ between these categories are considered and two mappings $\mathbb{PR}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{PR}(S)$ are defined by them between the classes of preradicals of the categories R-Mod and S-Mod.

The main results of this work show the properties of these mappings, in particular the preservation of order relations, of intersections and of heredity of preradicals. The special cases of preradicals defined by the trace ideals of the given context are studied. Some similar results were proved earlier for adjoint functors [4, 8, 9]. The torsions in Morita contexts are studied in [5–8].

Now we shall describe more precisely the notions and the situation to be studied. Let R be a ring with unity and R-Mod be the category of unitary left R-modules. The homomorphisms of left R-modules are written on the right of elements. The product (composition) of R-morphisms $f : M \to N$ and

$g: N \to P$ is denoted as $f \cdot g: M \to P$, where $(m)(f \cdot g) \stackrel{\text{def}}{=\!=} (mf)\,g$ for every $m \in M$.

A *preradical* r of R-Mod is a subfunctor of the identity functor $\mathbb{1}_R:$ R-Mod $\to R$-Mod, i.e. r is a function which separates in every module $M \in R$-Mod a submodule $r(M) \subseteq M$ such that $[r(M)]f \subseteq r(M')$ for every R-morphism $f: M \to M'$ of R-Mod. We denote by $\mathbb{PR}(R)$ the class of all preradicals of the category R-Mod [1,4].

The Morita context $(R, {}_RU_S, {}_SV_R, S)$ consists of two rings R and S, and of two bimodules ${}_RU_S$ and ${}_SV_R$ with bimodular morphisms:

$$(,): U \otimes_S V \to R, \qquad [,]: V \otimes_R U \to S,$$

which satisfy the relations of associativity:

$$(u,v)\,u' = u\,[v,u'], \qquad [v,u]\,v' = v\,(u,v')$$

for $u, u' \in U$ and $v, v' \in V$ [2,3,10,11].

The images of these morphisms $I = (U,V) \lhd R$ and $J = [V,U] \lhd S$ are ideals in R and S, respectively, and are called the *trace ideals* of the given Morita context.

The notion of Morita context is widely used in algebra with diverse goals and has its origin in the investigations of K. Morita on equivalence of module categories. Each bimodule defines a pair of adjoint functors. The preradicals in an adjoint situation are studied, in particular, in [4,8,9].

For Morita contexts in [5,6,8] a bijection is shown between two sublattices of the lattices of torsions of R-Mod and S-Mod. It is defined by the action of the functors H^U and H^V to the injective cogenerators of torsions.

In the present paper, we will analyze the connection between the preradicals of the categories R-Mod and S-Mod, determined by the pair of functors:

$$R\text{-Mod} \;\underset{H^V = Hom_S(V,\text{-})}{\overset{H^U = Hom_R(U,\text{-})}{\rightleftarrows}}\; S\text{-Mod},$$

where $(R, {}_RU_S, {}_SV_R, S)$ is an arbitrary Morita context. The pair (H^U, H^V) is accompanied by two natural transformations:

$$\varphi: \mathbb{1}_{R-\text{Mod}} \to H^V H^U, \qquad \psi: \mathbb{1}_{S-\text{Mod}} \to H^U H^V,$$

which are defined as follows. For every module $X \in R$-Mod we have the R-morphism $\varphi_X: X \to H^V H^U(X)$ determined by the rule:

$$u\left(v(x\,\varphi_X)\right) \stackrel{\text{def}}{=\!=} (u,v)\,x, \tag{1.1}$$

where $x \in X$, $v \in V$, $u \in U$. Similarly, for every module $Y \in S$-Mod the S-morphism $\psi_Y: Y \to H^U H^V(Y)$ is defined by the rule:

$$v\left(u(y\,\psi_Y)\right) \stackrel{\text{def}}{=\!=} [v,u]\,y, \tag{1.2}$$

where $y \in Y$, $u \in U$, $v \in V$.

The natural transformations φ and ψ are compatible with the functors H^U and H^V in the following sense. For every module $X \in R$-Mod we have the relation:

$$H^U(\varphi_X) \;=\; \psi_{H^U(X)}. \tag{1.3}$$

Similarly, for every module $Y \in S$-Mod the following holds:

$$H^V(\psi_Y) \;=\; \varphi_{H^V(Y)}. \tag{1.4}$$

2. MAPPINGS BETWEEN THE CLASSES OF PRERADICALS

Let $(R,\, {}_RU_S,\, {}_SV_R, S)$ be an arbitrary Morita context. We consider the functors $R\text{-Mod} \underset{H^V=Hom_S(V,\text{-})}{\overset{H^U=Hom_R(U,\text{-})}{\rightleftarrows}} S\text{-Mod}$ with the natural transformations $\varphi:\ \mathbb{1}_{R-\mathrm{Mod}} \to H^V H^U$ and $\psi:\ \mathbb{1}_{S-\mathrm{Mod}} \to H^U H^V$, defined by the rules (1.1) and (1.2). In this situation, we will define two mappings:

$$\mathbb{PR}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{PR}(S)$$

between the classes of preradicals of categories R-Mod and S-Mod.

The mapping $r \rightsquigarrow r^*$ from $\mathbb{PR}(R)$ to $\mathbb{PR}(S)$ is defined as follows. Let $r \in \mathbb{PR}(R)$ and $Y \in S$-Mod. Applying H^V and using r, we obtain in R-Mod the exact sequence:

$$0 \to r\big(H^V(Y)\big) \xrightarrow[\subseteq]{i^r_{H^V(Y)}} H^V(Y) \xrightarrow[\text{nat}]{\pi^r_{H^V(Y)}} H^V(Y)/r\big(H^V(Y)\big) \to 0,$$

where $i^r_{H^V(Y)}$ is the inclusion and $\pi^r_{H^V(Y)}$ is the natural epimorphism. Further, we apply the functor H^U, adding the morphism ψ_Y:

$$\begin{tikzcd}
 & & Y \arrow[d, "\psi_Y"] \arrow[drr, dashed] & & \\
0 \arrow[r] & H^U\Big(r\big(H^V(Y)\big)\Big) \arrow[r, "H^U\left(i^r_{H^V(Y)}\right)"] & H^U H^V(Y) \arrow[rr, "H^U\left(\pi^r_{H^V(Y)}\right)"] & & H^U\left[H^V(Y)/r\big(H^V(Y)\big)\right].
\end{tikzcd}$$

Definition. For every preradical $r \in \mathbb{PR}(R)$ we define in S-Mod the function r^* by the rule:

$$r^*(Y) \stackrel{\text{def}}{=\!=} \mathrm{Ker}\,[\psi_Y \cdot H^U(\pi^r_{H^V(Y)})] \tag{2.1}$$

for every module $Y \in S$-Mod.

The other form of the module $r^*(Y)$ is indicated in

LEMMA 2.1. $r^*(Y) = \{y \in Y \,|\, U(y\,\psi_Y) \subseteq r(H^V(Y))\}$.

Proof. By the definition we have:

$$y \in r^*(Y) \Leftrightarrow \forall u \in U, \quad u(y\,\psi_Y) + r(H^V(Y)) = \overline{0} \Leftrightarrow \forall u \in U,$$

$$u(y\,\psi_Y) \in r(H^V(Y)) \Leftrightarrow U(y\,\psi_Y) \subseteq r(H^V(Y)). \quad \square$$

REMARK. If $y \in r^*(Y)$, then $[V,U]\,y \subseteq V(r(H^V(Y)))$.

Indeed, by Lemma 2.1:

$$y \in r^*(Y) \Rightarrow U(y\,\psi_Y) \subseteq r(H^V(Y)) \Rightarrow V(U(y\,\psi_Y)) \subseteq V(r(H^V(Y))).$$

Now by the definition of ψ (see (1.2)) we obtain: $V(U(y\,\psi_Y)) = [V,U]\,y$, therefore $[V,U]\,y \subseteq V(r(H^V(Y)))$.

Further, the exactness of the sequence from the previous diagram permits us to show the other possibility to express the function r^*.

LEMMA 2.2. *For every preradical $r \in \mathbb{PR}(R)$ and every $Y \in S$-Mod the following relation holds:*

$$r^*(Y) = [Im\, H^U(i^r_{H^V(Y)})]\,\psi_Y^{-1}. \tag{2.2}$$

Proof. Since $Im\, H^U(i^r_{H^V(Y)}) = \mathrm{Ker}\, H^U(\pi^r_{H^V(Y)})$, we have:

$$\begin{aligned}[Im\, H^U(i^r_{H^V(Y)})]\,\psi_Y^{-1} &= [\mathrm{Ker}\, H^U(\pi^r_{H^V(Y)})]\,\psi_Y^{-1} \\ &= \mathrm{Ker}\,[\psi_Y \cdot H^V(\pi^r_{H^V(Y)})] \overset{\mathrm{def}}{=\!=} r^*(Y). \quad \square\end{aligned}$$

THEOREM 2.3. *For every preradical $r \in \mathbb{PR}(R)$ the function r^* defined by the rule* (2.1) (*or* (2.2)) *is a preradical of the category S-Mod.*

Proof. Let $r \in \mathbb{PR}(R)$ and $g : Y \to Y'$ be an arbitrary morphism of S-Mod. Using H^V and r we obtain in R-Mod the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & r(H^V(Y)) \arrow[r, "i^r_{H^V(Y)}", "\subseteq"'] \arrow[d, dashed, "r(H^V(g))"] & H^V(Y) \arrow[r, "\pi^r_{H^V(Y)}", "\mathrm{nat}"'] \arrow[d, "H^V(g)"] & H^V(Y)/r(H^V(Y)) \arrow[r] \arrow[d, dashed, "(1/r)(H^V(g))"] & 0 \\
0 \arrow[r] & r(H^V(Y')) \arrow[r, "i^r_{H^V(Y')}", "\subseteq"'] & H^V(Y') \arrow[r, "\pi^r_{H^V(Y')}", "\mathrm{nat}"'] & H^V(Y')/r(H^V(Y')) \arrow[r] & 0,
\end{tikzcd}$$

where $r(H^V(g))$ is the restriction of $H^V(g)$ (by the definition of preradical), which implies the morphism $(1/r)(H^V(g))$.

Now by H^U and ψ we have in S-Mod the commutative diagram:

$$\begin{tikzcd}
r^*(Y) \arrow[r, "\subseteq"'] \arrow[d, dashed, "\overline{g}"] & Y \arrow[r, "\psi_Y"] \arrow[d, "g"] & H^U H^V(Y) \arrow[rr, "H^U(\pi^r_{H^V(Y)})"] \arrow[d, "H^U H^V(g)"] & & H^U\big[H^V(Y)/r\big(H^V(Y)\big)\big] \arrow[d, dashed, "H^U[(1/r)(H^V(g))]"] \\
r^*(Y') \arrow[r, "\subseteq"'] & Y' \arrow[r, "\psi_{Y'}"] & H^U H^V(Y') \arrow[rr, "H^U(\pi^r_{H^V(Y')})"] & & H^U\big[H^V(Y')/r\big(H^V(Y')\big)\big].
\end{tikzcd}$$

By the definition of $r^*(Y)$ it is clear that $\big(r^*(Y)\big)\big[\psi_Y \cdot H^U(\pi^r_{H^V(Y)})\big] = 0$, therefore $\big(r^*(Y)\big)\big[\psi_Y \cdot H^U(\pi^r_{H^V(Y)}) \cdot H^U[(1/r)\,(H^V(g))]\big] = 0$. Since the diagram commutes, we obtain $\big(r^*(Y)\big)\big[g \cdot \psi_{Y'} \cdot H^U(\pi^r_{H^V(Y')})\big] = 0$. This means that $\big(r^*(Y)\big)g \subseteq \operatorname{Ker}\big[\psi_{Y'} \cdot H^U(\pi^r_{H^V(Y')})\big] \overset{\text{def}}{=\!=} r^*(Y')$ for every $g:\ Y \to Y'$. Therefore r^* is a preradical of S-Mod. $\square$

In that way for the given Morita context $(R,\ {}_RU_S,\ {}_SV_R, S)$ the mapping $\mathbb{PR}(R) \xrightarrow{\ (-)^*\ } \mathbb{PR}(S)$ is defined by the functors H^U and H^V, using the natural transformation ψ. In a completely similar manner we can define the inverse mapping $\mathbb{PR}(S) \xrightarrow{\ (-)^*\ } \mathbb{PR}(R)$ with the help of the same functors H^U and H^V, using the natural transformation φ.

Namely, if $s \in \mathbb{PR}(S)$ and $X \in R$-Mod, then in S-Mod we have the sequence:

$$0 \to s\big(H^U(X)\big) \xrightarrow[\subseteq]{i^s_{H^U(X)}} H^U(X) \xrightarrow[\text{nat}]{\pi^s_{H^U(X)}} H^U(X)/s\big(H^U(X)\big) \to 0.$$

Applying H^V and adding φ_X we obtain in R-Mod the situation:

$$\begin{tikzcd}
 & & X \arrow[d, "\varphi_X"] \arrow[dr, dashed] & \\
0 \arrow[r] & H^V\Big(s\big(H^U(X)\big)\Big) \arrow[r, "H^V\left(i^s_{H^U(X)}\right)"] & H^V H^U(X) \arrow[r, "H^V\left(\pi^s_{H^U(X)}\right)"] & H^V[H^U(X)/s\big(H^U(X)\big)].
\end{tikzcd}$$

We define the function s^* by the rule:

$$s^*(X) \overset{\text{def}}{=\!=} \operatorname{Ker}\big[\varphi_X \cdot H^V\big(\pi^s_{H^U(X)}\big)\big], \tag{2.3}$$

or

$$s^*(X) \overset{\text{def}}{=\!=} \big[Im\, H^V\big(i^s_{H^U(X)}\big)\big]\, \varphi_X^{-1}. \tag{2.4}$$

From the complete symmetry of the studied situation and of used methods, it is clear that the function s^* is a preradical of R-Mod.

Thus we have two mappings $\mathbb{PR}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{PR}(S)$ which are completely similar and this fact permits us to investigate only one of them, transferring without proofs the results from one mapping to the other.

3. PARTICULAR CASES

With the purpose of illustrating the previous constructions, we will show now the action of the defined mappings in some particular cases: for the extreme (trivial) preradicals $\mathbb{0}$ and $\mathbb{1}$, and also for the preradicals defined by the trace ideals $I \lhd R$ and $J \lhd S$ of the given Morita context. The ideal $I = (U, V)$ of R defines the preradical (pretorsion) $r_{(I)}$ of R-Mod by the rule:

$$r_{(I)}(X) = \{x \in X \mid Ix = 0\} = \operatorname{Ker} \varphi_X,$$

for every $X \in R$-Mod ([10]). Similarly, the ideal $J = [V, U]$ of S defines in S-Mod the preradical (pretorsion) $r_{(J)}$ such that:

$$r_{(J)}(Y) = \{y \in Y \mid Jy = 0\} = \operatorname{Ker} \psi_Y,$$

where $Y \in S$-Mod.

a) Let $r = \mathbb{0}_R$, i.e. $r(X) = 0$ for every $X \in R$-Mod. Then for $Y \in S$-Mod we have $r\big(H^V(Y)\big) = 0$, therefore $H^U\big(\pi^r_{H^V(Y)}\big) = 1_{H^U H^V(Y)}$ and by the definition $r^*(Y) = \operatorname{Ker} \psi_Y$, where $\psi_Y : Y \to H^U H^V(Y)$ and

$$\operatorname{Ker} \psi_Y = \{y \in Y \mid y\, \psi_Y = 0\} = \{y \in Y \mid [V, U]y = 0\} = \{y \in Y \mid Jy = 0\}.$$

This means that $\mathbb{0}_R^* = r_{(J)}$. By symmetry we have: $\mathbb{0}_S^* = r_{(I)}$. We remark that $r_{(J)}$ is the least preradical of the form r^* for some $r \in \mathbb{PR}(R)$.

b) Let $r = \mathbb{1}_R$, i.e. $r(X) = X$ for every $X \in R$-Mod. Then for $Y \in S$-Mod we have $H^V(Y) = r\big(H^V(Y)\big)$, so $\pi^r_{H^V(Y)} = 0$. Therefore $H^U\big(\pi^r_{H^V(Y)}\big) = 0$ and $\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big) = 0$. Then $\operatorname{Ker}\big[\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big)\big] = Y$ and by the definition $r^*(Y) = Y$ for every $Y \in S$-Mod, which means that $\mathbb{1}_R^* = \mathbb{1}_S$. By the symmetry: $\mathbb{1}_S^* = \mathbb{1}_R$.

c) Let $r = r_{(I)} \in \mathbb{PR}(R)$ and we show the corresponding preradical $r_{(I)}^* \in \mathbb{PR}(S)$. For $Y \in S$-Mod by the definition $r_{(I)}^*(Y) = \operatorname{Ker}\big[\psi_Y \cdot H^U\big(\pi^{r_{(I)}}_{H^V(Y)}\big)\big]$, where $\pi^{r_{(I)}}_{H^V(Y)} : H^V(Y) \to H^V(Y) \,/\, r_{(I)}\big(H^V(Y)\big)$ is the natural epimorphism. By the definition of $r_{(I)}$ for every $f \in H^V(Y)$ we have:

$$f \in r_{(I)}\big(H^V(Y)\big) \Leftrightarrow (U, V)\, f = 0. \tag{3.1}$$

Using this fact and the definition of ψ_Y (see (1.2)) we obtain:

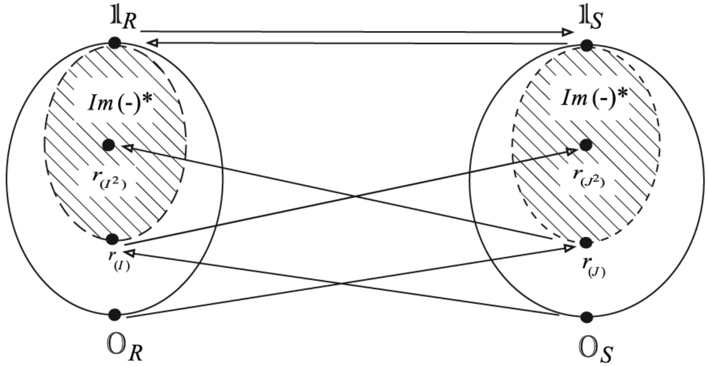
$$r_{(I)}^*(Y) \;=\; \big\{y \in Y \,\big|\, y\, \psi_Y \in \operatorname{Ker} H^U\big(\pi^{r_{(I)}}_{H^V(Y)}\big)\big\} = \big\{y \in Y \,\big|\, y\, \psi_Y \cdot \pi^{r_{(I)}}_{H^V(Y)} = 0\big\}$$

$$
\begin{aligned}
&= \quad \{y \in Y \mid U(y\,\psi_Y) \subseteq \operatorname{Ker} \pi^{r_{(I)}}_{H^V(Y)} = r_{(I)}\big(H^V(Y)\big)\}\\
&\overset{(3.1)}{=\!=} \quad \{y \in Y \mid (U,V)\big(U(y\,\psi_Y)\big) = 0\} = \{y \in Y \big| \big(U[V,U]\big)(y\,\psi_Y) = 0\}\\
&\overset{(1.2)}{=\!=} \quad \{y \in Y \mid \big[V, U[V,U]\big] y = 0\} = \{y \in Y \mid \big([V,U]\cdot[V,U]\big) y = 0\}\\
&= \quad r_{(J^2)}(Y).
\end{aligned}
$$

Therefore ${r_{(I)}}^* = r_{(J^2)}$. Similarly: ${r_{(J)}}^* = r_{(I^2)}$.

Summarizing the four cases mentioned above, now we can represent the general situation as follows:

$$
\mathbb{PR}(R) \quad \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \quad \mathbb{PR}(S)
$$



Proposition 3.1. a) ${\mathbb{0}_R}^* = r_{(J)}$, ${\mathbb{0}_S}^* = r_{(I)}$;
b) ${\mathbb{1}_R}^* = \mathbb{1}_S$, ${\mathbb{1}_S}^* = \mathbb{1}_R$;
c) ${r_{(I)}}^* = r_{(J^2)}$, ${r_{(J)}}^* = r_{(I^2)}$. $\square$

Further we will show the relation between the preradicals $r_{(I)} \in \mathbb{PR}(R)$ and $r_{(J)} \in \mathbb{PR}(S)$. For that we will define a mapping $r \rightsquigarrow r'$ such that to every preradical $r \in \mathbb{PR}(R)$ corresponds a function r' defined on $Im\, H^U \subseteq S$-Mod. Namely, having $r \in \mathbb{PR}(R)$ and $X \in R$-Mod we consider the inclusion $i^r_X:\ r(X) \xrightarrow{\subseteq} X$ and the image $H^U(i^r_X):\ H^U\big(r(X)\big) \to H^U(X)$ in S-Mod. We define the function r' by the rule:

$$
r'\big(H^U(X)\big) \overset{\text{def}}{=\!=} Im\,\big(H^U(i^r_X)\big). \tag{3.2}
$$

Then r' acts in $Im\, H^U \subseteq S$-Mod and separates the submodule $r'\big(H^U(X)\big)$ in $H(X)$ for every $X \in R$-Mod. This function possesses the following property, which shows its concordance with the morphisms of R-Mod.

Proposition 3.2. *For every preradical $r \in \mathbb{PR}(R)$ and every R-morphism $f:\ X \to X'$ the following relation holds*:

$$
\big[r'\big(H^U(X)\big)\big]\big(H^U(f)\big) \subseteq r'\big(H^U(X')\big).
$$

Proof. For $r \in \mathbb{PR}(R)$ and $f: \ X \to X'$ we have the diagram:

$$\begin{tikzcd} r(X) \arrow[r, "i^r_X", "\subseteq"'] \arrow[d, dashed, "\overline{f}"] & X \arrow[d, "f"] \\ r(X') \arrow[r, "i^r_{X'}", "\subseteq"'] & X', \end{tikzcd}$$

where $\overline{f}$ is the restriction of f. Then in S-Mod we obtain the situation:

$$\begin{tikzcd} H^U\big(r(X)\big) \arrow[r] \arrow[rr, bend left, "H^U(i^r_X)"] \arrow[d, dashed, "H^U(\overline{f}\,)"] & Im\, H^U(i^r_X) \arrow[r, "\subseteq"] \arrow[d, dashed, "\overline{H^U(f)}"] & H^U(X) \arrow[d, "H^U(f)"] \\ H^U\big(r(X')\big) \arrow[r] \arrow[rr, bend right, "H^U(i^r_{X'})"] & Im\, H^U(i^r_{X'}) \arrow[r, "\subseteq"] & H^U(X'). \end{tikzcd}$$

Therefore:

$$Im\left[H^U\!\big(i^{\,r}_X \cdot H^U(f)\big)\right] = Im\,[H^U(\overline{f}) \cdot H^U(i^{\,r}_{X'})] \subseteq H^U(i^{\,r}_{X'}),$$

i.e. $\big[Im\, H^U(i^{\,r}_X)\big]\big(H^U(f)\big) \subseteq Im\, H^U(i^{\,r}_{X'})$, which by the definition (see (3.2)) means that $\big[r'\big(H^U(X)\big)\big]\big(H^U(f)\big) \subseteq r'\big(H^U(X')\big)$. $\square$

Let (r, s) be a pair of preradicals, where $r \in \mathbb{PR}(R)$ and $s \in \mathbb{PR}(S)$. We will say that these preradicals are *conjugated* if $r'\big(H^U(X)\big) = s\big(H^U(X)\big)$ and $s'\big(H^V(Y)\big) = r\big(H^V(Y)\big)$ for every $X \in R$-Mod and $Y \in S$-Mod. Then the connection between the preradicals $r_{(I)}$ and $r_{(J)}$ can be expressed as follows.

PROPOSITION 3.3. *The preradicals* $r_{(I)} \in \mathbb{PR}(R)$ *and* $r_{(J)} \in \mathbb{PR}(S)$ *are conjugated, i.e.* $Im\, H^U(i_X^{r_{(I)}}) = r_{(J)}\big(H^U(X)\big)$ *and* $Im\, H^V(i_Y^{r_{(J)}}) = r_{(I)}\big(H^V(Y)\big)$ *for every* $X \in R$*-Mod and* $Y \in S$*-Mod.*

Proof. For $X \in R$-Mod by the definition $r_{(I)}(X) = \operatorname{Ker} \varphi_X$. We consider in R-Mod the exact sequence:

$$0 \longrightarrow \operatorname{Ker} \varphi_X \xrightarrow[\subseteq]{i_X^{r_{(I)}}} X \xrightarrow{\ \varphi_X\ } H^V H^U(X),$$

which implies in S-Mod the exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^U(\operatorname{Ker} \varphi_X) \arrow[r, "H^U(i_X^{r_{(I)}})"] & H^U(X) \arrow[r, shift left, "H^U(\varphi_X)"] \arrow[r, shift right, dashed, "\psi_{H^U(X)}"'] & H^U H^V H^U(X), \end{tikzcd}$$

where $H^U(\varphi_X) = \psi_{H^U(X)}$ (see (1.3)). Therefore:

$$Im\, H^U(i_X^{r_{(I)}}) = \operatorname{Ker} H^U(\varphi_X) = \operatorname{Ker} \psi_{H^U(X)} \stackrel{\text{def}}{=\!=} r_{(J)}\big(H^U(X)\big).$$

By symmetry the second relation of proposition also holds using (1.3). $\square$

For every preradical $r \in \mathbb{PR}(R)$ we denote by

$$\mathcal{P}(r) = \{X \in R\text{-Mod} \mid r(X) = 0\}$$

the class of all r-torsionfree modules. From the last proposition we get the

Corollary 3.4. $H^U(\mathcal{P}(r_{(I)})) \subseteq \mathcal{P}(r_{(J)}), \quad H^V(\mathcal{P}(r_{(J)})) \subseteq \mathcal{P}(r_{(I)})$.

Proof. If $X \in \mathcal{P}(r_{(I)})$, then $r_{(I)}(X) = \operatorname{Ker} \varphi_X = 0$ and $i_X^{r_{(I)}} = 0$. Therefore $Im\, H^U(i_X^{r_{(I)}}) = r_{(J)}\big(H^U(X)\big) = 0$, *i.e.* $H^U(X) \in \mathcal{P}(r_{(J)})$. $\square$

To conclude this section we show the connection between the mappings $r \rightsquigarrow r'$ and $r \rightsquigarrow r^*$ defined above.

Proposition 3.5. *For every preradical $r \in \mathbb{PR}(R)$ and for every module $X \in R$-Mod we have the relation*:

$$r'\big(H^U(X)\big) \subseteq r^*\big(H^U(X)\big).$$

Proof. In the given conditions we consider the diagram:

$$\begin{tikzcd}
X \arrow[rr, "\varphi_X"] & & H^V H^U(X) \\
r(X) \arrow[u, "\cup\!\mid\, i_X^r"'] \arrow[rr, dashed, "\overline{\varphi}_X"] & & r\big(H^V H^U(X)\big). \arrow[u, "\cup\!\mid\, i^r_{H^V H^U(X)}"']
\end{tikzcd}$$

By H^U it implies in S-Mod the situation:

$$\begin{tikzcd}
H^U(X) \arrow[rr, shift left, "H^U(\varphi_X)"] \arrow[rr, shift right, dashed, "\psi_{H^U(X)}"'] & & H^U H^V H^U(X) \\
H^U\big(r(X)\big) \arrow[u, "H^U(i_X^r)"'] \arrow[rr, dashed, "H^U(\overline{\varphi}_X)"] & & H^U\big[r\big(H^V H^U(X)\big)\big], \arrow[u, "H^U(i^r_{H^V H^U(X)})"']
\end{tikzcd}$$

where $H^U(\varphi_X) = \psi_{H^U(X)}$. By the definitions it follows that:

$$r^*\big(H^U(X)\big) \overset{\text{def}}{=\!=} \big[Im\, H^U\big(i^r_{H^V H^U(X)}\big)\big]\psi^{-1}_{H^U(X)},$$

$$r'\big(H^U(X)\big) \overset{\text{def}}{=\!=} Im\, H^U(i_X^r).$$

Since the diagram commutes, we have:

$$\begin{aligned}
Im\,[H^U(i_X^r) \cdot H^U(\varphi_X)] &= [Im\,[H^U(i_X^r) \cdot \psi_{H^U(X)}] \\
&= Im\,\big[H^U(\overline{\varphi}_X) \cdot H^U\big(i^r_{H^V H^U(X)}\big)\big] \subseteq Im\, H^U\big(i^r_{H^V H^U(X)}\big).
\end{aligned}$$

Applying $\psi^{-1}_{H^U(X)}$ we obtain:

$$Im\, H^U(i_X^r) \subseteq \{Im\,[H^U(i_X^r) \cdot \psi_{H^U(X)}]\}\, \psi^{-1}_{H^U(X)} \subseteq \big[Im\, H^U\big(i^r_{H^V H^U(X)}\big)\big]\psi^{-1}_{H^U(X)},$$

which by the definitions means that $r'\big(H^U(X)\big) \subseteq r^*\big(H^U(X)\big)$. $\square$

4. "STAR" MAPPINGS AND ORDER RELATIONS

Now we will verify the behavior of the mappings $\mathbb{PR}(R) \underset{(-)^*}{\overset{(-)^*}{\rightleftarrows}} \mathbb{PR}(S)$ defined above relative to the partial order in the classes of preradicals of R-Mod and S-Mod. Recall that the partial order in $\mathbb{PR}(R)$ is defined as follows:

$$r_1 \leq r_2 \ \Leftrightarrow \ r_1(X) \subseteq r_2(X)$$

for every module $X \in R$-Mod.

Theorem 4.1. *If $r_1, r_2 \in \mathbb{PR}(R)$ and $r_1 \leq r_2$, then in the class $\mathbb{PR}(S)$ we have ${r_1}^* \leq {r_2}^*$, i.e. the mapping $r \rightsquigarrow r^*$ is monotone.*

Proof. Let $Y \in S$-Mod. Then the relation $r_1 \leq r_2$ implies the inclusion $i: \ r_1\big(H^V(Y)\big) \xrightarrow{\subseteq} r_2\big(H^V(Y)\big)$ and we have in R-Mod the situation:

$$\begin{tikzcd}
r_1\big(H^V(Y)\big) \arrow[dr, "i_1", "\subseteq"'] \arrow[dd, dashed, "\cap|"', "i"] & & H^V(Y)/r_1\big(H^V(Y)\big) \arrow[dd, dashed, "\pi"] \\
& H^V(Y) \arrow[ur, "\pi_1", "nat"'] \arrow[dr, "\pi_2", "nat"'] & \\
r_2\big(H^V(Y)\big) \arrow[ur, "i_2", "\subseteq"'] & & H^V(Y)/r_2\big(H^V(Y)\big),
\end{tikzcd}$$

where π is defined by the inclusion i. By H^U and ψ we obtain in S-Mod the diagram:

$$\begin{tikzcd}
& & H^U\big[H^V(Y)/r_1\big(H^V(Y)\big)\big] \arrow[dd, dashed, "H^U(\pi)"] \\
Y \arrow[r, "\psi_Y"] & H^U H^V(Y) \arrow[ur, "H^U(\pi_1)"] \arrow[dr, "H^U(\pi_2)"] & \\
& & H^U\big[H^V(Y)/r_2\big(H^V(Y)\big)\big],
\end{tikzcd}$$

where $\psi_Y \cdot H^U(\pi_1) \cdot H^U(\pi) = \psi_Y \cdot H^U(\pi_2)$, so the kernels of these morphisms coincide. Therefore

$$\mathrm{Ker}\,[\psi_Y \cdot H^U(\pi_1)] \ \subseteq \ \mathrm{Ker}\,[\psi_Y \cdot H^U(\pi_1) \cdot H^U(\pi)] = \mathrm{Ker}\,[\psi_Y \cdot H^U(\pi_2)].$$

By the definition this means that ${r_1}^*(Y) \subseteq {r_2}^*(Y)$ for every $Y \in S$-Mod, *i.e.* ${r_1}^* \leq {r_2}^*$. $\square$

By symmetry we conclude that the inverse mapping $\mathbb{PR}(S) \xrightarrow{(-)^*} \mathbb{PR}(R)$ also is monotone: $s_1 \leq s_2 \Rightarrow {s_1}^* \leq {s_2}^*$.

In connection with the order relations in the classes of preradicals we mention one more fact on the "star" mappings.

Theorem 4.2. *For every preradical $r \in \mathbb{PR}(R)$ the following relation holds*: $r \leq r^{**}$.

Proof. Let $r \in \mathbb{PR}(R)$ and $X \in R$-Mod. By the rule (2.1) for $Y = H^U(X)$ we have $r^*\big(H^U(X)\big) \stackrel{\text{def}}{=\!=} \operatorname{Ker}\big[\psi_{H^U(X)} \cdot H^U(\pi^r_{H^V H^U(X)})\big]$, where the natural epimorphism

$$\pi^r_{H^V H^U(X)} \quad : \quad H^V H^U(X) \longrightarrow H^V H^U(X)/r\big(H^V H^U(X)\big)$$

implies in S-Mod the composition of morphisms:

$$H^U(X) \xrightarrow{\psi_{H^U(X)}} H^U H^V H^U(X) \xrightarrow{H^U(\pi^r_{H^V H^U(X)})} H^U\big[H^V H^U(X)/r\big(H^V H^U(X)\big)\big].$$

Now we apply the mapping $\mathbb{PR}(S) \xrightarrow{(-)^*} \mathbb{PR}(R)$ to the preradical $r^* \in \mathbb{PR}(S)$ and module $X \in R$-Mod (see (2.3) or (2.4)).

Using H^U and r^* we have in S-Mod the natural epimorphism $\pi^{r^*}_{H^U(X)}$: $H^U(X) \longrightarrow H^U(X)/r^*\big(H^U(X)\big)$. By H^V and φ now we obtain in R-Mod the composition of morphisms:

$$X \xrightarrow{\varphi_X} H^V H^U(X) \xrightarrow{H^V(\pi^{r^*}_{H^U(X)})} H^V\left[H^U(X)/r^*\big(H^U(X)\big)\right].$$

By the definition: $r^{**}(X) \stackrel{\text{def}}{=\!=} \operatorname{Ker}\left[\varphi_X \cdot H^V\big(\pi^{r^*}_{H^U(X)}\big)\right]$.

To establish the relation of this module with $r(X)$ we consider in R-Mod the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & r(X) \arrow[r, "i", "\subseteq"'] \arrow[d, dashed, "r(\varphi_X)"] & X \arrow[r, "\pi^r_X", "\text{nat}"'] \arrow[d, "\varphi_X"] & X/r(X) \arrow[r] \arrow[d, dashed, "(1/r)(\varphi_X)"] & 0 \\
0 \arrow[r] & r\big(H^V H^U(X)\big) \arrow[r, "j", "\subseteq"'] & H^V H^U(X) \arrow[r, "\pi^r_{H^V H^U(X)}", "\text{nat}"'] & H^V H^U(X)/r\big(H^V H^U(X)\big) \arrow[r] & 0,
\end{tikzcd}$$

where the lateral morphisms are implied by φ_X and r. Applying H^U we obtain in S-Mod the commutative diagram:

$$\begin{tikzcd}
r^*(H^U(X)) \arrow[r, "\subseteq"] & H^U(X) \arrow[r, shift left, "H^U(\varphi_X)"] \arrow[r, shift right, "\psi_{H^U(X)}"'] \arrow[d, "H^U(\pi^r_X)"] \arrow[dr, dashed, "f"] \arrow[rr, bend left, "\pi^{r^*}_{H^U(X)}"] & H^U H^V H^U(X) \arrow[d, "H^U(\pi^r_{H^V H^U(X)})"] & H^U(X)/r^*(H^U(X)) \arrow[dl, dashed, "g"'] \arrow[d, "\cong"] \\
& H^U(X/r(X)) \arrow[r, dashed, "H^U[(1/r)(\varphi_X)]"] & H^U[H^V H^U(X)/r(H^V H^U(X))] & Im\, f \arrow[l, "\supseteq"'] ,
\end{tikzcd}$$

where by f is denoted the composition $\psi_{H^U(X)} \cdot H^U(\pi^r_{H^V H^U(X)})$, therefore $r^*(H^U(X)) \stackrel{\text{def}}{=\!=} \operatorname{Ker} f$. By the first isomorphism theorem we have $H^U(X)/r^*(H^U(X)) \cong Im\, f$. We denote by g the composition of this isomorphism with the inclusion $Im\, f \subseteq H^U[H^V H^U(X)/r(H^V H^U(X))]$. Then it is clear that $\pi^{r^*}_{H^U(X)} \cdot g = f$ and that g is a monomorphism.

Now we observe that by the relation (2.3) we have $H^U(\varphi_X) = \psi_{H^U(X)}$ and using the commutativity of diagram it follows that:

$$\psi_{H^U(X)} \cdot H^U(\pi^r_{H^V H^U(X)}) = H^U(\varphi_X) \cdot H^U(\pi^r_{H^V H^U(X)}) = H^U(\pi^r_X) \cdot H^U[(1/r)(\varphi_X)],$$

therefore $\pi^{r^*}_{H^U(X)} \cdot g = H^U(\pi^r_X) \cdot H^U[(1/r)(\varphi_X)]$. Using H^V and φ we obtain now in R-Mod the commutative diagram:

$$\begin{tikzcd}
r^{**}(X) \arrow[r, "\subseteq"] & X \arrow[r, "\varphi_X"] \arrow[d, "\pi^r_X"] & H^V H^U(X) \arrow[r, "H^V(\pi^{r^*}_{H^U(X)})"] \arrow[d, "H^V H^U(\pi^r_X)"] & H^V[H^U(X)/r^*(H^U(X))] \arrow[d, dashed, "H^V(g)"] \\
& X/r(X) \arrow[r, "\varphi_{X/r(X)}"] & H^V H^U(X/r(X)) \arrow[r, dashed, "H^V H^U[(1/r)(\varphi_X)]"] & H^V H^U[H^V H^U(X)/r(H^V H^U(X))].
\end{tikzcd}$$

Since $H^V(g)$ is a monomorphism, from the commutativity it follows that:

$$\begin{aligned} r^{**}(X) \;&\stackrel{\text{def}}{=\!=}\; \operatorname{Ker}\left[\varphi_X \cdot H^V(\pi^{r^*}_{H^U(X)})\right] = \operatorname{Ker}\left[\varphi_X \cdot H^V(\pi^{r^*}_{H^U(X)}) \cdot H^V(g)\right] \\ &= \;\operatorname{Ker}\left[\pi^r_X \cdot \varphi_{X/r(X)} \cdot H^V H^U[(1/r)(\varphi_X)]\right] \supseteq \operatorname{Ker} \pi^r_X = r(X). \end{aligned}$$

Thus $r^{**}(X) \supseteq r(X)$ for every $X \in R$-Mod, *i.e.* $r^{**} \geq r$. $\square$

By symmetry we have the relation $s^{**} \geq s$ for every preradical $s \in \mathbb{PR}(S)$.

5. INTERSECTION AND HEREDITY FOR "STAR" MAPPINGS

In this section, we will show other examples of good behavior of "star" mappings, namely the preservation of intersection of preradicals, as well as of hereditary property for preradicals.

Let $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}(R)$ be an arbitrary family of preradicals of R-Mod. The intersection $\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha$ of these preradicals is defined by the rule:

$$\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha\Big)(X) = \bigcap_{\alpha \in \mathfrak{A}} r_\alpha(X)$$

for every $X \in R$-Mod. Further we intend to verify the preservation of this lattice operation by the "star" mappings defined above. For that we formulate two preliminary statements, which show the concordance of kernels and preimages with the intersection of submodules (Lemmas 5.1 and 5.2).

For the family of preradicals $\{r_\alpha \mid \alpha \in \mathfrak{A}\} \subseteq \mathbb{PR}(R)$ in the construction of ${r_\alpha}^*$ and $\big(\bigwedge\limits_{\alpha \in \mathfrak{A}} r_\alpha\big)^*$ the following natural epimorphisms are used:

$$\pi^{r_\alpha}_{H^V(Y)} \quad : \quad H^V(Y) \longrightarrow H^V(Y)/r_\alpha\big(H^V(Y)\big),$$

$$\pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)} \quad : \quad H^V(Y) \longrightarrow H^V(Y)/\Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha\Big)\big(H^V(Y)\big),$$

where $Y \in S$-Mod. For these morphisms the following relation holds.

Lemma 5.1. $\operatorname{Ker}\Big[H^U\big(\pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)}\big)\Big] = \bigcap\limits_{\alpha \in \mathfrak{A}} \Big[\operatorname{Ker} H^U\big(\pi^{r_\alpha}_{H^V(Y)}\big)\Big].$

Proof. By the definition of H^U we see that $H^U\big(\pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)}\big)$ transfers every morphism $f : U \to H^V(Y)$ in the composition $f \cdot \pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)}$. Therefore:

$$\begin{aligned} \operatorname{Ker} H^U\big(\pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)}\big) &= \Big\{f : U \to H^V(Y) \mid f \cdot \pi^{\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha}_{H^V(Y)} = 0\Big\} \\ &= \Big\{f : U \to H^V(Y) \mid Uf \subseteq \Big(\bigwedge_{\alpha \in \mathfrak{A}} r_\alpha\Big)\big(H^V(Y)\big) \stackrel{\text{def}}{=\!=} \bigcap_{\alpha \in \mathfrak{A}} r_\alpha\big(H^V(Y)\big)\Big\}. \end{aligned}$$

Similarly, for every r_α $(\alpha \in \mathfrak{A})$ we have:

$$\begin{aligned} \operatorname{Ker} H^U\big(\pi^{r_\alpha}_{H^V(Y)}\big) &= \Big\{f : U \to H^V(Y) \mid f \cdot \pi^{r_\alpha}_{H^V(Y)} = 0\Big\} \\ &= \Big\{f : U \to H^V(Y) \mid Uf \subseteq \operatorname{Ker} \pi^{r_\alpha}_{H^V(Y)} = r_\alpha\big(H^V(Y)\big)\Big\}. \end{aligned}$$

Therefore:

$$\begin{aligned} \bigcap_{\alpha \in \mathfrak{A}} \Big[\operatorname{Ker} H^U\big(\pi^{r_\alpha}_{H^V(Y)}\big)\Big] &= \Big\{f : U \to H^V(Y) \mid Uf \subseteq r_\alpha\big(H^V(Y)\big) \ \forall \alpha \in \mathfrak{A}\Big\} \\ &= \Big\{f : U \to H^V(Y) \mid Uf \subseteq \bigcap_{\alpha \in \mathfrak{A}} r_\alpha\big(H^V(Y)\big)\Big\}. \end{aligned}$$

Comparing the obtained expressions we see the relation indicated in the lemma. $\square$

A similar property holds for the preimages of morphisms.

Lemma 5.2. $\big[\bigcap\limits_{\alpha\in\mathfrak{A}} \operatorname{Ker} H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big]\psi_Y^{-1} = \bigcap\limits_{\alpha\in\mathfrak{A}} \big[\big(\operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big)\psi_Y^{-1}\big].$

Proof. By the definitions we have:

$$
\begin{aligned}
y &\in \Big[\bigcap_{\alpha\in\mathfrak{A}} \operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\Big]\psi_Y^{-1} \Leftrightarrow y\,\psi_Y \in \bigcap_{\alpha\in\mathfrak{A}} \operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big) \\
&\Leftrightarrow\ y\,\psi_Y \in \operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\ \forall\alpha\in\mathfrak{A} \Leftrightarrow y \in \big[\operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big]\psi_Y^{-1}\forall\alpha\in\mathfrak{A} \\
&\Leftrightarrow\ y \in \bigcap_{\alpha\in\mathfrak{A}} \big[\big(\operatorname{Ker}\, H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big)\psi_Y^{-1}\big]. \quad \square
\end{aligned}
$$

Theorem 5.3. *For every family of preradicals* $\{r_\alpha \mid \alpha\in\mathfrak{A}\} \subseteq \mathbb{PR}(R)$ *we have the relation:* $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)^* = \bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha^*.$

Proof. For every module $Y \in S$-Mod by the definition we have $r_\alpha^*(Y) = \operatorname{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big]$ and $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)^*(Y) = \operatorname{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}_{{}_H V_{(Y)}}\big)\big]$. Using Lemmas 5.1 and 5.2 we obtain:

$$
\begin{aligned}
\Big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha\Big)^*(Y) &= \operatorname{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}_{{}_H V_{(Y)}}\big)\big] = \big[\operatorname{Ker}\, H^U\big(\pi^{\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha}_{{}_H V_{(Y)}}\big)\big]\psi_Y^{-1} \\
&\overset{5.1}{=\!=} \Big[\bigcap_{\alpha\in\mathfrak{A}} \operatorname{Ker} H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\Big]\psi_Y^{-1} \overset{5.2}{=\!=} \bigcap_{\alpha\in\mathfrak{A}} \big[\big(\operatorname{Ker} H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big)\psi_Y^{-1}\big] \\
&= \bigcap_{\alpha\in\mathfrak{A}} \operatorname{Ker}\,\big[\psi_Y \cdot H^U\big(\pi^{r_\alpha}_{{}_H V_{(Y)}}\big)\big] \overset{\text{def}}{=\!=} \bigcap_{\alpha\in\mathfrak{A}} r_\alpha^*(Y) = \Big(\bigwedge_{\alpha\in\mathfrak{A}} r_\alpha^*\Big)(Y),
\end{aligned}
$$

for every $Y \in S$-Mod, which means that $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha\big)^* = \bigwedge\limits_{\alpha\in\mathfrak{A}} r_\alpha^*.$ $\square$

The similar property is true for the inverse "star" mapping: $\big(\bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha\big)^* = \bigwedge\limits_{\alpha\in\mathfrak{A}} s_\alpha{}^*$ for every family $\{s_\alpha \mid \alpha\in\mathfrak{A}\} \subseteq \mathbb{PR}(S)$.

Further we will study the behavior of "star" mappings relative to the hereditary property of preradicals. Recall that the preradical $r \in \mathbb{PR}(R)$ is called *hereditary* (or: r is a *pretorsion*) if $r(M) = r(X) \cap M$ for every module $X \in R$-Mod and every submodule $M \subseteq X$. This means that class of r-torsion modules $\mathcal{R}(r) = \{X \in R\text{-Mod} \mid r(X) = X\}$ is hereditary (*i.e.* is closed under submodules).

Theorem 5.4. *If the preradical $r \in \mathbb{PR}(R)$ is hereditary, then the corresponding preradical $r^* \in \mathbb{PR}(S)$ also is hereditary.*

Proof. Let $r \in \mathbb{PR}(R)$ be an hereditary preradical. We will prove that for every inclusion $n:\ N \xrightarrow{\subseteq} Y$ of S-Mod the relation $r^*(N) = r^*(Y) \cap N$ holds.

Applying the functor H^V to n and using r we obtain in R-Mod the commutative diagram with exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & r\big(H^V(N)\big) \arrow[r, "i^r_{H^V(N)}", "\subseteq"'] \arrow[d, dashed, "r(H^V(n))"] & H^V(N) \arrow[r, "\pi^r_{H^V(N)}", "\text{nat}"'] \arrow[d, "H^V(n)"] & H^V(N)/r\big(H^V(N)\big) \arrow[r] \arrow[d, dashed, "(1/r)(H^V(n))"] & 0 \\
0 \arrow[r] & r\big(H^V(Y)\big) \arrow[r, "i^r_{H^V(Y)}", "\subseteq"'] & H^V(Y) \arrow[r, "\pi^r_{H^V(Y)}", "\text{nat}"'] & H^V(Y)/r\big(H^V(Y)\big) \arrow[r] & 0,
\end{tikzcd}$$

where the lateral morphisms are defined by $H^V(n)$ and r, in particular $(1/r)\big(H^V(n)\big)$ acts as follows: for every $m + r\big(H^V(N)\big) \in H^V(N)/r\big(H^V(N)\big)$ we have:

$$\big(m + r\big(H^V(N)\big)\big)\big[(1/r)\big(H^V(n)\big)\big] \overset{\text{def}}{=\!=} (m)\big(H^V(n)\big) + r\big(H^V(Y)\big).$$

It is obvious that $H^V(n)$ is a monomorphism.

Using the heredity of $r \in \mathbb{PR}(R)$ now we will show that $(1/r)\big(H^V(n)\big)$ also is a monomorphism. Indeed, the left square of the above diagram can be completed as follows:

$$\begin{tikzcd}
r\big(H^V(N)\big) \arrow[rr, "i^r_{H^V(N)}", "\subseteq"'] \arrow[d, "\cong"] \arrow[dd, dashed, bend right, "r(H^V(n))"'] & & H^V(N) \arrow[d, "\cong"] \arrow[dd, bend right, "H^V(n)"'] \\
r\big(Im\, H^V(n)\big) \arrow[rr, "\subseteq"] \arrow[d, "\cap|"] & & Im\, H^V(n) \arrow[d, "\cap|"] \\
r\big(H^V(Y)\big) \arrow[rr, "i^r_{H^V(Y)}", "\subseteq"'] & & H^V(Y).
\end{tikzcd}$$

Since r is hereditary, for the inclusion $Im\, H^V(n) \subseteq H^V(Y)$ we have:

(5.1) $$r\big(Im\, H^V(n)\big) = Im\, H^V(n) \cap r\big(H^V(Y)\big).$$

Let $\overline{m} = m + r\big(H^V(N)\big) \in \operatorname{Ker}[(1/r)\big(H^V(n)\big)]$. Then $(m)\big(H^V(n)\big) \in r\big(H^V(Y)\big)$ and so $(m)\big(H^V(n)\big) \in Im\, H^V(n) \cap r\big(H^V(Y)\big) \overset{(5.1)}{=\!=} r\big(Im\, H^V(n)\big) \cong r\big(H^V(N)\big)$. Therefore $m \in r\big(H^V(N)\big)$, *i.e.* $\overline{m} = \overline{0}$ and $\operatorname{Ker}\big[(1/r)\big(H^V(n)\big)\big] = \overline{0}$, which means that $(1/r)\big(H^V(n)\big)$ is a monomorphism.

Now by the first diagram of this proof, applying H^U and using ψ, we obtain in S-Mod the commutative diagram:

$$\begin{tikzcd}
r^*(N) \arrow[r, "\subseteq"] \arrow[d, dashed, "\cap\!\mid n'"] & N \arrow[r, "\psi_N"] \arrow[d, "\cap\!\mid n"] & H^U H^V(N) \arrow[r, "H^U(\pi^r_{H^V(N)})"] \arrow[d, "H^U H^V(n)"] & H^U\big[H^V(N)/r\big(H^V(N)\big)\big] \arrow[d, "H^U\left[(1/r)\left(H^V(n)\right)\right]"] \\
r^*(Y) \arrow[r, "\subseteq"] & Y \arrow[r, "\psi_Y"] & H^U H^V(Y) \arrow[r, "H^U(\pi^r_{H^V(Y)})"] & H^U\big[H^V(Y)/r\big(H^V(Y)\big)\big],
\end{tikzcd}$$

where $r^*(N) \overset{\text{def}}{=\!=}$ Ker $\big[\psi_N \cdot H^U\big(\pi^r_{H^V(N)}\big)\big]$ and $r^*(Y) \overset{\text{def}}{=\!=}$ Ker $\big[\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big)\big]$. Since $(1/r)\big(H^V(n)\big)$ is a monomorphism, it is clear that $H^U\big[(1/r)\big(H^V(n)\big)\big]$ also is a monomorphism.

Now we can prove that the preradical r^* is hereditary, *i.e.* $r^*(N) = r^*(Y) \cap N$ for every inclusion $N \subseteq Y$. The relation $r^*(N) \subseteq r^*(Y) \cap N$ is trivial, so it is sufficient to verify that $r^*(Y) \cap N \subseteq r^*(N)$.

Let $y \in r^*(Y) \cap N$. Then:

$$(y)\big[n \cdot \psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big)\big] = (y)\big[\psi_Y \cdot H^U\big(\pi^r_{H^V(Y)}\big)\big] = 0.$$

By the commutativity of the diagram we have:

$$(y)\big[\psi_N \cdot H^U\big(\pi^r_{H^V(N)}\big) \cdot H^U[(1/r)\big(H^V(n)\big)]\big] = 0.$$

Since $H^U[(1/r)\left(H^V(n)\right)]$ is a monomorphism, it is obvious that $(y)\big[\psi_N \cdot H^U\big(\pi^r_{H^V(N)}\big)\big] = 0$, which means that $y \in r^*(N)$. So we have $r^*(Y) \cap N \subseteq r^*(N)$, therefore r^* is a hereditary preradical. $\square$

As a general conclusion now we can affirm that for every Morita context $(R,\ {}_RU_S,\ {}_SV_R, S)$ there exists a good connection between the preradicals of the categories R-Mod and S-Mod. It is obtained in the form of two ("star") mappings between the classes of preradicals $\mathbb{PR}(R)$ and $\mathbb{PR}(S)$, which are defined by the Hom-functors H^U and H^V (Theorem 2.3). These mappings possess some useful properties, in particular they preserve the order relation, intersection and hereditary (Theorems 4.1, 5.3, 5.4). The indicated results supplement the known facts on the preradicals in an adjoint situation.

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Institute of Mathematics and Computer, Science "Vladimir Andrunachievici"
Academiei str., 5, MD–2028, Chishinev, Moldova
alexei.kashu@math.md