WEAK CONVERGENCE AND TIGHTNESS OF PROBABILITY MEASURES IN AN ABSTRACT SKOROHOD SPACE

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In this article, we introduce the space $\mathbb{D}([0,1];\mathbb{D})$ of functions defined on [0,1] with values in the Skorohod space \mathbb{D} , which are right-continuous and have left limits with respect to the J_1 topology. This space is equipped with the Skorohod-type distance introduced in [13]. Following the classical approach of [4, 5], we give several criteria for tightness of probability measures on this space, by characterizing the relatively compact subsets of this space. In particular, one of these criteria has been used in the recent article [1] for proving the existence of a \mathbb{D} -valued α -stable Lévy motion. Finally, we give a criterion for weak convergence of random elements in $\mathbb{D}([0,1];\mathbb{D})$, and a criterion for the existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$ based on its finite-dimensional distributions.

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1. INTRODUCTION

The area of limit theorems for stochastic processes has been growing steadily in the last 50 years, especially after the publication of Billingsley's seminal monograph [4]. This area has developed from the original investigations of Donsker [7] and Skorohod [11,12] regarding the asymptotic behaviour of the partial sum process associated with independent identically distributed (i.i.d.) random variables. These results state that if the variables have finite variance, the partial sum process converges in distribution to the Brownian motion, whereas if the variables have regularly varying tail probabilities with index $\alpha \in (0, 2)$, the partial sum process converges in distribution to an α -stable Lévy motion.

In the recent article [1], we proved an extension of this later result to random elements with values in the Skorohod space $\mathbb{D} = \mathbb{D}[0,1]$ of càdlàg functions on [0,1] (*i.e.* right-continuous functions with left limits), the limit

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being an infinite-dimensional process called the \mathbb{D} -valued α -stable Lévy motion. This study was motivated by the fact that nowadays, data is no longer observed at fixed moments at time, but is recorded continuously over a fixed period of time (or a region in space), which can be modeled by the interval [0, 1]. This approach was initiated in article [6] in which the authors considered the example of the high-tide water level observed continuously at any location along the northern coast of the Netherlands. Another example is the evolution of a stock price which is monitored continuously between 9 a.m and 5 p.m. when the stock market operates; if the price is likely to exhibit a sudden drop or increase, then its behaviour over this 8-hour window can be viewed as the sample path of a random process with values in \mathbb{D} .

In [1], we proved that the \mathbb{D} -valued α -stable Lévy motion $\{Z(t)\}_{t\in[0,1]}$ arises as the limit in distribution of the partial sum process $\{S_n(t) = a_n^{-1} \sum_{i=1}^{[nt]} X_i\}_{t\in[0,1]}$ associated with a sequence $(X_i)_{i\geq 1}$ of i.i.d. random elements in \mathbb{D} , which are "regularly varying" in the sense introduced in [9]. Moreover, the sample paths of this process belong to the space $\mathbb{D}([0,1];\mathbb{D})$ of functions defined on [0, 1] with values in \mathbb{D} , which are right-continuous and have left-limits with respect to Skorohod J_1 -topology on \mathbb{D} . Therefore, the law of $\{Z(t)\}_{t\in[0,1]}$ is a probability measure on $\mathbb{D}([0,1];\mathbb{D})$. For any element $x = \{x(t)\}_{t\in[0,1]}$ in $\mathbb{D}([0,1];\mathbb{D}), x(t)$ is a càdlàg function denoted by $x(t) = \{x(t,s)\}_{s\in[0,1]}$. We interpret t as the time variable and s as the space variable.

The goal of the present article is to provide some of the technical details which are missing from the companion article [1], related to the weak convergence and tightness of probability measures on the space $\mathbb{D}([0,1];\mathbb{D})$, providing in this way some useful tools for developing new limit theorems for random elements in \mathbb{D} . In order to do this, we need first to develop a compactness criterion for subsets of $\mathbb{D}([0,1];\mathbb{D})$. We note that the space $\mathbb{D}([0,1];\mathbb{D})$ is endowed with a Skorohod-type topology which was introduced in [13] for spaces of the form $\mathbb{D}([0,1];S)$, where S is a Polish space, *i.e.* a complete separable metric space. The main result of the present article is Theorem 3.4 which gives a criterion for tightness of probability measures on $\mathbb{D}([0,1];\mathbb{D})$. This result is new in the literature and has been used in the recent article [1] for proving the existence of the D-valued α -stable Lévy motion with $\alpha > 1$ (see the proof of Theorem 3.14 of [1]). The problem of weak convergence and tightness for probability measures on the space $\mathbb{D}([0,\infty); S)$ of càdlàg functions defined on $[0,\infty)$ with values in a Polish space S was also studied in [8] (Chapter 3, Sections 5–9), but the particular result that we obtained when $S = \mathbb{D}$ is not discussed in this reference.

Although it does not have a direct relationship with the results that we present here, we should mention that a version of the Itô-Nisio theorem for the

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sum of i.i.d. random processes with sample paths in $\mathbb{D}([0,1]; E)$ was proved in [2], when E is a separable Banach space. The space \mathbb{D} equipped with the uniform norm is a Banach space, but is not separable, and therefore the result of [2] does not apply to $\mathbb{D}([0,1]; \mathbb{D})$.

This article is organized as follows. In Section 2, we introduce the space $\mathbb{D}([0,1];\mathbb{D})$ and discuss some of its properties. In Section 3, we present some criteria for tightness and weak convergence of probability measures on $\mathbb{D}([0,1];\mathbb{D})$. One of these criteria, namely Theorem 3.8 below, has been used in the proof of Theorem 3.14 of [1]. In Section 4, we refine the criterion for weak convergence and we derive a result about existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$. These results generalize classical results from [4,5] and may be useful in future investigations.

2. BASIC PROPERTIES OF $\mathbb{D}([0,1];\mathbb{D})$

In this section, we introduce the space $\mathbb{D}([0,1];\mathbb{D})$ of càdlàg functions on [0,1] with values in \mathbb{D} (equipped with the J_1 -topology), and we examine its properties following very closely the discussion contained in Section 12 of [5] for càdlàg functions with values in \mathbb{R} .

We begin by recalling some basic properties of the Skorohod space \mathbb{D} , the space of functions $x : [0, 1] \to \mathbb{R}$ which are right-continuous and have left limits. On this space, we consider the supremum norm: $||x|| = \sup_{s \in [0,1]} |x(s)|$.

The Skorohod distance d_{J_1} on \mathbb{D} is defined as follows: for any $x, y \in \mathbb{D}$,

$$d_{J_1}(x,y) = \inf_{\lambda \in \Lambda} \{ \|\lambda - e\| \lor \|x - y \circ \lambda\| \},\$$

where Λ the set of strictly increasing continuous functions from [0, 1] onto [0, 1]and e is the identity function on [0, 1]. The space \mathbb{D} equipped with distance d_{J_1} is separable, but it is not complete. There exists another distance $d_{J_1}^0$ on \mathbb{D} , which is equivalent to d_{J_1} , under which \mathbb{D} is complete and separable. This distance is given by (see (12.16) of [5]):

(1)
$$d_{J_1}^0(x,y) = \inf_{\lambda \in \Lambda} \{ \|\lambda\|^0 \vee \|x - y \circ \lambda\| \},$$

for any $x, y \in \mathbb{D}$, where $\|\lambda\|^0 = \sup_{s < s'} \left| \log \frac{\lambda(s') - \lambda(s)}{s' - s} \right|$. Note that:

(2)
$$\sup_{s \in [0,1]} |\lambda(s) - s| \le e^{\|\lambda\|^0} - 1 \quad \text{for all } \lambda \in \Lambda,$$

and therefore

(3)
$$d(x,y) \le e^{d_{J_1}^0(x,y)} - 1 \quad \text{for all} \quad x, y \in \mathbb{D}.$$

Taking $\lambda = e$ in (1), we obtain:

(4)
$$d_{J_1}^0(x,y) \le \|x-y\| \text{ for all } x,y \in \mathbb{D}.$$

Note that

(5)
$$d_{J_1}(x,0) = d^0_{J_1}(x,0) = ||x|| \text{ for all } x \in \mathbb{D}.$$

For functions $(x_n)_{n>1}$ and x in \mathbb{D} , we write $x_n \stackrel{J_1}{\to} x$ if $d_{J_1}^0(x_n, x) \to 0$.

For any set $T \subset [0,1]$ and for any $x \in \mathbb{D}(\mathbb{D})$, we let

$$w(x,T) = \sup_{s_1, s_2 \in T} |x(s_1) - x(s_2)|.$$

A set $\{t_i\}_{0 \le i \le v}$ with $0 = t_0 < t_1 < \ldots < t_v = 1$ is called δ -sparse if $\min_{1 \le i \le v} (t_i - t_{i-1}) > \delta$. For any $\delta \in (0, 1)$, we consider the following moduli of continuity of a function $x \in \mathbb{D}$:

(6)
$$w'(x,\delta) = \inf_{\{t_i\}} \max_{1 \le i \le v} w(x, [t_{i-1}, t_i)),$$

where the infimum is taken over all δ -sparse sets $\{t_i\}_{0 \le i \le v}$, and

(7)
$$w''(x,\delta) = \sup_{s_1 \le s \le s_2, s_2 - s_1 \le \delta} \left(|x(s) - x(s_1)| \wedge |x(s_2) - x(s)| \right).$$

We denote by \mathcal{D} the Borel σ -field of \mathbb{D} , which coincides with the σ -field generated by the projections $\pi_t : \mathbb{D} \to \mathbb{R}, t \in [0, 1]$ given by $\pi_t(x) = x(t)$.

We introduce now the set $\mathbb{D}([0,1];\mathbb{D})$ of functions $x:[0,1] \to \mathbb{D}$ such that: (i) x is right-continuous with respect to J_1 , *i.e.* for any $t \in [0, 1)$ and for any $(t_k)_{k\geq 1} \subset [0,1]$ with $t_k \to t$ and $t_k \geq t$ for all k, we have $x(t_k) \xrightarrow{J_1} x(t)$;

(ii) x has left limits with respect to J_1 , *i.e.* for any $t \in (0, 1]$, there exists $x(t-) \in \mathbb{D}$ such that for any $(t_k)_{k\geq 1} \subset [0,1]$ with $t_k \to t$ and $t_k < t$ for all k, we have $x(t_k) \xrightarrow{J_1} x(t_{-})$.

For any $t \in [0, 1]$, x(t) is an element of \mathbb{D} , which we denote by $\{x(t, s); s \in$ [0,1]. In applications, t may be interpreted as time variable, and s as space variable (see |1|).

The next result shows that a function in $\mathbb{D}([0,1];\mathbb{D})$ is uniformly bounded in t and s.

LEMMA 2.1. For any $x \in \mathbb{D}([0,1];\mathbb{D})$, the set $\{x(t); t \in [0,1]\}$ is relatively compact in (\mathbb{D}, J_1) , and therefore $\sup_{t \in [0,1]} ||x(t)|| < \infty$.

Proof. Let $A = \{x(t); t \in [0,1]\}$ and $\{x(t_n)\}_{n\geq 1}$ be an arbitrary sequence in A. There exists a monotone subsequence $(t_{n_k})_{k\geq 1}$: either $t_{n_k} \downarrow t$ or $t_{n_k} \uparrow t$. Then either $x(t_{n_k}) \xrightarrow{J_1} x(t)$ or $x(t_{n_k}) \xrightarrow{J_1} x(t-)$. This shows that any sequence in A has a J_1 -convergent subsequence. So, A is relatively compact in (\mathbb{D}, J_1) . The last part follows by the characterization of relative compactness in (\mathbb{D}, J_1) given by Theorem 12.3 of [5]. \Box

We denote by $\|\cdot\|_{\mathbb{D}}$ the super-uniform norm on $\mathbb{D}([0,1];\mathbb{D})$ given by:

$$||x||_{\mathbb{D}} = \sup_{t \in [0,1]} ||x(t)||.$$

We let $d_{\mathbb{D}}$ be the *Skorohod distance* on $\mathbb{D}([0,1];\mathbb{D})$, given by relation (2.1) of [13]:

(8)
$$d_{\mathbb{D}}(x,y) = \inf_{\lambda \in \Lambda} \{ \|\lambda - e\| \lor \rho_{\mathbb{D}}(x,y \circ \lambda) \},$$

where $\rho_{\mathbb{D}}$ is the uniform distance on $\mathbb{D}([0,1];\mathbb{D})$:

(9)
$$\rho_{\mathbb{D}}(x,y) = \sup_{t \in [0,1]} d^0_{J_1}(x(t),y(t)).$$

By relation (5), it follows that for any $x \in \mathbb{D}([0,1];\mathbb{D})$,

(10)
$$d_{\mathbb{D}}(x,0) = \rho_{\mathbb{D}}(x,0) = ||x||_{\mathbb{D}}.$$

Note that for any $x, y \in \mathbb{D}([0, 1]; \mathbb{D})$, we have:

(11)
$$d_{\mathbb{D}}(x,y) \le \rho_{\mathbb{D}}(x,y) \le \|x-y\|_{\mathbb{D}}.$$

By definition, $d_{\mathbb{D}}(x_n, x) \to 0$ if and only if there exists a sequence $(\lambda_n)_{n \ge 1} \subset \Lambda$ such that

(12)
$$\sup_{t \in [0,1]} |\lambda_n(t) - t| \to 0 \text{ and } \sup_{t \in [0,1]} d^0_{J_1}(x_n(\lambda_n(t)), x(t)) \to 0.$$

Similarly to \mathbb{D} , the uniform topology on $\mathbb{D}([0,1];\mathbb{D})$ is stronger than the Skorohod topology on this space: if $\rho_{\mathbb{D}}(x_n, x) \to 0$ then $d_{\mathbb{D}}(x_n, x) \to 0$ (take $\lambda_n = e$ in (12)). The following result is also similar to the classical case.

LEMMA 2.2. a) If $d_{\mathbb{D}}(x_n, x) \to 0$, then $x_n(t) \xrightarrow{J_1} x(t)$ for any continuity point t of x (with respect to J_1).

b) If $d_{\mathbb{D}}(x_n, x) \to 0$ and x is continuous on [0, 1] with respect to J_1 , then $\rho_{\mathbb{D}}(x_n, x) \to 0$.

Proof. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (12) holds. a) Then

$$d_{J_1}^0(x_n(t), x(t)) \le d_{J_1}^0(x_n(t), x(\lambda_n(t))) + d_{J_1}^0(x(\lambda_n(t)), x(t)) \to 0.$$

b) Since x is continuous on the compact set [0, 1], it is also uniformly continuous. Hence

$$\rho_{\mathbb{D}}(x_n, x) \le \sup_{t \in [0,1]} d^0_{J_1}(x_n(t), x(\lambda_n(x(t)))) + \sup_{t \in [0,1]} d^0_{J_1}(x(\lambda_n(x(t))), x(t)) \to 0. \quad \Box$$

The next result shows that the super-uniform norm is continuous on $\mathbb{D}([0,1];\mathbb{D})$.

LEMMA 2.3. If $(x_n)_{n\geq 1}$ and x are functions in $\mathbb{D}([0,1];\mathbb{D})$ such that $d_{\mathbb{D}}(x_n,x) \to 0$ as $n \to \infty$, then $||x_n||_{\mathbb{D}} \to ||x||_{\mathbb{D}}$ as $n \to \infty$.

Proof. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (12) holds. By (10), we have:

$$||x_n \circ \lambda_n||_{\mathbb{D}} - ||x||_{\mathbb{D}}| = |\rho_{\mathbb{D}}(x_n \circ \lambda_n, 0) - \rho_{\mathbb{D}}(x, 0)| \le \rho_{\mathbb{D}}(x_n \circ \lambda_n, x) \to 0.$$

The conclusion follows since $||x_n \circ \lambda_n||_{\mathbb{D}} = ||x_n||_{\mathbb{D}}$ (because λ_n is a one-to-one map). \Box

For any set $T \subset [0, 1]$ and for any $x \in \mathbb{D}([0, 1]; \mathbb{D})$, we let

$$w_{\mathbb{D}}(x,T) = \sup_{t_1,t_2 \in T} d^0_{J_1}(x(t_1),x(t_2)).$$

The following result is proved similarly to Lemma 1 (page 122) of [5].

LEMMA 2.4. For any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$, there exist $0 = t_0 < t_1 < \ldots < t_v = 1$ such that

$$w_{\mathbb{D}}(x, [t_{i-1}, t_i)) < \varepsilon \quad for \ all \quad i = 1, \dots, v.$$

A consequence of this result is that for $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$, there can be at most finitely many points $t \in [0,1]$ such that $d_{J_1}^0(x(t), x(t-)) > \varepsilon$. Hence, any function $x \in \mathbb{D}([0,1];\mathbb{D})$ has a countable set of discontinuities with respect to J_1 , which we denote by Disc(x). The maximum jump of x is defined by:

$$j(x) = \sup_{t \in [0,1]} d^0_{J_1}(x(t), x(t-))$$

For any $\delta \in (0, 1)$ and $x \in \mathbb{D}([0, 1]; \mathbb{D})$, we let

(13)
$$w_{\mathbb{D}}'(x,\delta) = \inf_{\{t_i\}} \max_{1 \le i \le v} w_{\mathbb{D}}(x,[t_{i-1},t_i)),$$

where the infimum is taken over all δ -sparse sets $\{t_i\}_{0 \le i \le v}$.

Clearly, the function $w'_{\mathbb{D}}(x, \cdot)$ is non-decreasing. The following two results give some further properties of $w'_{\mathbb{D}}(x, \delta)$.

(14)

$$\begin{aligned}
& \lim_{\delta \to 0} w'_{\mathbb{D}}(x, \delta) = 0 \\
& w'_{\mathbb{D}}(x, \delta) \leq w_{\mathbb{D}}(x, 2\delta) \quad \text{for any} \quad \delta \in (0, 1/2), \\
& w_{\mathbb{D}}(x, \delta) \leq 2w'_{\mathbb{D}}(x, \delta) + j(x) \quad \text{for any} \quad \delta \in (0, 1).
\end{aligned}$$

Proof. To prove the first relation, let $\varepsilon > 0$ be arbitrary and $\{t_i\}_{0 \le i \le v}$ be the sequence given by Lemma 2.4. Pick $0 < \delta_{\varepsilon} < \min_{0 \le i \le v} (t_i - t_{i-1})$. For any $\delta \in (0, \delta_{\varepsilon}), \{t_i\}_{0 \le i \le v}$ is δ -sparse, and hence $w'_{\mathbb{D}}(x, \delta) \le \max_{1 \le i \le v} w_{\mathbb{D}}(x, [t_{i-1}, t_i))$ $< \varepsilon$. The last two relations are proved similarly to (12.7) and (12.9) of [5], using the triangle inequality in $(\mathbb{D}, d^0_{J_1})$. We omit the details. \Box LEMMA 2.6. $w'_{\mathbb{D}}(\cdot, \delta)$ is upper-semicontinuous on $\mathbb{D}([0, 1]; \mathbb{D})$ equipped with $d_{\mathbb{D}}$.

Proof. Let $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\varepsilon > 0$ be arbitrary. We have to prove that there exists $\eta > 0$ such that $w'_{\mathbb{D}}(y,\delta) < w'_{\mathbb{D}}(x,\delta) + \varepsilon$ for any $y \in \mathbb{D}([0,1];\mathbb{D})$ such that $d_{\mathbb{D}}(x,y) < \eta$. This follows by the same argument as in Lemma 4 (page 130) of [5], replacing $|y(t) - x(\lambda(t))|$ by $d^0_{J_1}(y(t), x(\lambda(t)))$ and using the triangle inequality in $(\mathbb{D}, d^0_{J_1})$. \Box

The space $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$ is separable, but it is not complete. Similarly to the distance $d_{J_1}^0$ on \mathbb{D} , we consider another distance $d_{\mathbb{D}}^0$ on $\mathbb{D}([0,1];\mathbb{D})$, given by:

(15)
$$d^{0}_{\mathbb{D}}(x,y) = \inf_{\lambda \in \Lambda} \{ \|\lambda\|^{0} \lor \rho_{\mathbb{D}}(x,y \circ \lambda) \}.$$

Then $d_{\mathbb{D}}(x,y) \leq e^{d_{\mathbb{D}}^0(x,y)} - 1$ for all $x, y \in \mathbb{D}([0,1];\mathbb{D})$.

Similarly to Theorems 12.1 and 12.2 of [5], and using the fact that \mathbb{D} is separable and complete under $d_{J_1}^0$, we obtain the following result. (See also Theorem 2.6 of [13].)

THEOREM 2.7. The metrics $d_{\mathbb{D}}$ and $d_{\mathbb{D}}^{0}$ are equivalent. The space $\mathbb{D}([0,1]; \mathbb{D})$ is separable under $d_{\mathbb{D}}$ and $d_{\mathbb{D}}^{0}$, and is complete under $d_{\mathbb{D}}^{0}$.

The following result characterizes the relatively compact subsets of $\mathbb{D}([0,1];\mathbb{D})$, being the analogue of Theorem 12.3 of [5].

THEOREM 2.8. A set $A \subset \mathbb{D}([0,1];\mathbb{D})$ is relatively compact with respect to $d_{\mathbb{D}}$ if and only if it satisfies the following three conditions:

- (i) $\sup_{x \in A} \|x\|_{\mathbb{D}} < \infty;$
- (ii) $\lim_{\delta \to 0} \sup_{x \in A} \sup_{t \in [0,1]} w'(x(t), \delta) = 0;$
- (iii) $\lim_{\delta \to 0} \sup_{x \in A} w'_{\mathbb{D}}(x, \delta) = 0.$

Proof. Note that conditions (i) and (ii) are equivalent to saying that the set $U = \{x(t); x \in A, t \in [0,1]\}$ is relatively compact in (\mathbb{D}, J_1) (see Theorem 12.3 of [5]).

Suppose that A is relatively compact in $\mathbb{D}([0,1];\mathbb{D})$. We first prove that U is relatively compact in (\mathbb{D}, J_1) . Let $\{x_n(t_n)\}_{n\geq 1}$ be an arbitrary sequence in U, with $x_n \in A$ and $t_n \in [0,1]$. Since A is relatively compact, there exists a subsequence $N \subset \mathbb{N}$ such that $d_{\mathbb{D}}(x_n, x) \to 0$ as $n \to \infty, n \in N$. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ such that (12) hold as $n \to \infty, n \in N$. The sequence $(t_n)_{n\in N}$ has a monotone convergent sub-sequence $(t_n)_{n\in N'}$ with $N' \subset N$: either $t_n \uparrow t$ or $t_n \downarrow t$ as $n \to \infty, n \in N'$. Since λ_n^{-1} is strictly increasing, either $\lambda_n^{-1}(t_n) \uparrow t$ or $\lambda_n^{-1}(t_n) \downarrow t$ as $n \to \infty, n \in N'$. Therefore, either $x(\lambda_n^{-1}(t_n)) \xrightarrow{J} x(t-)$ or

 $\begin{aligned} x(\lambda_n^{-1}(t_n)) &\xrightarrow{J_1} x(t) \text{ as } n \to \infty, n \in N'. \text{ In the first case,} \\ d_{J_1}^0(x_n(t_n), x(t-)) &\leq d_{J_1}^0(x_n(t_n), x(\lambda_n^{-1}(t_n))) + d_{J_1}^0(x(\lambda_n^{-1}(t_n)), x(t-)) \to 0, \end{aligned}$

as $n \to \infty$, $n \in N'$. In the second case, $d_{J_1}^0(x_n(t_n), x(t)) \to 0$ as $n \to \infty$, $n \in N'$. This shows that the sequence $\{x_n(t_n)\}_{n\geq 1}$ has a J_1 -convergence subsequence.

To prove (iii), we apply Dini's theorem, as stated in Appendix M8 of [5]. Since $w'_{\mathbb{D}}(\cdot, 1/n)$ is upper semi-continuous for any n, and $w'_{\mathbb{D}}(x, 1/n) \downarrow 0$ for any $x \in \mathbb{D}([0, 1]; \mathbb{D})$, this convergence is uniform on compact sets. Hence $\sup_{x \in A} w'_{\mathbb{D}}(x, n^{-1}) \to 0$ as $n \to \infty$. Condition (iii) follows since $w'_{\mathbb{D}}(x, \cdot)$ is non-decreasing.

Next, suppose that the set A satisfies conditions (i)–(iii). Since $\mathbb{D}([0,1];\mathbb{D})$ is complete with respect to $d^0_{\mathbb{D}}$, the closure \overline{A} of A is also complete. To show that \overline{A} is compact, it suffices to show that \overline{A} is totally bounded with respect to $d^0_{\mathbb{D}}$ (see Theorem of Appendix M5 of [5]). This follows as in the sufficiency part of the proof of Theorem 12.3 of [5], by choosing H to be a finite ε -net of the set U in \mathbb{D} . \Box

To give a second characterization of the relatively compact subsets of $\mathbb{D}([0,1];\mathbb{D})$, we consider the following modulus of continuity: for any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $\delta \in (0,1)$,

(16)
$$w_{\mathbb{D}}''(x,\delta) = \sup_{t_1 \le t \le t_2, \ t_2 - t_1 \le \delta} \left(d_{J_1}^0(x(t), x(t_1)) \wedge d_{J_1}^0(x(t_2), x(t)) \right).$$

We have the following result.

LEMMA 2.9 (Lemma 2.2 of [1]). For any
$$x, y \in \mathbb{D}([0,1];\mathbb{D})$$
, we have:
$$w''_{\mathbb{D}}(x+y,\delta) \leq w''_{\mathbb{D}}(x,\delta) + 2\|y\|_{\mathbb{D}}.$$

As in the classical case, it follows that $w_{\mathbb{D}}''(x,\delta) \leq w_{\mathbb{D}}'(x,\delta)$ (see the proof of (12.28) of [5]). The following result is the analogue of Theorem 12.4 of [5].

THEOREM 2.10. A set $A \subset \mathbb{D}([0,1];\mathbb{D})$ is relatively compact with respect to $d_{\mathbb{D}}$ if and only if it satisfies the following three conditions:

(1)
$$\sup_{x \in A} ||x||_{\mathbb{D}} < \infty;$$

(ii')

$$\begin{cases}
(a) \quad \lim_{\delta \to 0} \sup_{t \in [0,1]} w''(x(t), \delta) = 0 \\
(b) \quad \lim_{\delta \to 0} \sup_{x \in A} \sup_{t \in [0,1]} |x(t, \delta), x(t, 0)| = 0 \\
(c) \quad \lim_{\delta \to 0} \sup_{x \in A} \sup_{t \in [0,1]} |x(t, 1-), x(t, 1-\delta)| = 0;
\end{cases}$$
(iii')

$$\begin{cases}
(a) \quad \lim_{\delta \to 0} w''_{\mathbb{D}}(x, \delta) = 0 \\
(b) \quad \lim_{\delta \to 0} \sup_{x \in A} d^{0}_{J_{1}}(x(\delta), x(0)) = 0 \\
(c) \quad \lim_{\delta \to 0} \sup_{x \in A} d^{0}_{J_{1}}(x(1-), x(1-\delta)) = 0.
\end{cases}$$

Proof. If A is relatively compact, then conditions (i)–(iii) of Theorem 2.8 hold. Condition (ii') follows by applying inequality (12.31) of [5] to the function $x(t) \in \mathbb{D}$, for any $t \in [0, 1]$. Condition (iii') follows by the following inequality (proved similarly to (12.31) of [5]):

(17)
$$w_{\mathbb{D}}''(x,\delta) \vee d_{J_1}^0(x(\delta),x(0)) \vee d_{J_1}^0(x(1-),x(1-\delta)) \leq w_{\mathbb{D}}'(x,2\delta).$$

Suppose that conditions (i), (ii') and (iii') hold. The fact that A is relatively compact will follow by Theorem 2.8, once we show that conditions (ii) and (iii) of this theorem hold. Condition (ii) follows from (ii') by applying inequality (12.32) of [5] to the function $x(t) \in \mathbb{D}$ for any $t \in [0, 1]$. Condition (iii') follows by the following inequality

(18)
$$w'_{\mathbb{D}}(x,\delta/2) \le 12\{w''_{\mathbb{D}}(x,\delta) + d^{0}_{J_{1}}(x(\delta),x(0)) + d^{0}_{J_{1}}(x(1-),x(1-\delta))\}$$

This is proved similarly to inequality (12.32) of [5], using the triangle inequality in \mathbb{D} and the fact that $x_n \xrightarrow{J_1} x$ implies that $d^0_{J_1}(x_n, y) \to d^0_{J_1}(x, y)$ for any $y \in \mathbb{D}$. \Box

We conclude this subsection with a discussion about measurability and finite-dimensional sets in $\mathbb{D}([0,1];\mathbb{D})$. Let $\mathcal{D}_{\mathbb{D}}$ be the Borel σ -field of $\mathbb{D}([0,1];\mathbb{D})$ with respect to $d_{\mathbb{D}}$. For any $t \in [0,1]$, we let $\pi_t^{\mathbb{D}} : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}$ be the projection given by $\pi_t^{\mathbb{D}}(x) = x(t)$. By Lemma 2.3 of [13], $\pi_t^{\mathbb{D}}$ is $\mathcal{D}_{\mathbb{D}}/\mathcal{D}$ -measurable for any $t \in [0,1]$. By Theorem 2.7 of [13], $\mathcal{D}_{\mathbb{D}}$ coincides with the σ -field generated by the projections $\pi_t^{\mathbb{D}}$ for $t \in [0,1]$. Similarly to the classical case, the function $\pi_t^{\mathbb{D}}$ has the following continuity properties.

LEMMA 2.11. a) $\pi_0^{\mathbb{D}}$ and $\pi_1^{\mathbb{D}}$ are continuous with respect to $d_{\mathbb{D}}$.

b) For any $t \in (0, 1)$, $\pi_t^{\mathbb{D}}$ is continuous at x with respect to $d_{\mathbb{D}}$ if and only if x is continuous at t with respect to J_1 .

Proof. a) Assume that $d_{\mathbb{D}}(x_n, x) \to 0$. Let $(\lambda_n)_{n\geq 1} \subset \Lambda$ be such that (12) holds. In particular, since $\lambda_n(0) = 0$, we obtain: $d_{J_1}^0(x_n(0), x(0)) \to 0$. This shows that $\pi_0^{\mathbb{D}}(x_n) \xrightarrow{J_1} \pi_0(x)$. Similarly, $\pi_1^{\mathbb{D}}(x_n) \xrightarrow{J_1} \pi_1(x)$.

b) Suppose that x is continuous at t with respect to J_1 . Assume that $d_{\mathbb{D}}(x_n, x) \to 0$. Then $\pi_t^{\mathbb{D}}(x_n) \xrightarrow{J_1} \pi_t^{\mathbb{D}}(x)$, by Lemma 2.2.a). Suppose next that x is discontinuous at t with respect to J_1 , *i.e.* $d_{J_1}^0(x(t-), x(t)) > 0$. Let $\lambda_n \in \Lambda$ be such that $\lambda_n(t) = t - 1/n$, and λ is linear on [0, t] and [t, 1]. Define $x_n(s) = x(\lambda_n(s))$. Then $d_{\mathbb{D}}(x_n, x) \to 0$, and $\pi_t^{\mathbb{D}}(x_n) = x_n(t) = x(\lambda_n(t)) = x(t-1/n) \xrightarrow{J_1} x(t-)$, and so $\pi_t^{\mathbb{D}}(x_n)$ does not converge in J_1 to x(t). This shows that $\pi_t^{\mathbb{D}}$ is discontinuous at x with respect to $d_{\mathbb{D}}$.

For an arbitrary set $T \subset [0, 1]$, we let $\mathcal{D}_{f,T}^{\mathbb{D}}$ be the class of finite-dimensional sets of the form $(\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1}(H)$ for some $0 \leq t_1 < \ldots < t_k \leq 1, t_i \in T, H \in \mathcal{D}^k$

and $k \geq 1$. Note that the σ -field generated by $\mathcal{D}_{f,T}^{\mathbb{D}}$ coincides with $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}$, the minimal σ -field with respect to which the maps $\pi_t^{\mathbb{D}}, t \in T$ are measurable.

THEOREM 2.12. If $T \subset [0,1]$ is such that $1 \in T$ and T is dense in [0,1], then:

a) $\mathcal{D}_{\mathbb{D}}$ is the σ -field generated by $\mathcal{D}_{f,T}^{\mathbb{D}}$;

b) $\mathcal{D}_{f,T}^{\mathbb{D}}$ is a separating class of $\mathcal{D}_{\mathbb{D}}$, i.e. if P and Q are two probability measures on $(\mathbb{D}, \mathcal{D}_{\mathbb{D}})$ such that P(A) = Q(A) for any $A \in \mathcal{D}_{f,T}^{\mathbb{D}}$, then P = Q.

Proof. a) Since $\pi_t^{\mathbb{D}}$ is $\mathcal{D}_{\mathbb{D}}$ -measurable, $\sigma\{\pi_t^{\mathbb{D}}; t \in T\} \subset \mathcal{D}_{\mathbb{D}}$. To prove the other inclusion, it suffices to show that the identity $i : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}([0,1];\mathbb{D})$ given by i(x) = x is $\sigma\{\pi_t^{\mathbb{D}}; t \in [0,1]\}/\mathcal{D}_{\mathbb{D}}$ -measurable. For this, we use the same argument as in the proof of Theorem 12.5.(iii) of [5]. For any $\sigma = \{t_i\}_{i=0,\dots,k}$ such that $0 = t_0 < t_1 < \dots < t_k = 1$, we define the map $A_{\sigma}: \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{D}([0,1];\mathbb{D})$ by $A_{\sigma}(x) = \sum_{i=1}^k x(t_{i-1})\mathbf{1}_{\{t_{i-1},t_i\}} + x(1)\mathbf{1}_{\{1\}}(t)$. Similarly to Lemma 3 (page 127) of [5], it can be proved that

(19) $\max_{1 \le i \le k} (t_i - t_{i-1}) \le \delta$ implies that $d_{\mathbb{D}}(A_{\sigma}(x), x) \le \delta \lor w'_{\mathbb{D}}(x, \delta)$.

For any σ as above, we consider also the map $V_{\sigma} : \mathbb{D}^{k+1} \to \mathbb{D}([0,1];\mathbb{D})$ given by $V_{\sigma}(\alpha) = \sum_{i=1}^{k} \alpha_{i-1} \mathbb{1}_{[t_{i-1},t_i)}(t) + \alpha_k \mathbb{1}_{\{1\}}(t)$, for $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbb{D}^{k+1}$.

The function $V_{\sigma} : \mathbb{D}^{k+1} \to \mathbb{D}([0,1];\mathbb{D})$ is $\rho_{\mathbb{D}}$ -continuous (hence $d_{\mathbb{D}}$ -continuous), where \mathbb{D}^{k+1} is endowed with the product topology: if $\alpha^n, \alpha \in \mathbb{D}^k$ are such that $\alpha_i^n \xrightarrow{J_1} \alpha_i$ as $n \to \infty$, for $i = 0, \ldots, k$, then

$$\rho_{\mathbb{D}}(V_{\sigma}(\alpha^{n}), V_{\sigma}(\alpha)) = \sup_{t \in [0,1]} d^{0}_{J_{1}}(V_{m}(\alpha^{n})(t), V_{m}(\alpha)(t)) = \max_{0 \le i \le k} d^{0}_{J_{1}}(\alpha^{n}_{i}, \alpha_{i}) \to 0.$$

It follows that V_{σ} is $\mathcal{D}^{k+1}/\mathbb{D}_{\mathbb{D}}$ -measurable. If $t_i \in T$ for all i, then A_{σ} is $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$ -measurable, since $A_{\sigma} = V_{\sigma} \circ \pi_{t_0,\ldots,t_k}^{\mathbb{D}}$ and $\pi_{t_0,\ldots,t_k}^{\mathbb{D}}$ is $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}^{k+1}$ -measurable.

For any $m \geq 1$, choose $\sigma_m = \{t_i^m\}_{i=0,\dots,k_m}$ such that $t_i^m \in T$ and $\max_i(t_i^m - t_{i-1}^m) < 1/m$. By (14) and (19), it follows that $d_{\mathbb{D}}(A_{\sigma_m}(x), x) \to 0$ as $m \to \infty$. This proves that the identity map i is the pointwise limit (with respect to $d_{\mathbb{D}}$) of the sequence $(A_{\sigma_m})_{m\geq 1}$ of $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$ -measurable maps. Since $\mathbb{D}_{\mathbb{D}}$ is the Borel σ -field corresponding to $d_{\mathbb{D}}$, it the map i is also $\sigma\{\pi_t^{\mathbb{D}}; t \in T\}/\mathcal{D}_{\mathbb{D}}$ -measurable.

b) This follows by Theorem 3.3 of [3], since $\mathcal{D}_{f,T}^{\mathbb{D}}$ is a π -system generating $\mathcal{D}_{\mathbb{D}}$. \Box

The characterization of tightness of probability measures on $\mathbb{D}([0, 1]; \mathbb{D})$ given in Section 3 relies on certain events involving the functions $w'_{\mathbb{D}}(\cdot, \delta)$ and $w''_{\mathbb{D}}(\cdot, \delta)$. Measurability of these functions is essential for this purpose. Before establishing this, we need the following simple result (which is valid in any metric space).

LEMMA 2.13. The map $\Phi : \mathbb{D} \times \mathbb{D} \to [0,\infty)$ given by $\Phi(x,y) = d_{J_1}^0(x,y)$ is continuous with respect to the product of J_1 -topologies on $\mathbb{D} \times \mathbb{D}$.

Proof. If
$$x_n \xrightarrow{J_1} x$$
 and $y_n \xrightarrow{J_1} y$, then $d^0_{J_1}(x_n, y_n) \to d^0_{J_1}(x, y)$ since
 $|d^0_{J_1}(x_n, y_n) - d^0_{J_1}(x, y)| \le |d^0_{J_1}(x_n, y_n) - d^0_{J_1}(x, y_n)| + |d^0_{J_1}(x, y_n) - d^0_{J_1}(x, y)|$
 $\le d^0_{J_1}(x_n, x) + d^0_{J_1}(y_n, y).$

LEMMA 2.14. The functions $w'_{\mathbb{D}}(\cdot, \delta)$ and $w''_{\mathbb{D}}(\cdot, \delta)$ are $\mathcal{D}_{\mathbb{D}}$ -measurable.

Proof. The measurability of $w'_{\mathbb{D}}(\cdot, \delta)$ follows by Lemma 2.6. For $w''_{\mathbb{D}}(\cdot, \delta)$, note that in the definition (16) of $w''_{\mathbb{D}}(x, \delta)$, we may take t_1, t, t_2 to be rational numbers. By Lemma 2.13, Φ is $\mathcal{D} \times \mathcal{D}$ -measurable, and so the map $\Phi \circ \pi^{\mathbb{D}}_{t,t_1}$ given by $x \mapsto d^0_{J_1}(x(t), x(t_1))$ is $\mathcal{D}_{\mathbb{D}}$ -measurable, for any $t_1, t \in [0, 1]$. Therefore, the map $x \mapsto d^0_{J_1}(x(t), x(t_1)) \wedge d^0_{J_1}(x(t_2), x(t))$ is $\mathcal{D}_{\mathbb{D}}$ -measurable, for any rational numbers $t_1, t, t_2 \in [0, 1]$ with $t_1 \leq t \leq t_2$. The conclusion follows since the supremum of a countable collection of measurable functions is measurable. \Box

Finally, we recall the definition of a random element in $\mathbb{D}([0,1];\mathbb{D})$.

Definition. Let (Ω, \mathcal{F}, P) be a probability space. A map $X : \Omega \to \mathbb{D}([0,1];\mathbb{D})$ is called a *random element* in $\mathbb{D}([0,1];\mathbb{D})$ if X is $\mathcal{F}/\mathcal{D}_{\mathbb{D}}$ -measurable, *i.e.* X(t) is \mathcal{F}/\mathcal{D} -measurable for any $t \in [0,1]$.

3. WEAK CONVERGENCE AND TIGHTNESS

In this section, we study the weak convergence and tightness of probability measures on the space $(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}})$, following the discussion contained in Section 13 of [5] for probability measures on (\mathbb{D},\mathcal{D}) . We provide some of the details which are missing from [5]) since they are more delicate and require special attention in our situation.

Recall that if $(P_n)_{n\geq 1}$ and P are probability measures on $(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}})$, we say that $(P_n)_{n\geq 1}$ converges weakly to P if $\int f dP_n \to \int f dP$ for any $d_{\mathbb{D}}$ continuous bounded function $f : \mathbb{D}([0,1];\mathbb{D}) \to \mathbb{R}$. In this case, we write $P_n \xrightarrow{w} P$. Since $\mathbb{D}([0,1];\mathbb{D})$ is separable, there is a distance on the set of probability measures on $(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}})$ (called the *Prohorov distance*), which gives rise to the topology of weak convergence (see page 72 of [5]).

If $(X_n)_{n\geq 1}$ and X are random elements in $\mathbb{D}([0,1];\mathbb{D})$ (possibly defined on different probability spaces) with respective laws denoted by $(P_n)_{n\geq 1}$ and P, we say that $(X_n)_{n\geq 1}$ converges in distribution to X if $P_n \xrightarrow{w} P$. In this case, we write $X_n \xrightarrow{d} X$.

For any probability measure P on $(\mathbb{D}([0,1];\mathbb{D}),\mathcal{D}_{\mathbb{D}})$, we let T_P be the set of $t \in [0,1]$ for which the projection $\pi_t^{\mathbb{D}}$ is $d_{\mathbb{D}}$ -continuous a.s. with respect to P. Note that $0, 1 \in T_P$. If $t \in (0,1)$, then $t \in T_P$ if and only if $P(J_t) = 0$, where $J_t = \{x \in \mathbb{D}([0,1];\mathbb{D}); t \in \text{Disc}(x)\}.$

Using the same argument as in the classical case (page 238 of [5]), it can be shown that $P(J_t) > 0$ is possible for at most countably many $t \in (0, 1)$. Hence, the complement of T_P in [0, 1] is countable. The following result follows by the continuous mapping theorem.

LEMMA 3.1. Let $(P_n)_{n\geq 1}$ and P be probability measures on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ such that $P_n \xrightarrow{w} P$. Then $P_n \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1} \xrightarrow{w} P \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1}$ for any $t_1,\ldots,t_k \in T_P$.

We recall the following definitions.

Definition. A family Π of probability measures on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ is tight if for every $\eta > 0$, there exists a $d_{\mathbb{D}}$ -compact set K in $\mathbb{D}([0,1];\mathbb{D})$ such that $P(K) \ge 1 - \eta$ for all $P \in \Pi$.

Definition. A family Π of probability measures on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ is relatively compact if for every sequence $(P_n)_{n\geq 1}$ in Π , there exists a subsequence $(P_{n_k})_{k\geq 1}$ which converges weakly to a probability measure Q (which is not necessarily an element of Π).

The following result follows by Prohorov's theorem, since $\mathbb{D}([0,1];\mathbb{D})$ is separable and complete (see Theorems 5.1 and 5.2 of [5]).

THEOREM 3.2. A family Π of probability measures on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ is tight if and only if it is relatively compact.

The next result is an important tool for proving weak convergence in $\mathbb{D}([0,1];\mathbb{D})$. Its proof is the same as in the classical case (see Theorem 13.1 of [5]). We include it for the sake of completeness.

THEOREM 3.3. Let $(P_n)_{n\geq 1}$ and P be probability measures on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ such that

(20)
$$P_n \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1} \xrightarrow{w} P \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1} \text{ in } \mathbb{D}^k, \text{ for any } t_1,\ldots,t_k \in T_P$$

and $(P_n)_{n\geq 1}$ is tight. Then $P_n \xrightarrow{w} P$.

Proof. It is enough to prove that for any subsequence $(n_k)_{k\geq 1}$, there exists a further sub-subsequence $(k_l)_{l\geq 1}$ such that $P_{n_{k_l}} \xrightarrow{w} P$ as $l \to \infty$ (see e.g. Appendix 5.1.2 of [10]).

Let $(n_k)_{k\geq 1}$ be an arbitrary subsequence. By Theorem 3.2, $(P_n)_{n\geq 1}$ is relatively compact. Hence, there exists a sub-subsequence $(k_l)_{l\geq 1}$ such that $P_{n_{k_l}} \stackrel{w}{\to} Q$ as $l \to \infty$, for some probability measure Q on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$. By hypothesis, $P_{n_{k_l}} \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1} \stackrel{w}{\to} P \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1}$ as $l \to \infty$, for any $t_1,\ldots,t_k \in$ T_P . By Lemma 3.1, $P_{n_{k_l}} \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1} \stackrel{w}{\to} Q \circ (\pi_{t_1,\ldots,t_k}^{\mathbb{D}})^{-1}$ as $l \to \infty$, for any $t_1,\ldots,t_k \in T_Q$. Uniqueness of the limit implies that:

$$P \circ (\pi_{t_1,\dots,t_k}^{\mathbb{D}})^{-1} = Q \circ (\pi_{t_1,\dots,t_k}^{\mathbb{D}})^{-1} \quad \text{for all } t_1,\dots,t_k \in T_P \cap T_Q$$

The set $T = T_P \cap T_Q$ contains 0 and 1, and is dense in [0, 1] (since its complement in [0, 1] is countable). By Theorem 2.12, $\mathcal{D}_{f,T}$ is a separating class of $\mathcal{D}_{\mathbb{D}}$, and hence P = Q. \Box

We continue now with a discussion about tightness. The next result gives a criterion for tightness, being the analogue of Theorem 13.2 of [5] for the space $\mathbb{D}([0,1];\mathbb{D})$. This result has been used in the recent article [1] for the construction of the \mathbb{D} -valued α -stable Lévy motion with $\alpha > 1$ (see the proof of Theorem 3.14 of [1]). Conditions (i) and (iii) of this result are similar to (13.4) and (13.5) of [5], but (ii) is a new condition, due to the space variable s of an element in $\mathbb{D}([0,1];\mathbb{D})$. Recall that $w'(x(t),\delta)$ is given by (6), whereas $w'_{\mathbb{D}}(x,\delta)$ is given by (13), for any $x \in \mathbb{D}([0,1];\mathbb{D})$ and $t \in [0,1]$.

THEOREM 3.4. A sequence $(P_n)_{n\geq 1}$ of probability measures on $(\mathbb{D}([0,1]; \mathbb{D}), \mathcal{D}_{\mathbb{D}})$ is tight if and only if it satisfies the following three conditions:

(i) We have:

(21)
$$\lim_{a \to \infty} \limsup_{n \to \infty} P_n(\{x; \|x\|_{\mathbb{D}} \ge a\}) = 0.$$

(ii) For any $\varepsilon > 0$,

(22)
$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; w'(x(t), \delta) \ge \varepsilon \text{ for some } t \in [0, 1]\}) = 0.$$

(iii) For any
$$\varepsilon > 0$$
,

(23)
$$\lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; w'_{\mathbb{D}}(x, \delta) \ge \varepsilon\}) = 0.$$

Proof. We use a similar argument as in the proof of Theorem 13.2 of [5] (see also the proof of Theorem 7.3 of [5]). Suppose that $(P_n)_{n\geq 1}$ is tight. Let $\eta > 0$ and $\varepsilon > 0$ be arbitrary. We have to prove that there exist $a > 0, \delta \in (0, 1)$ and an integer $n_0 \geq 1$ such that for all $n \geq n_0$,

(24)
$$\begin{cases} (a) \quad P_n(\{x; \|x\|_{\mathbb{D}} \ge a\}) \le \eta \\ (b) \quad P_n(\{x; w'(x(t), \delta) \ge \varepsilon \text{ for some } t \in [0, 1]\}) \le \eta \\ (c) \quad P_n(\{x; w'_{\mathbb{D}}(x, \delta) \ge \varepsilon) \le \eta. \end{cases}$$

We will show that (a)-(c) hold with $n_0 = 1$. By Theorem 3.2, $(P_n)_{n\geq 1}$ is relatively compact. Hence, there exists a compact set K in $\mathbb{D}([0,1];\mathbb{D})$ such that $P_n(K) \geq 1-\eta$ for all $n \geq 1$. The set K is characterized using Theorem 2.8. More precisely, we know that:

(25)
$$\begin{cases} (a') \quad \sup_{x \in K} ||x||_{\mathbb{D}} < \infty \\ (b') \quad \lim_{\delta \to 0} \sup_{x \in K} \sup_{t \in [0,1]} w'(x(t), \delta) = 0 \\ (c') \quad \lim_{\delta \to 0} \sup_{x \in K} w'_{\mathbb{D}}(x, \delta) = 0 \end{cases}$$

Due to (a'), we can choose $a > \sup_{x \in K} ||x||_{\mathbb{D}}$ arbitrary. Then $K \subset \{x; ||x||_{\mathbb{D}} < a\}$ and so,

$$P_n(\{x; \|x\|_{\mathbb{D}} \ge a\}) \le P_n(K^c) \le \eta \quad \text{for all } n \ge 1.$$

By (b'), there exists $\delta \in (0, 1)$ such that $w'(x(t), \delta) < \varepsilon$ for all $x \in K, t \in [0, 1]$. Hence, $K \subset \{x; w'(x(t), \delta) < \varepsilon$ for all $t \in [0, 1]\}$, and so

$$P_n(\{x; w'(x(t), \delta) < \varepsilon \text{ for some } t \in [0, 1]\}) \le P_n(K^c) \le \eta \quad \text{for all } n \ge 1.$$

By (c'), there exists $\delta \in (0,1)$ such that $w'_{\mathbb{D}}(x,\delta) < \varepsilon$ for all $x \in K$. Hence, $K \subset \{x; w'_{\mathbb{D}}(x,\delta) < \varepsilon\}$, and so

$$P_n(\{x; w'(x, \delta) < \varepsilon\}) \le P_n(K^c) \le \eta \quad \text{for all } n \ge 1.$$

Suppose next that conditions (i)–(iii) hold. Let $\eta > 0$ and $\varepsilon > 0$ be arbitrary. Then there exist a' > 0, $\delta' \in (0,1)$ and an integer $n_0 \ge 1$ such that (24) holds for all $n \ge n_0$ (with a' and δ' replacing a and δ). We first prove that (24) actually holds for all $n \ge 1$, for some values a and δ which will be given below. Fix $i \in \{1, \ldots, n_0 - 1\}$. Since $\mathbb{D}([0, 1]; \mathbb{D})$ is separable and complete, the single probability measure P_i is tight, and therefore it satisfies conditions (i)–(iii). Hence, there exists $a_i > 0$ and $\delta_i \in (0, 1)$ such that

$$\begin{cases} P_i(\{x; \|x\|_{\mathbb{D}} \ge a_i\}) \le \eta\\ P_i(\{x; w'(x(t), \delta_i) \ge \varepsilon \text{ for some } t \in [0, 1]\}) \le \eta\\ P_i(\{x; w'_{\mathbb{D}}(x, \delta_i) \ge \varepsilon) \le \eta. \end{cases}$$

Then (24) holds for all $n \ge 1$, with $a = \max\{a', \max_{i \le n_0-1} a_i\}$ and $\delta = \min\{\delta', \min_{i \le n_0-1} \delta_i\}$.

Let $B = \{x; \|x\|_{\mathbb{D}} < a\}$. Then $P_n(B) \ge 1 - \eta$ for all $n \ge 1$. By parts (b) and (c) of (24) with $\varepsilon = 1/k$ and η replaced by $\eta/2^k$, there exists $\delta_k \in (0, 1)$ such that for all $n \ge 1$,

$$P_n(B_k) \ge 1 - \frac{\eta}{2^k}$$
 and $P_n(C_k) \ge 1 - \frac{\eta}{2^k}$,

where $B_k = \{x; \sup_{t \in [0,1]} w'(x(t), \delta_k) < 1/k\}$ and $C_k = \{x; w'_{\mathbb{D}}(x, \delta_k) < 1/k\}$. Let $A = B \cap (\bigcap_{k \ge 1} B_k) \cap (\bigcap_{k \ge 1} C_k)$ and $K = \overline{A}$. For any $n \ge 1$, $P_n(K) \ge 1$ $P_n(A) \ge 1 - 3\eta$, since

$$P_n(A^c) \le P_n(B^c) + \sum_{k \ge 1} P_n(B^c_k) + \sum_{k \ge 1} P_n(C^c_k) \le \eta + \sum_{k \ge 1} \frac{\eta}{2^k} + \sum_{k \ge 1} \frac{\eta}{2^k} = 3\eta$$

We show that K is compact in $\mathbb{D}([0,1];\mathbb{D})$. By Theorem 2.8, this is equivalent to showing that K satisfies (25). Since $||x||_{\mathbb{D}} < a$ for any $x \in B$ and $A \subset B$, we have $\sup_{x \in A} \|x\|_{\mathbb{D}} < a$. This shows that (a') holds. Note that for any $k \geq 1$, $\sup_{x \in A} \sup_{t \in [0,1]} w'(x(t), \delta_k) < 1/k$ (since $A \subset B_k$), and so (b') holds. Finally, for any $k \geq 1$, $\sup_{x \in A} w'_{\mathbb{D}}(x, \delta_k) < 1/k$ (since $A \subset C_k$), and hence (c') holds. This proves that $(P_n)_{n \geq 1}$ is tight. \Box

The following result gives a replacement for condition (i) in Theorem 3.4. This condition is the analogue of (13.6) of [5].

COROLLARY 3.5. Condition (i) of Theorem 3.4 can be replaced by the following condition:

(i') for each t in a dense subset T of [0,1] which contains 1, we have:

(26)
$$\lim_{a \to \infty} \limsup_{n \to \infty} P_n(\{x; \|x(t)\| \ge a\}) = 0.$$

Proof. Suppose that condition (i) of Theorem 3.4 holds. Then (i') clearly holds, since $\{x; ||x(t)|| \ge a\} \subset \{x; ||x||_{\mathbb{D}} \ge a\}$ for any $t \in T$.

Suppose next that conditions (i') and (iii) hold. We prove that (i) holds, using a similar argument as in the Corollary on page 140 of [5]. Let $\eta > 0$ be arbitrary. By condition (iii), there exist $\delta \in (0, 1)$ and an integer $n_1 \ge 1$ such that

(27)
$$P_n(\{x; w_{\mathbb{D}}'(x,\delta) \ge 1\}) \le \eta \quad \text{for all } n \ge n_1.$$

Let $\{t_i\}_{i=1,\ldots,v}$ be a δ -sparse set with $0 = t_0 < t_1 < \ldots < t_v = 1$ such that $w_{\mathbb{D}}(x, [t_{i-1}, t_i)) \leq w'_{\mathbb{D}}(x, \delta) + 1$ for all $i = 1, \ldots, v$. Choose points $0 = s_0 < s_1 < \ldots < s_k = 1$ such that $s_j \in T$ and $s_j - s_{j-1} < \delta$ for all $k = 1, \ldots, k$. Let $m(x) = \max_{1 \leq j \leq k} ||x(s_j)||$. By (26), $\lim_{a \to \infty} \limsup_{n \to \infty} P_n(\{x; m(x) \geq a\}) = 0$. So, there exist a > 0 and $n_2 \geq 1$ such that

(28)
$$P_n(\{x; m(x) \ge a\}) \le \eta \quad \text{for all } n \ge n_2.$$

We claim that for any $x \in \mathbb{D}([0,1];\mathbb{D})$,

(29)
$$||x||_{\mathbb{D}} \le w'_{\mathbb{D}}(x,\delta) + 1 + m(x).$$

To see this, note that since $\{t_i\}_i$ is δ -sparse, each interval $[t_{i-1}, t_i)$ contains at least one point s_j , that we call s_{j_i} . For any $i = 1, \ldots, v$ and for any $t \in [t_{i-1}, t_i)$,

$$\begin{aligned} \|x(t)\| &= d_{J_1}^0\left(x(t), 0\right) \le d_{J_1}^0\left(x(t), x(s_{j_i})\right) + d_{J_1}^0\left(x(s_{j_i}), 0\right) \\ &= d_{J_1}^0\left(x(t), x(s_{j_i})\right) + \|x(s_{j_i})\|. \end{aligned}$$

Hence,

$$\sup_{t \in [t_{i-1}, t_i)} \|x(t)\| \le w_{\mathbb{D}}(x, [t_{i-1}, t_i)) + \|x(s_{j_i})\| \le w'_{\mathbb{D}}(x, \delta) + 1 + m(x).$$

Relation (29) follows since $||x||_{\mathbb{D}} = \max\{\max_{1 \le i \le v} \sup_{t \in [t_{i-1}, t_i)} ||x(t)||, ||x(1)||\}$. Let $n_0 = \max(n_1, n_2)$. From (27), (28) and (29), we infer that

 $P_n(\{x; \|x\|_{\mathbb{D}} \ge a+2\}) \le P_n(\{x; w'_{\mathbb{D}}(x, \delta) + m(x) \ge a+1\}) \le 2\eta \quad \text{for all } n \ge n_0.$ This concludes the proof of (i). \square

The following result is the analogue of relation (13.8) of [5] (or Theorem 15.3 of [4]), and it plays a crucial role in article [1] (see Theorem 2.4 of [1]).

THEOREM 3.6. A sequence $(P_n)_{n>1}$ of probability measures on $(\mathbb{D}([0,1];$ \mathbb{D}), $\mathcal{D}_{\mathbb{D}}$) is tight if and only if it satisfies condition (i) of Theorem 3.4 and the following two conditions:

(ii') For any $\varepsilon > 0$,

- $(a) \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; w''(x(t), \delta) \ge \varepsilon \text{ for some } t \in [0, 1]\}) = 0;$
- $\begin{cases} \text{(b)} & \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; |x(t,\delta) x(t,0)| \ge \varepsilon \text{ for some } t \in [0,1]\}) = 0; \\ \text{(c)} & \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; |x(t,1-) x(t,1-\delta)| \ge \varepsilon \end{cases} \end{cases}$ for some $t \in [0, 1]$ = 0.

(iii') For any $\varepsilon > 0$,

- $\begin{cases} (\mathbf{a}) & \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; w_{\mathbb{D}}''(x, \delta) \ge \varepsilon\}) = 0; \\ (\mathbf{b}) & \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; d_{J_1}^0(x(\delta), x(0)) \ge \varepsilon\}) = 0; \\ (\mathbf{c}) & \lim_{\delta \to 0} \limsup_{n \to \infty} P_n(\{x; d_{J_1}^0(x(1-), x(1-\delta)) \ge \varepsilon\}) = 0. \end{cases}$

Proof. This follows directly from Theorem 3.4. To see this, note that (ii') is equivalent to (ii) of Theorem 3.4, due to inequalities (12.31) and (12.32)of [5], whereas (iii') is equivalent to (iii) of Theorem 3.4, due to inequalities (17) and (18).

The following result is the analogue of Theorem 13.3 of [5].

THEOREM 3.7. Let $(P_n)_{n\geq 1}$ and P be probability measures on $\mathbb{D}([0,1];\mathbb{D})$ such that (20) holds, $(P_n)_{n\geq 1}$ satisfies parts (ii') and (iii'.a) of Theorem 3.6, and P satisfies

(30)
$$\lim_{\delta \to 0} P(\{x; d_{J_1}^0(x(1), x(1-\delta)) \ge \varepsilon\}) = 0 \quad for \ all \ \varepsilon > 0.$$

Then $P_n \xrightarrow{w} P$.

Proof. By Theorem 13.1, it is enough to prove that $(P_n)_{n\geq 1}$ is tight. For this, we use Theorem 3.6. We first check condition (i') given by Corollary 3.5, with $T = T_P$. Let $t \in T_P$ be arbitrary. The sequence $\{P_n \circ (\pi_t^{\mathbb{D}})^{-1}\}_{n \geq 1}$ is relatively compact in \mathbb{D} being weakly convergent. By Prohorov theorem, this sequence is tight. Hence, for any $\eta > 0$, there exists a compact set K in \mathbb{D} such that $[P_n \circ (\pi_t^{\mathbb{D}})^{-1}](K^c) \leq \eta$ for all $n \geq 1$. By Theorem 12.3 of [5], $M := \sup_{y \in K} \|y\| < \infty$. For any a > M, $\{y \in \mathbb{D}; \|y\| \geq a\} \subset K^c$ and

$$P_n\big(\{x; \|x(t)\| \ge a\}\big) \le [P_n \circ (\pi_t^{\mathbb{D}})^{-1}](K^c) \le \eta \quad \text{for all } n \ge 1$$

Next, we check that part (b) of (iii') holds. Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. By the right continuity of elements in $\mathbb{D}([0,1];\mathbb{D})$, $P(\{x; d_{J_1}^0(x(\delta), x(0)) \ge \varepsilon\}) \to 0$ as $\delta \to 0$. Choose $\delta \in T_P$ small such that $P(\{x; d_{J_1}^0(x(\delta), x(0))\}) < \eta$. By (20), $P_n \circ (\pi_{0,\delta}^{\mathbb{D}})^{-1} \xrightarrow{w} P \circ (\pi_{0,\delta}^{\mathbb{D}})^{-1}$ in \mathbb{D}^2 . By Lemma 2.13, the set $A = \{(y_1, y_2) \in \mathbb{D}^2; d_{J_1}^0(y_1, y_2) \ge \varepsilon\}$ is closed in \mathbb{D}^2 with respect to the product of J_1 -topologies. By Portmanteau theorem, it follows that

$$\limsup_{n \to \infty} P_n(\{x; d_{J_1}^0(x(\delta), x(0))\}) \le P(\{x; d_{J_1}^0(x(\delta), x(0))\}) < \eta$$

We prove that part (c) of (iii') holds. By the left continuity of elements in $\mathbb{D}([0,1];\mathbb{D}), P(\{x; d_{J_1}^0(x(1-), x(1-\delta)) \ge \varepsilon\}) \to 0$ as $\delta \to 0$, for any $\varepsilon > 0$. By (30), it follows that $P(\{x; d_{J_1}^0(x(1), x(1-)) \ge \varepsilon\}) = 0$, for any $\varepsilon > 0$. Hence, $P(\{x; d_{J_1}^0(x(1), x(1-)) > 0\}) = 0$. The rest of the argument is the same as for part (b). \Box

The previous theorem can also be stated in terms of random elements, as follows.

THEOREM 3.8. Let $(X_n)_{n\geq 1}$ and X be random elements in $\mathbb{D}([0,1];\mathbb{D})$ defined on the same probability space. Let $T_X = \{t \in [0,1]; P(X(t) = X(t-)) = 1\}$. Suppose that:

a) $(X_n(t_1), \ldots, X_n(t_k)) \xrightarrow{d} (X(t_1), \ldots, X(t_k))$ in \mathbb{D}^k , for any $t_1, \ldots, t_k \in T_X$;

b)
$$d_{J_1}^0(X(1), X(1-\delta)) \xrightarrow{P} 0 \text{ as } \delta \to 0;$$

c) for any $\varepsilon > 0$,

 $\begin{cases} \lim_{\delta \to 0} \limsup_{n \to \infty} P(\{w''(X_n(t), \delta) \ge \varepsilon \text{ for some } t \in [0, 1]\}) = 0, \\ \lim_{\delta \to 0} \limsup_{n \to \infty} P(|X_n(t, \delta) - X_n(t, 0)| \ge \varepsilon \text{ for some } t \in [0, 1]) = 0, \\ \lim_{\delta \to 0} \limsup_{n \to \infty} P(|X_n(t, 1-) - X_n(t, 1-\delta)| \ge \varepsilon \text{ for some } t \in [0, 1]) = 0; \end{cases}$

d) for any
$$\varepsilon > 0$$
,

(31)
$$\lim_{\delta \to 0} \limsup_{n \to \infty} P(w_{\mathbb{D}}''(X_n, \delta) \ge \varepsilon) = 0 \quad for \ all \ \varepsilon > 0.$$

Then $X_n \xrightarrow{d} X$ in $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$.

Remark. Hypothesis c) of Theorem 3.8 may be difficult to verify in practice. In the proof of Theorem 3.14 of [1], this hypothesis is verified by showing that

(32)
$$\inf_{n_0 \ge 1} \sup_{n \ge n_0} P(\|X_n - X_{n_0}\|_{\mathbb{D}} \ge \varepsilon) = 0 \quad \text{for all } \varepsilon > 0.$$

Since for any $n_0 \ge 1$, the single probability measure $P \circ X_{n_0}^{-1}$ is tight in $\mathbb{D}([0,1];\mathbb{D})$, part (ii') of Theorem 3.6 gives:

$$\lim_{\delta \to 0} P(\{w''(X_{n_0}(t), \delta) \ge \varepsilon \text{ for some } t \in [0, 1]\}) = 0, \\ \lim_{\delta \to 0} P(|X_{n_0}(t, \delta) - X_{n_0}(t, 0)| \ge \varepsilon \text{ for some } t \in [0, 1]) = 0, \\ \lim_{\delta \to 0} P(|X_{n_0}(t, 1-) - X_{n_0}(t, 1-\delta)| \ge \varepsilon \text{ for some } t \in [0, 1]) = 0;$$

Hypothesis c) then follows from (32), using the following inequalities:

$$w''(X_n(t),\delta) \le w''(X_{n_0}(t),\delta) + 2||X_n - X_{n_0}||_{\mathbb{D}}$$

$$|X_n(t,\delta) - X_n(t,0)| \le |X_{n_0}(t,\delta) - X_{n_0}(t,0)| + 2||X_n - X_{n_0}||_{\mathbb{D}}$$

$$|X_n(t,1-) - X_n(t,1-\delta)| \le |X_{n_0}(t,1-) - X_{n_0}(t,1-\delta)| + 2||X_n - X_{n_0}||_{\mathbb{D}}.$$

4. CRITERIA FOR EXISTENCE AND CONVERGENCE

In this section, we give a criterion for weak convergence of random elements in $\mathbb{D}([0,1];\mathbb{D})$, and a criterion for the existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$ based on its finite-dimensional distributions. Both these results rely on some maximal inequalities which are of independent interest.

The first two results are analogue of Theorems 10.3 and 10.4 of [5], stated in terms of the Skorohod distance $d_{J_1}^0$.

THEOREM 4.1. Let T be a Borel set in [0,1] and $\{X(t)\}_{t\in T}$ a collection of random elements in \mathbb{D} defined on the same probability space (Ω, \mathcal{F}, P) such that the map $T \ni t \mapsto X(\omega, t)$ is right-continuous with respect to J_1 , for any $\omega \in \Omega$. (If T is finite, this imposes no restriction.) For any $r, s, t \in T$ with $r \leq s \leq t$, let

(33)
$$m_{rst}^{J_1} = d_{J_1}^0 \left(X(r), X(s) \right) \wedge d_{J_1}^0 \left(X(s), X(t) \right)$$

and $L_{J_1}(X) = \sup_{r,s,t\in T; r\leq s\leq t} m_{rst}^{J_1}$. Suppose that there exist $\alpha > 1/2, \beta \geq 0$ and a finite measure μ on T such that for any $\lambda > 0$ and for any $r, s, t \in T$ with $r \leq s \leq t$,

(34)
$$P(m_{rst}^{J_1} \ge \lambda) \le \frac{1}{\lambda^{4\beta}} \{\mu(T \cap (r, t])\}^{2\alpha}.$$

Then there exists a constant K depending on α and β such that for any $\lambda > 0$,

(35)
$$P(L_{J_1}(X) > \lambda) \le \frac{K}{\lambda^{4\beta}} \mu^{2\alpha}(T).$$

Proof. We follow the same idea as in the proof of Theorem 10.3 of [5], replacing increments of the form |X(t) - X(s)| by $d_{J_1}^0(X(t), X(s))$.

Case 1. T = [0, 1] and μ is the Lebesgue measure. Let $D_k = \{i/2^k; 0 \le i \le 2^k\}$. Define B_k be the maximum of all $m_{t_1 t_2 t_3}^{J_1}$ for all $t_1, t_2, t_3 \in D_k$ with $t_1 \le t_2 \le t_3$ and A_k be the maximum of $m_{t_1 t_2 t_3}^{J_1}$ with $t_1 = (i-1)/2^k$, $t_2 = i/2^k$ and $t_3 = (i+1)/2^k$, for $i = 1, \ldots, 2^k - 1$. It can be proved that $B_k \le 2(A_1 + \ldots + A_k)$ for any $k \ge 1$. Note that $B_k \le B_{k+1}$ for all $k \ge 1$. We claim that:

(36)
$$L_{J_1}(X) = \lim_{k \to \infty} B_k.$$

To see this, let $\varepsilon > 0$ be arbitrary. Let $t_1, t_2, t_3 \in T$ be such that $t_1 \leq t_2 \leq t_3$. For each $k \geq 1$, there exist $t_1^k, t_2^k, t_3^k \in D_k$ with $t_1^k \leq t_2^k \leq t_3^k$ such that $t_i^k \downarrow t_i$ as $k \to \infty$, for i = 1, 2, 3. Since $t \mapsto X(t)$ is right-continuous with respect to $J_1, X(t_i^k) \xrightarrow{J_1} X(t_i)$ as $k \to \infty$, for i = 1, 2, 3. By Lemma 2.13, $a_k = d_{J_1}^0(X(t_1^k), X(t_2^k)) \to a = d_{J_1}^0(X(t_1), X(t_2))$ as $k \to \infty$ and $b_k = d_{J_1}^0(X(t_2^k), X(t_3^k)) \to b = d_{J_1}^0(X(t_2), X(t_3))$ as $k \to \infty$. Hence, there exists k_{ε} such that $a_{k_e} \geq a - \varepsilon$ and $b_{k_{\varepsilon}} \geq b - \varepsilon$. So, $a \land b \leq a_{k_e} \land b_{k_e} + \varepsilon \leq B_{k_e} + \varepsilon$. Since t_1, t_2, t_3 were arbitrary, we obtain that $L_{J_1}(X) \leq B_{k_e} + \varepsilon$.

From (36), it follows that $L_{J_1}(X) \leq 2 \sum_{k \geq 1} A_k$. From this, we deduce relation (35) using (34) to estimate the tail probability of A_k (see page 110 of [5]).

The other cases follow as in the proof of Theorem 10.3 of [5]. \Box

COROLLARY 4.2. If condition (34) of Theorem 4.1 only holds for $t - r < 2\delta$, then

$$P(L_{J_1}(X,\delta) > \lambda) \le \frac{2K}{\lambda^{4\beta}} \mu(T) \sup_{0 \le t \le 1-2\delta} \mu^{2\alpha-1} \big(T \cap [t,t+2\delta] \big),$$

where $L_{J_1}(X, \delta)$ is the supremum of $m_{rst}^{J_1}$ for all $r, s, t \in T$ with $r \leq s \leq t$ and $t - r < \delta$, and $m_{rst}^{J_1}$ is given by (33). In particular, if T = [0, 1], then $L_{J_1}(X, \delta) = w_{\mathbb{D}}''(X, \delta)$.

The following result gives a criterion for convergence in distribution in the space $\mathbb{D}([0,1];\mathbb{D})$. being the analogue of Theorem 13.5 of [5].

THEOREM 4.3. Let $(X_n)_{n\geq 1}$ and X be random elements in $\mathbb{D}([0,1];\mathbb{D})$ defined on the same probability space, such that hypotheses a, b, c) of Theorem 3.8 hold. If there exist $\alpha > 1/2$, $\beta \ge 0$ and a non-decreasing continuous function F on [0,1] such that for any $r, s, t \in [0,1]$ with $r \le s \le t$, for any $\lambda > 0$ and for any $n \ge 1$,

$$P(d_{J_1}^0(X_n(r), X_n(s)) \wedge d_{J_1}^0(X_n(s), X_n(t)) \ge \lambda) \le \frac{1}{\lambda^{4\beta}} [F(t) - F(r)]^{2\alpha},$$

then $X_n \xrightarrow{d} X$ in $\mathbb{D}([0,1];\mathbb{D})$ equipped with $d_{\mathbb{D}}$.

Proof. We apply Theorem 3.8. Hypothesis (31) of this theorem is verified using Corollary 4.2 with T = [0, 1]. \Box

The goal of the remaining part of this section is to give a criterion for the existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$. For this, we first need to state a variant of Theorem 4.1 using the uniform norm $\|\cdot\|$ on \mathbb{D} (instead of the Skorohod distance $d_{J_1}^0$).

THEOREM 4.4. Let T be a Borel set in [0,1] and $\{X(t)\}_{t\in T}$ a collection of random elements in \mathbb{D} defined on the same probability space (Ω, \mathcal{F}, P) such that the map $T \ni t \mapsto X(\omega, t)$ is right-continuous with respect to the uniform norm on \mathbb{D} , for any $\omega \in \Omega$. (If T is finite, this imposes no restriction.) For any $r, s, t \in T$ with $r \leq s \leq t$, let

(37)
$$m_{rst}^{u} = \|X(r) - X(s)\| \wedge \|X(s) - X(t)\|$$

and $L_u(X) = \sup_{r,s,t\in T; r\leq s\leq t} m_{rst}^u$. Suppose that there exist $\alpha > 1/2, \beta \geq 0$ and a finite measure μ on T such that for any $\lambda > 0$ and for any $r, s, t \in T$ with $r \leq s \leq t$,

(38)
$$P(m_{rst}^u \ge \lambda) \le \frac{1}{\lambda^{4\beta}} \{\mu(T \cap (r, t])\}^{2\alpha}.$$

Then there exists a constant K depending on α and β such that for any $\lambda > 0$,

(39)
$$P(L_u(X) > \lambda) \le \frac{K}{\lambda^{4\beta}} \mu^{2\alpha}(T).$$

COROLLARY 4.5. If condition (38) of Theorem 4.4 only holds for $t - r < 2\delta$, then

$$P(L_u(X,\delta) > \lambda) \le \frac{2K}{\lambda^{4\beta}} \mu(T) \sup_{0 \le t \le 1-2\delta} \mu^{2\alpha-1} \big(T \cap [t,t+2\delta] \big),$$

where $L_u(X,\delta)$ is the supremum of m_{rst}^u for all $r, s, t \in T$ with $r \leq s \leq t$ and $t - r < \delta$, and m_{rst}^u is given by (37). In particular, if T = [0,1], then $L_u(X,\delta) = w_u''(X,\delta)$, where

$$w_u''(x,\delta) = \sup_{t_1 \le t \le t_2, t_2 - t_1 \le \delta} \left(\|x(t) - x(t_1)\| \wedge \|x(t_2) - x(t)\| \right)$$

for any $x \in \mathbb{D}([0,1];\mathbb{D})$.

In the particular case when T is a finite set, we obtain the following result, which is of independent interest.

THEOREM 4.6. Let ξ_1, \ldots, ξ_n be random elements in $\mathbb{D}([0,1];\mathbb{D})$, $S_k = \xi_1 + \ldots + \xi_k$ for $k = 1, \ldots, n$, and $S_0 = 0$. Suppose that there exist $\alpha \ge 1/2$, $\beta > 0$ and $u_i \ge 0, i = 1, \ldots, n$ such that for any $\lambda > 0$,

$$P(\|S_j - S_i\| \wedge \|S_k - S_j\| \ge \lambda) \le \frac{1}{\lambda^{4\beta}} \Big(\sum_{j=i+1}^k u_j\Big)^{2\alpha}$$

Then there exist a constant K depending on α and β such that for any $\lambda > 0$,

$$P(M_n \ge \lambda) \le \frac{K}{\lambda^{2\beta}} \Big(\sum_{i=1}^n u_i\Big)^{2\alpha},$$

where $M_n = \max_{0 \le i \le j \le k \le n} (\|S_j - S_i\| \wedge \|S_k - S_j\|).$

We are now ready to state the criterion for existence of a process with sample paths in $\mathbb{D}([0,1];\mathbb{D})$.

THEOREM 4.7. Let $\{X(t)\}_{t\in[0,1]}$ be a collection of random elements in \mathbb{D} defined on the same probability space (Ω, \mathcal{F}, P) such that:

a) there exist $\alpha > 1/2$, $\beta \ge 0$ and a non-decreasing continuous function Fon [0,1] such that for any $t_1, t_2, t_3 \in [0,1]$ with $t_1 \le t_2 \le t_3$ and for any $\lambda > 0$,

$$P(||X(t) - X(t_1)|| \land ||X(t_2) - X(t)|| \ge \lambda) \le \frac{1}{\lambda^{4\beta}} [F(t_2) - F(t_1)]^{2\alpha}$$

b) for any $\varepsilon > 0$,

 $\lim_{\delta \to 0} \limsup_{n \to \infty} P(w'(X(i/2^n), \delta) \ge \varepsilon \text{ for some } 0 \le i \le 2^{-n}) = 0;$

c) for any $t \in [0,1)$ and for any sequence $(t_n)_{n\geq 1}$ in [0,1] with $t_n \to t$ and $t_{n+1} \leq t_n$ for any $n \geq 1$, $d_{J_1}^0(X(t_n), X(t)) \xrightarrow{p} 0$.

Then, there exists a collection $\{Y(t)\}_{t\in[0,1]}$ of random elements in \mathbb{D} defined on the another probability space $(\Omega', \mathcal{F}', P')$, such that the map $t \mapsto Y(\omega', t)$ is in $\mathbb{D}([0,1]; \mathbb{D})$ for any $\omega' \in \Omega'$, and the vectors $(X(t_1), \ldots, X(t_k))$ and $(Y(t_1), \ldots, Y(t_k))$ have the same distribution in \mathbb{D}^k , for any $t_1, \ldots, t_k \in [0,1]$ and for any $k \geq 1$.

Proof. We argue as in the proof of Theorem 13.6 of [5]. We consider two cases.

Case 1. Suppose that there exists $\delta_0 \in (0, 1/2)$ such that for all $h \in (0, \delta_0)$,

(40)
$$P(X(0) = X(h)) = 1$$
 and $P(X(1) = X(1-h)) = 1$.

Let $T = \bigcup_{n \ge 1} T_n$, where $T_n = \{t_i^n; i = 0, 1, \dots, 2^n\}$ and $t_i^n = i/2^n$. For any $n \ge 1$, we define

(41)
$$X_n(t) = X(t_i^n) \text{ for all } t \in [t_i^n, t_{i+1}^n)$$

for $i = 0, 1, ..., 2^n - 1$, and $X_n(1) = X(1)$. Note that

(42)
$$t \in T_k$$
 implies that $X_n(t) = X(t)$ for all $n \ge k$.

We will prove that

(43)
$$(P \circ X_n^{-1})_{n \ge 1} \text{ is tight in } \mathbb{D}([0,1];\mathbb{D}).$$

By Prohorov's theorem, it will follow that $(P \circ X_n^{-1})_{n \ge 1}$ is relatively compact in $\mathbb{D}([0,1];\mathbb{D})$. Hence, there exist a subsequence $(n_k)_{k\ge 1}$ and a probability measure Q on $(\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}})$ such that $P_{n_k} \stackrel{w}{\to} Q$. Let $\{Y(t)\}_{t\in[0,1]}$ be a collection of random elements in \mathbb{D} with law Q, defined on the other probability space $(\Omega', \mathcal{F}', P')$. For instance, we may take $(\Omega', \mathcal{F}', P') = (\mathbb{D}([0,1];\mathbb{D}), \mathcal{D}_{\mathbb{D}}, Q)$ and $Y(t) = \pi_t^{\mathbb{D}}$ for all $t \in [0,1]$. Then $(X(t_1), \ldots, X(t_k))$ and $(Y(t_1), \ldots, Y(t_k))$ have the same distribution in \mathbb{D}^k , for any $t_1, \ldots, t_k \in A$, and the same thing remains true for arbitrary points t_1, \ldots, t_k in [0,1] due to hypothesis c) and the right continuity of the sample paths of $\{Y(t)\}_{t\in[0,1]}$ with respect to J_1 , since for each $t_i \in [0,1), i = 1, \ldots, k$ there exists a sequence $(t_i^m)_{m\ge 1} \subset A$ such that $t_i^m \downarrow t_i$ as $m \to \infty$.

It remains to prove (43). For this, we apply Theorem 3.6 to $P_n = P \circ X_n^{-1}$. Condition (ii') of this theorem is equivalent to condition (ii) of Theorem 3.4, which is the same as our hypothesis b) (using definition (41) of $X_n(t)$).

We begin by checking condition (iii') of Theorem 3.6. Let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. We prove that there exist $\delta \in (0, 1)$ and an integer $n_0 \ge 1$ such that for all $n \ge n_0$,

(44)
$$\begin{cases} (a) \quad P(w_{\mathbb{D}}^{\prime\prime}(X_n,\delta) \ge \varepsilon) \le \eta \\ (b) \quad P(d_{J_1}^0(X_n(\delta),X_n(0)) \ge \varepsilon) \le \eta \\ (c) \quad P(d_{J_1}^0(X_n(1-),X_n(1-\delta)) \ge \varepsilon) \le \eta. \end{cases}$$

For part b), let $\delta < \delta_0$ and $n \ge 1$ be arbitrary. Choose k such that $k/2^n \le \delta < (k+1)/2^n$. Then $X_n(\delta) = X(k/2^n) = X(0)$ a.s. and $X_n(0) = X(0)$. Hence $d_{J_1}^0(X_n(\delta), X_n(0)) = 0$ a.s. For part c), let $\delta < \delta_0/2$ and $n \ge n_0$ where n_0 is such that $2^{-n_0} \le \delta_0/2$. Choose l such that $l/2^n \le 1 - \delta < (l+1)/2^n$. Then $X_n(1-\delta) = X(l/2^n) = X(1)$ a.s. since $1 - l/2^n = \delta + (1 - \delta - l/2^n) < \delta + 1/2^n < \delta_0$. Since $\delta < \delta_0$ is arbitrary, this also shows that $X_n(1-) = X(1)$ a.s. for any $n \ge n_0$. Hence, $d_{J_1}^0(X_n(1-), X_n(1-\delta)) = 0$ a.s.

To prove part (a) of (44), it suffices to show that $P(w''_u(X_n, \delta) \ge \varepsilon) \le \eta$ since $w''_{\mathbb{D}}(x, \delta) \le w''_u(x, \delta)$ for any $x \in \mathbb{D}([0, 1]; \mathbb{D})$. This can be proved exactly as on page 144 of [5], by applying Corollary 4.5 to the discrete-time process $\{Y_n(t)\}_{t\in T_n}$ given by $Y_n(t) = X_n(t) = X(t)$, and the measure μ_n on T_n given by $\mu_n(\{t_i^n\}) = F(t_i^n) - F(t_{i-1}^n)$. The process Y_n satisfies hypothesis (38) of Corollary 4.5, due to our hypothesis a). Note that $w''_u(X_n, \delta) \le L(Y_n, 2\delta)$.

Finally, we prove that condition (i) of Theorem 3.4 holds. Let $\eta > 0$ be arbitrary. We will prove that there exist a > 0 and an integer $n_0 \ge 1$ such that

$$P(||X_n||_{\mathbb{D}} \ge a) \le 2\eta$$
 for all $n \ge n_0$.

Let $\varepsilon > 0$ be arbitrary. Choose $\delta \in (0,1)$ and $n_0 \ge 1$ such that part (a)

of (44) holds. Choose $k \ge 1$ such that $2^{-k} \le \delta$. We claim that for all $n \ge k$,

(45)
$$||X_n||_{\mathbb{D}} \le w''_{\mathbb{D}}(X_n, \delta) + \max_{i \le 2^k} ||X(i/2^k)||$$

To see this, note that clearly $||X_n(1)|| \leq \max_{i \leq 2^k} ||X(i/2^k)||$. Let $t \in [0,1)$ be arbitrary. Say $i/2^k \leq t < (i+1)/2^k$. We have two situations: $d_{J_1}^0(X_n(t), X_n(i/2^k))$ is either smaller or larger than $d_{J_1}^0(X_n(t), X_n((i+1)/2^k))$. We consider only the case when it is smaller, the other case being similar. By (5) and the triangle inequality in \mathbb{D} , we have

$$||X_n(t)|| \le d_{J_1}^0 (X_n(t), X_n(i/2^k)) + ||X_n(i/2^k)|| \le w_{\mathbb{D}}''(X_n, \delta) + \max_{i \le 2^k} ||X_n(i/2^k)||.$$

From (45) and part (a) of (44), it follows that for all $n \ge n_0$,

$$P(||X_n||_{\mathbb{D}} > a) \le P(w''_{\mathbb{D}}(X_n, \delta) + \max_{i \le 2^k} ||X(i/2^k)|| > a, w''_{\mathbb{D}}(X_n, \delta) < \varepsilon) + P(w''_{\mathbb{D}}(X_n, \delta) \ge \varepsilon) \le P(\max_{i \le 2^k} ||X(i/2^k)|| > a - \varepsilon) + P(w''_{\mathbb{D}}(X_n, \delta) \ge \varepsilon) \le 2\eta,$$

for all $a > a_0$ and some $a_0 > \varepsilon$ large enough, since $\lim_{A\to\infty} P(\max_{i\leq 2^k} ||X(i/2^k)|| > A) = 0.$

Case 2. In the absence of condition (40), let $\delta_0 \in (0, 1/2)$ be arbitrary. For any $t \in [0, 1]$, define $\widetilde{X}(t) = X(f(t))$ where

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, \delta_0) \\ (t - \delta_0) / (1 - 2\delta_0) & \text{if } t \in [\delta_0, 1 - \delta_0] \\ 1 & \text{if } t \in (1 - \delta_0, 1] \end{cases}$$

Since the map $\phi : [\delta_0, 1 - \delta_0] \to [0, 1]$ given by $\phi(t) = (t - \delta_0)/(1 - 2\delta_0)$ is a bijection, $X(s) = \widetilde{X}(\delta_0 + (1 - 2\delta_0)s)$ for all $s \in [0, 1]$. The process \widetilde{X} satisfies hypotheses a), b), c) of the theorem, and also condition (40). Therefore, by Case 1, there exists a collection $\{\widetilde{Y}(t)\}_{t\in[0,1]}$ of random elements in \mathbb{D} defined on the another probability space $(\Omega', \mathcal{F}', P')$, such that the map $t \mapsto \widetilde{Y}(\omega', t)$ is in $\mathbb{D}([0,1];\mathbb{D})$ for any $\omega' \in \Omega'$, and the vectors $(\widetilde{X}(t_1), \ldots, \widetilde{X}(t_k))$ and $(\widetilde{Y}(t_1), \ldots, \widetilde{Y}(t_k))$ have the same distribution in \mathbb{D}^k , for any $t_1, \ldots, t_k \in [0,1]$ and for any $k \geq 1$. We define $Y(s) = \widetilde{Y}(\delta_0 + (1 - 2\delta_0)s)$ for all $s \in [0,1]$. \Box

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REFERENCES

[1] R.M. Balan and B. Saidani, Stable Lévy motion with values in the Skorokhod space: construction and approximation. J. Theoret. Probab. To appear.

- [2] A. Basse-O'Connor and J. Rosinski, On the uniform convergence of random series in Skorohod space and representation of càdlàg infinitely divisible processes. Ann. Probab. 41 (2013), 4317-4341.
- [3] P. Billingsley, Probability and Measure. Third Edition. John Wiley, New York, 1995.
- [4] P. Billingsley, Convergence of Probability Measures. John Wiley, New York, 1968.
- [5] P. Billingsley, Convergence of Probability Measures. Second Edition. John Wiley, New York, 1999.
- [6] L. de Haan and T. Lin, T, On convergence toward an extreme value distribution in C[0;1]. Ann. Probab. 29 (2001), 467–483.
- M. Donsker, An invariance principle for certain probability limit theorems. In:
 K.L. Chung et al. (Eds.), Four Papers on Probability. Mem. Amer. Math. Soc.
 6, Amer. Math. Soc., 1951.
- [8] S.N. Ethier and T.G. Kurtz, Markov Processes. Characterization and Convergence. John Wiley, New York, 1986.
- H. Hult and F. Lindskog, Extremal behaviour of regularly varying stochastic processes. Stochastic Process. Appl. 115 (2005), 249-274.
- [10] O. Kallenberg, Random Measures. Third edition. Academic Press, London, 1983.
- [11] A.V. Skorokhod, Limit theorems for stochastic processes. Theory Probab. Appl. 1 (1956), 261-290.
- [12] A.V. Skorokhod, Limit theorems for stochastic processes with independent increments. Theory Probab. Appl. 2 (1957), 138–171.
- [13] W. Whitt, Some useful functions for functional limit theorems. Math. Oper. Res. 5 (1980), 67-85.

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