Dedicated to Vasile Brînzănescu on his 75^{th} birthday

THE GEOMETRY OF A MODULI SPACE OF BUNDLES

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Let X be a class VII surface with $b_2(X) > 0$. Following ideas developed in previous articles, we study the moduli space

$$\mathcal{M}_X \coloneqq \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)\,,$$

where E is a differentiable rank 2 bundle on X with $c_2(E) = 0$, and $\det(E) = K_X$, the underlying differentiable line bundle of the canonical line bundle $\mathcal{K}_X$. In this article we are interested in the non-minimal case: assuming that the minimal model of X is a primary Hopf surface, we prove that any point in the moduli space is a line bundle extension, and we give explicit geometric descriptions of $\mathcal{M}_X$ for $b_2(X) \in \{1, 2\}$.

Our motivation comes from the classification theory of class VII surfaces. Let X_0 be a minimal class VII surface with positive b_2 which is the deformation in large of a family of blown up primary Hopf surfaces. In other words X_0 is the central fiber of a holomorphic family $(X_z)_{z\in D}$, where X_z is a blown up primary Hopf surface for any $z \neq 0$. The classification of minimal class VII surfaces with this property is still an open problem.

The moduli space $\mathcal{M}_{X_0}$ associated with an *unknown* such surface X_0 will be "the limit" of the family $(\mathcal{M}_{X_z})_{z\in D^\bullet}$ of moduli spaces associated with blown up primary Hopf surfaces.

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CONTENTS

1. INTRODUCTION

In this section we explain the notions, the terminology and the formalism used in this article, and we present the main results and their motivation.

1.1. Moduli spaces of polystable bundles on Gauduchon surfaces

By surface we mean a compact, connected, complex manifold of dimension 2 [3]. It is well known ([26], [11], [12]) that a complex surface is Kählerian, i.e. it admits a Kähler metric, if and only if $b_1(X)$ is even.

Let X be a complex surface. We will denote by $\mathcal{K}_X$ the canonical line bundle of X, i.e. the line bundle of holomorphic 2-forms. The Picard group of X is the group of isomorphism classes of holomorphic line bundles on X; it can be identified with $H^1(X, \mathcal{O}^*_X)$, has the natural structure of a complex Lie group, and will be denoted by $\mathrm{Pic}(X)$. The first Chern class map defines a group morphism

$$c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$$

whose image is the Neron-Severi group

$$\mathrm{NS}(X) := \ker(H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X)) .$$

For a class $c \in \mathrm{NS}(X)$ we denote by $\mathrm{Pic}^c(X) \subset \mathrm{Pic}(X)$ the fiber over c of the morphism c_1. The kernel $\mathrm{Pic}^0(X)$ of this morphism is the identity component of $\mathrm{Pic}(X)$; it can be identified with the quotient $H^1(X, \mathcal{O}_X)/2\pi i H^1(X, \mathbb{Z})$, and is compact if and only if $b_1(X)$ is even (see for instance [35, Appendix]).

A Hermitian metric g on X is called Gauduchon if $dd^c\omega_g = 0$, where $\omega_g \in A^{1,1}(X)$ is the Kähler form of g. An important result of Gauduchon [17] states that any conformal class of Hermitian metrics contains such a metric, so there is no obstruction to the existence of Gauduchon metrics. A Gauduchon metric g on X gives a degree map

$$\deg_g : \mathrm{Pic}(X) \to \mathbb{R} \tag{1}$$

defined by

$$\deg_g([\mathcal{L}]) = \int_X c_1(\mathcal{L}, h) \wedge \omega_g ,$$

where h is a Hermitian metric on $\mathcal{L}$, and $c_1(\mathcal{L}, h)$ is the first Chern form of the Chern connection associated with h. Since $dd^c\omega_g = 0$, the right hand term in

(1) is independent of h, so $\deg_g$ is well defined. The degree map is a morphism of real Lie groups; it is a topological invariant (i.e. it vanishes on $\mathrm{Pic}^0(X)$) if and only if $b_1(X)$ is even.

For a coherent sheaf $\mathcal{F}$ on X one defines

$$\deg_g(\mathcal{F}) := \deg_g(\det(\mathcal{F}))\,,$$

where $\det(\mathcal{F})$ is the determinant line bundle (invertible sheaf) of $\mathcal{F}$ (see [20]).

Definition 1.1. Let (X, g) be a complex surface endowed with a Gauduchon metric. A holomorphic bundle $\mathcal{E}$ on X is called

1. stable, if, for any non-trivial coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with torsion free quotient, one has

$$\frac{1}{\mathrm{rk}(\mathcal{F})}\deg_g(\mathcal{F}) < \frac{1}{\mathrm{rk}(\mathcal{E})}\deg_g(\mathcal{E})\,.$$

2. polystable, if it decomposes as a direct sum $\mathcal{E} = \bigoplus_{i=1}^k \mathcal{E}_i$, where $\mathcal{E}_i$ are stable bundles with $\frac{1}{\mathrm{rk}(\mathcal{E}_i)}\deg_g(\mathcal{E}_i) = \frac{1}{\mathrm{rk}(\mathcal{E})}\deg_g(\mathcal{E})$.

The Kobayashi-Hitchin correspondence ([15], [14], [10], [21], [22]) states that a holomorphic vector bundle $\mathcal{E}$ is polystable if and only if it admits a Hermitian metric h such that the associated Chern connection A_h is Hermite-Einstein.

Let (E, h) be a differentiable Hermitian vector bundle of rank r over X. Fix a holomorphic structure $\mathcal{D}$ on the determinant line bundle $\det(E)$. We will denote by $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$, $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ the moduli spaces of stable, respectively polystable holomorphic structures on E inducing the fixed holomorphic structure $\mathcal{D}$ on $\det(E)$, modulo the complex gauge group of SL-automorphisms of E.

Note that $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ has the natural structure of a complex space [22], but, in the non-Kählerian framework, $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is not always a complex space (see [29], [32], [9] and the results in this article).

Let a be the Chern connection of the pair $(\mathcal{D}, \det(h))$, and let $\mathcal{M}^{\mathrm{ASD}}_a(E, h)$ ($\mathcal{M}^{\mathrm{ASD}}_a(E, h)^*$) be the moduli space of (respectively irreducible) projectively ASD Hermitian connections on (E, h) inducing the fixed connection a on $\det(E)$, modulo the real gauge group of SU-automorphisms of E (see for instance [29], [34] for details).

The Kobayashi-Hitchin correspondence can be reformulated in terms of moduli spaces [22]: it gives a bijection

$$KH : \mathcal{M}^{\mathrm{ASD}}_a(E) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$$

which restricts to a real analytic isomorphism $\mathcal{M}_a^{\mathrm{ASD}}(E)^* \xrightarrow{\simeq} \mathcal{M}_{\mathcal{D}}^{\mathrm{st}}(E)$. We will endow $\mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E)$ with the topology which makes KH a homeomorphism; this topology is metrizable, because the instanton moduli space $\mathcal{M}_a^{\mathrm{ASD}}(E)$ has this property [14].

The complement

$$\mathcal{R} := \mathcal{M}_{\mathcal{D}}^{\mathrm{pst}}(E) \setminus \mathcal{M}_{\mathcal{D}}^{\mathrm{st}}(E)$$

is the space of isomorphism classes of split polystable bundles in the moduli space, and can be identified via the Kobayashi-Hitchin correspondence with the space of reducible projectively ASD Hermitian connections on (E, h). This complement will be called *the space of reductions* in the moduli space.

Stability theory for bundles has been studied intensively on projective algebraic surfaces: in this framework the theory found important applications in 4-dimensional differential topology, for instance the first computation of Donaldson invariants [15]. Moreover, in algebraic geometry we have classical tools which, in many cases, allow a complete classification of bundles with fixed topological invariants.

In the non-algebraic, and especially non-Kählerian framework, the situation is more difficult. A major difficulty is the appearance of non-filtrable bundles, for which we do not have a general classification method (see section 1.4). However nowadays several explicit descriptions of moduli spaces of stable and polystable bundles on non-Kählerian surfaces are known. We mention in particular [4], [28], [22, Sections 6.3, 6.4], [1], [2], [5]–[6].

This article is concerned with certain moduli spaces of polystable bundles on class VII surfaces which play an important role in our programme to complete the classification of this class of surfaces up to deformation equivalence [32], [34].

1.2. **Class VII surfaces**

We recall that, in the theory of complex surfaces [3], the class VII is the class of surfaces X with $b_1(X) = 1$ and $\kappa(X) = -\infty$. The former condition is topological, and implies that such a surface is not Kählerian. The latter is equivalent to the vanishing of $h^0(\mathcal{K}_X^{\otimes n})$, for all positive integers n.

Class VII surfaces are not classified yet. It is known [27] that any class VII surface with $b_2 = 0$ is biholomorphic to either a Hopf surface, or an Inoue surface, but the method of proof does not generalize to surfaces with positive b_2. We have an interesting class of "known" minimal class VII surfaces with $b_2 > 0$, namely the Kato surfaces. By definition, a Kato surface is a minimal class VII surface X with positive b_2 which contains a global spherical shell, i.e. an open submanifold which does not disconnect the surface, and is biholomorphic to a standard neighborhood of S^3 in $\mathbb{C}^2$. Note that Kato surfaces are well understood ([18], [19], [13] [35]).

The classification will be completed if the following fundamental conjecture of the theory is proved:

CONJECTURE. *Any minimal class VII surface with $b_2 > 0$ is a Kato surface.*

The conjecture has been proved for $b_2 = 1$ [29]. Note that all Kato surfaces with fixed b_2 are deformation equivalent, and they are deformations in large of blown up primary Hopf surfaces; more precisely for any Kato surface X there exists a proper holomorphic submersion $p : \mathcal{X} \to D$ such that $p^{-1}(0) \simeq X$, and $p^{-1}(z)$ is a blown up primary Hopf surface for any $z \neq 0$.

A weaker conjecture, which will solve the classification problem up to deformation equivalence, states that *any minimal class VII surface with $b_2 > 0$ is a deformation in large of blown up primary Hopf surfaces.*

This weaker conjecture has been proved for $b_2 \leqslant 3$ (see [32], [34]) using ideas introduced in [29]: For an *unknown minimal* class VII surface X one studies geometric properties of the moduli space $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$, where E is a rank 2 differentiable bundle on X with $c_2(E) = 0$, $\det(E) = K_X$, where K_X is the underlying differentiable line bundle of the canonical holomorphic line bundle $\mathcal{K}_X$.

In this article we will study the same moduli space, but on a surface which is *a blown up primary Hopf surface*, hence a known surface. We believe that the obtained results are useful for understanding $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ on an unknown class VII surface X which is a deformation in large of a family of blown up primary Hopf surfaces.

1.3. **Standard properties of class VII surfaces**

In this section we explain briefly standard results on the intersection form, Chern numbers, the Picard group and the Gauduchon degree of a class VII surface.

Let X be a class VII surface. The condition $\kappa(X) = 0$ implies the vanishing of the geometric genus $p_g(X) := h^0(\mathcal{K}_X)$. Since X is non-Kählerian it follows that $b_+(X) = 0$, so the intersection form

$$(H^2(X,\mathbb{Z})/\mathrm{Tors}) \times (H^2(X,\mathbb{Z})/\mathrm{Tors}) \to \mathbb{Z}$$

of the underlying oriented, differentiable 4-manifold is negative definite. Therefore, by Donaldson's first theorem ([16], [14]), this intersection form is standard over $\mathbb{Z}$, i.e. there exists a basis $(e_1, \ldots, e_b)$ of $H^2(X,\mathbb{Z})/\mathrm{Tors}$ (with $b := b_2(X)$)

such that $e_i \cdot e_j = -\delta_{ij}$. Using the fact that $c_1(\mathcal{K}_X)$ is a lift of $w_2(X)$, one can prove that there exists a basis $(e_1, \dots, e_b)$ of $H^2(X,\mathbb{Z})/\mathrm{Tors}$ such that

$$e_i \cdot e_j = -\delta_{ij},\ c_1(\mathcal{K}_X) + \mathrm{Tors} = \sum_{i=1}^{b} e_i \,.$$

A basis satisfying these two conditions is unique up to order, and will be called *a standard basis* of $H^2(X,\mathbb{Z})/\mathrm{Tors}$.

In general, the second Chern number $c_2(X)$ of a complex surface coincides with its topological Euler characteristic. For a class VII surface X we obtain $c_2(X) = b_2(X)$. On the other hand, for a class VII surface X we obtain $h^1(\mathcal{O}_X) = 1$ (see for instance [35, Appendix]) which, together with $p_g(X) = 0$, gives

$$\chi(\mathcal{O}_X) = 0\,. \tag{2}$$

By the Noether formula we obtain the following formula for the Chern numbers:

$$c_1^2(X) = -c_2(X) = -b_2(X)\,. \tag{3}$$

Note also that, by Serre duality $H^0(\mathcal{K}_X) = 0$ implies $H^2(\mathcal{O}_X) = 0$, which shows that on a class VII surface we have

$$\mathrm{NS}(X) = H^2(X,\mathbb{Z})\,. \tag{4}$$

In other words any class $c \in H^2(X,\mathbb{Z})$ is the Chern class of a holomorphic line bundle.

We also need an explicit description of the group $\mathrm{Pic}^0(X)$. Note first that, for a class VII surface X, the canonical morphism $H^1(X,\mathbb{C}) \to H^1(X,\mathcal{O})$ is an isomorphism ([29], [35]). Fixing an isomorphism $\beta : H_1(X,\mathbb{Z})/\mathrm{Tors} \to \mathbb{Z}$, we obtain associated isomorphisms $\gamma : \mathbb{Z} \xrightarrow{\simeq} H^1(X,\mathbb{Z})$, $\gamma^{\mathbb{C}} : \mathbb{C} \xrightarrow{\simeq} H^1(X,\mathbb{C})$ given by the universal coefficients formula, and also an induced isomorphism

$$\bar{\gamma}^{\mathbb{C}} : \mathbb{C}^* = \mathbb{C}/2\pi i\mathbb{Z} \xrightarrow{\simeq} H^1(X,\mathbb{C})/2\pi i H^1(X,\mathbb{Z}) = \mathrm{Pic}^0(X).$$

Fix $x_0 \in X$. For $\zeta \in \mathbb{C}^*$ the element $\bar{\gamma}^{\mathbb{C}}(\zeta) \in \mathrm{Pic}^0(X)$ is the isomorphism class of the flat line bundle $\mathcal{L}_\zeta$ associated with the representation $\rho_\zeta : \pi_1(X,x_0) \to \mathbb{C}^*$ which corresponds to ζ via the composition

$$\begin{aligned}\mathbb{C}^* = \mathrm{Hom}(\mathbb{Z},\mathbb{C}^*) \xrightarrow{\beta^* \simeq} &\mathrm{Hom}(H_1(X,\mathbb{Z})/\mathrm{Tors},\mathbb{C}^*) \hookrightarrow \\ &\hookrightarrow \mathrm{Hom}(H_1(X,\mathbb{Z}),\mathbb{C}^*) = \mathrm{Hom}(\pi_1(X,x_0),\mathbb{C}^*).\end{aligned}$$

Using the identification

$$\mathbb{C}^* \ni \zeta \mapsto [\mathcal{L}_\zeta] = \bar{\gamma}^{\mathbb{C}}(\zeta) \in \mathrm{Pic}^0(X)\,,$$

the restriction to $\mathrm{Pic}^0(X)$ of the degree map associated with a Gauduchon metric g on X is given by

$$\deg_g(\mathcal{L}_\zeta) = \nu_g \ln|\zeta|\,, \tag{5}$$

where $\nu_g \in \mathbb{R}^*$ is a constant depending smoothly on g, and whose sign is independent of g [22]. We will choose the initial isomorphism β such that $\nu_g > 0$, and with this choice the identification $\mathrm{Pic}^0(X) \simeq \mathbb{C}^*$ becomes canonical (intrinsically associated with X).

1.4. **Main results**

We recall that a holomorphic rank r bundle $\mathcal{E}$ is filtrable if it admits a filtration $(\mathcal{E}_i)_{1\leqslant i\leqslant r}$ by coherent subsheaves such that $\mathrm{rk}(\mathcal{E}_i) = i$ for $1 \leqslant i \leqslant r$ (see [7, section 4.2]). On a projective surface any bundle is filtrable, but on non-algebraic surfaces this is no longer true. This is a substantial difficulty in understanding moduli spaces of bundles over non-algebraic surfaces, because we have no general classification method for non-filtrable bundles.

Remark 1.2. Let $\mathcal{E}$ be a filtrable bundle of rank $r = 2$ on a complex surface X. Any rank 1 subsheaf of $\mathcal{E}$ is contained in a rank 1 subsheaf $\mathcal{M} \subset \mathcal{E}$ with torsion free quotient. Such a subsheaf $\mathcal{M}$ is invertible, and $\mathcal{E}/\mathcal{M}$ is isomorphic to $\mathcal{L}\otimes\mathcal{I}_Z$, where $\mathcal{L}$ is invertible and $Z \subset X$ is a 0-dimensional locally complete intersection. Therefore $\mathcal{E}$ fits in a short exact sequence of the form

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L}\otimes\mathcal{I}_Z \to 0\,,$$

where $\mathcal{L}$, $\mathcal{M}$ are invertible sheaves, and $Z \subset X$ is a 0-dimensional locally complete intersection.

The main result of this article is (see section 2.2):

Theorem 2.7. *Let X be complex surface whose minimal model is a primary Hopf surface. Let $\{e_1, \dots, e_b\}$ be a standard basis of $H^2(X,\mathbb{Z})$. Then any holomorphic rank 2 bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) \simeq \mathcal{K}_X$ on X fits in a short exact sequence of the form*

$$0 \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{E} \to \mathcal{L} \to 0\,, \tag{6}$$

where $\mathcal{L}$ is a line bundle on X with $c_1(\mathcal{L}) = e_I \coloneqq \sum_{i\in I} e_i$ for a subset $I \subset \{1, \dots, b\}$. In particular any such bundle is filtrable.

Note that the conclusion of the theorem in not true on *minimal* class VII surfaces. For instance, using the results of [29] one can prove:

Example 1.3. Let X be an Enoki surface with $b_2 = 1$, and g be a Gauduchon metric on X with $\deg_g(\mathcal{K}_X) < 0$. The moduli space $\mathcal{M}^{\rm pst}_{\mathcal{K}_X}(E)$ can be identified with a compact disk, whose boundary corresponds to the reduction space $\mathcal{R}$. The center of this disk corresponds to a twisted reduction [31, section 1.4], [32, section 1.3] which is not filtrable.

For the proof of Theorem 2.7 we make use of techniques developed by Brussee in the algebraic framework. As in [8] we construct 2-bundles on a blown up surface using an elementary transformation along the exceptional divisor. Brussee's main result identifies the moduli stable of stable bundles (with suitable topological invariants) over the blown up $\hat{Y}_y$ of a *projective algebraic* surface Y with a fiber bundle over a moduli space of stable bundles over the original surface Y. Unfortunately such an identification cannot be obtained in our (non-Kählerian) framework.

Let again X be a blown up primary Hopf surface. For any $I \subset \{1,\dots,b\}$, any line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = e_I$ and any extension class $\varepsilon \in \mathrm{Ext}^1(\mathcal{L}, \mathcal{K}\otimes\mathcal{L}^\vee) = H^1(\mathcal{K}\otimes\mathcal{L}^{\otimes -2})$ we obtain a rank 2 bundle $\mathcal{E}$ which is the central term of an extension of the form (6) with extension class ε. Theorem 2.7 states that, on blown up primary Hopf surfaces, all bundles $\mathcal{E}$ with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) \simeq \mathcal{K}_X$ can be obtained in this way. The result can be used to describe explicitly the moduli space $\mathcal{M}^{\rm pst}_{\mathcal{K}}(E)$ at least for surfaces with small b_2, but important difficulties remain: first, one has to select the pairs $(\mathcal{L}, \varepsilon)$ as above which give a polystable bundle; second, one has to control isomorphisms between bundles associated with different pairs. The main tool used here is [30, Proposition 4.8], which allows one to classify all line subbundles of the 2-bundle associated with a given extension class.

We will illustrate the method and the difficulties in section 3, in which we describe explicitly the moduli space on blown up Hopf surfaces with $b_2 \in \{1,2\}$ under the assumption $\deg_g(\mathcal{K}_X) < 0$. This assumption is not restrictive: using the construction of [22, p. 163], one obtains Gauduchon metrics on X for which this inequality holds.

The case $b_2 = 1$ will be treated in detail. The final result is simple (see Theorem 3.9): denoting by D the exceptional divisor, and putting $\nu := \nu_g$, $\delta := \deg_g(\mathcal{O}_X(D))$, the moduli space $\mathcal{M}^{\rm pst}_{\mathcal{K}_X}(E)$ can be naturally identified with the disk $\bar{D}(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})$ bounded by the ellipse $\Gamma(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})$ of semi-axes $(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})$. The boundary of the disk corresponds to the unique circle of reductions.

In the case $b_2 = 2$, the moduli space $\mathcal{M}^{\rm st}_{\mathcal{K}_X}(E)$ can be identified with the complement of an ellipse in the product $D(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1$, and

$\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ with the space obtained from $\bar{D}(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1$ by collapsing the fiber over each point $z \in \Gamma(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})$ to a point p_z (see Theorem 3.12, Proposition 3.13).

2. GENERAL RESULTS

2.1. Filtrable bundles

We start with the following theorem, which concerns 2-bundles $\mathcal{E}$ with $c_1^{\mathbb{Q}}(\mathcal{E}) = c_1^{\mathbb{Q}}(\mathcal{K}_X)$ on an arbitrary class VII surface X. The theorem states that any such a bundle has $c_2 \geqslant 0$, and any such bundle with $c_2 = 0$ which is filtrable is a line bundle extension of a very special type.

THEOREM 2.1. *Let X be a class VII surface, and $(e_1, \ldots, e_b)$ be a standard basis of $H^2(X, \mathbb{Z})/\mathrm{Tors}$. Let $\mathcal{E}$ be a holomorphic rank 2 vector bundle on X with*

$$c_1(\mathcal{E}) + \mathrm{Tors} = c_1(\mathcal{K}_X) + \mathrm{Tors}$$

in $H^2(X, \mathbb{Z})/\mathrm{Tors}$. Then

1. *$c_2(\mathcal{E}) \geqslant 0$.*
2. *Suppose that $c_2(\mathcal{E}) = 0$, $\mathcal{E}$ is filtrable, and let $\mathcal{M}$ be a rank 1 subsheaf of $\mathcal{E}$ with torsion free quotient (see Remark 1.2). Then $\mathcal{L} := \mathcal{E}/\mathcal{M}$ is locally free, and $c_1(\mathcal{L}) + \mathrm{Tors} = e_I := \sum_{i\in I} e_i$ for a subset $I \subset \{1, \ldots, b\}$. Therefore $\mathcal{E}$ fits in a short exact sequence*

$$0 \to \det(\mathcal{E}) \otimes \mathcal{L}^{\vee} \to \mathcal{E} \to \mathcal{L} \to 0\,,$$

where $\mathcal{L}$ is an invertible sheaf with $c_1(\mathcal{L}) + \mathrm{Tors} = e_I$ for a subset $I \subset \{1, \ldots, b\}$.

Proof. (1) Taking into account that $\chi(\mathcal{O}_X) = 0$ (see section 1.3), the Hirzebruch-Riemann-Roch theorem gives

$$\chi(\mathcal{E}) = \frac{1}{2}(c_1(\mathcal{E})^2 - 2c_2(\mathcal{E})) - \frac{1}{2}c_1(\mathcal{E})c_1(\mathcal{K}_X) = -c_2(\mathcal{E})\,.$$

If $c_2(\mathcal{E}) < 0$, we get $\chi(\mathcal{E}) > 0$, so $h^0(\mathcal{E}) > 0$ or $h^2(\mathcal{E}) = h^0(\mathcal{K}_X \otimes \mathcal{E}^{\vee}) > 0$. In both cases it follows that $\mathcal{E}$ is filtrable so, by Remark 1.2, it fits in an exact sequence of the form

$$0 \to \mathcal{M} \to \mathcal{E} \to \mathcal{L} \otimes \mathcal{I}_Z \to 0\,, \tag{7}$$

where $Z \subset X$ is a 0-dimensional local complete intersection, and $\mathcal{L}$, $\mathcal{M}$ are line bundles. Put $\bar{l} := c_1(\mathcal{L}) + \mathrm{Tors}$, $\bar{m} := c_1(\mathcal{M}) + \mathrm{Tors}$, $\bar{k} = c_1(\mathcal{K}_X) + \mathrm{Tors}$. The short exact sequence (7) gives

$$\bar{l} + \bar{m} = \bar{k},\ \bar{l} \cdot \bar{m} + |Z| = c_2(\mathcal{E})\,. \tag{8}$$

Expanding with respect to $(e_1, \ldots, e_b)$ we obtain $\bar{l} = \sum_{i=1}^b x_i e_i$ with $x_i \in \mathbb{Z}$, and (8) gives

$$\sum_{i=1}^{b} x_i(x_i - 1) + |Z| = c_2(\mathcal{E})\,.$$

But the left-hand side is non-negative, so the assumption $c_2(\mathcal{E}) < 0$ leads to a contradiction.

(2) By Remark 1.2 the quotient $\mathcal{E}/\mathcal{M}$ is isomorphic to $\mathcal{L} \otimes \mathcal{I}_Z$, where $\mathcal{L}$ is invertible and $Z \subset X$ is a 0-dimensional local complete intersection. Therefore we obtain a short exact sequence of the form (7). The same computation as above gives

$$\sum_{i=1}^{b} x_i(x_i - 1) + |Z| = 0\,,$$

so $x_i \in \{0, 1\}$ for $1 \leqslant i \leqslant b$, and $Z = \emptyset$. On the other hand (7) gives $\mathcal{M} = \det(\mathcal{E}) \otimes \mathcal{L}^\vee$. Putting $I := \{i \in \{1, \ldots, b\} |\ x_i = 1\}$ the claim is proved.

□

2.2. **2-Bundles on blown up surfaces**

In this section we make use of complex geometric analogues of the techniques developed in [8, section 2] for bundles on projective algebraic surfaces. Note that Brussee's main result ([8, Corollary 5]) cannot be extended to our non-Kählerian framework.

Proposition 2.2. *Let Y be a complex surface, $y \in Y$ and $\pi : X \to Y$ be the blow up at y with exceptional divisor $D \subset X$. Let $\mathcal{L}$ be a holomorphic line bundle on Y, and $\mathcal{E}$ be a holomorphic 2-bundle on X with* $\det(\mathcal{E}) = \pi^*(\mathcal{L})(D)$. *One has*

1. $\det(\pi_*(\mathcal{E})^{\vee\vee}) \simeq \mathcal{L}$.

2. $c_2(\pi_*(\mathcal{E})^{\vee\vee}) = c_2(\mathcal{E}) - (h^0(R^1\pi_*(\mathcal{E})) + h^0(\mathcal{Q}))$, *where* $\mathcal{Q} := \pi_*(\mathcal{E})^{\vee\vee}/\pi_*(\mathcal{E})$.

Proof. (1) The restriction of the line bundles $\det(\pi_*(\mathcal{E})^{\vee\vee})$ and $\mathcal{L}$ to $Y\backslash\{y\}$, are isomorphic, so, by Hartogs theorem, they are isomorphic on Y.

(2) The Grothendieck-Riemann-Roch theorem for proper holomorphic morphisms [24] gives:

$$\mathrm{ch}(\pi_!\mathcal{E})\mathrm{td}(Y) = \pi_*\big(\mathrm{ch}(\mathcal{E})\mathrm{td}(X)\big). \tag{9}$$

Let $\mathfrak{Y} \in H^4(Y,\mathbb{Z})$, $\mathfrak{X} \in H^4(X,\mathbb{Z})$ be the standard generators of $H^4(Y,\mathbb{Z})$, $H^4(X,\mathbb{Z})$ (the Poincaré duals of the standard generators of $H_0(Y,\mathbb{Z})$, $H_0(X,\mathbb{Z})$). Putting $l \coloneqq c_1(\mathcal{L})$, $d \coloneqq c_1(\mathcal{O}(D))$, the hypothesis $\det(\mathcal{E}) = \pi^*(\mathcal{L})(D)$ gives

$$c_1(\mathcal{E}) = \pi^*(l) + d.$$

The Chern character of a 2-bundle F on Y with $c_1(F) = l$, $c_2(F) = c_2(\mathcal{E})$ is

$$\mathrm{ch}(F) = 2 + l + \frac{1}{2}(l^2 - 2c_2(\mathcal{E}))\,. \tag{10}$$

Using the formulae

$$\mathrm{td}(X) = \pi^*(\mathrm{td}(Y)) - \frac{1}{2}d,\ \pi^*(H^2(Y,\mathbb{Q}))\cdot d = 0,\ d^2 = -\mathfrak{X}\,,$$

$$\mathrm{ch}(\mathcal{E}) = \pi^*(\mathrm{ch}(F)) + d + \frac{1}{2}d^2\,,$$

we obtain:

$$\pi_*\big(\mathrm{ch}(\mathcal{E})\mathrm{td}(X)\big) = \pi_*\big(\mathrm{ch}(\mathcal{E})(\pi^*(\mathrm{td}(Y)) - \frac{1}{2}d)\big) = \pi_*(\mathrm{ch}(\mathcal{E}))\mathrm{td}(Y) - \frac{1}{2}\pi_*\big(\mathrm{ch}(\mathcal{E})d\big) =$$
$$= \mathrm{ch}(F)\mathrm{td}(Y) - \frac{1}{2}\mathfrak{Y}\mathrm{td}(Y) + \frac{1}{2}\mathfrak{Y} = \mathrm{ch}(F)\mathrm{td}(Y)\,.$$

Formula (9) becomes:

$$\mathrm{ch}(\pi_!(\mathcal{E})) = \mathrm{ch}(F)\,. \tag{11}$$

On the other hand one has

$$\mathrm{ch}(\pi_!\mathcal{E}) = \mathrm{ch}(\pi_*(\mathcal{E})) - \mathrm{ch}(R^1\pi_*(\mathcal{E}))\,.$$

The sheaf $\pi_*(\mathcal{E})$ is torsion free, so its singularity set Z is 0-dimensional; it fits in an exact sequence

$$0 \to \pi_*(\mathcal{E}) \to \pi_*(\mathcal{E})^{\vee\vee} \to \mathcal{Q} \to 0\,,$$

where $\mathcal{Q}$ is a torsion sheaf whose support is Z. This exact sequence gives:

$$\mathrm{ch}(\pi_*(\mathcal{E})) = \mathrm{ch}(\pi_*(\mathcal{E})^{\vee\vee}) - \mathrm{ch}(\mathcal{Q})\,.$$

For a torsion sheaf $\mathcal{S}$ on Y with 0-dimensional support, the Hirzebruch-Riemann-Roch theorem for coherent sheaves [23] gives the general formula

$$\mathrm{ch}(\mathcal{S}) = -c_2(\mathcal{S}) = h^0(\mathcal{S})\mathfrak{Y}\,.$$

Therefore

(12)
$$\mathrm{ch}(\pi_*(\mathcal{E})^{\vee\vee}) = \mathrm{ch}(\pi_*(\mathcal{E})) + h^0(\mathcal{Q})\mathfrak{Y} = \mathrm{ch}(\pi_!\mathcal{E}) + \big(h^0(R^1\pi_*(\mathcal{E})) + h^0(\mathcal{Q})\big)\mathfrak{Y} =$$
$$= \mathrm{ch}(F) + \big(h^0(R^1\pi_*(\mathcal{E})) + h^0(\mathcal{Q})\big)\mathfrak{Y}\,.$$

By part (1) proved above we have $c_1(\pi_*(\mathcal{E})^{\vee\vee}) = l$, so

$$\mathrm{ch}(\pi_*(\mathcal{E})^{\vee\vee}) = 2 + l + \frac{1}{2}(l^2 - 2c_2(\pi_*(\mathcal{E})^{\vee\vee}))\,. \tag{13}$$

Formulae (10), (13) (12) give (via the usual identification $H^4(Y,\mathbb{Z}) = \mathbb{Z}$):

$$c_2(\pi_*(\mathcal{E})^{\vee\vee}) = c_2(\mathcal{E}) - (h^0(R^1\pi_*(\mathcal{E})) + h^0(\mathcal{Q}))\,.$$

□

Corollary 2.3. *Under the assumptions of Proposition 2.2, one has $\det(\pi_*(\mathcal{E})^{\vee\vee}) \simeq \mathcal{L}$, and $c_2(\pi_*(\mathcal{E})^{\vee\vee}) \leqslant c_2(\mathcal{E})$ with equality if only if $R^1\pi_*(\mathcal{E}) = 0$ and $\pi_*(\mathcal{E})$ is locally free.*

Let Y be a complex surface, and $\pi : X = \hat{Y}_y \to Y$ be the blow up at $y \in Y$. Let $\mathcal{F}$ be a holomorphic rank 2 vector bundle on Y with $\det(\mathcal{F}) = \mathcal{L}$, let $w \subset \mathcal{F}(y)$ be a 1-dimensional linear subspace, and $q_w : \mathcal{F}(y) \to Q_w \coloneqq \mathcal{F}(y)/w$ be the associated quotient. Lifting to X we obtain an epimorphism

$$q_{w,D} : \pi^*(\mathcal{F})_D = \mathcal{O}_D \otimes \mathcal{F}(y) \to \mathcal{O}_D \otimes Q_w\,,$$

so an epimorphism

$$q_{w,D}(D) : \pi^*(\mathcal{F})(D)_D = \mathcal{O}(D)_D \otimes \mathcal{F}(y) \to \mathcal{O}(D)_D \otimes Q_w\,.$$

The kernel

$$\mathcal{F}_w \coloneqq \ker(\mathfrak{q}_w)$$

of the composition

$$\begin{tikzcd} \pi^*(\mathcal{F})(D) \arrow[r] \arrow[rr, bend right, "\mathfrak{q}_w"'] & \pi^*(\mathcal{F})(D)_D \arrow[r, "q_{w,D}(D)"] & \mathcal{O}(D)_D \otimes Q_w \end{tikzcd}$$

is locally free of rank 2, and comes with a canonical epimorphism

$$\mathfrak{q}' : \mathcal{F}_w \to \ker(q_{w,D}(D)) = \mathcal{O}(D)_D \otimes w$$

(see [9, section 6.1.1]). In the terminology of [9] the pair $(\mathcal{F}_w, \mathfrak{q}')$ is the elementary transformation of the pair $(\pi^*(\mathcal{F})(D), \mathfrak{q}_w)$. The short exact sequence

$$0 \to \mathcal{F}_w \hookrightarrow \pi^*(\mathcal{F})(D) \to \mathcal{O}(D)_D \otimes Q_w \to 0$$

gives

$$\det(\mathcal{F}_w) = \det(\pi^*(\mathcal{F})(D))(-D) = \pi^*(\det(\mathcal{F}))(D),\ c_2(\mathcal{F}_w) = c_2(\mathcal{F})\,.$$

Using [9, Proposition 6.3] it follows that the restriction $\mathcal{F}_{wD}$ fits in a short exact sequence

$$0 \to \mathcal{O}_D \otimes Q_w \to \mathcal{F}_{wD} \to \mathcal{O}(D)_D \otimes w \to 0\,.$$

Note that $\mathcal{O}(D)_D \otimes w \simeq \mathcal{O}_D(-1)$, $\mathcal{O}_D \otimes Q_w \simeq \mathcal{O}_D$. Since $\mathrm{Ext}^1(\mathcal{O}_D(-1), \mathcal{O}_D) = H^1(\mathcal{O}_D(1))$ vanishes, it follows that this exact sequence splits, so

$$\mathcal{F}_{wD} \simeq \mathcal{O}_D \oplus \mathcal{O}_D(-1)\,.$$

PROPOSITION 2.4. *Let Y be a complex surface, $y \in Y$ and $\pi : X \to Y$ be the blow up at y with exceptional divisor $D \subset X$. Let $\mathcal{L}$ be a holomorphic bundle on Y, and $\mathcal{E}$ be a holomorphic 2-bundle on X with* $\det(\mathcal{E}) = \pi^*(\mathcal{L})(D)$. *If $R^1\pi_*(\mathcal{E}) = 0$, then*

1. $\mathcal{E}_D \simeq \mathcal{O}_D \oplus \mathcal{O}_D(-1)$.
2. $\pi_*(\mathcal{E})$ *is locally free.*
3. *Putting $\mathcal{F} := \pi_*(\mathcal{E}) = \pi_*(\mathcal{E})^{\vee\vee}$, there exists $w \in \mathbb{P}(\mathcal{F}(y))$ such that $\mathcal{E} \simeq \mathcal{F}_w$.*

Proof. (1) One has $\det(\mathcal{E})_D \simeq \pi^*(\mathcal{L})(D)_D = \mathcal{O}(D)_D \simeq \mathcal{O}_D(-1)$. Let $\mathcal{M}$ be the maximal destabilizing line bundle of $\mathcal{E}_D$, and

(14) $$0 \to \mathcal{M} \hookrightarrow \mathcal{E}_D \xrightarrow{q} \mathcal{E}_D/\mathcal{M} \to 0$$

be the corresponding short exact sequence. Since $\mathcal{M}$ destabilizes $\mathcal{E}_D$ and $\deg(\mathcal{E}_D) = -1$, one has $\deg(\mathcal{M}) \geqslant 0$, so $\mathcal{M} \simeq \mathcal{O}_D(k)$ with $k \geqslant 0$. Denoting by $\mathfrak{q}$ the composition

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[rr, bend right, "\mathfrak{q}"'] & \mathcal{E}_D \arrow[r, "q"] & \mathcal{E}_D/\mathcal{M}\,, \end{tikzcd}$$

one obtains the short exact sequence

(15) $$0 \to \mathcal{U} := \ker(\mathfrak{q}) \hookrightarrow \mathcal{E} \xrightarrow{\mathfrak{q}} \mathcal{E}_D/\mathcal{M} \to 0\,,$$

which gives the exact sequence

$$R^1\pi_*(\mathcal{E}) \to R^1\pi_*(\mathcal{E}_D/\mathcal{M}) \to R^2\pi_*(\mathcal{U})\,.$$

Since the all the fibers of π have dimension $\leqslant 1$, we have $R^2\pi_*(\mathcal{U}) = 0$, so the hypothesis $R^1\pi_*(\mathcal{E}) = 0$ implies $R^1\pi_*(\mathcal{E}_D/\mathcal{M}) = 0$, where

$$\mathcal{E}_D/\mathcal{M} \simeq \det(\mathcal{E}_D) \otimes \mathcal{M}^\vee \simeq \mathcal{O}_D(-1-k).$$

But the stalk of $R^1\pi_*(\mathcal{E}_D/\mathcal{M})$ at y is

$$H^1(\mathcal{E}_D/\mathcal{M}) \simeq H^1(\mathcal{O}_D(-1-k)) \simeq H^0(\mathcal{O}_D(k-1))^\vee\,,$$

where for the last isomorphism we used Serre duality. Since $k \geqslant 0$, we see that the vanishing of this space implies $k = 0$. Therefore in the exact sequence (14) we have $\mathcal{M} \simeq \mathcal{O}_D$, $\mathcal{E}_D/\mathcal{M} \simeq \mathcal{O}_D(-1)$. Since $\mathrm{Ext}^1_D(\mathcal{O}_D(-1), \mathcal{O}_D) = 0$, this short exact sequence splits, which proves the claim.

(2) The short exact sequence (15) gives the exact sequence

$$0 \to \pi_*(\mathcal{U}) \to \pi_*(\mathcal{E}) \to \pi_*(\mathcal{E}_D/\mathcal{M}) .$$

Since $\mathcal{E}_D/\mathcal{M} \simeq \mathcal{O}_D(-1)$ we obtain $\pi_*(\mathcal{E}_D/\mathcal{M}) = 0$, so $\pi_*(\mathcal{U}) = \pi_*(\mathcal{E})$. Using again [9, section 6.1.1] it follows that $\mathcal{U} \coloneqq \ker(\mathfrak{q})$ comes with an epimorphism $\mathfrak{q}' : \mathcal{U} \to \ker(q) = \mathcal{M}$, the pair $(\mathcal{U}, \mathfrak{q}')$ is the elementary transformation associated with the pair $(\mathcal{E}, \mathfrak{q})$, and the restriction $\mathcal{U}_D$ fits in a short exact sequence of the form

$$0 \to (\mathcal{E}_D/\mathcal{M})(-D) \to \mathcal{U}_D \to \mathcal{M} \to 0 .$$

We have $(\mathcal{E}_D/\mathcal{M})(-D) \simeq \mathcal{O}_D$, $\mathcal{M} \simeq \mathcal{O}_D$, so this short exact sequence splits, and $\mathcal{U}_D \simeq \mathcal{O}_D^{\oplus 2}$. Therefore, by [25, Theorem] it follows that $\pi_*(\mathcal{U})$ is locally free, and the canonical morphism

$$\pi^*(\pi_*(\mathcal{U})) \to \mathcal{U}$$

is an isomorphism. Thus $\pi_*(\mathcal{E})$ is locally free, too.

(3) [9, Proposition 6.3 (3)] gives an isomorphism $\ker(\mathfrak{q}') \simeq \mathcal{E}(-D)$. Tensorizing with $\mathcal{O}(D)$ the short exact sequence $0 \to \ker(\mathfrak{q}') \to \mathcal{U} \xrightarrow{\mathfrak{q}'} \mathcal{M} \to 0$, and putting $\mathcal{F} \coloneqq \pi_*(\mathcal{U}) = \pi_*(\mathcal{E})$, we obtain a short exact sequence

$$0 \to \ker(\mathfrak{q}')(D) \simeq \mathcal{E} \to \pi^*(\mathcal{F})(D) \to \mathcal{M}(D) \simeq \mathcal{O}(D)_D \to 0.$$

It suffices to note that the kernel of any epimorphism $\pi^*(\mathcal{F})_D \to \mathcal{O}_D$ has the form $\mathcal{O}_D \otimes w$ for a line $w \subset \mathcal{F}(y)$. $\square$

Proposition 2.5. *Let Y be a class VII surface, $y \in Y$, and $\pi : X \to Y$ be the blow up at y with exceptional divisor $D \subset X$. Let $\mathcal{E}$ be a holomorphic 2-bundle on X with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) = \mathcal{K}_X$. The sheaf $\mathcal{F} \coloneqq \pi_*(\mathcal{E})$ is locally free, has $c_2(\mathcal{F}) = 0$, $\det(\mathcal{F}) = \mathcal{K}_Y$, and there exists $w \in \mathbb{P}(\mathcal{F}(y))$ such that $\mathcal{E} \simeq \mathcal{F}_w$.*

Proof. Note that $\det(\mathcal{E}) = \pi^*(\mathcal{K}_Y)(D)$ so Corollary 2.3 applies with $\mathcal{L} = \mathcal{K}_Y$. By this corollary the bundle $\mathcal{F} \coloneqq \pi_*(\mathcal{E})^{\vee\vee}$ on Y has $\det(\mathcal{F}) = \mathcal{K}_Y$, and $c_2(\mathcal{F}) \leqslant c_2(\mathcal{E}) = 0$. By Theorem 2.1 (1) we have $c_2(\mathcal{F}) = 0$ so, using Corollary 2.3 again, we obtain that $R^1\pi_*(\mathcal{E}) = 0$ and $\pi_*(\mathcal{E})$ is locally free. The claim follows by Proposition 2.4. $\square$

Lemma 2.6. *Let Y be a primary Hopf surface, and $\mathcal{E}$ be a holomorphic 2-bundle on Y with $c_2 = 0$, $\det(\mathcal{E}) = \mathcal{K}_Y$. Then $\mathcal{E}$ fits in a short exact sequence of the form*

$$0 \to \mathcal{K}_Y \otimes \mathcal{L}^\vee \to \mathcal{E} \to \mathcal{L} \to 0 ,$$

with $c_1(\mathcal{L}) = 0$.

Proof. Taking into account Theorem 2.1 (2), it suffices to prove that $\mathcal{E}$ is filtrable. If $\mathcal{E}$ is not filtrable, it will be stable with respect to any Gauduchon metric g on X. Therefore, fixing such a metric g, the Kobayashi-Hitchin correspondence (see section 1.1) gives a Hermitian metric h on $\mathcal{E}$ such that the associated Chern connection A_h on the underlying differentiable bundle E is Hermite-Einstein. Since $\Delta(E) = 4c_2(E) - c_1^2(E) = 0$, it follows that A_h is projectively flat (see [34, Remark 1.4]). Let $y_0 \in Y$, $\gamma : [0,1] \to Y$ be a smooth loop representing a generator of the fundamental group $\pi_1(Y, y_0) \simeq \mathbb{Z}$, and $\chi_\gamma \in \mathrm{U}(E_{y_0}, h_{y_0})$ be the holonomy automorphism along γ. Let $w \subset E_{y_0}$ be the line generated by an eigenvector of h_γ. Therefore w is invariant with respect to the whole holonomy group of A_h at y_0. Applying to w parallel transport with respect to A_h along smooth paths starting at y_0, one obtains a line subbundle $M \subset \mathcal{E}$, which will be parallel with respect to A_h, in particular holomorphic. This contradicts the assumption "$\mathcal{E}$ is not filtrable". $\square$

Now we can prove our main result:

THEOREM 2.7. *Let X be a complex surface whose minimal model is a primary Hopf surface. Let $\{e_1, \dots, e_b\}$ be a standard basis of $H^2(X, \mathbb{Z})$. Then any holomorphic rank 2 bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) \simeq \mathcal{K}_X$ on X fits in a short exact sequence of the form*

$$0 \to \mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{E} \to \mathcal{L} \to 0\,,$$

where $\mathcal{L}$ is a line bundle on X with $c_1(\mathcal{L}) = e_I$ for a subset $I \subset \{1, \dots, b\}$. In particular any such bundle is filtrable.

Proof. Induction with respect to $b_2(X)$: If $b_2(X) = 0$, X is a primary Hopf surface, so the claim follows by Lemma 2.6.

If $b_2(X) > 0$, then $X = \hat{Y}_y$ for a surface Y whose minimal model is still a primary Hopf surface, and one has $b_2(Y) = b_2(X) - 1$. By Proposition 2.5 we have $\mathcal{E} \simeq \mathcal{F}_w$, where $\mathcal{F} := \pi_*(\mathcal{E}) = \pi_*(\mathcal{E})^{\vee\vee}$ and $w \in \mathbb{P}(\mathcal{F}(y))$. Moreover, we have $\det(\mathcal{F}) = \mathcal{K}_Y$, $c_2(\mathcal{F}) = 0$ so, by induction, we know that $\mathcal{F}$ is filtrable. Let $j : \mathcal{M} \to \mathcal{F}$ be a sheaf monomorphism, where $\mathcal{M}$ is a line bundle on Y. The image of the composition

$$\pi^*(\mathcal{M}) \xrightarrow{\pi^*(j)} \pi^*(\mathcal{F}) \to \pi^*(\mathcal{F})(D)$$

is contained in $\ker(\mathfrak{q}_w) = \mathcal{F}_w$, so $\mathcal{E}$ is filtrable. The claim follows now from Theorem 2.1 (2).

$\square$

2.3. **Regularity results**

We refer to [34, Propositions 2.6 (1)] for the following regularity result:

Theorem 2.8. *Let (X, g) be a complex surface endowed with a Gauduchon metric g such that $\deg_g(\mathcal{K}_X) < 0$. For any polystable holomorphic bundle $\mathcal{E}$ on X one has $H^2(\mathcal{E}nd_0(\mathcal{E})) = 0$.*

Corollary 2.9. *Under the assumptions of Theorem 2.8, let E be a differentiable vector bundle on X, and $\mathcal{D}$ a holomorphic structure on $\det(E)$. The moduli space $\mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E)$ is a smooth complex manifold.*

Moreover, using the comparison theorem [34, Propositions 2.6 (2)], it follows that, under the assumption $\deg_g(\mathcal{K}_X) < 0$, any reducible point in $\mathcal{M}^{\mathrm{ASD}}_a(E) = \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ is regular in the sense of [34, Definition 1.3]. Using this result one obtains an explicit topological description of $\mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E)$ around the reducible locus

$$\mathcal{R} \coloneqq \mathcal{M}^{\mathrm{pst}}_{\mathcal{D}}(E) \setminus \mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E).$$

Let X be a class VII surface with $b \coloneqq b_2(X) > 0$, and let $(e_1, \dots, e_b)$ be a standard basis of $H^2(X, \mathbb{Z})/\mathrm{Tors}$. The set

$$\mathcal{H}_X \coloneqq \{c \in H^2(X, \mathbb{Z}) |\ \exists I \subset \{1, \dots, b\},\ c + \mathrm{Tors} = e_I\}$$

comes with an obvious surjection $\mathfrak{p} : \mathcal{H}_X \to \mathcal{P}(\{1, \dots, b\})$ which identifies the power set $\mathcal{P}(\{1, \dots, b\})$ with the quotient $\mathcal{H}_X/\mathrm{Tors}$. The involution $\mathcal{H}_X \to \mathcal{H}_X$ given by $c \mapsto \bar{c} \coloneqq c_1(\mathcal{K}_X) - c$ lifts the involution $I \mapsto \bar{I} \coloneqq \{1, \dots, b\} \setminus I$ on $\mathcal{P}(\{1, \dots, b\})$. Denote by $\mathfrak{H}_X$ the quotient of $\mathcal{H}_X$ by the involution $c \mapsto \bar{c}$.

Recall that the (topological) cone of a topological space B is

$$C_B \coloneqq {[0,1] \times B)}\big/{\{0\} \times B} \cdot$$

The vertex of the cone C_B is the point $v_B \in C_B$ which corresponds to the collapsed end $\{0\} \times B$. With these preparations we can state (see [33] [34]):

Corollary 2.10. *Let X be a class VII surface with $b \coloneqq b_2(X) > 0$, and g be a Gauduchon metric on X. Let E be a rank 2-bundle with $c_2(E) = 0$, $\det(E) = K_X$, and $\mathcal{R}$ be the reducible locus in $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$. Then*

1. *One has a natural bijection $\mathfrak{H}_X \xrightarrow{\simeq} \pi_0(\mathcal{R})$.*

2. *The connected component $\mathcal{R}_{\{c,\bar{c}\}}$ associated with an element $\{c, \bar{c}\} \in \mathfrak{H}_X$ is homeomorphic to a circle.*

3. *Suppose $\deg_g(\mathcal{K})_X) < 0$. For any connected component $\mathcal{R}_{\{c,\bar{c}\}} \subset \mathcal{R}$ there exists an open neighborhood $\mathcal{U}_{\{c,\bar{c}\}}$ of $\mathcal{R}_{\{c,\bar{c}\}}$ in $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$, and a homeomorphism $\mathcal{R}_{\{c,\bar{c}\}} \times C_{\mathbb{P}^{b-1}_{\mathbb{C}}} \to \mathcal{U}_{\{c,\bar{c}\}}$ which induces the obvious identification $\mathcal{R}_{\{c,\bar{c}\}} \times \{v_{\mathbb{P}^{b-1}_{\mathbb{C}}}\} \to \mathcal{R}_{\{c,\bar{c}\}}$.*

Remark 2.11. In the special case when $H_1(X,\mathbb{Z}) \simeq \mathbb{Z}$, for instance when the minimal model of X is a Hopf surface or a Kato surface, one has $\mathrm{Tors} = 0$, so $\mathfrak{p}$ is a bijection. In this case we can identify $\mathcal{H}_X$ with $\mathcal{P}(\{1,\dots,b\})$, and $\mathfrak{H}_X$ with the quotient $\mathfrak{P}(\{1,\dots,b\})$ of $\mathcal{P}(\{1,\dots,b\})$ by the involution $I \mapsto \bar{I}$. $\mathfrak{P}(\{1,\dots,b\})$ is just the set of unordered two-term partitions of $\{1,\dots,b\}$. Therefore in this case the circles of reductions in the moduli space are parameterized by unordered pairs $\{I,\bar{I}\}$.

We will denote by $\mathcal{R}_{I,\bar{I}}$ the circle associated with the unordered pair $\{I,\bar{I}\}$.

3. EXAMPLES OF MODULI SPACES

In this section we give explicit geometric descriptions of the moduli space $\mathcal{M}_{\mathcal{K}_X}(E)$ for a blown up primary Hopf surface X with $b_2(X) \in \{1,2\}$.

3.1. $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ on blown up primary Hopf surfaces with $b_2 = 1$

Let Y be a primary Hopf surface of the form

$$Y = {}^{\mathbb{C}^2\setminus\{0\}}\!\big/_{\langle f\rangle}\,, \text{ where } f(z_1,z_2) = (\alpha_1 z_1, \alpha_2 z_2) \text{ with } |\alpha_i| < 1\,. \tag{16}$$

The images of the coordinate lines $\mathbb{C}\times\{0\}$, $\{0\}\times\mathbb{C}$ in Y are elliptic curves, which will be denoted by C_1, C_2 respectively. One has a canonical isomorphism $\mathcal{K}_Y \simeq \mathcal{O}(-C_1 - C_2)$.

Recall that (see for instance [7, Theorem 2.7, 2.13, Proposition 2.26]):

Remark 3.1. Let Y be a primary Hopf surface of the form (16). The following conditions are equivalent:

1. There exists $(k_1,k_2) \in \mathbb{N}_{>0}\times\mathbb{N}_{>0}$ such that $\alpha_1^{k_1} = \alpha_2^{k_2}$.
2. $a(Y) > 0$, where $a(Y)$ is the algebraic dimension on Y.
3. There exists a non-constant meromorphic function on Y.
4. Y is an elliptic surface.
5. There exists a line bundle $\mathcal{N}$ on Y such that $h^0(\mathcal{N}) \geqslant 2$.

In order to avoid technical (but unessential) complications, we will assume that Y is not elliptic, so that $h^0(\mathcal{N}) \leqslant 1$ for any line bundle $\mathcal{N}$ on Y. Moreover, this assumption also implies that the only irreducible curves on Y are C_1 and C_2.

Fix a point $y \in Y\backslash(C_1 \cup C_2)$, let $\pi : X = \hat{Y}_y \to Y$ be the blow up at y, and $D \subset Y$ be the exceptional divisor. We will denote by the same symbols the lifts of the curves C_i to X. Put $d \coloneqq c_1(\mathcal{O}_X(D)) = c_1(\mathcal{K}_X)$.

Remark 3.2. The assignment $(m, n_1, n_2) \mapsto mD + n_1C_1 + n_2C_2$ defines a bijection between $\mathbb{N}^3$ and the set of effective divisors on X. In particular $H^0(\mathcal{L}) = 0$ for any line bundle on X with $c_1(\mathcal{L}) \in \mathbb{Z}_{<0}d$.

Using the isomorphism $\mathcal{K}_Y \simeq \mathcal{O}_Y(-C_1 - C_2)$ we obtain

$$\mathcal{K}_X \simeq \mathcal{O}_X(D - C_1 - C_2)\,. \tag{17}$$

Denote by $\mathfrak{k}$, $\mathfrak{d} \in \mathrm{Pic}^d(X)$, $\mathfrak{c}_i \in \mathrm{Pic}^0(X)$ the isomorphism classes of the line bundles $\mathcal{K}_X$, $\mathcal{O}_X(D)$, $\mathcal{O}_X(C_i)$ respectively. Let g be a Gauduchon metric on X, and

$$\kappa \coloneqq \deg_g(\mathcal{K}_X)\,,\ \delta = \deg_g(\mathcal{O}_X(D))\,,\ \gamma_i \coloneqq \deg_g(\mathcal{O}_X(C_i))\,.$$

Using [22, Proposition 1.3.5] we obtain $\delta > 0$, $\gamma_i > 0$. Formula (17) implies

$$\mathfrak{k} = \mathfrak{d} \otimes \mathfrak{c}_1^{-1} \otimes \mathfrak{c}_2^{-1}\,,\ \kappa = \delta - \gamma_1 - \gamma_2\,. \tag{18}$$

By Theorem 2.7 we know that any rank 2-bundle $\mathcal{E}$ on X with $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) = \mathcal{K}_X$ fits in a short exact sequence

$$0 \to \mathcal{K}_X \otimes \mathcal{L}^\vee \xrightarrow{i} \mathcal{E} \xrightarrow{p} \mathcal{L} \to 0 \tag{19}$$

with $c_1(\mathcal{L}) \in \{d, 0\}$. For fixed $\mathcal{L}$, the extensions of the form (19) are classified by elements in

$$\mathrm{Ext}^1(\mathcal{L}, \mathcal{K}_X \otimes \mathcal{L}^\vee) = H^1(\mathcal{K}_X \otimes \mathcal{L}^{-2})\,.$$

The Riemann-Roch and Serre duality theorems combined with formula (2) give:

$$h^1(\mathcal{K}_X \otimes \mathcal{L}^{-2}) = -\frac{1}{2}\big(c_1^2(\mathcal{K}_X \otimes \mathcal{L}^{-2}) - c_1(\mathcal{K}_X \otimes \mathcal{L}^{-2})c_1(\mathcal{K}_X)\big) + \\ + h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}) + h^0(\mathcal{L}^2)\,.$$

Taking into account Remark 3.2, we obtain

$$h^1(\mathcal{K}_X \otimes \mathcal{L}^{-2}) = \left\{ \begin{array}{ll} 1 + h^0(\mathcal{L}^2) & \text{if} \quad c_1(\mathcal{L}) = d \\ h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}) + h^0(\mathcal{L}^2) & \text{if} \quad c_1(\mathcal{L}) = 0 \end{array} \right. . \tag{20}$$

The component $\mathrm{Pic}^d(X)$ of the Picard group comes with a natural involution $\iota : \mathrm{Pic}^d(X) \to \mathrm{Pic}^d(X)$ given by

$$\iota(\mathfrak{l}) \coloneqq \mathfrak{k} \otimes \mathfrak{d} \otimes \mathfrak{l}^{-1}\,,$$

and whose fixed points are the two square roots $\mathfrak{l}'$, $\mathfrak{l}''$ of the equation $\mathfrak{l}^2 = \mathfrak{k} \otimes \mathfrak{d}$. Consider the analytic subset

$$A \coloneqq \{\mathfrak{l} = [\mathcal{L}] \in \mathrm{Pic}^d(X) |\ h^0(\mathcal{L}^2 \otimes \mathcal{K}_X^\vee(-D)) > 0\} \subset \mathrm{Pic}^d(X)\,.$$

Note that

$$A \cap \iota(A) = \{\mathfrak{l}', \mathfrak{l}''\} . \tag{21}$$

Put $B := A \backslash \{\mathfrak{l}', \mathfrak{l}''\}$. Using Remark 3.2, it follows easily that this set is empty when $\delta \leqslant \min(\gamma_1, \gamma_2)$.

PROPOSITION 3.3. *Let* $\mathfrak{l} := [\mathcal{L}] \in \mathrm{Pic}^d(X)$ *with* $h^0(\mathcal{L}^2) = 0$. *Then*

1. *All non-trivial extensions of* $\mathcal{L}$ *by* $\mathcal{K} \otimes \mathcal{L}^\vee$ *are isomorphic.*

2. *Let* $\mathcal{E}$ *be a non-trivial extension of* $\mathcal{L}$ *by* $\mathcal{K} \otimes \mathcal{L}^\vee$. *Then*

   (a) $\mathcal{E}$ *has a line subbundle* $\mathcal{L}'$ *isomorphic to* $\mathcal{L}(-D)$, *if and only if* $\mathfrak{l} \notin \iota(B)$.

   (b) *For* $\mathfrak{l} \notin \iota(B)$ *the line subbundle* $\mathcal{L}'$ *is unique. The corresponding extension of* $\mathcal{K}_X \otimes \mathcal{L}'^\vee \simeq \mathcal{K}_X \otimes \mathcal{L}^\vee(D)$ *by* $\mathcal{L}' \simeq \mathcal{L}(-D)$ *is split if and only if* $\mathfrak{l} \in B$.

Proof. (1) Taking into account the hypothesis, formula (20) gives

$$\dim(\mathrm{Ext}^1(\mathcal{L}, \mathcal{K}_X \otimes \mathcal{L}^\vee)) = 1 .$$

Recalling that the canonical $\mathbb{C}^*$-action on the extension space of two bundles does not change the isomorphism class of the central term, the claim follows.

(2) (a) Suppose first that $\mathcal{E}$ has a line subbundle isomorphic to $\mathcal{L}(-D)$. In other words there exists a bundle embedding $j : \mathcal{L}(-D) \hookrightarrow \mathcal{E}$. We have to prove that $\mathfrak{l} \notin \iota(B)$. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K}_X \otimes \mathcal{L}^\vee \arrow[r, "i"] & \mathcal{E} \arrow[r, "p"] & \mathcal{L} \arrow[r] & 0 \\ & & \mathcal{L}(-D) \arrow[u, hook, "j"] & & \end{tikzcd}$$

in which the horizontal row is the non-trivial extension defining $\mathcal{E}$, and denote by $\varepsilon \in H^1(\mathcal{K}_X \otimes \mathcal{L}^{-2})$ the corresponding extension class.

If $\mathfrak{l} \in \{\mathfrak{l}', \mathfrak{l}''\}$ we have $\mathfrak{l} \notin \iota(B)$, so the claim is proved.

If $\mathfrak{l} \notin \{\mathfrak{l}', \mathfrak{l}''\}$, then $\mathfrak{l}$ is not a fixed point of ι, so $\mathcal{L}(-D) \not\simeq \mathcal{K}_X \otimes \mathcal{L}^\vee$. Therefore j does not factorize through i (if it did factorize, the obtained morphism $\mathcal{L}(-D) \to \mathcal{K}_X \otimes \mathcal{L}^\vee$ would be an isomorphism), so $p \circ j$ does not vanish. Rescaling if necessary, we may suppose that $p \circ j$ is the canonical monomorphism $\mathcal{L}(-D) \to \mathcal{L}$, so j is a lift of this canonical monomorphism.

By [30, Proposition 4.8] this canonical sheaf monomorphism has a lift j to $\mathcal{E}$ if and only if there exists $\sigma \in H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D)$ which is mapped to ε

via the connecting morphism μ in the canonical exact sequence

$$
(22)\quad \begin{aligned} 0 \to H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}) \to H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) \xrightarrow{u} H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D) \xrightarrow{\mu} \\ \xrightarrow{\mu} H^1(\mathcal{K}_X \otimes \mathcal{L}^{-2}) \to \dots \end{aligned}
$$

If, by reductio ad absurdum $\mathfrak{l} \in \iota(B)$, then $\mathfrak{l} \in \iota(A)$, i.e. $h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) > 0$. This gives $\mathcal{K}_X \otimes \mathcal{L}^{-2}(D) \simeq \mathcal{O}(n_1C_1 + n_2C_2)$ with $n_i \in \mathbb{N}$, and $\mathcal{K}_X \otimes \mathcal{L}^{-2} \simeq \mathcal{O}(-D + n_1C_1 + n_2C_2)$, which implies

$$h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}) = 0\,,\ h^1(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) = 1.$$

On the other hand $c_1(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) = 0$, so (since D is a projective line) we get $(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D \simeq \mathcal{O}_D$, and $h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D) = 1$. This shows that in the canonical exact sequence above u is an isomorphism, so μ vanishes. Since $\varepsilon \neq 0$ by assumption, it follows that ε cannot have a lift via μ.

Conversely, suppose that $\mathfrak{l} \notin \iota(B)$. Therefore either $\mathfrak{l} \in \{\mathfrak{l}', \mathfrak{l}''\}$ or $\mathfrak{l} \notin \iota(A)$, i.e. $h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) = 0$. In the former case one has $\mathcal{K} \otimes \mathcal{L}^\vee \simeq \mathcal{L}(-D)$, so it suffices to take $\mathcal{L}' := i(\mathcal{K} \otimes \mathcal{L}^\vee)$. In the latter case it follows that the morphism μ in (22) is an isomorphism. Therefore ε has a lift in $\sigma \in H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D)$, so the canonical monomorphism $\mathcal{L}(-D) \to \mathcal{L}$ has a lift $j : \mathcal{L}(-D) \to \mathcal{E}$. Since σ is nowhere vanishing (because $(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)_D \simeq \mathcal{O}_D)$, it follows by [30, Proposition 4.8 (5)] that j is a bundle embedding.

(2)(b) Suppose $\mathfrak{l} \notin \iota(B)$. We prove first the unicity of $\mathcal{L}'$. We have two cases:

(i) $\mathfrak{l} \in \{\mathfrak{l}', \mathfrak{l}''\}$. Suppose by reductio ad absurdum that $\mathcal{E}$ admits a line subbundle $\mathcal{L}' \simeq \mathcal{L}(-D)$ which does not coincide with $i(\mathcal{K}_X \otimes \mathcal{L}_X^\vee)$. We get a bundle embedding $j : \mathcal{L}(-D) \to \mathcal{E}$ with $p \circ j \neq 0$. As in the proof of (3)(a) we may assume that $p \circ j$ is the canonical monomorphism $\mathcal{L}(-D) \to \mathcal{L}$. But in this case $\mathcal{K}_X \otimes \mathcal{L}^{-2}(D) \simeq \mathcal{O}_X$, $\mathcal{K}_X \otimes \mathcal{L}^{-2} \simeq \mathcal{O}_X(-D)$, and the same method as above shows that this canonical monomorphism does not admit a lift to $\mathcal{E}$.

(ii) $\mathfrak{l} \notin \iota(A)$. Since in this case $\mathfrak{l} \notin \{\mathfrak{l}', \mathfrak{l}''\}$, any line subbundle $\mathcal{L}' \simeq \mathcal{L}(-D)$ of $\mathcal{E}$ is the image of a lift of the canonical monomorphism $\mathcal{L}(-D) \to \mathcal{L}$. The difference of two such lifts belongs to

$$\mathrm{Hom}(\mathcal{L}(-D), \mathcal{K}_X \otimes \mathcal{L}^\vee) = H^0((\mathcal{L}(-D))^\vee \otimes (\mathcal{K}_X \otimes \mathcal{L}^\vee)) = H^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) = 0\,,$$

which proves the claim.

It remains to prove that the obtained extension of $\mathcal{K}_X \otimes \mathcal{L}'^\vee$ by $\mathcal{L}'$ is split if and only if $\mathfrak{l} \in B$. Suppose $\mathfrak{l} \in B$. We have

$$(23)\quad \mathcal{H}om(\mathcal{K}_X \otimes \mathcal{L}^\vee, \mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D))) = (\mathcal{K}_X^\vee \otimes \mathcal{L}^2(-D)) \oplus \mathcal{O}(D)\,.$$

The condition $\mathfrak{l} \in B$ is equivalent to $\mathcal{K}_X^\vee \otimes \mathcal{L}^2(-D) \simeq \mathcal{O}(C)$, where $C = n_1C_1 + n_2C_2$ with $(n_1, n_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$. Since $C \cap D = \varnothing$, formula (23) shows that there exists a bundle embedding

$$\mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D)),$$

so the direct sum $\mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D))$ can be written as an extension of $\mathcal{L}$ by $\mathcal{K}_X \otimes \mathcal{L}^\vee$, which must be non-trivial, because, since $C \neq \emptyset$,

$$\mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D)) \not\simeq (\mathcal{K}_X \otimes \mathcal{L}^\vee) \oplus \mathcal{L}.$$

Taking into account part (1) proved above, it follows

$$\mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D)) \simeq \mathcal{E}.$$

Conversely, if $\mathcal{E}$ is a trivial extension of $\mathcal{K}_X \otimes \mathcal{L}'^\vee \simeq \mathcal{K}_X \otimes \mathcal{L}^\vee(D)$ by $\mathcal{L}' \simeq \mathcal{L}(-D)$, there exists a bundle embedding $\mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{L}(-D) \oplus (\mathcal{K}_X \otimes \mathcal{L}^\vee(D))$, so, using (23), we obtain $H^0(\mathcal{K}_X^\vee \otimes \mathcal{L}^2(-D)) \neq 0$, so $\mathfrak{l} \in A$. Moreover, $\mathfrak{l} \notin \{\mathfrak{l}', \mathfrak{l}''\}$, because, for $\mathfrak{l} \in \{\mathfrak{l}', \mathfrak{l}''\}$ we know that $\mathcal{L}' = i(\mathcal{K}_X \otimes \mathcal{L}^\vee)$, so the extension associated with $\mathcal{L}'$ can be identified with the initial extension defining $\mathcal{E}$. But this initial extension was non-trivial. Therefore $\mathfrak{l} \in A \setminus \{\mathfrak{l}', \mathfrak{l}''\} = B$. □

Put

$$S := \{\mathfrak{l} = [\mathcal{L}] \in \mathrm{Pic}^d(X) |\ \mathfrak{l}^2 \otimes \mathfrak{k}^{-1} \otimes \mathfrak{d}^{-1} = \mathfrak{c}_1^{n_1} \otimes \mathfrak{c}_2^{-n_2} \text{ with } (n_1, n_2) \in \mathbb{N}^* \times \mathbb{N}^*\}.$$

Note that the sets $\{\mathfrak{l}', \mathfrak{l}''\}$, B, $\iota(B)$, S, $\iota(S)$ are pairwise disjoint. Using the same method as in the proof of Proposition 3.3, we obtain:

Proposition 3.4. *Let $\mathfrak{l} := [\mathcal{L}] \in \mathrm{Pic}^d(X)$ with $h^0(\mathcal{L}^2) = 0$, and $\mathcal{E}$ be a non-trivial extension of $\mathcal{L}$ by $\mathcal{K}_X \otimes \mathcal{L}^\vee$. Then*

1. *If $\mathfrak{l} \notin \{\mathfrak{l}', \mathfrak{l}''\} \cup B \cup \iota(B) \cup S \cup \iota(S)$, $\mathcal{E}$ has exactly two line subbundles, which are isomorphic to $\mathcal{K}_X \otimes \mathcal{L}^\vee$ and $\mathcal{L}(-D)$ respectively.*

2. *If $\mathfrak{l} \in \{\mathfrak{l}', \mathfrak{l}''\} \cup \iota(B) = \iota(A)$, $\mathcal{E}$ has exactly one line subbundle which is isomorphic to $\mathcal{K}_X \otimes \mathcal{L}^\vee$.*

3. *Let $\mathfrak{l} \in S$, and $(n_1, n_2) \in \mathbb{N}^* \times \mathbb{N}^*$ be such that $\mathfrak{l}^2 \otimes \mathfrak{k}^{-1} \otimes \mathfrak{d}^{-1} = \mathfrak{c}_1^{n_1} \otimes \mathfrak{c}_2^{-n_2}$. Then $\mathcal{E}$ has exactly three line subbundles, which are isomorphic to $\mathcal{K}_X \otimes \mathcal{L}^\vee$, $\mathcal{L}(-D)$ and $\mathcal{L}(-n_1 C_1)$ respectively.*

4. *Let $\mathfrak{l} \in \iota(S)$, and $(n_1, n_2) \in \mathbb{N}^* \times \mathbb{N}^*$ be such that $\mathfrak{l}^2 \otimes \mathfrak{k}^{-1} \otimes \mathfrak{d}^{-1} = \mathfrak{c}_1^{-n_1} \otimes \mathfrak{c}_2^{n_2}$. Then $\mathcal{E}$ has exactly three line subbundles, which are isomorphic to $\mathcal{K}_X \otimes \mathcal{L}^\vee$, $\mathcal{L}(-D)$ and $\mathcal{L}(-n_2 C_2)$ respectively.*

We can now prepare our geometric description of the moduli space. The open subset

$$\Omega := \left\{\mathfrak{l} = [\mathcal{L}] \in \mathrm{Pic}^d(X) |\ \frac{\kappa}{2} < \deg(\mathcal{L}) < \frac{\kappa}{2} + \delta\right\} \subset \mathrm{Pic}^d(X)$$

is ι-invariant, and contains the fixed point locus $\{\mathfrak{l}', \mathfrak{l}''\}$ of ι.

Remark 3.5. Suppose $\deg_g(\mathcal{K}_X) < 0$. For any $[\mathcal{L}] \in \Omega$ one has $h^0(\mathcal{L}^2) = 0$.

Proof. If $h^0(\mathcal{L}^2) > 0$, Remark 3.2 gives $\mathcal{L}^2 = \mathcal{O}(2D + n_1 C_1 + n_2 C_2)$ with $n_i \in \mathbb{N}$, so $\deg_g(\mathcal{L}) \geqslant \delta$, which, under the assumption $\kappa < 0$, contradicts the second inequality in the definition of Ω. ☐

THEOREM 3.6. *Suppose* $\deg_g(\mathcal{K}_X) < 0$.

1. *Let* $\mathfrak{l} := [\mathcal{L}] \in \mathrm{Pic}^d(X)$ *with* $h^0(\mathcal{L}^2) = 0$, *and* $\mathcal{E}$ *be a non-trivial extension of* $\mathcal{L}$ *by* $\mathcal{K}_X \otimes \mathcal{L}^\vee$. $\mathcal{E}$ *is stable if and only of* $\mathfrak{l} \in \Omega \backslash B$.

2. *Let* $\mathcal{E}$ *be a holomorphic bundle on* X *with* $c_2(\mathcal{E}) = 0$, $\det(\mathcal{E}) = \mathcal{K}_X$ *which is stable. Then there exists* $[\mathcal{L}] \in \Omega \backslash B$ *such that* $\mathcal{E}$ *is a non-trivial extension of* $\mathcal{L}$ *by* $\mathcal{K}_X \otimes \mathcal{L}^\vee$.

Proof. (1) Suppose that $\mathcal{E}$ is stable. If $\mathfrak{l} \in B$, $\mathcal{E}$ splits as a direct sum of line bundles by Proposition 3.3 (2)(b), so it cannot be stable. Therefore $\mathfrak{l} \notin B$. We now show that $\mathfrak{l} \in \Omega$. For $\mathfrak{l} \notin \iota(A)$, $\mathcal{E}$ has two line subbundles isomorphic to $\mathcal{K}_X \otimes \mathcal{L}^\vee$, $\mathcal{L}(-D)$ respectively, and the stability condition corresponding to these subsheaves gives

$$\frac{\kappa}{2} < \deg(\mathcal{L}) < \frac{\kappa}{2} + \delta\,,$$

so $\mathfrak{l} \in \Omega$. Finally, if $\mathfrak{l} \in \iota(A)$, one has $h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}(D)) > 0$, which gives

$$\deg_g(\mathcal{L}) \leqslant \frac{\kappa}{2} + \frac{\delta}{2} < \frac{\kappa}{2} + \delta\,.$$

On the other hand, the stability condition for the subbundle $\mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{E}$ gives $\frac{\kappa}{2} < \deg(\mathcal{L})$.

Conversely, let $\mathfrak{l} \in \Omega \backslash B$. We prove that $\mathcal{E}$ is stable. By Theorem 2.1(2) we know that any non-trivial subsheaf with torsion free quotient of $\mathcal{E}$ is a line subbundle. Therefore, to check the stability condition it suffices to verify the inequality required in Definition 1.1 for all line subbundles of $\mathcal{E}$, which are classified by Proposition 3.4. It is easy to see that none of these subbundles destabilize $\mathcal{E}$. Note that the condition $\deg_g(\mathcal{K}_X) < 0$ plays a crucial role in the case $\mathfrak{l} \in S \cup \iota(S)$.

(2) We know that $\mathcal{E}$ fits in an exact sequence of the form (19) with $c_1(\mathcal{L}) \in \{0, d\}$. Since $\mathcal{E}$ is stable, this extension is non-trivial. We have three cases:

(i) $c_1(\mathcal{L}) = d$, $h^0(\mathcal{L}^2) = 0$. In this case the claim follows from (1).

(ii) $c_1(\mathcal{L}) = 0$. Since the extension is non-trivial, formula (20) gives $h^0(\mathcal{K}_X \otimes \mathcal{L}^{-2}) = 1$ or $h^0(\mathcal{L}^2) = 1$. The stability condition for the subbundle

$\mathcal{K}_X \otimes \mathcal{L}^\vee \to \mathcal{E}$ gives $\deg_g(\mathcal{K}_X \otimes \mathcal{L}^{-2}) < 0$, so the first case is ruled out. Therefore $h^0(\mathcal{L}^2) = 1$, and Lemma 3.7 stated below shows that $\mathcal{E}$ is also a non-trivial extension of $\mathcal{L}_i$ by $\mathcal{K} \otimes \mathcal{L}_i^\vee$ (for $i \in \{1, 2\}$), where $c_1(\mathcal{L}_i) = d$, $h^0(\mathcal{L}_i^2) = 0$. The claim follows from (1).

(iii) $c_1(\mathcal{L}) = d$, $h^0(\mathcal{L}^2) = 1$. Since $\mathcal{E}$ is stable, it cannot be isomorphic to a direct sum of line bundles, so, by Lemma 3.8 stated below, $\mathcal{E}$ is a non-trivial extension of a line bundle $\mathcal{L}'$ by $\mathcal{K} \otimes \mathcal{L}'^\vee$, where $c_1(\mathcal{L}') = 0$. The claim follows from (2)(ii).

$\square$

The proofs of Lemmata 3.7, 3.8 below follow the method used for Proposition 3.3, so they will be omitted.

Lemma 3.7. *Let $[\mathcal{L}] \in \mathrm{Pic}^0(X)$ with $h^0(\mathcal{L}^2) = 1$, and n_1, $n_2 \in \mathbb{N}$ be such that $\mathcal{L}^2 \simeq \mathcal{O}(n_1 C_1 + n_2 C_2)$. Let $\mathcal{E}$ be a non-trivial extension of $\mathcal{L}$ by $\mathcal{K} \otimes \mathcal{L}^\vee$. Then $\mathcal{E}$ is also a non-trivial extension of $\mathcal{K} \otimes \mathcal{L}^\vee((n_i + 1)C_i)$ by $\mathcal{L}(-(n_i + 1)C_i)$ for $i \in \{1, 2\}$.*

Putting $\mathcal{L}_i := \mathcal{K} \otimes \mathcal{L}^\vee((n_i + 1)C_i)$, one has $c_1(\mathcal{L}_i) = d$, $h^0(\mathcal{L}_i^2) = 0$.

Lemma 3.8. *Let $[\mathcal{L}] \in \mathrm{Pic}^d(X)$ with $h^0(\mathcal{L}^2) = 1$, and n_1, $n_2 \in \mathbb{N}$ be such that $\mathcal{L}^2 \simeq \mathcal{O}(2D + n_1 C_1 + n_2 C_2)$. Define the lines Λ, Λ_1, $\Lambda_2 \subset H^1(\mathcal{K} \otimes \mathcal{L}^2)$ by*

$$\Lambda := \mathrm{im}(H^0(\mathcal{K} \otimes \mathcal{L}^2(D)_D) \to H^1(\mathcal{K} \otimes \mathcal{L}^2)),$$

$$\Lambda_i := \mathrm{im}(H^0(\mathcal{K} \otimes \mathcal{L}^2((n_i + 1)C_i)_{(n_i+1)C_i}) \to H^1(\mathcal{K} \otimes \mathcal{L}^2)).$$

Let $\mathcal{E}$ be a non-trivial extension of $\mathcal{L}$ by $\mathcal{K} \otimes \mathcal{L}^\vee$, and let $\varepsilon \in H^1(\mathcal{K} \otimes \mathcal{L}^2)\setminus\{0\}$ be the corresponding extension class.

1. *If $\varepsilon \in \Lambda$, then $\mathcal{E} \simeq \mathcal{L}(-D) \oplus (\mathcal{K} \otimes \mathcal{L}^\vee(D))$.*

2. *If $\varepsilon \in \Lambda_i$, then $\mathcal{E} \simeq \mathcal{L}(-(n_i + 1)C_i) \oplus (\mathcal{K} \otimes \mathcal{L}^\vee((n_i + 1)C_i))$.*

3. *If $\varepsilon \in H^1(\mathcal{K} \otimes \mathcal{L}^2)\setminus(\Lambda \cup \Lambda_1 \cup \Lambda_2)$, then $\mathcal{E}$ is a non-trivial extension of $\mathcal{K} \otimes \mathcal{L}^\vee((n_1 + 1)C_1 + (n_2 + 1)C_2)$ by $\mathcal{L}(-(n_1 + 1)C_1 - (n_2 + 1)C_2)$.*

Recall (see section 1.3) that we defined a *canonical* identification

$$\mathbb{C}^* \ni \zeta \xrightarrow{\simeq} [\mathcal{L}_\zeta] \in \mathrm{Pic}^0(X)$$

and, with respect to this identification, we have the identity $\deg_g(\mathcal{L}_\zeta) = \nu \ln|z|$ for a positive constant $\nu = \nu_g$ depending smoothly on g. The map

$$F : \mathrm{Pic}^d(X) \to \mathrm{Pic}^0(X) = \mathbb{C}^*, \ F(\mathfrak{l}) = \mathfrak{k} \otimes \mathfrak{l}^{-1}$$

is a biholomorphism and maps the pair (Ω, ι) on the pair $(A(e^{\frac{\kappa}{2\nu} - \frac{\delta}{\nu}}, e^{\frac{\kappa}{2\nu}}), I)$, where $A(e^{\frac{\kappa}{2\nu} - \frac{\delta}{\nu}}, e^{\frac{\kappa}{2\nu}}) \subset \mathbb{C}^*$ is the annulus of biradius $(e^{\frac{\kappa}{2\nu} - \frac{\delta}{\nu}}, e^{\frac{\kappa}{2\nu}})$, and I is the involution $z \mapsto (\mathfrak{k} \otimes \mathfrak{d}^{-1}) z^{-1}$. Denoting by $D(\alpha, \beta)$ the interior of the compact disk $\bar{D}(\alpha, \beta)$ bounded by the ellipse $\Gamma(\alpha, \beta)$ of semi-axes α, β, we obtain a biholomorphism

$$\Phi : {}^{\Omega}\!/_{\langle \iota \rangle} \xrightarrow{\simeq} D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}})$$

given explicitly by

$$\mathfrak{l} \mapsto \mu^{-1} e^{-\frac{\kappa}{2\nu}} (\mathfrak{k} \otimes \mathfrak{l}^{-1} + \mathfrak{d}^{-1} \otimes \mathfrak{l}), \tag{24}$$

where $\mu \in \mathrm{S}^1$ is a square root of $\mathfrak{k} \otimes \mathfrak{d}^{-1} / |\mathfrak{k} \otimes \mathfrak{d}^{-1}|$. Note that

$$|\Omega \cap B| = 2 |\{(n_1, n_2) \in \mathbb{N} \times \mathbb{N} \setminus \{(0,0)\} |\ n_1 \gamma_1 + n_2 \gamma_2 < \delta\}|,$$

in particular this set is finite. Let R be the equivalence relation on $\Omega \setminus B$ induced by the $\langle \iota \rangle$-congruence on Ω. The map

$${}^{\Omega \setminus B}\!/_{R} \to {}^{\Omega}\!/_{\langle \iota \rangle}$$

induced by the inclusion $\Omega \setminus B \hookrightarrow \Omega$ is obviously a biholomorphism, because any $\langle \iota \rangle$-orbit in Ω intersects $\Omega \setminus B$. Therefore formula (24) also defines a biholomorphism

$$\Psi : {}^{\Omega \setminus B}\!/_{R} \xrightarrow{\simeq} D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}).$$

With these preparations we can prove:

Theorem 3.9. *Suppose* $\deg_g(\mathcal{K}_X) < 0$. *For* $\mathfrak{l} = [\mathcal{L}] \in \Omega$ *denote by* $\mathfrak{E}_{\mathfrak{l}}$ *the isomorphism type of a non-trivial extension of* $\mathcal{L}$ *by* $\mathcal{K}_X \otimes \mathcal{L}^\vee$ *(see Proposition 3.3(1), Remark 3.5). The map* $\Omega \setminus B \ni \mathfrak{l} \stackrel{f}{\mapsto} \mathfrak{E}_{\mathfrak{l}} \in \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ *is holomorphic, surjective, R-invariant, and induces a biholomorphism* $F : {}^{\Omega \setminus B}\!/_{R} \to \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$. *The composition*

$$F \circ \Psi^{-1} : D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$$

is biholomorphic, and extends to a homeomorphism

$$\bar{D}(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E).$$

Proof. Let $\mathscr{L}$ be a Poincaré line bundle on $\mathrm{Pic}^d(X) \times X$. Therefore, for any $\mathfrak{l} \in \mathrm{Pic}^d(X)$, the restriction $\mathcal{L}_{\mathfrak{l}}$ of $\mathscr{L}$ to the fiber $\{\mathfrak{l}\} \times X \simeq X$ belongs to the isomorphism class $\mathfrak{l}$. Denote by $\mathscr{L}_\Omega$ the restriction of $\mathscr{L}$ to $\Omega \times X$.

Since Ω is Stein, we have $H^1(p_{\Omega *}(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2})) = 0$, so, using the Leray spectral sequence associated with the coherent sheaf $p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2}$ and the proper holomorphic map p_Ω, it follows that the canonical morphism

$$H^1(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2}) \to H^0(R^1 p_{\Omega *}(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2})) \tag{25}$$

is an isomorphism. By formula (20) and Remark 3.5, we have $h^1(\mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}}^{-2}) = 1$ for any $\mathfrak{l} \in \Omega$, so, by Grauert's local freeness theorem, it follows that the sheaf

$R^1 p_{\Omega *}(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2})$ is invertible. Since $H^1(\mathcal{O}_\Omega) = 0$ (because Ω is Stein), and $H^2(\Omega, \mathbb{Z}) = 0$, we obtain (using the cohomology exact sequence associated with the exponential sequence) $H^1(\mathcal{O}_\Omega^*) = 0$, so any holomorphic line bundle on Ω is trivial. Therefore $R^1 p_{\Omega *}(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2})$ has a nowhere vanishing section σ. Let

$$\Sigma \in H^1(p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^{-2}) = \mathrm{Ext}^1(\mathscr{L}_\Omega, p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^\vee)$$

be the pre-image of σ via the isomorphism (25), and

$$0 \to p_X^*(\mathcal{K}_X) \otimes \mathscr{L}_\Omega^\vee \to \mathscr{E} \to \mathscr{L}_\Omega \to 0$$

be the associated extension. Regarding $\mathscr{E}|_{(\Omega \setminus B) \times X}$ as a holomorphic family of stable bundles parameterized by $\Omega \setminus B$, we obtain a holomorphic map $\Omega \setminus B \to \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ which coincides with f; indeed, the restriction $\mathcal{E}_{\mathfrak{l}}$ of $\mathscr{E}$ to $\{\mathfrak{l}\} \times X \simeq X$ is a non-trivial extension of $\mathcal{L}_{\mathfrak{l}}$ by $\mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}}^\vee$, so it belongs to the isomorphism class $\mathfrak{E}_{\mathfrak{l}}$. The surjectivity of f follows from Theorem 3.5 (2), and the R-invariance from Proposition 3.3. Therefore f induces a holomorphic, surjective map

$$F : {}^{(\Omega \setminus B)}/_{R} \to \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)\,.$$

We claim that F is injective. Indeed, if $\mathcal{E}_{\mathfrak{l}_1} \simeq \mathcal{E}_{\mathfrak{l}_2}$, then $\mathcal{E}_{\mathfrak{l}_1}$ contains a line subbundle isomorphic to $\mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}_2}^\vee$. By Proposition 3.4 this implies $\mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}_2}^\vee \simeq \mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}_1}^\vee$, or $\mathcal{K}_X \otimes \mathcal{L}_{\mathfrak{l}_2}^\vee \simeq \mathcal{L}_{\mathfrak{l}_1}(-D)$. Therefore $\mathfrak{l}_2 = \mathfrak{l}_1$ or $\mathfrak{l}_2 = \iota(\mathfrak{l}_1)$, so $\mathfrak{l}_1 \; R \; \mathfrak{l}_2$.

Therefore F is a holomorphic bijection between smooth complex manifolds, so it is a biholomorphism. The homeomorphic extension

$$\bar{D}(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$$

of $F \circ \Psi^{-1}$ is constructed using elliptic semicontinuity, as in the proof of [29, Proposition 4.4].

Remark 3.10. The obtained homeomorphism $\bar{D}(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}}) \xrightarrow{\simeq} \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ maps the boundary $\Gamma(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}})$ of the disk on the unique circle of reductions $\mathcal{R} = \mathcal{R}_{\varnothing, \{1\}}$ in the moduli space (see Remark 2.11). The result is compatible with Corollary 2.10(3) which describes the topology of $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ around a circle of reductions.

Remark 3.11. For $\deg_g(\mathcal{K}_X) > 0$ the moduli space $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ contains the isomorphism class of the nontrivial extension on $\mathcal{K}_X$ by $\mathcal{O}_X$, and $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ is singular at this point. This follows using the arguments of [29, Proposition 3.3 (5)]. This shows that the condition $\deg_g(\mathcal{K}_X) < 0$ is crucial in our regularity Theorem 2.8.

□

3.2. $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ on blown up primary Hopf surfaces with $b_2 = 2$

Let now $X = \hat{Y}_{y_1,y_2}$ be the blown up of a non-elliptic primary Hopf surface Y of the form (16) at two distinct points y_1, y_2 which do not belong to $C_1 \cup C_2$. Let D_1, D_2 be the corresponding exceptional divisors, $D := D_1 + D_2$, $d_i := c_1(\mathcal{O}_X(D_i))$, $d := d_1 + d_2$, $\delta_i := \deg_g(\mathcal{O}_X(D_i))$, and $\delta := \delta_1 + \delta_2$.

We define the involution $\iota : \mathrm{Pic}^d(X) \to \mathrm{Pic}^d(X)$, and the subsets Ω, B of $\mathrm{Pic}^d(X)$ as in the case $b_2 = 1$. Let ζ', ζ'' be the images of the ι-fixed points $\mathfrak{l}'$, $\mathfrak{l}'' \in \Omega$, and S be the image of $\iota(B) \cap \Omega$ via the composition

$$\Omega \to {}^{\Omega}\!/_{\langle\iota\rangle} \xrightarrow{\Phi \simeq} D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}),$$

where $\Phi : {}^{\Omega}\!/_{\langle\iota\rangle} \xrightarrow{\simeq} D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}})$ is the biholomorphism given by (24).

The following statements give explicit geometric descriptions of the moduli spaces $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$, $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ in the case $\deg_g(\mathcal{K}_X) < 0$. The proofs use the methods explained in detail in the previous section, combined with results specific to the case $b_2 = 2$ [32].

Theorem 3.12. *Suppose* $\deg_g(\mathcal{K}_X) < 0$, $\delta_1 \neq \delta_2$. *There exists:*

1. *A divisor* $\Delta \subset D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1$ *such that the restriction*

$$p|_\Delta : \Delta \to D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}})$$

of the projection on the first factor is a flat finite map of degree 2 which identifies $\mathrm{Sing}(\Delta)$ *with* S *and its ramification locus with* $\{\zeta', \zeta''\}$.

2. *A lift* $\Gamma \subset \Delta$ *of the ellipse* $\Gamma(e^{-\frac{\delta_1}{\nu}} + e^{-\frac{\delta_2}{\nu}}, |e^{-\frac{\delta_1}{\nu}} - e^{-\frac{\delta_2}{\nu}}|)$ *via* $p|_\Delta$.

3. *A biholomorphism*

$$\mathfrak{F} : \big(D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1\big)\backslash\Gamma \to \mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$$

which maps

   (a) *the set* $\big(D(1 + e^{-\frac{\delta}{\nu}}, 1 - e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1\big)\backslash\Delta$ *onto the set of isomorphism classes of stable extensions of the form (19) with* $c_1(\mathcal{L}) = d$.

   (b) *the set* $\Delta\backslash\Gamma$ *onto the set of isomorphism classes of stable extensions of the form (19) with* $c_1(\mathcal{L}) \in \{d_1, d_2\}$.

To obtain $\mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ we have to compactify $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ by adding two circles of reductions. Using the notation introduced in Remark 2.11 we have:

Proposition 3.13. *Under the assumptions of Theorem 3.12, let* Π *be the space obtained from* $\bar{D}(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}}) \times \mathbb{P}^1$ *by collapsing the fiber* $\{z\} \times \mathbb{P}^1$ *over each point* $z \in \Gamma(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})$ *to a point* p_z*. Then* $\mathfrak{F}$ *extends to a homeomorphism* $\Pi \to \mathcal{M}^{\mathrm{pst}}_{\mathcal{K}_X}(E)$ *which maps the ellipse*

$$\Gamma' := \{p_z |\ z \in \Gamma(1+e^{-\frac{\delta}{\nu}}, 1-e^{-\frac{\delta}{\nu}})\} \subset \Pi$$

onto the circle of reductions $\mathcal{R}_{\varnothing,\{1,2\}}$*, and the ellipse* $\Gamma \subset \Pi$ *onto the circle of reductions* $\mathcal{R}_{\{1\},\{2\}}$*.*

Remark 3.14. The obtained descriptions of the moduli space $\mathcal{M}^{\mathrm{st}}_{\mathcal{K}_X}(E)$ for $b_2 \in \{1,2\}$ confirm the main result of [36], which states that, on any blown up primary Hopf surface, this moduli space does not contain any compact connected component.

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