

Dedicated to Vasile Brînzănescu on his 75th birthday

AN INJECTIVITY THEOREM II

FLORIN AMBRO

We extend the injectivity theorem of Esnault and Viehweg to a class of non-normal log varieties, which contains normal crossings log varieties, and is closed under the operation of taking the LCS locus.

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1. INTRODUCTION

The birational classification of complex manifolds rests on vanishing theorems for Cartier divisors of the form $L \sim_{\mathbb{Q}} K_X + B$, where (X, B) is a *log smooth variety* (i.e. X is a smooth complex variety and $B = \sum_i b_i E_i$ is a boundary with coefficients $b_i \in \mathbb{Q} \cap [0, 1]$, such that $\sum_i E_i$ is a normal crossings divisor on X). In the order in which one may prove these vanishing theorems, they are Esnault-Viehweg injectivity, Tankeev-Kollár injectivity, Kollár's torsion freeness, Ohsawa-Kollár vanishing, Kawamata-Viehweg vanishing. The injectivity theorems imply the rest. Modulo cyclic covering tricks and Hironaka's desingularization, the injectivity theorems are a direct consequence of the E_1 -degeneration of the Hodge to de Rham spectral sequence associated to an open manifold.

To study the category of log smooth varieties, it is necessary to enlarge it to allow certain non-normal, even reducible objects, which appear in inductive arguments in the study of linear systems, or in compactification problems for moduli spaces of manifolds. The smallest such enlargement is the category of *normal crossings log varieties* (X, B) , which may be thought as glueings of log smooth varieties, in the simplest possible way. By definition, they are locally analytically isomorphic to the local model $0 \in X = \cup_{i \in I} \{z_i = 0\} \subset \mathbb{A}^N$, endowed with the boundary $B = \sum_{j \in J} b_j \{z_j = 0\}|_X$, where I, J are disjoint subsets of $\{1, \dots, N\}$ and $b_j \in \mathbb{Q} \cap [0, 1]$. Since X has Gorenstein singularities, the dualizing sheaf ω_X is an invertible \mathcal{O}_X -module. We denote by K_X a Cartier divisor on X such that $\omega_X \simeq \mathcal{O}_X(K_X)$. By definition, B is \mathbb{Q} -Cartier. Normal crossings varieties are build up of their lc centers, closed irreducible

subvarieties, which on the local analytic model correspond to (unions of) affine subspaces $\cap_{i \in I'} \{z_i = 0\} \cap \cap_{j \in J'} \{z_j = 0\} \subset \mathbb{A}^N$, where $I' \subset I$ is a non-empty subset, and $J' \subset \{j \in J; b_j = 1\}$ is a possibly empty subset. For example, the irreducible components of X are lc centers of (X, B) . Inside the category of normal crossings log varieties, log smooth varieties are exactly those with normal ambient space. The aim of this paper is to show that the above mentioned vanishing theorems remain true in the category of normal crossings varieties.

THEOREM 1. *Let (X, B) be a normal crossings log variety, L a Cartier divisor on X , and $f: X \rightarrow Y$ a proper morphism.*

- 1) (*Esnault-Viehweg injectivity*) *Suppose $L \sim_{\mathbb{Q}} K_X + B$. Let D be an effective Cartier divisor supported by B . Then the natural homomorphisms $R^q f_* \mathcal{O}_X(L) \rightarrow R^q f_* \mathcal{O}_X(L + D)$ are injective.*
- 2) (*Tankeev-Kollár injectivity*) *Suppose $L \sim_{\mathbb{Q}} K_X + B + H$, where H is an f -semiample \mathbb{Q} -divisor. Let D be an effective Cartier divisor which contains no lc center of (X, B) , and such that $D \sim_{\mathbb{Q}} uH$ for some $u > 0$. Then the natural homomorphisms $R^q f_* \mathcal{O}_X(L) \rightarrow R^q f_* \mathcal{O}_X(L + D)$ are injective.*
- 3) (*Kollár's torsion freeness*) *Suppose $L \sim_{\mathbb{Q}} K_X + B$. Let s be a local section of $R^q f_* \mathcal{O}_X(L)$ whose support does not contain $f(C)$, for every lc center C of (X, B) . Then $s = 0$.*
- 4) (*Ohsawa-Kollár vanishing*) *Let $g: Y \rightarrow Z$ be a projective morphism. Suppose $L \sim_{\mathbb{Q}} K_X + B + f^*A$, where A is a g -ample \mathbb{Q} -Cartier divisor on Y . Then $R^p g_* R^q f_* \mathcal{O}_X(L) = 0$ for $p \neq 0$.*

The notation $L \sim_{\mathbb{Q}} M$ means that there exists a positive integer r such that both rL and rM are Cartier divisors, and $\mathcal{O}_X(rL) \simeq \mathcal{O}_X(rM)$. Kawamata-Viehweg vanishing is the case $\dim Z = 0$ of the Ohsawa-Kollár vanishing.

Theorem 1.2)-4) was proved by Kawamata [7] if B has coefficients strictly less than 1, and it was proved for *embedded* normal crossings varieties (X, B) in [1, Section 3]. We remove the global embedded assumption in this paper, as expected in [1, Remark 2.9]. Theorem 1.1) is implicit in the proof of [1, Theorem 3.1], in the case when (X, B) is embedded normal crossings and D is supported by the part of B with coefficients strictly less than 1, which is the original setting of Esnault and Viehweg. We observed in [2] that the same results holds if D is supported by B , and Theorem 1.1) extends [2] to the normal crossings case.

Theorem 1 is proved by reduction to the log smooth case. There are two known methods of proof. Let $\bar{X} \rightarrow X$ be the normalization, let $X_n = (\bar{X}/X)^{n+1}$ for $n \geq 0$. With the natural projections and diagonals, we obtain a simplicial algebraic variety X_\bullet , together with a natural augmentation $\epsilon: X_\bullet \rightarrow X$. The key point is that each X_n is smooth, so we may really think of ϵ as a resolution of singularities. The method in [7] is to use the descent spectral sequence to deduce a statement on X from the same statement on each X_n . The method in [1] is to lift the statement from X to a statement on X_\bullet , and imitate the proof used in the log smooth case in this simplicial setting. In this paper we use the method in [7]. The new idea is an adjunction formula

$$(K_X + B)|_{X_n} \sim_{\mathbb{Q}} K_{X_n} + B_n,$$

for a suitable log smooth structure (X_n, B_n) , for each n . Moreover, (X_n, B_n) glue to a log smooth simplicial variety. To achieve this, we observe that each irreducible component of X_n is the normalization of some lc center of (X, B) . Then the adjunction formula follows from the theory of residues for normal crossings varieties developed in [4]. To construct residues for normal crossings varieties we have to deal with slightly more general singularities, namely *generalized normal crossings log varieties*. The motivation for this enlargement, is that if X has normal crossings singularities, then $\text{Sing } X$ may not have normal crossings singularities. But if X has generalized normal crossings singularities, so does $\text{Sing } X$. We actually prove Theorem 1 in the category of generalized normal crossings singularities (Theorems 12, 14, 15, 16). The same proof works in the category of normal crossings log varieties, provided their residues to lc centers are taken for granted. Note that generalized normal crossings singularities in our sense are more general than those defined by Kawamata [7]. For example, every seminormal curve is generalized normal crossings.

To illustrate how generalized normal crossings appear, let us consider two examples of residues. First, consider the log smooth variety $(\mathbb{A}^2, H_1 + H_2)$, where H_1, H_2 are the standard hyperplanes, intersecting at the origin 0. We want to perform adjunction from $(\mathbb{A}^2, H_1 + H_2)$ to its lc center 0. We may first take residue onto H_1 , and end up with the log structure $(H_1, 0)$, and then take residue from $(H_1, 0)$ to 0. But we may also restrict to $(H_2, 0)$, and then to 0. The two chains of residues do not coincide; they differ by -1 . Since an analytic isomorphism interchanges the two hyperplanes, none of the above compositions of residues is canonical. But they become canonical if raised to even powers. We obtain a canonical residue isomorphism

$$\text{Res}_{\mathbb{A}^2 \rightarrow 0}^{[2]}: \omega_{\mathbb{A}^2}(\log H_1 + H_2)^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}$$

Now we construct the same residue isomorphism, without coordinates. Denote

$C = H_1 + H_2$. Let ω_C be the sheaf whose sections are rational differential forms which are regular outside 0, and on the normalization $H_1 \sqcup H_2$ of C induce forms with logarithmic poles along the two points O_1, O_2 above the origin, and have the same residues at O_1, O_2 . One checks that ω_C is an invertible \mathcal{O}_C -module. The residues from \mathbb{A}^2 to the irreducible components of normalization of C glue to a residue isomorphism

$$\text{Res}_{\mathbb{A}^2 \rightarrow C}^{[2]}: \omega_{\mathbb{A}^2}(\log C)^{\otimes 2}|_C \xrightarrow{\sim} \omega_C^{\otimes 2}.$$

Since the forms of ω_C have the same residues above the origin, we also obtain a residue isomorphism

$$\text{Res}_{C \rightarrow 0}^{[2]}: \omega_C^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}.$$

The composition $\text{Res}_{C \rightarrow 0}^{[2]} \circ \text{Res}_{\mathbb{A}^2 \rightarrow C}^{[2]}$ is exactly $\text{Res}_{\mathbb{A}^2 \rightarrow 0}^{[2]}$. It is intrinsic, independent of the choice of coordinates, or analytic isomorphisms. Note that ω_C differs from the Rosenlicht dualizing sheaf Ω_C , but $\omega_C^{\otimes m} = \Omega_C^{\otimes m}$ for $m \in 2\mathbb{Z}$ (at the origin, the local generator for ω_C is $(\frac{dz_1}{z_1}, \frac{dz_2}{z_2})$, and for Ω_C is $(\frac{dz_1}{z_1}, -\frac{dz_2}{z_2})$).

Second, let S be the normal crossings surface $(xyz = 0) \subset \mathbb{A}^3$, set $B = 0$. We want to perform adjunction from S to its lc center the origin. As above, we may first restrict to a plane, then to a line, and then to the origin. There are several choices of chains, which coincide up to a sign. If we raise to an even power, we obtain residue isomorphisms from S to 0. These are invariant under analytic isomorphisms, since we can also define them in the following invariant way. Let $C = \text{Sing } S$. Then C is the union of coordinate axis in \mathbb{A}^3 , a seminormal curve which is not Gorenstein. The usual dualizing sheaf is useless in this situation. We may define ω_C as above (requiring same residues over the origin), and then ω_C is an invertible \mathcal{O}_C -module (at the origin, the local generator is $(\frac{dz_1}{z_1}, \frac{dz_2}{z_2}, \frac{dz_3}{z_3})$), and residues from S to the irreducible components of the normalization of C glue to a residue isomorphism

$$\text{Res}_{S \rightarrow C}^{[2]}: \omega_S^{\otimes 2}|_C \xrightarrow{\sim} \omega_C^{\otimes 2}.$$

The singular locus of C is 0, and we again obtain a residue isomorphism

$$\text{Res}_{C \rightarrow 0}^{[2]}: \omega_C^{\otimes 2}|_0 \xrightarrow{\sim} \omega_0^{\otimes 2}.$$

The composition $\text{Res}_{C \rightarrow 0}^{[2]} \circ \text{Res}_{S \rightarrow C}^{[2]}$ is exactly $\text{Res}_{S \rightarrow 0}^{[2]}$, defined from coordinates.

The conclusion we draw from these two examples is that we must redefine the powers of log canonical sheaf $\omega_{(X,B)}^{[n]}$ ($n \in \mathbb{Z}$) (without dualizing property), and we must allow singularities which are not normal crossings, but very close. In [4], we constructed residues for so called n-wlc varieties. Generalized normal crossings varieties are a special case of n-wlc varieties.

The goal of this paper is to have a simple technique which glues injectivity theorems on log smooth varieties to injectivity theorems on glueings of log smooth varieties. To keep the paper short, we do not present new applications. But the reader should easily extend the applications in Sections 5 and 6 of [2] to the setting of quasi-log varieties, as a corollary of Theorem 1. An application of Section 3 is that the usual vanishing theorems are valid for the simplest examples of weakly log canonical varieties, introduced in [4] as a natural generalization of semi-log canonical singularities. It is an interesting question if the Base Point Free and the Cone Theorem are valid in the category of weakly log canonical varieties.

We outline the structure of this paper. In Section 1, we construct the simplicial log variety induced by a n -wlc log variety. The reader should be familiar with [4, Sections 3 and 5]. In Section 2, we define generalized normal crossings log varieties, and analyze the induced simplicial log variety. In Section 3, we prove the vanishing theorems. The injectivity theorems are reduced to the smooth case, using the simplicial log structure induced. The torsion freeness and vanishing theorems are deduced then by standard arguments. In Section 4, we collect some inductive properties of generalized normal crossings varieties. The key inductive property is that the LCS-locus of a generalized normal crossings log variety is again a generalized normal crossings log variety, for a suitable boundary, and we can perform adjunction onto the LCS-locus. We hope that in the future one may be able to use these inductive properties to reprove the vanishing theorems in Section 3.

2. PRELIMINARIES

All varieties are defined over an algebraically closed field k , of characteristic zero.

A *log smooth variety* is a pair (X, B) , where X is a smooth k -variety and $B = \sum_i b_i E_i$ is a boundary such that $b_i \in \mathbb{Q} \cap [0, 1]$ and $\sum_i E_i$ is a NC divisor.

We refer the reader to [4] for the definition and basic properties of wlc varieties $(X/k, B)$, and some special cases: toric and n -wlc. We will remove the fixed ground field k from notation; for example we denote $\omega_{(X/k, B)}^{[n]}$ by $\omega_{(X, B)}^{[n]}$.

LEMMA 2. *Let $(X', B_{X'})$ and (X, B) be normal log pairs, let $f: (X', B_{X'}) \rightarrow (X, B)$ be étale and log crepant. Let $Z' \subset X'$ be a closed irreducible subset. Then Z' is an lc center of $(X', B_{X'})$ if and only if $f(Z')$ is an lc center of (X, B_X) .*

Proof. Cutting $f(Z')$ with general hyperplane sections, we may suppose Z' is a closed point P' . Since f is open, we may replace X by the image of f and

suppose f is surjective. After removing from X' the finite set $f^{-1}f(P') \setminus P'$, we may also suppose $f^{-1}f(P') = P'$. Then the claim follows from [8, page 46, 2.14.(2)]. \square

2.1. Simplicial log structure induced by a n-wlc log variety

Let (X, B) be a n-wlc log variety (see [4, Section 5]). Let $r \in (2\mathbb{Z})_{>0}$ such that rB has integer coefficients and $\omega_{(X,B)}^{[r]}$ is an invertible \mathcal{O}_X -module. Let $\pi: \bar{X} \rightarrow X$ be the normalization. Then $X_n = (\bar{X}/X)^{n+1}$ ($n \geq 0$) are the components of a simplicial k -algebraic variety X_\bullet , endowed with a natural augmentation $\epsilon: X_\bullet \rightarrow X$.

PROPOSITION 3. *The following properties hold:*

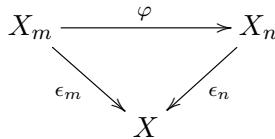
- a) *Each X_n is normal. Let Z_n be an irreducible component of X_n . Then $\epsilon_n: Z_n \rightarrow X$ is the normalization of an lc center of (X, B) . Let (Z_n, B_{Z_n}) be the n-wlc log variety structure induced by the residue isomorphism $\text{Res}^{[r]}: \omega_{(X,B)}^{[r]}|_{Z_n} \xrightarrow{\sim} \omega_{(Z_n, B_{Z_n})}^{[r]}$ (see [4, Theorem 5.9]).*

Let $(X_n, B_n) = \sqcup_{Z_n} (Z_n, B_{Z_n})$ be the induced structure of normal log variety, with n-wlc singularities (independent of the choice of r). We obtain isomorphisms

$$\text{Res}_{X \rightarrow X_n}^{[r]}: \epsilon_n^* \omega_{(X,B)}^{[r]} \xrightarrow{\sim} \omega_{(X_n, B_n)}^{[r]}.$$

Moreover, each lc center of (X, B) is the image of some lc center of (X_n, B_n) .

- b) *Let $\varphi: X_m \rightarrow X_n$ be the simplicial morphism induced by an order preserving morphism $\Delta_n \rightarrow \Delta_m$, for some $m, n \geq 0$. It induces a commutative diagram*



Let Z_m be an irreducible component of X_m . Then $\varphi: Z_m \rightarrow X_n$ is the normalization of an lc center of (X_n, B_n) . Let $\text{Res}^{[r]}: \omega_{(X_n, B_n)}^{[r]}|_{Z_m} \xrightarrow{\sim} \omega_{(Z_m, B_{Z_m})}^{[r]}$ be the induced residue isomorphism. Let $\text{Res}_\varphi^{[r]}: \varphi^ \omega_{(X_n, B_n)}^{[r]} \xrightarrow{\sim} \omega_{(X_m, B_m)}^{[r]}$ be the induced isomorphism. Then*

$$\text{Res}_\varphi^{[r]} \circ \varphi^* \text{Res}_{X \rightarrow X_n}^{[r]} = \text{Res}_{X \rightarrow X_m}^{[r]}.$$

In particular, $\omega_{(X_n, B_n)}^{[r]}$ and $\text{Res}_\varphi^{[r]}$ form an \mathcal{O}_{X_\bullet} -module $\omega_{(X_\bullet, B_\bullet)}^{[r]}$, endowed with an isomorphism $\epsilon^ \omega_{(X,B)}^{[r]} \xrightarrow{\sim} \omega_{(X_\bullet, B_\bullet)}^{[r]}$.*

Proof. By [5], we may suppose (X, B) coincides with a local model. That is $X = \text{Spec } k[\mathcal{M}]$ is the toric variety associated with a monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$, X has normal irreducible components, B is an effective boundary supported by invariant prime divisors at which X is smooth, and (X, B) has wlc singularities.

Let $X = \cup_F X_F$ be the decomposition into irreducible components, where the union runs after all facets F of Δ . Let $\psi \in \cap_F \frac{1}{r} S_F$ be the log discrepancy function of (X, B) . By assumption, each irreducible component X_F is normal. Therefore $\bar{X} = \sqcup_F X_F$. We obtain

$$X_n = \sqcup_{F_0, \dots, F_n} X_{F_0 \cap \dots \cap F_n}.$$

Since $\psi \in F_0 \cap \dots \cap F_n$, each $X_{F_0 \cap \dots \cap F_n}$ is an lc center of (X, B) . The toric log structure induced via residues on $X_{F_0 \cap \dots \cap F_n}$ is that induced by the log discrepancy function $\psi \in F_0 \cap \dots \cap F_n$.

An lc center of (X, B) is of the form X_γ , with $\psi \in \gamma \in \Delta$. If F is a facet of Δ which contains γ , then X_γ is also an lc center of the irreducible component $(X_{F \cap \dots \cap F}, B_n)$ of (X_n, B_n) . This proves a).

For b), recall that any simplicial morphism is a composition of face morphisms $\delta_i: X_{n+1} \rightarrow X_n$ and degeneracy morphisms $s_i: X_n \rightarrow X_{n+1}$. Hence suffices to verify b) for face and degeneracy morphisms. For our local model, δ_i embeds $X_{F_0 \cap \dots \cap F_{n+1}}$ into $X_{F_0 \cap \dots \widehat{F}_i \dots \cap F_{n+1}}$, and s_i maps $X_{F_0 \cap \dots \cap F_n}$ isomorphically onto $X_{F_0 \cap \dots \cap F_i \cap F_i \cap \dots \cap F_n}$. Then b) holds in our case, since all log structures involved have the same log discrepancy function ψ . \square

3. GNC LOG VARIETIES

Recall first some standard notation. The set $\{1, 2, \dots, N\}$ is denoted by $[N]$, the k -affine space \mathbb{A}_k^N has coordinates $(z_i)_{i \in [N]}$, and $H_i = \{z \in \mathbb{A}^N; z_i = 0\}$ is the standard i -th hyperplane. For a subset $F \subseteq [N]$, denote $\mathbb{A}_F = \cap_{i \in [N] \setminus F} \{z \in \mathbb{A}^N; z_i = 0\}$. It is an affine space with coordinates $(z_i)_{i \in F}$.

Definition 4. A *GNC (generalized normal crossings) local model* is a pair (X, B) , of the following form:

- a) $X = \cup_F \mathbb{A}_F \subset \mathbb{A}_k^N$, where the union is indexed after finitely many subsets $F \subseteq [N]$ (called facets), not contained in one another. We assume X satisfies Serre's property S_2 , that is for any two facets $F \neq F'$, there exists a chain of facets $F = F_0, F_1, \dots, F_l = F'$ such that for every $0 \leq i < l$, $F_i \cap F_{i+1}$ contains $F \cap F'$ and it has codimension one in both F_i and F_{i+1} .

- b) Denote $\sigma = \cap_F F$. If $\sigma \prec \tau \prec F$ and τ has codimension one in F , then there exists a facet F' such that $\tau = F \cap F'$.
- c) $B = (\sum_{i \in \sigma} b_i H_i)|_X$, where $b_i \in \mathbb{Q} \cap [0, 1]$ and $H_i = \{z \in \mathbb{A}^N; z_i = 0\}$. We may rewrite $B = \sum_F \sum_{i \in \sigma} b_i \mathbb{A}_{F \setminus i}$.

We claim that (X, B) is a toric wlc log variety. Note first that X is the toric variety $\text{Spec } k[\mathcal{M}]$ associated to the monoidal complex $\mathcal{M} = (M, \Delta, (S_\sigma)_{\sigma \in \Delta})$, where $M = \mathbb{Z}^N$, Δ is the fan consisting of the cones $\sum_{i \in F} \mathbb{R}_{\geq 0} m_i$ and all their faces, and $S_\sigma = \mathbb{Z}^N \cap \sigma$ for $\sigma \in \Delta$. Here m_1, \dots, m_N denotes the standard basis of the semigroup \mathbb{N}^N . Each irreducible component of X is smooth. The normalization of X is $\bar{X} = \sqcup_F \mathbb{A}_F$. Denote $\psi = \sum_{i \in \sigma} (1 - b_i) m_i$. On \mathbb{A}_F , ψ induces the log structure with boundary

$$B_{\mathbb{A}_F} = \sum_{i \in F \setminus \sigma} \mathbb{A}_{F \setminus i} + \sum_{i \in \sigma} b_i \mathbb{A}_{F \setminus i}.$$

Let $\bar{C} \subset \bar{X}$ be the conductor subscheme. By a), $\bar{C}|_{\mathbb{A}_F} \leq \sum_{i \in F \setminus \sigma} \mathbb{A}_{F \setminus i}$. Equality holds if and only if b) holds. Therefore

$$B_{\mathbb{A}_F} = \bar{C}|_{\mathbb{A}_F} + \sum_{i \in \sigma} b_i \mathbb{A}_{F \setminus i} = (\bar{C} + \bar{B})|_{\mathbb{A}_F}.$$

We conclude that the irreducible components of $(\bar{X}, \bar{C} + \bar{B})$ have the same log discrepancy function ψ , and therefore (X, B) is a toric wlc log variety, by [4, Proposition 4.10]. Note that X is \mathbb{Q} -orientable by [4, Lemma 4.7 and Example 4.8.(2)]. If $2 \mid r$ and $rb_i \in \mathbb{Z}$ for all $i \in \sigma$, then $\omega_{(X, B)}^{[r]} \simeq \mathcal{O}_X$. Given a), properties b) and c) are equivalent to

- b') $(X, 0)$ is a toric wlc log variety.
- c') B is a torus-invariant boundary whose support contains no lc center of $(X, 0)$.

The \mathbb{Q} -divisors $B, B^{=1}, B^{<1}$ are \mathbb{Q} -Cartier (so is the part of B with coefficients in a given interval in \mathbb{R}).

Example 5. A *NC (normal crossings) local model* is a pair (X, B) , where $X = \cup_{i \in I} H_i \subset \mathbb{A}_k^N$ and $B = (\sum_{i \notin I} b_i H_i)|_X$, where I is a non-empty subset of $[N]$ and $b_i \in \mathbb{Q} \cap [0, 1]$. If we set $F = [N] \setminus i$ ($i \in I$), we see that (X, B) is a GNC local model. Here we have $\sigma = [N] \setminus I$.

Example 6. Let $\sigma \subsetneq [N]$, let $|\sigma| \leq p < N$. Let $\{F\}$ consist of all subsets of $[N]$ which have cardinality p , and contain σ . Let $b_i \in \mathbb{Q} \cap [0, 1]$, for $i \in \sigma$. Then $(X = \cup_F \mathbb{A}_F \subset \mathbb{A}_k^N, (\sum_{i \in \sigma} b_i H_i)|_X)$ is a GNC local model.

Example 7. Let $X = \mathbb{A}_{12} \cup \mathbb{A}_{23} \cup \mathbb{A}_{34} \subset \mathbb{A}_k^4$ and $B = \mathbb{A}_1 + \mathbb{A}_4$. Then (X, B) is a toric wlc log variety (with log discrepancy function $\psi = 0$), but not a GNC local model.

Definition 8. A *GNC (NC) log variety* (X, B) is a wlc log variety such that for every closed point $x \in X$, there exists a GNC (NC) local model (X', B') and an isomorphism of complete local k -algebras $\mathcal{O}_{X,x}^\wedge \simeq \mathcal{O}_{X',0}^\wedge$, such that $(\omega_{(X,B)}^{[r]})_x^\wedge$ corresponds to $(\omega_{(X',B')}^{[r]})_0^\wedge$ for r sufficiently divisible.

By [5], there exists a common étale neighborhood

$$\begin{array}{ccc} & (U, y) & \\ i \swarrow & & \searrow i' \\ (X, x) & & (X', 0) \end{array}$$

and a wlc log variety structure (U, B_U) on U such that $i^* \omega_{(X,B)}^{[n]} = \omega_{(U,B_U)}^{[n]} = i'^* \omega_{(X',B')}^{[n]}$ for all $n \in \mathbb{Z}$.

It follows that $(X, 0)$ is a GNC (NC) log variety, and $B, B^{=1}, B^{<1}$ are \mathbb{Q} -Cartier divisors.

Remark 9. Let (X, B) be a NC log variety. Let ω_X be the canonical choice of dualizing sheaf, defined by Rosenlicht. It is an invertible \mathcal{O}_X -module, since X is locally complete intersection. If rB has integer coefficients and r is divisible by 2, then $\omega_X^{\otimes r} \otimes \mathcal{O}_X(rB) = \omega_{(X,B)}^{[r]}$ (see [4]).

3.1. Simplicial log structure induced by a GNC log variety

Let (X, B) be a GNC log variety. Let $\epsilon: X_\bullet \rightarrow X$ be the simplicial resolution induced by the normalization of X . A GNC log variety is n-wlc. By Proposition 3, residues induce a natural simplicial log variety structure (X_\bullet, B_\bullet) . In this case (X_n, B_n) is a disjoint union of log smooth log varieties, and we have residue isomorphisms

$$\text{Res}_{X \rightarrow X_n}^{[r]} : \epsilon_n^* \omega_{(X,B)}^{[r]} \xrightarrow{\sim} \omega_{(X_n, B_n)}^{[r]}$$

for $r \in (2\mathbb{Z})_{>0}$ such that rB has integer coefficients.

LEMMA 10. *The following properties hold:*

- 1) $\epsilon: X_\bullet \rightarrow X$ is a smooth simplicial resolution, and $\mathcal{O}_X \rightarrow R\epsilon_* \mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.

- 2) The lc centers of $(X, 0)$ are the images of the irreducible components of X_n ($n \geq 0$).
- 3) (X_n, B_n) is a log smooth variety, for all n .
- 4) The support of B contains no lc center of $(X, 0)$, and each $\epsilon_n^* B$ is supported by B_n .

Proof. We may suppose (X, B) is a GNC local model. Then

$$(X_n, B_n) = \sqcup_{F_0, \dots, F_n} (\mathbb{A}_{F_0 \cap \dots \cap F_n}, \sum_{i \in F_0 \cap \dots \cap F_n} \mathbb{A}_{F_0 \cap \dots \cap F_n \setminus i} + \sum_{i \in \sigma} b_i \mathbb{A}_{F_0 \cap \dots \cap F_n \setminus i}).$$

1) Each X_n is smooth, so $\epsilon: X_\bullet \rightarrow X$ is a smooth simplicial resolution. By [3, Theorem 0.1.b)], $\mathcal{O}_X \rightarrow R\epsilon_* \mathcal{O}_{X_\bullet}$ is a quasi-isomorphism.

2) The log variety $(X, 0)$ has log discrepancy function $\psi = \sum_{i \in \sigma} m_i \in \text{reint } \sigma$. Therefore its lc centers are X_γ , where $\sigma \prec \gamma \in \Delta$. We claim that each such γ is an intersection of facets of Δ . Indeed, if γ is a facet, the claim holds. Else, choose a facet F which contains γ . Since $\gamma \subsetneq F$, γ is the intersection after all codimension one faces $\tau \prec F$ which contain γ . Each τ contains the core σ . Therefore $\tau = F \cap F'$ for some facet F' , by axiom b) in the definition of GNC local models. We conclude that $\gamma = F_0 \cap \dots \cap F_n$ for some $n \geq 0$. Therefore X_γ appears as an irreducible component of X_n .

3) This is clear from the explicit formula for (X_n, B_n) .

4) The support of B does not contain the core X_σ . Since the image on X of an irreducible component of X_n does contain X_σ , we obtain that $\epsilon_n^* B$ is well \mathbb{Q} -Cartier defined for all n . Moreover,

$$(B_n - \epsilon_n^* B)|_{\mathbb{A}_{F_0 \cap \dots \cap F_n}} = \sum_{i \in F_0 \cap \dots \cap F_n} \mathbb{A}_{F_0 \cap \dots \cap F_n \setminus i}.$$

□

4. VANISHING THEOREMS

LEMMA 11. *Let (X, B) be a log smooth variety. Let L be a Cartier divisor on X such that $L \sim_{\mathbb{Q}} K_X + B$. Let D be an effective Cartier divisor supported by B . Let $f: X \rightarrow Z$ be a proper morphism. Then the natural homomorphisms $R^q f_* \mathcal{O}_X(L) \rightarrow R^q f_* \mathcal{O}_X(L + D)$ are injective.*

Proof. We may suppose X is irreducible, f is surjective, and Z is affine. Let $Z \hookrightarrow \mathbb{A}^N$ be a closed embedding into an affine space. Compactify $\mathbb{A}^N \subset \mathbb{P}^N$ by adding the hyperplane at infinity H_0 . Let $Z' \subset \mathbb{P}^N$ be the closure of Z . Let

$H = H_0|_{Z'}$. Then $Z \subset Z'$ is an open dense embedding, whose complement H is a hyperplane section.

By Nagata, there exists an open dense embedding $X \subset X''$ such that X'' is proper. The induced rational map $f: X'' \dashrightarrow Z'$ is regular on X . By Hironaka's desingularization, there exists a birational contraction $X' \rightarrow X''$, which is an isomorphism over X , such that X' is smooth and f induces a regular map $f': X' \rightarrow Z'$. We may also suppose $\Sigma = X' \setminus X$ is a NC divisor, and $(X', B' + \Sigma)$ is log smooth, where $B' = \sum_i b_i(E_i)'$ is the closure of B in X' (defined componentwise). We obtained a diagram

$$\begin{array}{ccc} X & \longrightarrow & X' \\ f \downarrow & & \downarrow f' \\ Z & \longrightarrow & Z' \end{array}$$

where the vertical arrows are open dense embeddings, Z' is projective and X' is proper. The properness of f is equivalent to $X = f'^{-1}(Z)$, so the diagram is also cartesian.

We represent L by a Weil divisor on X . Let L' be its closure in X' . Then $L' \sim_{\mathbb{Q}} K_{X'} + B' + N$, where N is a \mathbb{Q} -divisor supported by Σ . Denote $P = L' - \lfloor N \rfloor$ and $\Delta = B' + \{N\}$. Then $P \sim_{\mathbb{Q}} K_{X'} + \Delta$ and (X', Δ) is log smooth. The closure D' of D in X' is supported by B' , hence it is supported by Δ .

Let m be a positive integer. Let S be a general member of the free linear system $|f'^*(mH)|$. Then $P + f'^*(mH) \sim_{\mathbb{Q}} K_{X'} + \Delta + S$, $(X', \Delta + S)$ is log smooth, and D' is supported by $\Delta + S$. Denote $\mathcal{F} = \mathcal{O}_{X'}(P)$. By [2, Theorem 0.1], the natural homomorphism

$$H^n(X', \mathcal{F}(f'^*(mH))) \rightarrow H^n(X', \mathcal{F}(f'^*(mH) + D')) \quad (n \geq 0)$$

is injective. We have the Leray spectral sequence

$$E_2^{pq} = H^p(Z', R^q f'_* \mathcal{F}(m)) \implies H^{p+q}(X', \mathcal{F}(f'^*(mH))).$$

Suppose m is sufficiently large. Serre vanishing gives $E_2^{pq} = 0$ if $p \neq 0$. Therefore we obtain a natural isomorphism

$$H^0(Z', R^n f'_* \mathcal{F}(m)) \xrightarrow{\sim} H^n(X', \mathcal{F}(f'^*(mH) + D')).$$

By the same argument, we have a natural isomorphism

$$H^0(Z', R^n f'_* \mathcal{F}(D')(m)) \xrightarrow{\sim} H^n(X', \mathcal{F}(D' + f'^*(mH))).$$

The injective homomorphism above becomes the injective homomorphism

$$H^0(Z', R^n f'_* \mathcal{F}(m)) \rightarrow H^0(Z', R^n f'_* \mathcal{F}(D')(m)).$$

Since $\mathcal{O}_{Z'}(m)$ is very ample, this means that $R^n f'_* \mathcal{F} \rightarrow R^n f'_* \mathcal{F}(D')$ is injective. But $X = f'^{-1}(Z)$, $P|_X = L$, $\mathcal{F}|_X = \mathcal{O}_X(L)$ and $D'|_X = D$, so the restriction of this injective homomorphism to Z is just the injective homomorphism $R^n f_* \mathcal{O}_X(L) \rightarrow R^n f_* \mathcal{O}_X(L + D)$. \square

THEOREM 12 (Esnault-Viehweg injectivity). *Let (X, B) be a GNC log variety. Let \mathcal{L} be an invertible \mathcal{O}_X -module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X, B)}^{[r]}$ for some $r \geq 1$ such that rB has integer coefficients. Let D be an effective Cartier divisor supported by B . Let $f: X \rightarrow Z$ be a proper morphism. Then the natural homomorphism $R^i f_* \mathcal{L} \rightarrow R^i f_* \mathcal{L}(D)$ is injective, for every i .*

Proof. We may suppose Z is affine. Denote $\Sigma = \text{Supp } B$ and $U = X \setminus \Sigma$. Since rB is Cartier, we have an isomorphism

$$\varinjlim_{m \in \mathbb{N}} H^i(X, \mathcal{O}_X(mrB)) \xrightarrow{\sim} H^i(U, \mathcal{L}|_U).$$

The claim for all D is thus equivalent to the injectivity of the restriction homomorphisms

$$H^i(X, \mathcal{L}) \rightarrow H^i(U, \mathcal{L}|_U).$$

Let $\epsilon: X_\bullet \rightarrow X$ be the smooth simplicial resolution induced by the normalization of X . Let $\Sigma_n = \epsilon_n^{-1}(\Sigma)$ and $U_n = X_n \setminus \Sigma_n$. The restriction $\epsilon: U_\bullet \rightarrow U$ is also a smooth simplicial resolution. By Lemma 10, $\mathcal{L} \rightarrow R\epsilon_* \mathcal{L}_\bullet$ and $\mathcal{L}|_U \rightarrow R\epsilon_* \mathcal{L}_\bullet|_{U_\bullet}$ are quasi-isomorphisms. Therefore the claim is equivalent to the injectivity of the restriction homomorphisms

$$\alpha: H^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}).$$

Both spaces are endowed with simplicial filtrations S . The Godement resolutions $\mathcal{L}_p \rightarrow \mathcal{K}_p^*$ ($p \geq 0$) glue to a simplicial resolution $\mathcal{L}_\bullet \rightarrow \mathcal{K}_\bullet^*$. Denote $A_p^q = \Gamma_{\Sigma_p}(X_p, \mathcal{K}_p^q)$, $B_p^q = \Gamma(X_p, \mathcal{K}_p^q)$ and $C_p^q = \Gamma(U_p, \mathcal{K}_p^q)$. The associated simple complexes fit into a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

which induces in homology the long exact sequence

$$\cdots \rightarrow H_{\Sigma_\bullet}^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow H^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \rightarrow \cdots$$

Let S be the simplicial filtration (naive with respect to p) on A, B, C . For each p , the short exact sequence

$$0 \rightarrow \Gamma_{\Sigma_p}(X_p, \mathcal{K}_p^*) \rightarrow \Gamma(X_p, \mathcal{K}_p^*) \rightarrow \Gamma(U_p, \mathcal{K}_p^*) \rightarrow 0$$

is split. That is $0 \rightarrow E_0A \rightarrow E_0B \rightarrow E_0C \rightarrow 0$ is a split short exact sequence. Passing to homology, we obtain that $0 \rightarrow E_1A \rightarrow E_1B \rightarrow E_1C \rightarrow 0$ is a split short exact sequence. Iterating this argument, we conclude that $0 \rightarrow E_rA \rightarrow$

$E_r B \rightarrow E_r C \rightarrow 0$ is a split short exact sequence, for every r . Therefore $0 \rightarrow E_\infty A \rightarrow E_\infty B \rightarrow E_\infty C \rightarrow 0$ is a short exact sequence, which induces in homology the long exact sequence

$$\cdots \rightarrow \mathrm{Gr}_S H_{\Sigma_\bullet}^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow \mathrm{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow \mathrm{Gr}_S H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \rightarrow \cdots$$

Step 1. $H_{\Sigma_p}^q(X_p, \mathcal{L}_p) \rightarrow H^q(X_p, \mathcal{L}_p)$ is zero for all p, q . Indeed,

$$\mathcal{L}_p^{\otimes 2r} = \mathcal{L}^{\otimes 2r}|_{X_p} \simeq \omega_{(X,B)}^{[2r]}|_{X_p} \xrightarrow{\sim} \omega_{(X_p, B_p)}^{[2r]},$$

(X_p, B_p) is a log smooth variety, $U_p \supseteq X_p \setminus B_p$ by Lemma 10.4), and $X_p \rightarrow Z$ is proper. By Lemma 11, $H^q(X_p, \mathcal{L}_p) \rightarrow H^q(U_p, \mathcal{L}_p|_{U_p})$ is injective for all p, q . Equivalently,

$$H_{\Sigma_p}^q(X_p, \mathcal{L}_p) \rightarrow H^q(X_p, \mathcal{L}_p)$$

is zero for all p, q .

Step 2. $\mathrm{Gr}_S \alpha$ is injective. Indeed, $E_1 A \rightarrow E_1 B$ is the direct sum of $H_{\Sigma_p}^q(X_p, \mathcal{L}_p) \rightarrow H^q(X_p, \mathcal{L}_p)$. By Step 1, $E_1 A \rightarrow E_1 B$ is zero. Step by step, we deduce that $E_r A \rightarrow E_r B$ is zero for every $r \geq 1$. Then $E_\infty A \rightarrow E_\infty B$ is zero, that is $\mathrm{Gr}_S H_{\Sigma_\bullet}^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow \mathrm{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet)$ is zero. Therefore the last long exact sequence breaks up into short exact sequences

$$0 \rightarrow \mathrm{Gr}_S H^i(X_\bullet, \mathcal{L}_\bullet) \rightarrow \mathrm{Gr}_S H^i(U_\bullet, \mathcal{L}_\bullet|_{U_\bullet}) \rightarrow \mathrm{Gr}_S H_{\Sigma_\bullet}^{i+1}(X_\bullet, \mathcal{L}_\bullet) \rightarrow 0.$$

Step 3. Since $S_{i+1} H^i(X_\bullet, \mathcal{L}_\bullet) = 0$, the filtration S on $H^i(X_\bullet, \mathcal{L}_\bullet)$ is finite. Therefore the injectivity of $\mathrm{Gr}_S \alpha$ means that α is injective and strict with respect to the filtration S . \square

LEMMA 13. *Let (X, B) be a log smooth variety, let $f: X \rightarrow Z$ be a proper morphism. Let L be a Cartier divisor such that the \mathbb{Q} -divisor $A = L - (K_X + B)$ is f -semiample. Let D be an effective Cartier divisor on X such that $D \sim_{\mathbb{Q}} uA$ for some $u > 0$, and D contains no lc center of (X, B) . Then the natural homomorphism $R^q f_* \mathcal{O}_X(L) \rightarrow R^q f_* \mathcal{O}_X(L + D)$ is injective, for all q .*

Proof. We may suppose Z is affine, and A is f -semiample.

Step 1. Suppose $(X, B + \epsilon D)$ is log smooth, for some $0 < \epsilon < \frac{1}{u}$. We have

$$L = K_X + B + \epsilon D + (A - \epsilon D) \sim_{\mathbb{Q}} K_X + B + \epsilon D + (1 - \epsilon u)A.$$

Let $n \geq 1$ such that $\mathcal{O}_X(nA)$ is generated by global sections. Let S be the zero locus of a generic global section. Then

$$L \sim_{\mathbb{Q}} K_X + B + \epsilon D + \frac{1 - \epsilon u}{n} S,$$

the log variety $(X, B + \epsilon D + \frac{1 - \epsilon u}{n} S)$ is log smooth, and its boundary supports D . By Lemma 11, $H^q(X, \mathcal{O}_X(L)) \rightarrow H^q(X, \mathcal{O}_X(L + D))$ is injective, for all q .

Step 2. By Hironaka, there exists a desingularization $\mu: X' \rightarrow X$ such that the exceptional locus of μ and the proper transforms of B and D are supported by a NC divisor on X' . Let $\mu^*(K_X + B) = K_{X'} + B_{X'}$, let $E = \lceil -B_{X'}^{\leq 0} \rceil$. Then

$$\mu^*L + E = K_{X'} + B_{X'}^{\geq 0} + \{B_{X'}^{\leq 0}\} + \mu^*A.$$

The log variety $(X', B_{X'}^{\geq 0} + \{B_{X'}^{\leq 0}\} + \epsilon\mu^*D)$ is log smooth for $0 < \epsilon \ll 1$, by the choice of the resolution, and since D contains no lc centers of (X, B) . We also have $\mu^*D \sim_{\mathbb{Q}} u\mu^*A$. By Step 1, the natural homomorphisms

$$H^q(X', \mathcal{O}_{X'}(\mu^*L + E)) \rightarrow H^q(X', \mathcal{O}_{X'}(\mu^*L + E + \mu^*D))$$

are injective. Consider now the commutative diagram

$$\begin{array}{ccc} H^q(X', \mathcal{O}_{X'}(\mu^*L + E)) & \xrightarrow{\alpha'} & H^q(X', \mathcal{O}_{X'}(\mu^*L + E + \mu^*D)) \\ \beta \uparrow & & \uparrow \\ H^q(X, \mathcal{O}_X(L)) & \xrightarrow{\alpha} & H^q(X, \mathcal{O}_X(L + D)) \end{array}$$

From above, α' is injective. If β is injective, it follows that α is injective. To show that β is injective, suffices to show that $\mathcal{O}_X \rightarrow R\mu_*\mathcal{O}_{X'}(E)$ has a left inverse. The Cartier divisor $E' = K_{X'} - \mu^*K_X$ is effective, and $-B_{X'} \leq E'$. Therefore $E \leq E'$. We obtain homomorphisms

$$\mathcal{O}_X \rightarrow R\mu_*\mathcal{O}_{X'}(E) \rightarrow R\mu_*\mathcal{O}_{X'}(E').$$

Suffices to show that the composition has a left inverse. Tensoring with ω_X , this is just the homomorphism $\omega_X \rightarrow R\mu_*\omega_{X'}$, which admits a left inverse defined by trace (see the proof of [6, Proposition 4.3]). \square

THEOREM 14 (Tankeev-Kollár injectivity). *Let (X, B) be a GNC log variety, let $f: X \rightarrow Z$ be a proper morphism. Let \mathcal{L} be an invertible \mathcal{O}_X -module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X,B)}^{[r]} \otimes \mathcal{H}$, where $r \geq 1$ and rB has integer coefficients, and \mathcal{H} is an invertible \mathcal{O}_X -module such that $f^*f_*\mathcal{H} \rightarrow \mathcal{H}$ is surjective. Let $s \in \Gamma(X, \mathcal{H})$ be a global section which is invertible at the generic point of each lc center of (X, B) , let D be the effective Cartier divisor defined by s . In particular, D contains no lc center of (X, B) . Then the natural homomorphism $R^q f_*\mathcal{O}_X(L) \rightarrow R^q f_*\mathcal{O}_X(L + D)$ is injective, for all q .*

Proof. We may suppose Z is affine. In particular, \mathcal{H} is generated by global sections. Let $U = X \setminus \text{Supp } D$. The claim for D and all its multiples is equivalent to the injectivity of the restriction homomorphisms $H^i(X, \mathcal{L}) \rightarrow H^i(U, \mathcal{L}|_U)$.

The proof is the same as that of Theorem 12, except that in Step 1 we use Lemma 13 instead of Lemma 11. Indeed, $\mathcal{L}_p^{\otimes 2r} \xrightarrow{\sim} \omega_{(X_p, B_p)}^{[2r]} \otimes \mathcal{H}_p^{\otimes 2}$, (X_p, B_p) is log smooth, \mathcal{H}_p is generated by global sections, and $\epsilon_p^* D \in |\mathcal{H}_p|$ contains no lc center of (X_p, B_p) . Therefore $H^q(X_p, \mathcal{L}_p) \rightarrow H^q(U_p, \mathcal{L}_p|_{U_p})$ is injective, where $U_p = \epsilon_p^{-1}(U) = X_p \setminus \text{Supp } \epsilon_p^* D$. \square

THEOREM 15 (Kollár's torsion freeness). *Let (X, B) be a GNC log variety. Let \mathcal{L} be an invertible \mathcal{O}_X -module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X, B)}^{[r]}$ for some $r \geq 1$ such that rB has integer coefficients. Let $f: X \rightarrow Z$ be a proper morphism. Let s be a local section of $R^q f_* \mathcal{L}$ whose support does not contain $f(C)$, for every lc center C of (X, B) . Then $s = 0$.*

Proof. Suppose by contradiction that $s \neq 0$. Choose a closed point $P \in \text{Supp}(s)$. We shrink Z to an affine neighborhood of P . There exists a non-zero divisor $h \in \mathcal{O}_{Z, P}$ which vanishes on $\text{Supp}(s)$, but does not vanish identically on $f(C)$, for every lc center C of (X, B) . There exists $n \geq 1$ such that $h^n s = 0$ in $(R^q f_* \mathcal{L})_P$.

After shrinking Z near P , we may suppose that $0 \neq s \in \Gamma(Z, R^q f_* \mathcal{L})$, $h \in \Gamma(Z, \mathcal{O}_Z)$ is a non-zero divisor, $h^n s = 0$, and h is invertible at the generic point of $f(C)$, for every lc center C of (X, B) . Since Z is affine, we have an isomorphism $\Gamma(Z, R^q f_* \mathcal{L}) \simeq H^q(X, \mathcal{L})$. Therefore the multiplication $\otimes f^* h^n: H^q(X, \mathcal{L}) \rightarrow H^q(X, \mathcal{L})$ is not injective. But $f^* h \in \Gamma(X, \mathcal{O}_X)$ is invertible at the generic point of each lc center of (X, B) . By Theorem 14 with $\mathcal{H} = \mathcal{O}_X$, the multiplication $\otimes f^* h: H^q(X, \mathcal{L}) \rightarrow H^q(X, \mathcal{L})$ is injective. Contradiction! \square

THEOREM 16 (Ohsawa-Kollár vanishing). *Let (X, B) be a GNC log variety, let $f: X \rightarrow Y$ be a proper morphism and $g: Y \rightarrow Z$ a projective morphism. Let \mathcal{L} be an invertible \mathcal{O}_X -module such that $\mathcal{L}^{\otimes r} \simeq \omega_{(X, B)}^{[r]} \otimes f^* \mathcal{A}$, where $r \geq 1$ and rB has integer coefficients, and \mathcal{A} is a g -ample invertible \mathcal{O}_Y -module. Then $R^p g_* R^q f_* \mathcal{L} = 0$ for $p \neq 0$.*

Proof. We use induction on the dimension of X . We may suppose Z is affine. Replacing r by a multiple, we may suppose \mathcal{A} is g -generated. Let m be a sufficiently large integer, to be chosen later. Let S be the zero locus of a general global section of $\mathcal{A}^{\otimes m}$. Denote $T = f^* S$.

Consider the short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{L}(T) \rightarrow \mathcal{L}(T)|_T \rightarrow 0.$$

The connecting homomorphism $\partial: R^q f_* \mathcal{L}(T)|_T \rightarrow R^{q+1} f_* \mathcal{L}$ is zero by Theorem 15, since the image is supported by T , which contains no lc center of

(X, B) , and $\mathcal{L}^{\otimes r} \simeq \omega_{(X,B)}^{[r]}$ locally over Y . Therefore the long exact sequence in cohomology breaks up into short exact sequences

$$0 \rightarrow R^q f_* \mathcal{L} \rightarrow R^q f_* \mathcal{L}(T) \rightarrow R^q f_* \mathcal{L}(T)|_T \rightarrow 0.$$

We have $R^p g_* R^q f_* \mathcal{L}(T) \simeq R^p g_* (R^q f_* \mathcal{L}(S)) \simeq R^p g_* (R^q f_* \mathcal{L} \otimes \mathcal{A}^m)$. If m is sufficiently large, Serre vanishing gives $R^p g_* R^q f_* \mathcal{L}(T) = 0$ for $p \neq 0$. By [4], $(X, B + T)$ is a GNC log variety, T is S_2 and there exists a natural boundary $B_T = B|_T$ such that (T, B_T) is a GNC log variety, and codimension one residues glue to residue isomorphisms

$$\text{Res}_{X \rightarrow T}^{[2r]}: \omega_{(X, B+T)}^{[2r]}|_T \xrightarrow{\sim} \omega_{(T, B_T)}^{[2r]}.$$

From $\mathcal{L}(T)^{\otimes r} \simeq \omega_{(X, B+T)}^{[r]} \otimes f^* \mathcal{A}$ we obtain $\mathcal{L}(T)|_T^{\otimes 2r} \simeq \omega_{(T, B_T)}^{[2r]} \otimes (f|_T)^*(\mathcal{A}|_T^{\otimes 2})$. Since $\dim T < \dim X$, we obtain by induction $R^p g_* R^q f_* \mathcal{L}(T)|_T = 0$ for $p \neq 0$.

From the short exact sequence above, we deduce $R^p g_* R^q f_* \mathcal{L} = 0$ for $p \geq 2$. For $p = 1$, consider the commutative diagram

$$\begin{array}{ccc} R^{1+q}(g \circ f)_* \mathcal{L} & \xrightarrow{\beta} & R^{1+q}(g \circ f)_* \mathcal{L}(T) \\ \uparrow & & \uparrow \\ R^1 g_* R^q f_* \mathcal{L} & \longrightarrow & R^1 g_* R^q f_* \mathcal{L}(T) = 0 \end{array}$$

The vertical arrows are injective, from the Leray spectral sequence. The homomorphism β is injective by Theorem 14, since $T \in |f^* \mathcal{A}^{\otimes m}|$ contains no lc centers of (X, B) . A diagram chase gives $R^1 g_* R^q f_* \mathcal{L} = 0$. \square

5. INDUCTIVE PROPERTIES OF GNC LOG VARIETIES

PROPOSITION 17. *Let (X, B) be a GNC log variety. Let $Y = \text{LCS}(X, B)$ and (Y, B_Y) the n -wlc structure induced by glueing of codimension one residues. Then $B_Y = (B - B^{-1})|_Y$ and (Y, B_Y) is a GNC log variety. If $2 \mid r$ and rB has integer coefficients, then $\text{Res}^{[r]}: \omega_{(X,B)}^{[r]}|_Y \xrightarrow{\sim} \omega_{(Y,B_Y)}^{[r]}$ is an isomorphism. Moreover,*

- 1) *Let $\pi: (\bar{X}, B_{\bar{X}}) \rightarrow (X, B)$ be the normalization of X , with induced log variety structure (with log smooth support). Let $\bar{Y} = \text{LCS}(\bar{X}, B_{\bar{X}})$. Let $n: Y^n \rightarrow Y$ and $\bar{n}: \bar{Y}^n \rightarrow \bar{Y}$ be the normalizations. In the commutative diagram*

$$\begin{array}{ccccc} \bar{X} & \longleftarrow & \bar{Y} & \longleftarrow & \bar{n} & \longrightarrow & \bar{Y}^n \\ \pi \downarrow & & \pi \downarrow & & & & \downarrow g \\ X & \longleftarrow & Y & \longleftarrow & n & \longrightarrow & Y^n \end{array}$$

each square is both cartesian and a push-out, and g is an étale covering. With the log structures induced by glueing of codimension one residues, we obtain a commutative diagram of GNC log varieties and log crepant morphisms

$$\begin{array}{ccccc}
 (\bar{X}, B_{\bar{X}}) & \longleftarrow & (\bar{Y}, B_{\bar{Y}}) & \xleftarrow{\bar{n}} & (\bar{Y}^n, B_{\bar{Y}^n}) \\
 \pi \downarrow & & \pi \downarrow & & \downarrow g \\
 (X, B) & \longleftarrow & (Y, B_Y) & \xleftarrow{n} & (Y^n, B_{Y^n})
 \end{array}$$

2) The lc centers of (X, B) are the irreducible components of X and the lc centers of (Y, B_Y) .

Proof. 1) We may suppose (X, B) is a GNC local model. Let $X = \cup_F \mathbb{A}_F \hookrightarrow \mathbb{A}^N$ and $B = \sum_{i \in \sigma} b_i H_i|_X$, with core $\sigma = \cap_F F$. Denote $\sigma' = \{i \in \sigma; b_i < 1\}$. Then $\psi = \sum_{i \in \sigma'} (1 - b_i) m_i = \sum_{i \in \sigma'} (1 - b_i) m_i$, which belongs to the relative interior of σ' . We deduce that X_γ is an lc center of (X, B) if and only if $\sigma' \prec \gamma \in \Delta$. Therefore $Y = \cup_\tau \mathbb{A}_\tau \hookrightarrow \mathbb{A}^N$ is an irreducible decomposition, where the union is taken after all codimension one faces $\tau \in \Delta$ which contain σ' . In particular, the core of Y is σ' . One checks that $(Y, 0)$ satisfies properties a) and b) of the GNC local model. The boundary induced by codimension one residues is $B_Y = \sum_{i \in \sigma'} b_i H_i|_Y = (B - B^{\neq 1})|_Y$, which satisfies c). The commutative diagram becomes

$$\begin{array}{ccccc}
 \sqcup_F \mathbb{A}_F & \longleftarrow & \sqcup_F \cup_{\tau \prec F} \mathbb{A}_\tau & \xleftarrow{\bar{n}} & \sqcup_F \sqcup_{\tau \prec F} \mathbb{A}_\tau \\
 \pi \downarrow & & \pi \downarrow & & \downarrow g \\
 \cup_F \mathbb{A}_F & \longleftarrow & \cup_\tau \mathbb{A}_\tau & \xleftarrow{n} & \sqcup_\tau \mathbb{A}_\tau
 \end{array}$$

and one checks that both squares are push-outs and cartesian, using axioms a) and b) of the GNC local models. Over \mathbb{A}_τ , g consists of several identical copies of \mathbb{A}_τ , one for each facet F which contains τ . Therefore g is an étale covering. All log structures have the same log discrepancy function ψ , hence the morphisms of the diagram are log crepant.

2) *Step 1.* The claim holds if (X, B) is a GNC local model. Indeed, the lc centers of (X, B) are the invariant cycles X_γ such that $\psi \in \gamma$ and $\gamma \in \Delta$, and the lc centers of (Y, B_Y) are the invariant cycles X_γ such that $\psi \in \gamma$ and $\gamma \in \Delta$ is a face of positive codimension.

Step 2. We reduce the claim to the case when (X, B) has log smooth support. Indeed, consider the commutative diagram of log structures in 1). The log structure on the normalization $(\bar{X}, B_{\bar{X}})$ has log smooth support. By Lemma 2 for g and a diagram chase, the claim for (X, B) and its LCS-locus is equivalent to the claim for $(\bar{X}, B_{\bar{X}})$ and its LCS-locus.

Step 3. Let (X, B) have log smooth support. Then $Y = B^{-1}$ and the induced boundary is $B_Y = (B - Y)|_Y$. We have to show that for a closed subset $Z \subseteq Y$, Z is an lc center of (X, B) if and only if Z is an lc center of (Y, B_Y) , i.e. the image of an lc center of the normalization (Y^n, B_{Y^n}) . We may cut with general hyperplane sections, and suppose Z is a closed point P . Note that if $f: (X', B_{X'}) \rightarrow (X, B)$ is étale log crepant, then $Y' = f^*Y$, and since normalization commutes with étale base change, we obtain a cartesian diagram

$$\begin{CD} (Y'^n, B_{Y'^n}) @>g>> (Y^n, B_{Y^n}) \\ @Vn'VV @VVnV \\ (X', B_{X'}) @>f>> (X, B) \end{CD}$$

with f, g étale log crepant. By Lemma 2 for f and g , the claim holds for n if and only if it holds for n' . By the existence of a common étale neighborhood [5] and Step 1, we are done. \square

COROLLARY 18. *Let X be a GNC log variety. Then $S = \text{Sing } X$ coincides with the non-normal locus of X , and with $\text{LCS}(X, 0)$. The n -wlc structure induced by glueing of codimension one residues is $(S, 0)$, a GNC log variety, and $\text{Res}^{[2]}: \omega_X^{[2]}|_S \xrightarrow{\sim} \omega_S^{[2]}$ is an isomorphism. Moreover,*

- 1) *Let $\pi: (\bar{X}, \bar{C}) \rightarrow (X, 0)$ be the normalization of X , with induced log variety structure (with log smooth support). Note that $\bar{C} = \text{LCS}(\bar{X}, \bar{C})$. Let $n: S^n \rightarrow S$ and $\bar{n}: \bar{C}^n \rightarrow \bar{C}$ be the normalizations. In the commutative diagram*

$$\begin{CD} \bar{X} @<< \bar{C} @<< \bar{n} @<< \bar{C}^n \\ @V \pi VV @V \pi VV @VV g V \\ X @<< S @<< n @<< S^n \end{CD}$$

each square is both cartesian and a push-out, and g is an étale covering. With the log structures induced by glueing of codimension one residues, we obtain a commutative diagram of GNC log varieties and log crepant morphisms

$$\begin{CD} (\bar{X}, \bar{C}) @<< (\bar{C}, 0) @<< \bar{n} @<< (\bar{C}^n, \text{Cond } \bar{n}) \\ @V \pi VV @V \pi VV @VV g V \\ (X, 0) @<< (S, 0) @<< n @<< (S^n, \text{Cond } n) \end{CD}$$

- 2) *The lc centers of X are the irreducible components of X and the lc centers of S .*

Proof. It remains to check that $S = C = \text{LCS}(X, 0)$. First of all, we claim that $S = C$. Indeed, let $x \in X$. We show that $\mathcal{O}_{X,x}$ is normal if and only if $\mathcal{O}_{X,x}$ is nonsingular. We may suppose $x \in X$ is a local model $X = \cup_F X_F$ and x belongs to the closed orbit of X . Then $\mathcal{O}_{X,x}$ is normal if and only if there is only one facet F . As X_F is smooth, the latter is equivalent to $\mathcal{O}_{X,x}$ being smooth.

Since $X \setminus S$ is smooth, $\text{LCS}(X, 0) \subseteq S$. On the other hand, each irreducible component Q of S is an irreducible component of C . Therefore Q is an lc center. We conclude that $\text{LCS}(X, 0) = S$. \square

Remark 19. Let (X, B) be a GNC log variety. Let $S = \text{Sing } X$ and $B_S = B|_S$. One can also show that (S, B_S) is a GNC log variety, induced by codimension one residues. If $2 \mid r$ and rB has integer coefficients, the glueing of codimension one residues induces an isomorphism $\text{Res}^{[r]}: \omega_{(X,B)}^{[r]}|_S \xrightarrow{\sim} \omega_{(S,B_S)}^{[r]}$.

LEMMA 20. *Let (X, B) be a GNC log variety. Let $\pi: (\bar{X}, B_{\bar{X}}) \rightarrow (X, B)$ be the normalization of X , with the induced log variety structure. Let $Y = \text{LCS}(X, B)$. Let Z be a union of lc centers of (X, B) .*

- 1) $Z \cap Y$ is a union of lc centers of (Y, B_Y) .
- 2) $\pi^{-1}(Z)$ is a union of lc centers of $(\bar{X}, B_{\bar{X}})$.
- 3) We have a short exact sequence $0 \rightarrow \mathcal{I}_{Z \cup Y \subset X} \rightarrow \mathcal{I}_{Z \subset X} \xrightarrow{|_Y} \mathcal{I}_{Z \cap Y \subset Y} \rightarrow 0$.

Proof. 1) We may suppose Z is an lc center. If $Z \subseteq Y$, the claim is clear. Therefore we may suppose Z is an irreducible component of X . Then the normalization \bar{Z} of Z is an irreducible component of the normalization \bar{X} of X . We have $\pi^{-1}(Y) = \bar{Y} = \text{LCS}(\bar{X}, B_{\bar{X}})$. Therefore $Z \cap Y = \pi(\bar{Z} \cap \bar{Y})$. We have $\bar{Z} \cap \bar{Y} = \text{LCS}(\bar{X}, B_{\bar{X}})|_{\bar{Z}}$, we deduce that $\bar{Z} \cap \bar{Y}$ is a union of lc centers of $(\bar{X}, B_{\bar{X}})$ contained in \bar{Y} . Therefore $Z \cap Y$ is a union of lc centers of (X, B) contained in Y , hence lc centers of (Y, B_Y) , by Proposition 17.

2) We use induction on $\dim X$. We may suppose Z is an lc center. If Z is an irreducible component of X , then its normalization \bar{Z} is an irreducible component of \bar{X} , and $\pi^{-1}(Z) = \bar{Z} \cup \pi^{-1}(Z \cap Y)$, since Y contains the non-normal locus of X . By induction, the claim holds.

Suppose Z is not an irreducible component of X . Then $Z \subseteq Y$, by Proposition 17. By induction, $n^{-1}(Z)$ is a union of lc centers of (Y^n, B_{Y^n}) . Let W be such an lc center. Since g is finite flat, each irreducible component of $g^{-1}(W)$ dominates W . Therefore $g^{-1}n^{-1}(Z)$ is a union of lc centers of $(\bar{Y}^n, B_{\bar{Y}^n})$, by Lemma 2. Equivalently, $\bar{n}^{-1}\pi^{-1}(Z)$ is a union of lc centers of

$(\bar{Y}^n, B_{\bar{Y}^n})$. Therefore $\pi^{-1}(Z)$ is a union of lc centers of $(\bar{Y}, B_{\bar{Y}})$. The latter lc centers are also lc centers of $(\bar{X}, B_{\bar{X}})$.

3) The sequence is exact if and only if $|_Y: \mathcal{I}_{Z \subset X} \rightarrow \mathcal{I}_{Z \cap Y \subset Y}$ is surjective, if and only if $|_Y: \mathcal{I}_{Z \subset Z \cup Y} \rightarrow \mathcal{I}_{Z \cap Y \subset Y}$ is surjective, if and only if the diagram

$$\begin{array}{ccc} Y & \longleftarrow & Y \cap Z \\ \downarrow & & \downarrow \\ Y \cup Z & \longleftarrow & Z \end{array}$$

is a push-out. By [9], this diagram is a push-out if $Y \cup Z$ is weakly normal. To show this, consider the normalization $\pi: \bar{X} \rightarrow X$. Denote $W = \pi^{-1}(Y \cup Z)$. Since

$$\begin{array}{ccc} \bar{X} & \longleftarrow & \bar{Y} \\ \downarrow & & \downarrow \\ X & \longleftarrow & Y \end{array}$$

is a push-out and $Y \cup Z$ contains Y , the diagram

$$\begin{array}{ccc} \bar{X} & \longleftarrow & W \\ \downarrow & & \downarrow \\ X & \longleftarrow & Y \cup Z \end{array}$$

is also a push-out. But \bar{X} is smooth, and W is the union of \bar{Y} with the irreducible components of \bar{X} which are mapped into Z . Therefore the singularities of W are at most normal crossings. We conclude that X, \bar{X}, W are weakly normal. From the last push-out diagram, we deduce that $Y \cup Z$ is weakly normal as well. \square

The results of this section can be used to reduce Kollár’s torsion freeness theorem and Ohsawa-Kollár vanishing theorem from the GNC varieties to log smooth varieties. This is done by a using the push-out and cartesian diagram obtained from normalization and restriction to the LCS-locus. We were unable to use the same argument to reduce the injectivity theorems from GNC varieties to log smooth varieties, but we expect this is possible.

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*“Simion Stoilow” Institute of Mathematics of the
Romanian Academy
P.O. BOX 1-764, RO-014700 Bucharest, Romania
florin. ambro@imar.ro*