Dedicated to Vasile Brînzănescu on his 75^{th} birthday

(TE)-STRUCTURES OVER THE IRREDUCIBLE 2-DIMENSIONAL GLOBALLY NILPOTENT F-MANIFOLD GERM

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We find formal and holomorphic normal forms for a class of meromorphic connections (the so-called (TE)-structures) over the irreducible 2-dimensional globally nilpotent F-manifold germ $\mathcal{N}_2$. We find normal forms for Euler fields on $\mathcal{N}_2$ and we characterize the Euler fields on $\mathcal{N}_2$ which are induced by a (TE)-structure.

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1. INTRODUCTION

An important topic in modern mathematics is the theory of Frobenius manifolds. Originally introduced in [8] by B. Dubrovin as a geometrization of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV)-equations, they received much interest from the mathematical community owing to their relation with various research fields, like singularity theory, integrable systems, Gromov-Witten invariants and the theory of meromorphic connections. A Frobenius manifold is a complex manifold M together with a commutative, associative, with unit multiplication $\circ$ on the holomorphic tangent bundle TM, a flat holomorphic metric g and a holomorphic vector field E (the Euler field), satisfying certain compatibility conditions. In particular, in flat coordinates (t_i) for the metric the tensor field $c(X, Y, Z) = g(X \circ Y, Z)$ can be written in terms of the third derivatives of a certain function, the so-called potential:

$$c(\partial_i, \partial_j, \partial_k) = \partial_i \partial_j \partial_k F.$$

where $F = F(t_i)$ is holomorphic. The associativity equations for the multiplication $\circ$ reduce to the WDVV-equations for F. The Euler field rescales the multiplication and metric by constants and imposes a quasi-homogeneity condition on the potential. In dimension two, the associativiy equations are empty and in dimension three they are related to Painlevé VI equations [8]. An important class of Frobenius manifolds is represented by the orbit spaces of Coxeter groups [23].

Later on, C. Hertling and Y. Manin defined the weaker notion of an F-manifold [10], which does not involve any potential or metric, but only a multiplication with similar properties as the multiplication of a Frobenius manifold [10]. An F-manifold is a complex manifold M together with a commutative, associative, multiplication $\circ$ on TM, with unit e, which satisfies a certain integrability condition (see Definition 5). An Euler field on $(M, \circ, e)$ is a holomorphic vector field $E \in \mathcal{T}_M$ which satisfies $L_E(\circ) = \circ$. Any Frobenius manifold without metric is an F-manifold with Euler field. But there are F-manifolds which cannot be enriched to a Frobenius manifold (see e.g. [12], where the multiplication of such an F-manifold is described in terms of its spectral cover). A way to produce a new F-manifolds from older ones is Dubrovin's duality developed in [9], or its generalizations developed in [21] and [7]. Endowing an F-manifold with a real structure or Hermitian metric leads to the notions of harmonic Higgs bundles, CV-structures or Hodge structures, which are central objects in tt^*-geometry [3]. F-manifolds endowed with purely holomorphic objects (holomorphic metrics or compatible holomorphic connections) lead to notions like Frobenius manifolds, flat F-manifolds, bi-flat F-manifolds, Riemannian F-manifolds etc, which were largely considered in the literature, see e.g. [2, 16, 17, 21] (and Section 3 of [6] for a survey). F-manifolds as submanifolds of Frobenius manifolds were considered in [24].

F-manifolds endowed with various objects as above arise naturally in the theory of meromorphic connections. More precisely, the parameter space of a certain meromorphic connection, a so-called (TE)-structure, inherits, when the so-called unfolding condition is satisfied, the structure of an F-manifold with Euler field. If, moreover, the (TE)-structure comes with various additional flat objects (holomorphic metrics, Hermitian metrics, real structures) its parameter space becomes an F-manifold with various additional objects mentioned above (see [15] for the way tt^*-geometry arises in this setting or Section 4 of [6] for a survey on the holomorphic theory). The fundamental example of this construction is represented by Frobenius manifolds and their structure connections (see [8] or [14]). Conversely, if one wants to enrich an F-manifold with Euler field to a Frobenius manifold, the most important step (of several steps) is the construction of a (TE)-structure (with additional good properties) over the F-manifold with Euler field. This stepwise construction is discussed in general in [6]. Examples of (TE)-structures with unfolding condition are universal unfoldings of certain germs of meromorphic connections (see e.g. [11]).

In this paper we are only interested in the relation between F-manifolds (with or without Euler fields) and (TE)-structures (with no additional metrics or real structures on either side of the correspondence). A natural question which arises is to classify (formally and holomorphically) the (TE)-structures which lie over (or induce) a given germ $((M,0),\circ,e)$ of an F-manifold. As the classification of F-manifolds in dimension bigger than two is still unknown, it is natural to address this question in two dimensions. Recall that the irreducible germs of 2-dimensional F-manifold are classified: any such germ is either isomorphic to a generically semisimple germ $I_2(m)$ (for $m \in \mathbb{Z}_{\geq 3}$) or to the globally nilpotent germ $\mathcal{N}_2$ (see Theorem 4.7 of [14]). As germs of manifolds, both $I_2(m)$ and $\mathcal{N}_2$ are $(\mathbb{C}^2,0)$ with standard coordinates (t_1,t_2). The multiplication of $I_2(m)$ is given by

$$\partial_1 \circ \partial_1 = \partial_1,\ \partial_1 \circ \partial_2 = \partial_2,\ \partial_2 \circ \partial_2 = t_2^{m-2}\partial_1$$

while the multiplication of $\mathcal{N}_2$ is given by

$$\partial_1 \circ \partial_1 = \partial_1,\ \partial_1 \circ \partial_2 = \partial_2,\ \partial_2 \circ \partial_2 = 0,$$

where $\{\partial_1,\partial_2\}$ are the vector fields associated to the standard coordinates (t_1,t_2). Theorem 8.5 of [6] answers the above question for the generically semisimple germ $I_2(m)$. It states that any (TE)-structure over $I_2(m)$ is formally isomorphic to a unique (TE)-structure which belongs to a short list of (TE)-structures (called the normal forms) and that the formal isomorphism between a (TE)-structure and its normal form is holomorphic. Therefore, the formal and holomorphic classifications for (TE)-structures over $I_2(m)$ coincide. As a consequence, any Euler field on $I_2(m)$ is induced by a (TE)-structure (from Theorem 8.5 of [6] combined with Theorem 4.7 of [14]).

Our aim in this paper is to develop similar results for the globally nilpotent germ $\mathcal{N}_2$. Namely, we classify (formally and holomorphically) the (TE)-structures over $\mathcal{N}_2$, we classify the Euler fields on $\mathcal{N}_2$ and we characterize the Euler fields on $\mathcal{N}_2$ which are induced by a (TE)-structure. Like for $I_2(m)$, the classifications for (TE)-structures are done by determining formal and holomorphic normal forms. The formal normal forms are obtained by computations similar to those from the $I_2(m)$ case. In order to obtain the holomorphic normal forms we prove that the restriction of any (TE)-structure ∇ over $\mathcal{N}_2$ at the origin $0 \in \mathcal{N}_2$ is either regular singular (in which case ∇ is holomorphically isomorphic to its formal normal forms) or is holomorphically isomorphic to a Malgrange universal connection (in rank two, with pole of Poincaré rank one, with residue a regular endomorphism with one eigenvalue). By developing a careful treatment for such Malgrange universal connections we obtain the complete list of holomorphic normal forms for (TE)-structures over $\mathcal{N}_2$.

As opposed to (TE)-structures over $I_2(m)$, the (TE)-structures over $\mathcal{N}_2$ have the following features: the (formal or holomorphic) normal form for a given (TE)-structure is not always unique; there are (TE)-structures over $\mathcal{N}_2$ which are not holomorphically isomorphic to their formal normal form(s); there are Euler fields on $\mathcal{N}_2$ which are not induced by a (TE)-structure.

Formal and holomorphic classifications are important topics of research in the theory of meromorphic connections. The results of this paper add to the existing knowledge in this field, using down-to-earth arguments rather than the abstract, more commonly used theory of Stokes structures. This paper is a natural continuation of [5], where a formal classification of (T)-structures (rather than (TE)-structures) over $\mathcal{N}_2$ was developed.

Structure of the paper. In Section 2 we recall basic definitions on (formal and holomorphic) (TE)-structures and F-manifolds. In Section 3 we find the formal normal forms for (TE)-structures over $\mathcal{N}_2$ and in Section 4 we find the holomorphic normal forms. In Section 5 we find normal forms for Euler fields on $\mathcal{N}_2$ and we characterize the Euler fields on $\mathcal{N}_2$ which are induced by a (TE)-structure.

In the appendix we study some classes of differential equations which are useful in our treatment. To keep our paper self-contained, we recall well-known general results on the theory of meromorphic connections, which we use along the paper (Fuchs criterion, irreducible bundles, Birkhoff normal forms and Malgrange universal connections).

2. PRELIMINARY MATERIAL

We preserve the notation used in [5], which we now recall.

Notation 1. For a complex manifold M, we denote by $\mathcal{O}_M$, $\mathcal{T}_M$, Ω^k_M the sheaves of holomorphic functions, holomorphic vector fields and holomorphic k-forms on M respectively. For an holomorphic vector bundle H, we denote by $\mathcal{O}(H)$ the sheaf of its holomorphic sections. We denote by $\Omega^1_{\mathbb{C}\times M}(\log\{0\}\times M)$ the sheaf of meromorphic 1-forms on $\mathbb{C}\times M$, logarithmic along $\{0\}\times M$. Locally, in a neighborhood of $(0,p)$, where $p\in M$, any $\omega\in\Omega^1_{\mathbb{C}\times M}(\log\{0\}\times M)$ is of the form

$$\omega = \frac{f(z,t)}{z}dz + \sum_i f_i(z,t)dt_i$$

where $t=(t_1,\cdots,t_m)$ is a coordinate system of M around p and f, f_i are holomorphic. The ring of holomorphic functions defined on a neighbourhood of $0\in\mathbb{C}$ will be denoted by $\mathbb{C}\{z\}$, the ring of formal power series $\sum_{n\geq 0} a_n z^n$ will be denoted by $\mathbb{C}[[z]]$, the subring of formal power series $\sum_{n\geq 0} a_n z^n$ with $a_n=0$ for any $n\leq k-1$ will be denoted by $\mathbb{C}[[z]]_{\geq k}$ and the vector space of polynomials of degree at most k in the variables $(t_1,\cdots,t_m)$ will be denoted by $\mathbb{C}[t]_{\leq k}$. Finally, we denote by $\mathbb{C}\{t,z]]$ the ring of formal power series $\sum_{n\geq 0} a_n z^n$ where all $a_n=a_n(t)$ are holomorphic on the *same* neighbourhood of $0\in\mathbb{C}$ and by $\mathbb{C}[[z]][t]_{\leq k}$ the vector space of formal power series $\sum_{n\geq 0} a_n z^n$ with a_n polynomials of degree at most k in t. For a function $f\in\mathbb{C}\{t,z]]$ and matrix

$A \in M_{k\times k}(\mathbb{C}\{t,z]])$, we often write $f = \sum_{n\geq 0} f^{(n)} z^n$ and $A = \sum_{n\geq 0} A^{(n)} z^n$ where $f^{(n)}$ and $A^{(n)}$ are independent on z. The ring of meromorphic functions defined on a neighborhood of the origin $0 \in \mathbb{C}$, with pole at the origin only, will be denoted by $\boldsymbol{k}$.

2.1. Basic facts on $(\boldsymbol{TE})$-structures

Let M be a complex manifold.

Definition 2. i) A (T)-structure over M is a pair $(H \to \mathbb{C}\times M, \nabla)$ where ∇ is a map

$$\nabla : \mathcal{O}(H) \to \frac{1}{z}\mathcal{O}_{\mathbb{C}\times M}\cdot \Omega^1_M \otimes \mathcal{O}(H) \tag{1}$$

such that, for any $z \in \mathbb{C}^*$, the restriction of ∇ to $H|_{\{z\}\times M}$ is a flat connection.

ii) A (TE)-structure over M is a pair $(H \to \mathbb{C}\times M, \nabla)$ where $H \to \mathbb{C}\times M$ is a holomorphic vector bundle and ∇ is a flat connection on $H|_{\mathbb{C}^*\times M}$ with poles of Poincaré rank 1 along $\{0\}\times M$:

$$\nabla : \mathcal{O}(H) \to \frac{1}{z}\Omega^1_{\mathbb{C}\times M}(\log(\{0\}\times M)) \otimes \mathcal{O}(H). \tag{2}$$

As we only consider (T) or (TE)-structures over germs of F-manifolds, we assume that $H = (\mathcal{O}_{(\mathbb{C}^m,0)})^r$ is the trivial rank r vector bundle and $M = (\mathbb{C}^m, 0)$ with coordinates $(t_1,\cdots,t_m)$ (in fact, in our computations $m = r = 2$, but we prefer to present the local formulae below in any rank or dimension).

With respect to the standard basis $\underline{s} = (s_1,\cdots,s_r)$ of H,

$$\nabla \underline{s} = \underline{s}\cdot \Omega,\ \Omega = \sum_{i=1}^{m} z^{-1}A_i(z,t)dt_i + z^{-2}B(z,t)dz, \tag{3}$$

where A_i, B are holomorphic,

$$A_i(z,t) = \sum_{k\geq 0} A_i^{(k)} z^k,\ B(z,t) = \sum_{k\geq 0} B^{(k)} z^k \tag{4}$$

and $A_i^{(k)}$ and $B^{(k)}$ depend only on (t_i). The flatness of ∇ gives, for any $i \neq j$,

$$z\partial_i A_j - z\partial_j A_i + [A_i, A_j] = 0, \tag{5}$$

$$z\partial_i B - z^2\partial_z A_i + zA_i + [A_i, B] = 0. \tag{6}$$

(When ∇ is a (T)-structure, the summand $z^{-2}B(t,z)dz$ in Ω and relations (6) are dropped). Relations (5), (6) split according to the powers of z as follows: for any $k \geq 0$,

$$\partial_i A_j^{(k-1)} - \partial_j A_i^{(k-1)} + \sum_{l=0}^{k} [A_i^{(l)}, A_j^{(k-l)}] = 0, \tag{7}$$

$$\partial_i B^{(k-1)} - (k-2) A_i^{(k-1)} + \sum_{l=0}^{k} [A_i^{(l)}, B^{(k-l)}] = 0, \tag{8}$$

where $A_i^{(-1)} = B^{(-1)} = 0$.

Let (H, ∇) and $(H, \tilde{\nabla})$ be two (TE)-structures over $(\mathbb{C}^m, 0)$, with underlying bundle $H = (\mathcal{O}_{(\mathbb{C}^m,0)})^r$, defined by matrices A_i, B and $\tilde{A}_i$, $\tilde{B}$ respectively. An isomorphism T between (H, ∇) and $(\tilde{H}, \tilde{\nabla})$ which covers $h : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$, $h = (h^1, \cdots, h^m)$, is given by a matrix

$$(T_{ij}) = \sum_{k \geq 0} T^{(k)} z^k \in M_{r \times r}(\mathcal{O}_{\Delta \times U})$$

(where $\Delta \subset \mathbb{C}$ is a small disc around the origin and $U := (\mathbb{C}^m, 0)$), with $T^{(k)} \in M_{r \times r}(\mathcal{O}_U)$, $T^{(0)}$ invertible, such that

$$z \partial_i \tilde{T} + \sum_{j=1}^{m} (\partial_i h^j)(A_j \circ h) \tilde{T} - \tilde{T} \tilde{A}_i = 0, \ \forall i \tag{9}$$

$$z^2 \partial_z \tilde{T} + (B \circ h) \tilde{T} - \tilde{T} \tilde{B} = 0, \tag{10}$$

where $\tilde{T} := T \circ h$ (relation (10) has to be omitted when $\tilde{\nabla}$ and ∇ are (T)-structures). Relations (9), (10) split according to the powers of z as

$$\partial_i \tilde{T}^{(r-1)} + \sum_{l=0}^{r} \Big(\sum_{j=1}^{m} (\partial_i h^j)(A_j^{(l)} \circ h) \tilde{T}^{(r-l)} - \tilde{T}^{(r-l)} \tilde{A}_i^{(l)} \Big) = 0 \tag{11}$$

$$(r-1) \tilde{T}^{(r-1)} + \sum_{l=0}^{r} \big((B^{(l)} \circ h) \tilde{T}^{(r-l)} - \tilde{T}^{(r-l)} \tilde{B}^{(l)} \big) = 0, \tag{12}$$

for any $r \geq 0$, where $\tilde{T}^{(-1)} = 0$. When $h = \mathrm{Id}_{(\mathbb{C}^m,0)}$, the isomorphism T is called a gauge isomorphism. It satisfies

$$z \partial_i T + A_i T - T \tilde{A}_i = 0 \tag{13}$$

$$z^2 \partial_z T + BT - T \tilde{B} = 0, \tag{14}$$

or

$$\partial_i T^{(r-1)} + \sum_{l=0}^{r} (A_i^{(l)} T^{(r-l)} - T^{(r-l)} \tilde{A}_i^{(l)}) = 0, \tag{15}$$

$$(r-1) T^{(r-1)} + \sum_{l=0}^{r} (B^{(l)} T^{(r-l)} - T^{(r-l)} \tilde{B}^{(l)}) = 0. \tag{16}$$

for any $r \geq 0$.

Remark 3. i) A formal (T) or (TE)-structure ∇ over $(\mathbb{C}^m, 0)$ is given by a connection form (3), where A_i and B (the latter only when ∇ is a formal (TE)-structure) are matrices with entries in $\mathbb{C}\{t, z]]$, satisfying relations (5), (6) or (7), (8) (relations (6), (8) only when ∇ is a formal (TE)-structure).

ii) A formal isomorphism between two formal (T) or (TE)-structures ∇ and $\tilde{\nabla}$ which covers a biholomorphic map $h : (\mathbb{C}^m, 0) \to (\mathbb{C}^m, 0)$, is given by a matrix $T = (T_{ij})$ with entries $T_{ij} \in \mathbb{C}\{t, z]]$, such that relations (9), (10) or (11), (12) are satisfied with $\tilde{T} = T \circ h$ (relations (10), (12) only when ∇ and $\tilde{\nabla}$ are formal (TE)-structures). Formal gauge isomorphisms between formal (T) or (TE)-structures are formal isomorphisms which cover the identity map. They are given by matrices $T = (T_{ij})$ with entries in $\mathbb{C}\{t, z]]$ such that relations (13), (14) or (15), (16) are satisfied (relations (14) and (16) only for formal (TE)-structures).

2.2. (*TE*)-structures and (*F*)-manifolds

2.2.1. General results

Let (H, ∇) be a (T)-structure over a complex manifold M. It induces a Higgs field $C \in \Omega^1(M, \mathrm{End}(K))$ on the restriction $K := H_{|\{0\}\times M}$, defined by

$$C_X[a] := [z\nabla_X a], \ \forall X \in \mathcal{T}_M, a \in \mathcal{O}(H), \tag{17}$$

where [ ] means the restriction to $\{0\} \times M$ and $X \in \mathcal{T}_M$ is lifted canonically to $\mathbb{C}\times M$. If (H, ∇) is a (TE)-structure then there is in addition an endomorphism $\mathcal{U} \in \mathrm{End}(K)$,

$$\mathcal{U} := [z\nabla_{z\partial_z}] : \mathcal{O}(K) \to \mathcal{O}(K). \tag{18}$$

Definition 4 ([15]). The (T)-structure (or (TE)-structure) (H, ∇) satisfies the *unfolding condition* if there is an open cover $\mathcal{V}$ of M and for any $U \in \mathcal{V}$ a section $\zeta_U \in \mathcal{O}(K|_U)$ (called a local primitive section) with the property that the map $TU \ni X \to C_X\zeta_U \in K$ is an isomorphism.

When $(H \to \mathbb{C} \times M, \nabla)$ satisfies the unfolding condition the rank of H coincides with the dimension of M.

Definition 5 ([10]). A complex manifold M with a (fiber-preserving) commutative, associative multiplication $\circ$ on the holomorphic tangent bundle TM and unit field $e \in \mathcal{T}_M$ is an *F-manifold* if

$$L_{X\circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ), \ \forall X, Y \in \mathcal{T}_M, \tag{19}$$

where L_X denotes the Lie derivative in the direction of $X \in \mathcal{T}_M$. A vector field $E \in \mathcal{T}_M$ is called an *Euler field* (of weight 1) if

$$L_E(\circ) = \circ. \tag{20}$$

The following theorem was proved in Theorem 3.3 of [13].

THEOREM 6. *A (T)-structure $(H \to \mathbb{C} \times M, \nabla)$ with unfolding condition induces a multiplication $\circ$ on TM which makes M an F-manifold. A (TE)-structure $(H \to \mathbb{C} \times M, \nabla)$ with unfolding condition induces in addition a vector field E on M, which, together with $\circ$, makes M an F-manifold with Euler field. The multiplication $\circ$, unit field e and Euler field E (the latter, in the case of a (TE)-structure), are defined by*

$$C_{X \circ Y} = C_X C_Y, \; C_e = \mathrm{Id}, \; C_E = -\mathcal{U} \tag{21}$$

where C and $\mathcal{U}$ are the Higgs field and endomorphism defined by ∇ as above.

A (T)- or (TE)-structure as in Theorem 6 is said to lie over the F-manifold $(M, \circ, e)$. F-manifold isomorphisms lift naturally to isomorphisms between the spaces of (T) or (TE)-structures lying over the respective F-manifolds. Theorem 6 and similar statements hold for formal (T) and (TE)-structures. In particular, the spaces of (formal or holomorphic) (T)- or (TE)-structures over isomorphic germs of F-manifolds are isomorphic.

2.2.2. $(\boldsymbol{T})$-structures over $\mathcal{N}_2$

Following [5], we recall the formal normal forms of (T)-structures over $\mathcal{N}_2$. They are the starting point in our treatment of (TE)-structures over $\mathcal{N}_2$.

Notation 7. We define matrices C_1, C_2, D and E, by

$$C_1 := \mathrm{Id}_2, \; C_2 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; D := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{22}$$

We remark that

$$(C_2)^2 = 0, \; D^2 = C_1, \; E^2 = 0, \tag{23}$$

$$C_2 D = C_2 = -DC_2, \; DE = E = -ED, \tag{24}$$

$$C_2 E = \frac{1}{2}(C_1 - D), \; EC_2 = \frac{1}{2}(C_1 + D), \tag{25}$$

$$[C_2, D] = 2C_2, \; [C_2, E] = -D, \; [D, E] = 2E. \tag{26}$$

Theorem 8 ([5]). *Any (T)-structure over $\mathcal{N}_2$ is formally isomorphic to a (T)-structure of the form*

$$A_1 = C_1,\ A_2 = C_2 + zE \tag{27}$$

$$A_1 = C_1,\ A_2 = C_2 + zt_2E \tag{28}$$

$$A_1 = C_1,\ A_2 = C_2, \tag{29}$$

or to a formal (T)-structure of the form

$$A_1 = C_1,\ A_2 = C_2 + z(t_2^r + \sum_{k\geq 1} P_k z^k)E, \tag{30}$$

where $r \in \mathbb{Z}_{\geq 2}$ and $P_k \in \mathbb{C}[t_2]_{\leq r-2}$ are polynomials of degree at most $r-2$.

The (T)-structures from Theorem 8 are called formal normal forms. They are pairwise formally gauge non-isomorphic (i.e. there is no formal gauge isomorphism between any two distinct formal normal forms). However, there exist (distinct) formal normal forms which are formally isomorphic (by a formal isomorphism which is not a formal gauge isomorphism). For a precise statement, see Theorem 21 of [5]. Recall that the automorphism group $\mathrm{Aut}(\mathcal{N}_2)$ of the F-manifold germ $\mathcal{N}_2$ is the group of all biholomorphic maps

$$(t_1, t_2) \to (t_1, \lambda(t_2)), \tag{31}$$

where $\lambda \in \mathbb{C}\{t_2\}$, with $\lambda(0) = 0$ and $\dot{\lambda}(0) \neq 0$ (i.e. $\lambda \in \mathrm{Aut}(\mathbb{C}, 0)$).

3. FORMAL CLASSIFICATION OF (TE)-STRUCTURES

Our aim in this section is to prove the following two theorems, which classify formally the (TE)-structures over $\mathcal{N}_2$.

Theorem 9. *Any formal (TE)-structure over $\mathcal{N}_2$ is formally isomorphic to a (TE)-structure of the following forms:*

i) *for $c, \alpha, c_0 \in \mathbb{C}$,*

$$A_1 = C_1,\ A_2 = C_2 + zE,$$
$$B = (-t_1 + c + \alpha z)C_1 + (-\frac{t_2}{2} + c_0)C_2 - \frac{z}{4}D + z(-\frac{t_2}{2} + c_0)E; \tag{32}$$

ii) *for $c, \alpha \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 1}$,*

$$A_1 = C_1,\ A_2 = C_2 + zt_2^rE,$$
$$B = (-t_1 + c + \alpha z)C_1 - \frac{t_2}{r+2}C_2 - \frac{z(r+1)}{2(r+2)}D - \frac{zt_2^{r+1}}{r+2}E, \tag{33}$$

iii) *a (TE)-structure with underlying (T)-structure $A_1 = C_1$, $A_2 = C_2$ and matrix B of one of the following forms:*

$$\begin{aligned}
B =&(-t_1 + c + \alpha z)C_1 - \frac{z}{2}D;\\
B =&(-t_1 + c + \alpha z)C_1 + t_2^2 C_2 - z(t_2 + \frac{1}{2})D - z^2 E;\\
B =&(-t_1 + c + \alpha z)C_1 + \lambda t_2 C_2 - \frac{z}{2}(\lambda + 1)D,\ \lambda \notin \mathbb{Z} \setminus \{0\};\\
B =&(-t_1 + c + \alpha z)C_1 + (\lambda t_2 + 1)C_2 - \frac{z}{2}(\lambda + 1)D,\ \lambda \notin \mathbb{Z} \setminus \{0\};\\
B =&(-t_1 + c + \alpha z)C_1 + (\lambda t_2 + 1 + \gamma t_2^2 z^\lambda)C_2\\
&- \frac{z}{2}(\lambda + 1 + 2\gamma t_2 z^\lambda)D - \gamma z^{\lambda+2}E,\ \lambda \in \mathbb{Z}_{\geq 1},\\
B =&(-t_1 + c + \alpha z)C_1 + t_2(\lambda + t_2 z^\lambda)C_2 - \frac{z}{2}(\lambda + 1 + 2t_2 z^\lambda)D\\
&- z^{\lambda+2}E,\ \lambda \in \mathbb{Z}_{\geq 1},\\
B =&(-t_1 + c + \alpha z)C_1 + t_2 \lambda C_2 - \frac{z}{2}(\lambda + 1)D,\ \lambda \in \mathbb{Z}_{\geq 1};\\
B =&(-t_1 + c + \alpha z)C_1 + (\lambda t_2 + z^{-\lambda})C_2 - \frac{z}{2}(\lambda + 1)D,\ \lambda \in \mathbb{Z}_{\leq -1}\\
B =&(-t_1 + c + \alpha z)C_1 + \lambda t_2 C_2 - \frac{z}{2}(\lambda + 1)D,\ \lambda \in \mathbb{Z}_{\leq -1},
\end{aligned} \tag{34}$$

where $c, \alpha, \gamma \in \mathbb{C}$.

The (TE)-structures from the above theorem are called formal normal forms. The next theorem studies when two formal normal forms are formally isomorphic.

THEOREM 10. *Any two (distinct) (TE)-structures ∇ and $\tilde{\nabla}$ in formal normal form are formally non-isomorphic, except when*

i) *both ∇ and $\tilde{\nabla}$ are of the form (32), with constants c, α, c_0, respectively $\tilde{c}$, $\tilde{\alpha}$, $\tilde{c}_0$ and $c_0\tilde{c}_0 \neq 0$. Then they are formally isomorphic if and only if $\tilde{c} = c$, $\tilde{\alpha} = \alpha$ and $\tilde{c}_0 = -c_0$ and they are formally gauge non-isomorphic;*

ii) *both ∇ and $\tilde{\nabla}$ have underlying (T)-structure $A_1 = C_1$, $A_2 = C_2$ and their matrices B and $\tilde{B}$ are of the fourth form in (34), with constants c, α, λ, respectively $\tilde{c}$, $\tilde{\alpha}$, $\tilde{\lambda}$ and $\lambda\tilde{\lambda} \neq 0$. Then they are formally isomorphic if and only if $\tilde{c} = c$, $\tilde{\alpha} = \alpha$ and $\tilde{\lambda} = -\lambda$ and they are formally gauge non-isomorphic.*

In order to prove Theorems 9 and 10, we begin by determining the formal (TE)-structures which extend the formal (T)-structures from Theorem 8. This is done in Section 3.1 below.

3.1. (*TE*)-structures with normal formal (*T*)-structures

Let ∇ be a formal (TE)-structure, with underlying formal (T)-structure $A_1 = C_1$, $A_2 = C_2 + zfE$, where $f \in \mathbb{C}\{t_2, z]]$.

LEMMA 11. *The formal (TE)-structure ∇ is formally isomorphic to a formal (TE)-structure with $A_1 = C_1$, $A_2 = C_2 + zfE$ and matrix B of the form*

$$B = (-t_1 + c + \alpha z)C_1 + b_2 C_2 + z b_3 D + z b_4 E, \tag{35}$$

where $\alpha, c \in \mathbb{C}$, $b_2, b_3, b_4 \in \mathbb{C}\{t_2, z]]$,

$$b_3 = -\frac{1}{2}(\partial_2 b_2 + 1),\ b_4 = -\frac{z}{2}\partial_2^2 b_2 + f b_2 \tag{36}$$

and

$$-\frac{z}{2}\partial_2^3 b_2 + (\partial_2 f) b_2 + 2f\partial_2 b_2 - z\partial_z f + f = 0. \tag{37}$$

Proof. Relation (6) with $i = 1$ gives $\partial_1 B = -C_1$. Relation (6) with $i = 2$ gives $[C_2, B^{(0)}] = 0$ i.e. $B^{(0)}$ is a linear combination of C_1 and C_2. We obtain

$$B = (-t_1 + b_1)C_1 + b_2 C_2 + z b_3 D + z b_4 E, \tag{38}$$

where $b_1, b_2, b_3, b_4 \in \mathbb{C}\{t_2, z]]$. With B given by (38) and $A_2 = C_2 + zfE$, relation (6) for $i = 2$ is equivalent to $\partial_2 b_1 = 0$, (36) and (37). It remains to show that b_1 can be chosen of the form $b_1(z, t_2) = c + \alpha z$, for $c \in \mathbb{C}$. Since $\partial_2 b_1 = 0$ and $b_1 \in \mathbb{C}\{t_2, z]]$, we can write $b_1 = c + \alpha z + \sum_{k\geq 2} b_1^{(k)} z^k$, for $\alpha, b_1^{(k)} \in \mathbb{C}$. Define

$$T := \exp(-\sum_{k\geq 2} \frac{b_1^{(k)}}{k-1} z^{k-1})C_1. \tag{39}$$

The isomorphism T maps ∇ to a new formal (TE)-structure which has the same underling formal (T)-structure $A_1 = C_1$, $A_2 = C_2 + zfE$ and the only change in B is that b_1 is replaced by $\tilde{b}_1(z, t_2) = c + \alpha z$. □

To simplify terminology we introduce the next definition.

Definition 12. A formal (TE)-structure ∇ as in Lemma 11 is said to be in pre-normal form, determined by (f, b_2, c, α). The functions (f, b_2) are called associated to ∇.

In order to find all formal (TE)-structures which extend the formal (T)-structures from Theorem 8, we need to determine their associated functions b_2, i.e. to solve equation (37) in the unknown function b_2, for various classes of functions f, which correspond to the various classes of formal (T)-structures from Theorem 8. This is done in the next proposition.

Proposition 13. i) *The first* (T)*-structure from Theorem 8 extends to formal* (TE)*-structures, with matrices* B *given by*

$$B=(-t_1+c+\alpha z)C_1+(-\frac{t_2}{2}+\sum_{k\geq 0}c_k z^k)C_2-\frac{z}{4}D+z(-\frac{t_2}{2}+\sum_{k\geq 0}c_k z^k)E, \tag{40}$$

where $\alpha, c, c_k \in \mathbb{C}$.

ii) *The second* (T)*-structure from Theorem 8 extends to formal* (TE)*-structures, with matrices* B *given by*

$$B=(-t_1+c+\alpha z)C_1-\frac{t_2}{3}C_2-\frac{z}{3}D-\frac{zt_2^2}{3}E, \tag{41}$$

where $c, \alpha \in \mathbb{C}$.

iii) *The third* (T)*-structure from Theorem 8 extends to formal* (TE)*-structures with matrices* B *as in (35), functions* b_3 *and* b_4 *given by (36) with* $f=0$ *and function* $b_2=\sum_{n\geq 0}b_2^{(n)}z^n$*, such that* $b_2^{(n)}\in\mathbb{C}\{t_2\}$ *satisfies* $\partial_2^3 b_2^{(n)}=0$*, for any* $n\geq 0$.

iv) *The fourth formal* (T)*-structure from Theorem 8 extends to a formal* (TE)*-structure if and only if* $P_k=0$*, for any* $k\geq 1$*. When* $P_k=0$ *for any* $k\geq 1$*, the extended formal* (TE)*-structures have matrices* B *given by*

$$B=(-t_1+c+\alpha z)C_1-\frac{t_2}{r+2}C_2-\frac{z(r+1)}{2(r+2)}D-\frac{zt_2^{r+1}}{r+2}E, \tag{42}$$

where $\alpha, c\in\mathbb{C}$ *(and* $r\in\mathbb{Z}_{\geq 2}$*).*

Proof. We only prove claim iv) (which is more involved), since the other claims can be proved similarly. Let

$$f(z,t_2)=t_2^r+\sum_{k\geq 1}P_k(t_2)z^k, \tag{43}$$

where P_k are polynomials of degree at most $r-2$. Equation (37) with f given by (43) becomes

$$\begin{aligned}&-\frac{z}{2}\partial_2^3 b_2+(rt_2^{r-1}+\sum_{k\geq 1}\dot{P}_k(t_2)z^k)b_2\\ &+2(t_2^r+\sum_{k\geq 1}P_k(t_2)z^k)\partial_2 b_2+\sum_{k\geq 1}(1-k)P_k(t_2)z^k+t_2^r=0.\end{aligned} \tag{44}$$

We write $b_2=\sum_{k\geq 0}b_2^{(k)}z^k$ with $b_2^{(k)}$ independent on z. Identifying the coefficients of z^0 in (44) we obtain

$$rb_2^{(0)}+2t_2\partial_2 b_2^{(0)}+t_2=0,$$

which implies

$$b_2^{(0)} = -\frac{t_2}{r+2}. \tag{45}$$

Identifying the coefficients of z^1 in (44) and using (45) we obtain

$$rt_2^{r-1}b_2^{(1)} + 2t_2^r\partial_2 b_2^{(1)} - \frac{1}{r+2}\left(\dot{P}_1(t_2)t_2 + 2P_1(t_2)\right) = 0. \tag{46}$$

The first two terms in (46) have degree at least $r-1$ and the last two terms have degree at most $r-2$. We obtain that (46) is equivalent to

$$\begin{aligned} &rb_2^{(1)} + 2t_2\partial_2 b_2^{(1)} = 0\\ &\dot{P}_1(t_2)t_2 + 2P_1(t_2) = 0, \end{aligned}$$

which imply $b_2^{(1)} = 0$ and $P_1 = 0$. Identifying the coefficients of z^n for $n \geq 2$ in (44) and using an induction argument we obtain that $b_2^{(k)} = 0$ for any $k \geq 2$ and $P_k = 0$ for any $k \geq 1$. From (45) and (36) we obtain

$$b_2 = -\frac{t_2}{r+2},\ b_3 = -\frac{r+1}{2(r+2)},\ b_4 = -\frac{t_2^{r+1}}{r+2},$$

which implies claim iv). $\square$

3.2. **Proof of Theorem 9**

The existence of a formal isomorphism between an arbitrary (TE)-structure and one from Theorem 9 will be proved by applying to the formal (TE)-structures from Proposition 13 formal automorphisms of their underlying (T)-structures. We shall proceed in two steps: I) we start with the formal (TE)-structures from Proposition 13 i) and we obtain the (TE)-structures from Theorem 9 i); II) we start with the formal (TE)-structures from Proposition 13 iii) and we obtain the (TE)-structures from Theorem 9 iii). (The (TE)-structures from Proposition 13 ii) and iv) are written in Theorem 9 ii) in a unified way).

3.2.1. **The first step**

The proof of the next lemma is straightforward and will be omitted.

LEMMA 14. *A formal automorphism of the (T)-structure $A = C_1$, $A_2 = C_2 + zE$ is either a formal gauge automorphism, given by*

$$T = (\sum_{n\geq 0}\tau_1^{(n)}z^n)C_1 + (\sum_{n\geq 0}\tau_2^{(n)}z^n)C_2 + (\sum_{n\geq 1}\tau_2^{(n-1)}z^n)E \tag{47}$$

where $\tau_1^{(n)}, \tau_2^{(n)} \in \mathbb{C}$ *and* $\tau_1^{(0)} \neq 0$, *or covers the map* $h(t_1, t_2) = (t_1, -t_2)$ *and is given by*

$$T = (\sum_{n\geq 0} \tau_2^{(n)} z^n) C_2 + (\sum_{n\geq 0} \tau_3^{(n)} z^n) D - (\sum_{n\geq 1} \tau_2^{(n-1)} z^n) E, \tag{48}$$

where $\tau_2^{(n)}, \tau_3^{(n)} \in \mathbb{C}$ *and* $\tau_3^{(0)} \neq 0$.

Lemma 15. *Let* ∇ *and* $\tilde{\nabla}$ *be two formal* (TE)*-structures as in* Proposition 13 i)*, with constants* c, α, c_k $(k \geq 0)$ *and, respectively,* $\tilde{c}, \tilde{\alpha}, \tilde{c}_k$ $(k \geq 0)$. *Then* ∇ *and* $\tilde{\nabla}$ *are formally gauge isomorphic if and only if*

$$c = \tilde{c}, \ \alpha = \tilde{\alpha}, \ c_0 = \tilde{c}_0. \tag{49}$$

In particular, ∇ *is formally gauge isomorphic to the* (TE)*-structure (32).*

Proof. Let T be a formal gauge isomorphism between ∇ and $\tilde{\nabla}$. As ∇ and $\tilde{\nabla}$ have the same underlying (T)-structure $A_1 = C_1$, $A_2 = C_2 + zE$, T is a formal gauge automorphism of this (T)-structure. From Lemma 14, T is of the form (47), and must satisfy relations (16), with B and $\tilde{B}$ of the form (40), with constants c, α, c_k and $\tilde{c}, \tilde{\alpha}, \tilde{c}_k$.

Replacing $T^{(l)}$, $B^{(l)}$ and $\tilde{B}^{(l)}$ in (16) and identifying the coefficients of $\{C_1, C_2, D, E\}$ we obtain, from a straightforward computation which uses relations (23)-(26),

$$\begin{aligned}
&c = \tilde{c}, \ \alpha = \tilde{\alpha}, \ c_0 = \tilde{c}_0;\\
&\frac{\tau_2^{(0)}}{2} + \tau_1^{(0)}(c_1 - \tilde{c}_1) = 0;\\
&(n-1)\tau_1^{(n-1)} + \sum_{l=2}^{n} \tau_2^{(n-l)}(c_{l-1} - \tilde{c}_{l-1}) = 0;\\
&(n - \frac{1}{2})\tau_2^{(n-1)} + \sum_{l=1}^{n} \tau_1^{(n-l)}(c_l - \tilde{c}_l) = 0,
\end{aligned} \tag{50}$$

for any $n \geq 2$. (In all. relations (16) the coefficients of D vanish; the coefficients of E in (16), with $r = 0, 1$, vanish as well and the coefficient of E in (16), with $r \geq 2$, coincides with the coefficient of C_2 in (16), with r replaced by $r - 1$. Thus, relations (16) are equivalent to the vanishing of their coefficients of C_1 and C_2, which leads to relations (50)). In particular, if ∇ and $\tilde{\nabla}$ are formally gauge isomorphic, then (49) is satisfied. Conversely, assume that (49) is satisfied. We aim to construct a formal gauge isomorphism between ∇ and $\tilde{\nabla}$, i.e. to find $\tau_1^{(n)}, \tau_2^{(n)} \in \mathbb{C}$, with $\tau_1^{(0)} \neq 0$, such that relations (50) hold. Let $\tau_1^{(0)} \in \mathbb{C}^*$ be arbitrary. The second relation (50) determines $\tau_2^{(0)}$ and then, $\tau_1^{(1)}$

is determined by the third relation (50) with $n = 2$:

$$\tau_1^{(1)} = \tau_2^{(0)}(\tilde{c}_1 - c_1).$$

Knowing $\tau_1^{(0)}$ and $\tau_1^{(1)}$, the fourth relation (50) with $n = 2$ determines $\tau_2^{(1)}$:

$$\tau_2^{(1)} = \frac{2}{3}\left(\tau_1^{(1)}(\tilde{c}_1 - c_1) + \tau_1^{(0)}(\tilde{c}_2 - c_2)\right).$$

Repeating the argument we obtain inductively $\tau_1^{(l)}$ and $\tau_2^{(l)}$, for all $l \geq 1$. □

3.2.2. **The second step**

We use a similar argument for the formal (TE)-structures from Proposition 13 iii). As before, we begin by finding the automorphisms of their underlying (T)-structure.

Lemma 16. i) *Any formal automorphism T of the (T)-structure $A_1 = C_1$, $A_2 = C_2$ covers an automorphism $h \in \mathrm{Aut}(\mathcal{N}_2)$ of the form*

$$h(t_1, t_2) = (t_1, \frac{kt_2}{et_2 + d}), \tag{51}$$

where $e \in \mathbb{C}$, $k, d \in \mathbb{C}^*$ and

$$\tilde{T} := T \circ h = \sum_{n\geq 0} \tilde{T}^{(n)} z^n,\ \tilde{T}^{(n)} = \tau_1^{(n)} C_1 + \tau_2^{(n)} C_2 + \tau_3^{(n)} D + \tau_4^{(n)} E \tag{52}$$

is given by: for any $n \geq 0$, $\tau_4^{(n)} \in \mathbb{C}$, with $\tau_4^{(0)} = 0$, $\tau_4^{(1)} = e$, and

$$\begin{aligned}
\tau_1^{(n)} &= \frac{t_2(et_2 + d - k)}{2(et_2 + d)}\tau_4^{(n+1)} + (\tau_1^{(n)})_0\\
\tau_2^{(n)} &= -\frac{t_2^2 k}{et_2 + d}\tau_4^{(n+2)} + \frac{t_2(et_2 + d - k)}{et_2 + d}(\tau_1^{(n+1)})_0\\
&\quad + (\tau_2^{(n)})_0 - \frac{t_2(et_2 + d + k)}{et_2 + d}(\tau_3^{(n+1)})_0,\\
\tau_3^{(n)} &= \frac{t_2(et_2 + d + k)}{2(et_2 + d)}\tau_4^{(n+1)} + (\tau_4^{(n)})_0,
\end{aligned} \tag{53}$$

where $(\tau_1^{(n)})_0, (\tau_2^{(n)})_0, (\tau_3^{(n)})_0 \in \mathbb{C}$ and $(\tau_1^{(0)})_0 = \frac{1}{2}(d + k)$, $(\tau_3^{(0)})_0 = \frac{1}{2}(d - k)$.

ii) *Any formal gauge automorphism of the (T)-structure $A_1 = C_1$, $A_2 = C_2$ is of the form*

$$T = (\sum_{k\geq 0} \tau_1^{(n)} z^n) C_1 + \sum_{n\geq 0} (\tau_2^{(n)} z^n) C_2 + (\sum_{n\geq 0} \tau_3^{(n)} z^n) D + (\sum_{n\geq 0} \tau_4^{(n)} z^n) E$$

where $\tau_1^{(n)} \in \mathbb{C}$ *with* $\tau_1^{(0)} \neq 0$, $\tau_2^{(n)}, \tau_3^{(n)}, \tau_4^{(n)} \in \mathbb{C}\{t_2\}$, $\tau_2^{(n)}$ *satisfies* $\partial_2^3 \tau_2^{(n)} = 0$ *for any* $n \geq 0$ *and*

$$\tau_3^{(n)} = -\frac{1}{2}\partial_2 \tau_2^{(n-1)}, \ \tau_4^{(n)} = -\frac{1}{2}\partial_2^2 \tau_2^{(n-2)}, \tag{54}$$

with the convention $\tau_2^{(n)} = 0$ *for* $n < 0$.

Proof. i) Let T be a formal automorphism of the (T)-structure $A_1 = C_1$, $A_2 = C_2$ and $h(t_1, t_2) = (t_1, \lambda(t_2))$ the automorphism of $\mathcal{N}_2$ covered by T. Let $\tau_1^{(n)}$, $\tau_2^{(n)}$, $\tau_3^{(n)}$, $\tau_4^{(n)}$ be the functions defined by (52). From relation (11) with $i = 1$ and $A_1 = \tilde{A}_1 = C_1$, we obtain that they are independent on t_1. Relation (11) with $i = 2$ is

$$\partial_2 \tilde{T}^{(n-1)} + \dot{\lambda} C_2 \tilde{T}^{(n)} - \tilde{T}^{(n)} C_2 = 0, \ n \geq 0. \tag{55}$$

Using relations (23)-(26) we obtain that (55) is equivalent to

$$\tau_4^{(0)} = 0, \ \dot{\lambda} = \frac{\tau_1^{(0)} - \tau_3^{(0)}}{\tau_1^{(0)} + \tau_3^{(0)}}, \tag{56}$$

$\tau_4^{(n)} \in \mathbb{C}$ (for $n \geq 1$) and, for any $n \geq 0$,

$$\begin{aligned} \partial_2 \tau_1^{(n)} &= (\frac{1-\dot{\lambda}}{2}) \tau_4^{(n+1)}, \\ \partial_2 \tau_2^{(n)} &= (1-\dot{\lambda}) \tau_1^{(n+1)} - (1+\dot{\lambda}) \tau_3^{(n+1)}, \\ \partial_2 \tau_3^{(n)} &= (\frac{1+\dot{\lambda}}{2}) \tau_4^{(n+1)}. \end{aligned} \tag{57}$$

(Since $\tilde{T}^{(0)}$ is invertible and $\tau_4^{(0)} = 0$, we obtain that $\tau_1^{(0)} - \tau_3^{(0)}$, $\tau_1^{(0)} + \tau_3^{(0)}$ are units in $\mathbb{C}\{t_2\}$). Using that $\tau_4^{(n)}$ are constant, we obtain from the first and third relation (57) that $\tau_1^{(n)}$ and $\tau_3^{(n)}$ are given by

$$\begin{aligned} \tau_1^{(n)} &= \frac{\tau_4^{(n+1)}}{2}(t_2 - \lambda) + (\tau_1^{(n)})_0 \\ \tau_3^{(n)} &= \frac{\tau_4^{(n+1)}}{2}(t_2 + \lambda) + (\tau_3^{(n)})_0, \end{aligned} \tag{58}$$

where $(\tau_1^{(n)})_0, (\tau_3^{(n)})_0 \in \mathbb{C}$. From (58) and the second relation (57), we obtain $\tau_2^{(n)} = -\tau_4^{(n+2)} t_2 \lambda - ((\tau_1^{(n+1)})_0 + (\tau_3^{(n+1)})_0)\lambda + ((\tau_1^{(n+1)})_0 - (\tau_3^{(n+1)})_0) t_2 + (\tau_2^{(n)})_0$, where $(\tau_2^{(n)})_0 \in \mathbb{C}$. Replacing the expressions of $\tau_1^{(0)}$ and $\tau_3^{(0)}$ provided by (58) in the second relation (56) we obtain that λ satisfies the differential equation

$$\dot{\lambda} = \frac{-\tau_4^{(1)}\lambda + k}{\tau_4^{(1)} t_2 + d}.$$

Let $e := \tau_4^{(1)}$. Solving this differential equation for λ we obtain (51). Finally, replacing λ in the above expressions for $\tau_1^{(n)}$, $\tau_2^{(n)}$ and $\tau_3^{(n)}$ we conclude the proof of claim i).

Claim ii) follows from relations (57) with $\dot{\lambda} = 1$. The condition $\partial_2^3 \tau_2^{(n)} = 0$ follows from $\partial_2^2 \tau_3^{(n)} = 0$ (from the third relation (57) and $\tau_4^{(n+1)} \in \mathbb{C}$). $\square$

Corollary 17. i) *Consider two formal (TE)-structures ∇ and $\tilde{\nabla}$ as in* Proposition 13 iii), *with associated functions b_2 and $\tilde{b}_2$ respectively. Assume that there is a formal isomorphism T between ∇ and $\tilde{\nabla}$ which is given by* Lemma 16 i). *In the notation of that lemma,*

$$\tilde{b}_2^{(0)}(t_2) = b_2^{(0)}\left(\frac{kt_2}{et_2 + d}\right)\frac{(et_2 + d)^2}{kd}. \tag{59}$$

In particular, $b_2^{(0)}$ is a formal gauge invariant of ∇.

ii) *Let ∇ be a formal (TE)-structure as in* Proposition 13 iii). *There is a formal isomorphism which maps ∇ to another formal (TE)-structure as in* Proposition 13 iii), *with associated function $\tilde{b}_2$, such that $\tilde{b}_2^{(0)}$ is of one of the following forms:*

$$\tilde{b}_2^{(0)} = 0,\ \tilde{b}_2^{(0)} = 1,\ \tilde{b}_2^{(0)} = \lambda t_2 + 1,\ \tilde{b}_2^{(0)} = \beta t_2,\ \tilde{b}_2^{(0)} = t_2^2, \tag{60}$$

where $\lambda \in \mathbb{C}$ and $\beta \in \mathbb{C}^*$.

Proof. i) Relation (59) follows by identifying the coefficients of z^0 in relation (10) with B, $\tilde{B}$ as in Proposition 13 iii) and $\tilde{T}$ given in Lemma 16 i).

ii) Let b_2 be the associated function of ∇. Since $\partial_2^3 b_2^{(0)} = 0$ and $b_2^{(0)}$ is independent on t_1, we can write $b_2^{(0)} = at_2^2 + bt_2 + c$ for $a, b, c \in \mathbb{C}$. Let T be any formal automorphism of the (T)-structure $A_1 = C_1$, $A_2 = C_2$, as in Lemma 16 i), and $\tilde{\nabla} := T \cdot \nabla$, with associated function $\tilde{b}_2$. From (59),

$$\tilde{b}_2^{(0)}(t_2) = \left(a\frac{k}{d} + b\frac{e}{d} + c\frac{e^2}{kd}\right)t_2^2 + \left(b + 2c\frac{e}{k}\right)t_2 + c\frac{d}{k}. \tag{61}$$

Suitable choices of $k, d \in \mathbb{C}^*$ and $e \in \mathbb{C}$ in (61) show that $\tilde{b}_2^{(0)}$ can be reduced to one of the forms (60). Any formal automorphism of the (T)-structure $A_1 = C_1$, $A_2 = C_2$, as in Lemma 16 i), with such constants k, d and e, maps ∇ to a formal (TE)-structure with the required property. $\square$

Corollary 18. i) *Two formal (TE)-structures ∇ and $\tilde{\nabla}$ as in* Proposition 13 iii), *with associated functions b_2 and $\tilde{b}_2$ respectively, such that $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are distinct, of the form (60), are formally non-isomorphic, unless*

$$b_2^{(0)} = \lambda t_2 + 1,\ \tilde{b}_2^{(0)} = -\lambda t_2 + 1,\ \lambda \in \mathbb{C}^*. \tag{62}$$

ii) *A formal* (TE)*-structure* ∇ *as in* Proposition 13 iii)*, with associated function* b_2 *such that* $b_2^{(0)} = \lambda t_2 + 1$*, where* $\lambda \in \mathbb{C}^*$*, can be mapped by a formal isomorphism to another formal* (TE)*-structure as in* Proposition 13 iii)*, with associated function* $\tilde{b}_2$*, such that* $\tilde{b}_2^{(0)} = -\lambda t_2 + 1$.

Proof. i) We consider two formal (TE)-structures ∇ and $\tilde{\nabla}$ as in Proposition 13 iii), with associated functions b_2 and $\tilde{b}_2$, but such that $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are not necessarily of the form (60). As $\partial_2^3 b_2^{(0)} = \partial_2^3 \tilde{b}_2^{(0)} = 0$,

$$b_2^{(0)} = at_2^2 + bt_2 + c, \ \tilde{b}_2^{(0)} = \tilde{a}t_2^2 + \tilde{b}t_2 + \tilde{c}, \tag{63}$$

for $a, b, c, \tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{C}$. From (61), if there is a formal isomorphism between ∇ and $\tilde{\nabla}$ then the system

$$a\frac{k}{d} + b\frac{e}{d} + c\frac{e^2}{kd} = \tilde{a}, \ b + 2c\frac{e}{k} = \tilde{b}, \ c\frac{d}{k} = \tilde{c} \tag{64}$$

in the unknown constants $k, d \in \mathbb{C}^*$ and $e \in \mathbb{C}$, has a solution. When $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are distinct, of the form (60), a solution of (64) exists only when $b_2^{(0)}$ and $\tilde{b}_2^{(0)}$ are of the form (62).

ii) Consider the automorphism given by Lemma 16 i), with $k = d = 1$, $e = \lambda$, $\tau_4^{(n)} = 0$ for any $n \neq 1$, $(\tau_1^{(n)})_0 = (\tau_3^{(n)})_0 = 0$ for any $n \geq 1$ and $(\tau_2^{(n)})_0 = 0$ for any $n \geq 0$. It maps ∇ to a formal (TE)-structure $\tilde{\nabla}$ with associated function $\tilde{b}_2$ and $\tilde{b}_2^{(0)} = -\lambda t_2 + 1$. □

Lemma 19. *Let* ∇ *be a formal* (TE)*-structure as in Proposition 13 iii). Then* ∇ *is formally isomorphic to a* (TE)*-structure with underlying* (T)*-structure* $A_1 = C_1$*,* $A_2 = C_2$ *and whose matrix* B *either belongs to the first five lines in (34) or is of one of the forms*

$$\begin{aligned} B =& (-t_1 + c + \alpha z)C_1 + t_2(\lambda + \gamma t_2 z^{\lambda})C_2 - \frac{z}{2}(\lambda + 1 + 2\gamma t_2 z^{\lambda})D \\ &- \gamma z^{\lambda+2}E, \ \lambda \in \mathbb{Z}_{\geq 1} \\ B =& (-t_1 + c + \alpha z)C_1 + (\lambda t_2 + \gamma z^{-\lambda})C_2 - \frac{z}{2}(\lambda + 1)D, \ \lambda \in \mathbb{Z}_{\leq -1}, \end{aligned} \tag{65}$$

where $c, \gamma \in \mathbb{C}$.

Proof. Let ∇, $\tilde{\nabla}$ be two formal (TE)-structures as in Proposition 13 iii), with constants c, α and associated function $b_2 = \sum_{n\geq 0} b_2^{(n)} z^n$, respectively constants $\tilde{c}$, $\tilde{\alpha}$ and associated function $\tilde{b}_2 = \sum_{n\geq 0} \tilde{b}_2^{(n)} z^n$. Recall that $b_2, \tilde{b}_2 \in \mathbb{C}\{t_2, z]]$ satisfy $\partial_2^3 b_2 = \partial_2^3 \tilde{b}_2 = 0$. We determine conditions on $b_2^{(n)}$ and $\tilde{b}_2^{(n)}$ such that ∇ and $\tilde{\nabla}$ are formally gauge isomorphic. This happens if and only if there is a formal gauge automorphism T of their underling (T)-structure $A_1 = C_1$, $A_2 = C_2$, such that (16), with matrices B and $\tilde{B}$ of ∇ and $\tilde{\nabla}$, is satisfied. The automorphism T is given by Lemma 16 ii). Relation (16) for $r = 0$ is equivalent to $b_2^{(0)} = \tilde{b}_2^{(0)}$ (which we already know, from Corollary 17) and $c = \tilde{c}$. For $r = 1$ it is equivalent to $\alpha = \tilde{\alpha}$ (by identifying the coefficients of C_1) together with

$$b_2^{(0)}\partial_2\tau_2^{(0)} - (\partial_2 b_2^{(0)} + 1)\tau_2^{(0)} + \tau_1^{(0)}(\tilde{b}_2^{(1)} - b_2^{(1)}) = 0 \tag{66}$$

(by identifying the coefficients of C_2). The coefficients of D and E give no relations in (16) with $r = 1$.

We now consider relation (16) with $r = n \geq 2$. Identifying the coefficients of C_1 in this relation we obtain

$$\begin{aligned} &(n-1)\tau_1^{(n-1)} - \frac{1}{4}\sum_{l=1}^{n}\partial_2^2\tau_2^{(n-l-2)}(b_2^{(l)} - \tilde{b}_2^{(l)}) \\ &+ \frac{1}{4}\sum_{l=2}^{n-1}\partial_2\tau_2^{(n-l-1)}\partial_2(b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) \\ &- \frac{1}{4}\sum_{l=2}^{n-1}\tau_2^{(n-l-1)}\partial_2^2(b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) = 0 \end{aligned} \tag{67}$$

(with the convention $\tau_2^{(l)} := 0$ for $l \in \mathbb{Z}_{\leq -1}$). Identifying the coefficients of C_2 in the same relation we obtain

$$\begin{aligned} &n\tau_2^{(n-1)} - b_2^{(0)}\partial_2\tau_2^{(n-1)} + \tau_2^{(n-1)}\partial_2 b_2^{(0)} + \sum_{l=1}^{n}\tau_1^{(n-l)}(b_2^{(l)} - \tilde{b}_2^{(l)}) \\ &- \frac{1}{2}\sum_{l=1}^{n}\partial_2\tau_2^{(n-l-1)}(b_2^{(l)} + \tilde{b}_2^{(l)}) + \frac{1}{2}\sum_{l=1}^{n-1}\tau_2^{(n-l-1)}\partial_2(b_2^{(l)} + \tilde{b}_2^{(l)}) = 0. \end{aligned} \tag{68}$$

Identifying the coefficients of D in the same relation we obtain:

$$\begin{aligned} &(n-1)\partial_2\tau_2^{(n-2)} - b_2^{(0)}\partial_2^2\tau_2^{(n-2)} - \frac{1}{2}\sum_{l=1}^{n}\partial_2^2\tau_2^{(n-l-2)}(b_2^{(l)} + \tilde{b}_2^{(l)}) \\ &+ \sum_{l=2}^{n}\tau_1^{(n-l)}\partial_2(b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) + \frac{1}{2}\sum_{l=0}^{n-2}\tau_2^{(n-l-2)}\partial_2^2(b_2^{(l)} + \tilde{b}_2^{(l)}) = 0. \end{aligned} \tag{69}$$

Identifying the coefficients of E in the same relation we obtain:

$$
(70)\quad \begin{aligned} &(n-2)\partial_2^2\tau_2^{(n-3)} - \partial_2 b_2^{(0)}\partial_2^2\tau_2^{(n-3)} - \frac{1}{2}\sum_{l=1}^{n-1}\partial_2^2\tau_2^{(n-l-3)}\partial_2(b_2^{(l)} + \tilde{b}_2^{(l)}) \\ &r + \frac{1}{2}\sum_{l=0}^{n-2}\partial_2\tau_2^{(n-l-3)}\partial_2^2(b_2^{(l)} + \tilde{b}_2^{(l)}) + \sum_{l=2}^{n}\tau_1^{(n-l)}\partial_2^2(b_2^{(l-2)} - \tilde{b}_2^{(l-2)}) = 0. \end{aligned}
$$

A long but straightforward computation shows that for $n \geq 3$, relation (69) is the derivative of relation (68) with n replaced by $n-1$. Similarly, for $n \geq 4$, relation (70) is the second derivative of (68) with n replaced by $n-2$. Therefore, relations (67)-(70) are equivalent to relations (67), (68), together with relation (69) with $n = 2$ and relation (70) with $n = 2, 3$. But relation (69) with $n = 2$ is the derivative of (66), relation (70) with $n = 2$ follows from $b_2^{(0)} = \tilde{b}_2^{(0)}$ and relation (70) with $n = 3$ is the second derivative of (66).

To summarize: we proved that ∇ and $\tilde{\nabla}$ are formally gauge isomorphic if and only if $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $b_2^{(0)} = \tilde{b}_2^{(0)}$ and relations (66), (67) and (68) are satisfied (the last two for any $n \geq 2$).

Using the above considerations, we now prove our claim. Let ∇ be a formal (TE)-structure as in Proposition 13 iii). We aim to construct a (TE)-structure $\tilde{\nabla}$ formally isomorphic to ∇, as required by the lemma. From Corollary 17 ii) we can assume, without loss of generality, that $b_2^{(0)} = \tilde{b}_2^{(0)}$ is of one of the forms (60). From Corollary 18 ii)), we can further assume that the constant λ from (60) belongs to $(\mathbb{C}\setminus\mathbb{Z})\cup\mathbb{Z}_{\geq 0}$. Let $\tau_1^{(0)} \in \mathbb{C}^*$. We choose $\tilde{b}_2^{(1)}$ in a suitable way such that relation (66), considered as an equation in the unknown function $\tau_2^{(0)}$, has a solution with $\partial_2^3\tau_2^{(0)} = 0$. More precisely, when $b_2^{(0)} = 0$ we choose $\tilde{b}_2^{(1)} = 0$ and $\tau_2^{(0)} = -\tau_1^{(0)}b_2^{(1)}$. When $b_2^{(0)} \neq 0$ we use Lemma 56 and we choose $\tilde{b}_2^{(1)}$ as follows: if $b_2^{(0)} = \lambda t_2$ (with $\lambda \notin \{-1, 1\}$) or $b_2^{(0)} = \lambda t_2 + 1$ (with $\lambda \neq 1$) or $b_2^{(0)} = t_2^2$ we choose $\tilde{b}_2^{(1)} = 0$; if $b_2^{(0)} = t_2$ or $b_2^{(0)} = t_2 + 1$ we choose $\tilde{b}_2^{(1)} = (b_2^{(1)})_2 t_2^2$ where $(b_2^{(1)})_i$ denotes the coefficient of t_2^i in $b_2^{(1)}$; if $b_2^{(0)} = -t_2$ we choose $\tilde{b}_2^{(1)} = (b_2^{(1)})_0$. Suppose now that $\tau_1^{(i)}$ and $\tau_2^{(i)} \in \mathbb{C}$ (with $i \leq n-1$) and $\tilde{b}_2^{(i)}$ (with $i \leq n$) are known (and satisfy $\partial_2^3\tau_2^{(i)} = \partial_2^3\tilde{b}_2^{(i)} = 0$). Relation (67) with n replaced by $n+1$ determines $\tau_1^{(n)}$:

$$
(71)\quad \begin{aligned} n\tau_1^{(n)} = &\frac{1}{4}\sum_{l=1}^{n+1}\partial_2^2\tau_2^{(n-l-1)}(b_2^{(l)} - \tilde{b}_2^{(l)}) - \frac{1}{4}\sum_{l=2}^{n}\partial_2\tau_2^{(n-l)}\partial_2(b_2^{(l-1)} - \tilde{b}_2^{(l-1)}) \\ &+ \frac{1}{4}\sum_{l=2}^{n}\tau_2^{(n-l)}\partial_2^2(b_2^{(l-1)} - \tilde{b}_2^{(l-1)}). \end{aligned}
$$

We remark that $\tau_1^{(n)} \in \mathbb{C}$: straightforward computation, which uses $\partial_2^3 b_2^{(l)} =$

$\partial_2^3\tilde b_2^{(l)} = \partial_2^3\tau_2^{(i)} = 0$ (for $l \leq n-1$ and $i \leq n-2$) shows that the right hand side of (71) is constant. Relation (68) with n replaced by $n+1$ is

$$
\begin{aligned}
&(n+1)\tau_2^{(n)} + \partial_2 b_2^{(0)}\tau_2^{(n)} - b_2^{(0)}\partial_2\tau_2^{(n)} + \tau_1^{(0)}(b_2^{(n+1)} - \tilde b_2^{(n+1)})\\
&+ \sum_{l=1}^{n} \tau_1^{(n+1-l)}(b_2^{(l)} - \tilde b_2^{(l)}) - \frac{1}{2}\sum_{l=1}^{n}\partial_2\tau_2^{(n-l)}(b_2^{(l)} + \tilde b_2^{(l)})\\
&+ \frac{1}{2}\sum_{l=1}^{n}\tau_2^{(n-l)}\partial_2(b_2^{(l)} + \tilde b_2^{(l)}) = 0. \qquad (72)
\end{aligned}
$$

The second and third lines from the left hand side of (72) are known. When $b_2^{(0)} \neq 0$ we choose as before (using Lemma 56) $\tilde b_2^{(n+1)}$ such that (72) has a solution $\tau_2^{(n)}$ and $\partial_2^3\tilde b_2^{(n+1)} = \partial_2^3\tau_2^{(n)} = 0$. When $b_2^{(0)} = 0$ we choose $\tilde b_2^{(n+1)} = 0$ and $\tau_2^{(n)}$ to satisfy (72) (with $\tilde b_2^{(n+1)} = 0$) and $\partial_2^3\tau_2^{(n)} = 0$. Using an induction procedure we define an automorphism T of the (T)-structure $A_1 = C_1$, $A_2 = C_2$, such that the associated function $\tilde b_2$ of $\tilde\nabla := T\cdot\nabla$ is of one of the following forms:

$\tilde b_2 = 0$, $\tilde b_2 = t_2^2$, $\tilde b_2 = \lambda t_2$ ($\lambda \notin \mathbb{Z}\setminus\{0\}$), $\tilde b_2 = \lambda t_2 + 1$ ($\lambda \notin \mathbb{Z}\setminus\{0\}$), $\tilde b_2 = \lambda t_2 + 1 + \gamma t_2^2 z^{\lambda}$ ($\lambda \in \mathbb{Z}_{\geq 1}$, $\gamma \in \mathbb{C}$), $\tilde b_2 = \lambda t_2 + \gamma t_2^2 z^{\lambda}$ ($\lambda \in \mathbb{Z}_{\geq 1}$, $\gamma \in \mathbb{C}$), $\tilde b_2 = \lambda t_2 + \gamma z^{-\lambda}$ ($\lambda \in \mathbb{Z}_{\leq -1}$, $\gamma \in \mathbb{C}$).

The first five forms above of $\tilde b_2$ give the (TE)-structures with $A_1 = C_1$, $A_2 = C_2$ and matrix B as in the first five lines from (34). The last two forms of $\tilde b_2$ give the (TE)-structures with $A_1 = C_1$, $A_2 = C_2$ and matrix B of either of the two forms (65). □

The next lemma concludes the proof of Theorem 9. It shows that the (TE)-structures from Lemma 19, with matrices B given in (65), are formally isomorphic to the last four classes of (TE)-structures from Theorem 9 iii).

Lemma 20. *The (TE)-structures with $A_1 = C_1$, $A_2 = C_2$ and matrices B given by (65) are formally isomorphic to the (TE)-structures of the same form (with the same constants c, α, λ), but with $\gamma \in \{0,1\}$.*

Proof. Let $T^{[1]}$ be a formal automorphism of the (T)-structure $A_1 = C_1$, $A_2 = C_2$ given by Lemma 16 i), with $\tau_4^{(n)} = (\tau_2^{(n)})_0 = 0$ ($n \geq 0$), $(\tau_1^{(n)})_0 = (\tau_3^{(n)})_0 = 0$ ($n \geq 1$), $k, d \in \mathbb{C}^*$ and $e = 0$. It maps the (TE)-structures from Lemma 19, with matrices B given by (65), to (TE)-structures of the same form, with the same $c, \alpha, \lambda \in \mathbb{C}$ but with γ replaced by $\tilde\gamma := \frac{k}{d}\gamma$. When $\gamma \neq 0$ we can choose $k, d \in \mathbb{C}^*$ such that $\tilde\gamma = 1$. □

3.2.3. **Proof of Theorem 10**

Consider two distinct (TE)-structures ∇ and $\tilde{\nabla}$ from Theorem 9, and suppose that they are formally isomorphic. Then their underlying (T)-structures are also formally isomorphic. From Theorem 21 ii) of [5], there are two cases, namely: a) ∇ and $\tilde{\nabla}$ are as in Theorem 9 i); b) ∇ and $\tilde{\nabla}$ are as in Theorem 9 iii). The next lemma treats the first possibility.

Lemma 21. *Consider two distinct (TE)-structures as in Theorem 9 i), with constants c, α, c_0, respectively $\tilde{c}$, $\tilde{\alpha}$, $\tilde{c}_0$. Then ∇ and $\tilde{\nabla}$ are formally isomorphic if and only if $\tilde{c} = c$, $\tilde{\alpha} = \alpha$ and $\tilde{c}_0 = -c_0$.*

Proof. Let T be a formal automorphism the (T)-structure $A_1 = C_1$, $A_2 = C_2 + zE$. From Lemma 15, T is not a formal gauge automorphism. From Lemma 14, T covers the map $h(t_1, t_2) = (t_1, -t_2)$ and $\tilde{T} = T \circ h = T$ is of the form (48). As $\tilde{B} = \tilde{B} \circ h$, relation (10) becomes

$$z^2 \partial_z T + \tilde{B} T - T B = 0. \tag{73}$$

By identifying the coefficients of z^0 and z in (73) we obtain that $\tilde{c} = c$, $\tilde{\alpha} = \alpha$ and $\tilde{c}_0 = -c_0$. Moreover, if these relations are satisfied then the isomorphism $T(z, t_1, t_2) := D$ which covers h maps ∇ to $\tilde{\nabla}$. □

It remains to study the second case. We begin with the next simple lemma.

Lemma 22. *Let ∇ be a (TE)-structure as in Theorem 9 iii). Then the constants c and α are formal invariants.*

Proof. We notice that c and α remain unchanged under the automorphisms from Lemma 16 i), for which $\tau_4^{(n)} = 0$ for any $n \neq 1$, $(\tau_1^{(n)})_0 = (\tau_3^{(n)})_0 = 0$ for any $n \geq 1$, $(\tau_2^{(n)})_0 = 0$ for any $n \geq 0$ (and $k, d \in \mathbb{C}^*$, $e \in \mathbb{C}$ arbitrary). These automorphisms cover all automorphisms of $\mathcal{N}_2$ which lift to automorphisms of the (T)-structure $A_1 = C_1$, $A_2 = C_2$. They can be used to reduce the statement we need to prove to showing that c and α are formal gauge invariant This was shown in the proof of Lemma 19. □

Lemma 23. *Consider two distinct (TE)-structures ∇ and $\tilde{\nabla}$ as in Theorem 9 iii). They are formally isomorphic if and only if their matrices B and $\tilde{B}$ are of the fourth form in (34), with constants c, α, λ and $\tilde{c}$, $\tilde{\alpha}$, $\tilde{\lambda}$, such that $\tilde{c} = c$, $\tilde{\alpha} = \alpha$ and $\tilde{\lambda} = -\lambda$.*

Proof. Assume that ∇ and $\tilde{\nabla}$ are formally isomorphic. Then, from Lemma 22, $c = \tilde{c}$ and $\alpha = \tilde{\alpha}$. Using Corollary 18 i), we deduce (exchanging ∇ with $\tilde{\nabla}$ if necessary) that one of the following cases holds: I) ∇ and $\tilde{\nabla}$ belong to the fourth class in Theorem 9 iii) and $\tilde{\lambda} = -\lambda$; II) ∇ and $\tilde{\nabla}$ belong to the fifth class in Theorem 9 iii) and $\tilde{\lambda} = \lambda$, $\tilde{\gamma} \neq \gamma$; III) ∇ belongs to the sixth class and $\tilde{\nabla}$ belongs to the seventh class in Theorem 9 iii) and $\tilde{\lambda} = \lambda$; IV) ∇ belongs to the eighth class and $\tilde{\nabla}$ to the nineth class in Theorem 9 iii) and $\tilde{\lambda} = \lambda$. In case I) ∇ and $\tilde{\nabla}$ are formally isomorphic by the isomorphism T used in Corollary 18 ii). It turns out that in the remaining cases ∇ and $\tilde{\nabla}$ are in fact formally non-isomorphic. Let us sketch the argument for case III). Assume, by contradiction, that ∇ and $\tilde{\nabla}$ are formally isomorphic. Then the B-matrices of ∇, $\tilde{\nabla}$ are of the first form (65), with $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $\lambda = \tilde{\lambda}$, $\gamma = 1$ and $\tilde{\gamma} = 0$. Since $\tilde{b}_2^{(0)} = b_2^{(0)} = \lambda t_2$, from (64) with $a = \tilde{a} = 0$, $b = \tilde{b} = \lambda$ and $c = \tilde{c} = 0$ we deduce that $e = 0$ and T covers a map of the form $h(t_1, t_2) = (t_1, \frac{k}{d} t_2)$, with $k, d \in \mathbb{C}^*$. The automorphism $T^{[1]}$ used in the proof of Lemma 20 maps ∇ to a (TE)-structure $\nabla^{[1]}$ with $A_1^{[1]} = C_1$, $A_2^{[1]} = C_2$ and matrix $B^{[1]}$ of the first form (65) with $\gamma^{[1]} = \frac{k}{d}$. Since ∇ and $\tilde{\nabla}$ are formally isomorphic, $\nabla^{[1]}$ and $\tilde{\nabla}$ are formally gauge isomorphic (and are both of the first form (65)). Going through the computations of Lemma 19 and using Lemma 56 we obtain that $\nabla^{[1]} = \tilde{\nabla}$ which is a contradiction (as $\gamma^{[1]} \neq 0$ while $\tilde{\gamma} = 0$). $\square$

4. HOLOMORPHIC CLASSIFICATION OF (TE)-STRUCTURES

4.1. Restriction of (TE)-structures at the origin; holomorphic classification in the non-elementary case

By an *elementary model* we mean a meromorphic connection ∇^0 on the germ $(\mathcal{O}_{(\mathbb{C},0)})^2$ with connection form $\Omega^0 = \frac{1}{z^2} B^0 dz = \frac{1}{z^2} \sum_{k\geq 0} B_0^{(k)} z^k dz$ (where $B_0^{(k)} \in M_{2\times 2}(\mathbb{C})$), such that $\nabla^1 := \mathcal{E}^{\frac{\mathrm{tr}\,(B_0^{(0)})}{2z}} \otimes \nabla^0$ is regular singular (we denote by $\mathcal{E}^{\rho}$ the connection in rank one with connection form $d\rho$). Obviously, ∇^0 is an elementary model if and only if $\mathcal{E}^{\frac{\mathrm{tr}\,(B^0)}{2z}} \otimes \nabla^0$ is regular singular. The property of a meromorphic connection to be an elementary model is invariant under holomorphic isomorphisms: if $\tilde{\nabla}^0$, with connection form $\tilde{\Omega}^0 = \frac{1}{z^2} \tilde{B}^0 dz = \frac{1}{z^2} \sum_{k\geq 0} \tilde{B}_0^{(k)} z^k dz$ ($\tilde{B}_0^{(k)} \in M_{2\times 2}(\mathbb{C})$) is isomorphic to ∇^0 by means of an holomorphic isomorphism $T^0 = \sum_{k\geq 0} T_0^{(k)} z^k$, then $\mathrm{tr}\,(\tilde{B}_0^{(0)}) = \mathrm{tr}\,(B_0^{(0)})$ and T^0 is an isomorphism also between ∇^1 and $\tilde{\nabla}^1$ (the latter defined as ∇^1 starting with $\tilde{\nabla}^0$ instead of ∇^0).

Definition 24. A (TE)-structure ∇ over $\mathcal{N}_2$ is called elementary if its restriction to the slice $(\mathbb{C}, 0) \times \{0\} \subset (\mathbb{C}, 0) \times \mathcal{N}_2$ is an elementary model. A (TE)-structure which is not elementary is called non-elementary.

Our aim in this section is to prove the next proposition (for Malgrange universal connections and Birkhoff normal forms, see appendix).

PROPOSITION 25. *Let ∇ be a (TE)-structure in pre-normal form, determined by (f, b_2, c, α).*

i) *Then ∇ is elementary if and only if $f(0,0)b_2(0,0) = 0$.*

ii) *If ∇ is elementary then ∇ is holomorphically isomorphic to its formal normal form(s).*

iii) *If ∇ is non-elementary then ∇ is holomorphically isomorphic to the Malgrange universal deformation of a meromorphic connection in Birkhoff normal form, with residue a regular endomorphism.*

We divide the proof of the above proposition into several steps. In the setting of Proposition 25, let ∇^{restr} be the restriction of ∇ to the slice $\Delta \times \{0\}$ (where Δ is a small disc around the origin in $\mathbb{C}$) and $\eta := b_2$, $\lambda := \partial_2 b_2$, $\beta := \partial_2^2 b_2$ and $\gamma := f$, all restricted to this slice. They are functions on z only and are holomorphic. From (35), (36), the connection form of ∇^{restr} is

$$\Omega^{\text{restr}} = \frac{1}{z^2}\left((c + z\alpha)C_1 + \eta C_2 - \frac{z(\lambda+1)}{2}D + z(-\frac{z\beta}{2} + \gamma\eta)E\right)dz. \tag{74}$$

LEMMA 26. *The connection ∇^{restr} is an elementary model if and only if $\eta(0)\gamma(0) = 0$.*

Proof. We need to show that ∇^{restr}, with $c = \alpha = 0$, is regular singular. Assume from now on that $c = \alpha = 0$. When $\eta(0) = 0$, Ω^{restr} has a logarithmic pole and the regular singularity of ∇^{restr} is obvious. Assume assume that $\eta(0) \neq 0$. Let $\{v_1, v_2\}$ be the standard basis of $(\mathcal{O}_{(\mathbb{C},0)})^2$, so that

$$\begin{aligned} \nabla^{\text{restr}}_{\partial z}(v_1) &= (\Omega^{\text{restr}}_{\partial z})_{11} v_1 + (\Omega^{\text{restr}}_{\partial z})_{21} v_2 \\ \nabla^{\text{restr}}_{\partial z}(v_2) &= (\Omega^{\text{restr}}_{\partial z})_{12} v_1 + (\Omega^{\text{restr}}_{\partial z})_{22} v_2. \end{aligned} \tag{75}$$

Using the definitions of the matrices C_2, D and E, we rewrite (75) as

$$\begin{aligned} \nabla^{\text{restr}}_{\partial z}(v_1) &= -\frac{\lambda+1}{2z} v_1 + \frac{\eta}{z^2} v_2 \\ \nabla^{\text{restr}}_{\partial z}(v_2) &= \left(-\frac{\beta}{2} + \frac{\eta\gamma}{z}\right) v_1 + \frac{\lambda+1}{2z} v_2. \end{aligned} \tag{76}$$

Since $\eta(0) \neq 0$, v_1 is a cyclic vector (see appendix). Let $\tilde{v}_2 := \nabla^{\text{restr}}_{\partial_z}(v_1)$. Then

$$\begin{aligned} \nabla^{\text{restr}}_{\partial z}(\tilde{v}_2) = &\left(-\partial_z(\frac{\lambda+1}{2z}) + \frac{\eta}{z^2}(-\frac{\beta}{2} + \frac{\eta\gamma}{z}) + \frac{(\lambda+1)}{2\eta}(\frac{\dot{\eta}}{z} - \frac{2\eta}{z^2}) + \frac{(\lambda+1)^2}{4z^2}\right) v_1 \\ &+ \frac{1}{\eta}(\dot{\eta} - \frac{2\eta}{z})\tilde{v}_2. \end{aligned} \tag{77}$$

The valuation of the coefficient of $\tilde{v}_2$ in $\nabla^{\mathrm{restr}}_{\partial z}(\tilde{v}_2)$ is equal to -1, while the valuation of the coefficient of v_1 is greater or equal to -2 if and only if $\gamma(0)=0$ (we used that $\eta(0)\neq 0$). From the Fuchs criterion (see appendix), we obtain our claim. □

Corollary 27. i) *The property of a (TE)-structure over $\mathcal{N}_2$ to be elementary is a formal invariant. Any formal isomorphism between elementary (TE)-structures is holomorphic.*

ii) *Any elementary (TE)-structure over $\mathcal{N}_2$ is holomorphically isomorphic to its formal normal form(s).*

Proof. Let ∇ and $\tilde{\nabla}$ be two formally isomorphic (TE)-structures in pre-normal form, determined by (f, b_2, c, α) and $(\tilde{f}, \tilde{b}_2, \tilde{c}, \tilde{\alpha})$. From Theorem 19 i) of [5] we know that $f(0,0)=0$ if and only if $\tilde{f}(0,0)=0$. Since ∇ and $\tilde{\nabla}$ are formally isomorphic, $B^{(0)}$ and $\tilde{B}^{(0)}$ are conjugated, which implies that $b_2(0,0)=0$ if and only if $\tilde{b}_2(0,0)=0$. We proved that ∇ is elementary if and only if $\tilde{\nabla}$ is elementary, i.e. being elementary is a formal invariant. Assume now that ∇ and $\tilde{\nabla}$ are elementary. Let $T=\sum_{\geq 0} T^{(k)} z^k$ be a formal isomorphism between them and T^0 be the restriction of T to $\Delta\times\{0\}$. As explained at the beginning of this section, T^0 is an isomorphism between the connections $(\nabla^{\mathrm{restr}})^1$ and $(\tilde{\nabla}^{\mathrm{restr}})^1$. Since these are regular singular, we deduce that T^0 is holomorphic. Claim i) is concluded by Theorem 5.6 of [6]. Claim ii) follows trivially from claim i). □

Claims i) and ii) from Proposition 25 are proved. It remains to prove claim iii). We do this in several lemmas. Recall the definition of η, λ, β and γ stated before Lemma 26.

Lemma 28. *Let ∇ be a (TE)-structure in pre-normal form, determined by (f, b_2, c, α). If $\eta(0)\gamma(0)\neq 0$, then ∇^{restr} can be put in Birkhoff normal form.*

Proof. In the standard basis $\{v_1, v_2\}$ of $(\mathcal{O}_{(\mathbb{C},0)})^2$, ∇^{restr} is given by

$$\nabla^{\mathrm{restr}}_{\partial z}(v_1) = (\frac{c+\alpha z}{z^2} - \frac{\lambda+1}{2z})v_1 + \frac{\eta}{z^2}v_2$$

$$\nabla^{\mathrm{restr}}_{\partial z}(v_2) = \left(-\frac{\beta}{2}+\frac{\eta\gamma}{z}\right)v_1 + (\frac{c+\alpha z}{z^2}+\frac{\lambda+1}{2z})v_2. \tag{78}$$

(Remark that these relations with $c=\alpha=0$ reduce to relations (76)). We apply the irreducibility criterion as stated in Lemma 58 (see appendix). Suppose, by absurd, that there is a section w such that

$$\nabla^{\mathrm{restr}}_{\partial_z}(w) = hw, \tag{79}$$

for a function $h \in \boldsymbol{k}$. Since $\eta(0) \neq 0$, the first relation (78) shows that w cannot be a multiple of v_1. Rescaling w if necessary, we can assume that $w = gv_1 + v_2$, where $g \in \boldsymbol{k}$. A straightforward computation which uses (78) shows that (79) is equivalent to

$$h = \frac{g\eta}{z^2} + \frac{\lambda + 1}{2z} + \frac{c + \alpha z}{z^2}$$

and

$$z^2\dot{g} - ((\lambda + 1)z + \eta g)g - \frac{\beta z^2}{2} + \eta\gamma z = 0. \tag{80}$$

We will show that (80) leads to a contradiction. Since $g \in \boldsymbol{k}$, we can write it as $g(z) = z^k r(z)$ for $k \in \mathbb{Z}$ and $r \in \mathbb{C}\{z\}$ a unit. Relation (80) is equivalent to (81)

$$kz^{k+1}r(z) + z^{k+2}\dot{r}(z) - z^{k+1}r(z)(\lambda(z)+1) - z^{2k}\eta(z)r(z)^2 - \frac{\beta(z)z^2}{2} + \eta(z)\gamma(z)z = 0.$$

If $k \leq 0$ then, multiplying the above relation by z^{-2k}, we obtain

$$kz^{-k+1}r(z) + z^{-k+2}\dot{r}(z) - z^{-k+1}r(z)(\lambda(z) + 1) - \eta(z)r(z)^2 - \frac{\beta(z)}{2}z^{-2k+2} + \eta(z)\gamma(z)z^{-2k+1} = 0.$$

All terms, except $\eta(z)r(z)^2$, contain z as a factor. Since $r, \lambda, \eta, \beta, \gamma \in \mathbb{C}\{z\}$ and η, r, are units we obtain a contradiction. If $k \geq 1$ the argument is similar: we multiply (81) by z^{-1} and we use that η, γ are units in $\mathbb{C}\{z\}$. □

To conclude Proposition 25 we notice that if $b_2(0, 0) \neq 0$ then the 'residue' $cC_1 + \eta(0)C_2$ of the restriction ∇^{restr} of ∇ to $\Delta \times \{0\}$ is a regular endomorphism. Therefore, the Birkhoff normal form provided by Lemma 28 also has a regular residue and admits a (unique, up to holomorphic isomorphisms) Malgrange universal deformation. The latter is (holomorphically) isomorphic to ∇.

4.2. **Holomorphic classification: non-elementary case**

The holomorphic classification of elementary (TE)-structures follows from Corollary 27 ii): the formal normal forms for elementary (TE)-structures coincide with the holomorphic normal forms. It remains to determine the holomorphic normal forms for non-elementary (TE)-structures. This will be done in the next sections.

4.2.1. **Classification of non-elementary models in Birkhoff normal form**

Lemma 29. i) *Any non-elementary (TE)-structure ∇ is isomorphic to the Malgrange universal deformation of a connection $\nabla^{B_0^0, B_\infty}$ in Birkhoff normal* *form, with connection form*

$$\Omega^{B_0^o,B_\infty} = \frac{1}{z^2}(B_0^o + B_\infty z)dz, \tag{82}$$

where

$$B_0^o = \begin{pmatrix} c & 0 \\ c_0 & c \end{pmatrix},\ B_\infty = \begin{pmatrix} B_{11}^\infty & B_{12}^\infty \\ B_{21}^\infty & B_{22}^\infty \end{pmatrix} \tag{83}$$

with $B_{ij}^\infty \in \mathbb{C}$, $c, c_0 \in \mathbb{C}$ *and* $c_0 B_{12}^\infty \neq 0$.

ii) *Two non-elementary* (TE)*-structures* ∇, $\tilde{\nabla}$ *are isomorphic if and only if the corresponding connections in Birkhoff normal form* $\nabla^{B_0^0,B_\infty}$ *and* $\nabla^{\tilde{B}_0^0,\tilde{B}_\infty}$ *are isomorphic.*

Proof. From Lemma 28, we know that the restriction ∇^{restr} of ∇ to the origin of $\mathcal{N}_2$ can be put in Birkhoff normal form. Let $\nabla^{B_0^o,B_\infty}$ be a connection in Birkhoff normal form, with connection form given by (82) (for some matrices $B_0^o, B_\infty \in M_{2\times 2}(\mathbb{C})$), isomorphic to ∇^{restr}. The connection $\nabla^{B_0^o,B_\infty}$ has two properties: it is not an elementary model and its 'residue' B_0^o is a regular endomorphism, with only one eigenvalue (these two properties are satisfied by ∇^{restr} and are invariant under holomorphic isomorphisms). From the second property, the 'residue' B_0^o is as in (83), with $c_0 \neq 0$. A direct check (using e.g. the Fuchs criterion), shows that $\nabla^{B_0^o,B_\infty}$ is not an elementary model if and only if $B_{12}^\infty \neq 0$. This proves claim i). Claim ii) follows from the unicity of Malgrange universal deformations. □

In order to use Lemma 29 for the classification of non-elementary (TE)-structures, we need to establish when two meromorphic connections in Birkhoff normal form, as in Lemma 29, are isomorphic. We start with the next lemma.

LEMMA 30. i) *Any meromorphic connection* $\nabla^{B_0^o,B_\infty}$ *in Birkhoff normal form, with connection form given by (82) where*

$$B_0^o = cC_1 + c_0C_2,\ B_\infty = \alpha C_1 + c_1C_2 + yD + fE, \tag{84}$$

and $c_0 f \neq 0$, *can be mapped, by means of a constant isomorphism, to a connection* $\nabla^{\tilde{B}_0^0,\tilde{B}_\infty}$ *in Birkhoff normal form with*

$$\tilde{B}_0^o = cC_1 + c_0C_2,\ \tilde{B}_\infty = \alpha C_1 + c_1C_2 - \frac{1}{4}D + c_0E, \tag{85}$$

where the constants c *and* α *are the same as in (84),* c_0 *and* c_1 *are possibly different from those in (84) and* $c_0 \neq 0$.

ii) *The constant isomorphism* $T := \mathrm{diag}(1,-1)$ *maps the connection in Birkhoff normal form* $\nabla^{\tilde{B}_0^0,\tilde{B}_\infty}$, *with matrices* $\tilde{B}_0^o$, $\tilde{B}_\infty$ *given in (85), to a connection of the same form (85), with constants* $(\tilde{c}, \tilde{c}_0, \tilde{\alpha}, \tilde{c}_1)$ *satisfying* $\tilde{c} = c$, $\tilde{\alpha} = \alpha$, $\tilde{c}_0 = -c_0$ *and* $\tilde{c}_1 = -c_1$.

Proof. i) First we map $\nabla^{B_0^o,B_\infty}$ to a connection of the same form (82), (84), with the coefficient of D equal to $-\frac{1}{4}$. This is realized using the constant isomorphism $T_1 := C_1 - \frac{1}{f}(y+\frac{1}{4})C_2$. Therefore, without loss of generality we may (and will) assume that $\nabla^{B_0^o,B_\infty}$ is given by (82), (84), with $y = -\frac{1}{4}$. Under this assumption, if $c_0 \neq f$ in (84), let $\tilde{c}_0$ such that $(\tilde{c}_0)^2 = c_0 f$ and $T_2 := -2\mathrm{diag}(\frac{\tilde{c}_0}{c_0-\tilde{c}_0}, \frac{c_0}{c_0-\tilde{c}_0})$. The isomorphism T_2 maps $\nabla^{B_0^o,B_\infty}$ to the connection $\nabla^{\tilde{B}_0^o,\tilde{B}_\infty}$ with

$$\tilde{B}_0^o = cC_1 + \tilde{c}_0 C_2,\ \tilde{B}_\infty = \alpha C_1 + \frac{c_1\tilde{c}_0}{c_0}C_2 - \frac{1}{4}D + \tilde{c}_0 E.$$

This proves claim i). Claim ii) follows from a direct check. □

Proposition 31. *Consider two distinct connections ∇ and $\tilde{\nabla}$ in Birkhoff normal form (82), with matrices B_0^o, B_∞, respectively $\tilde{B}_0^o$, $\tilde{B}_\infty$ as in (85), with constants c, α, c_0, c_1 and, respectively $\tilde{c}, \tilde{\alpha}, \tilde{c}_0, \tilde{c}_1$. Assume that $c_0\tilde{c}_0 \neq 0$.*

i) *If ∇ and $\tilde{\nabla}$ are formally isomorphic, then $c = \tilde{c}$, $\alpha = \tilde{\alpha}$ and $c_0 = \epsilon\tilde{c}_0$ where $\epsilon \in \{\pm 1\}$.*

ii) *Assume that the conditions from i) are satisfied. Then ∇ is isomorphic to $\tilde{\nabla}$ if and only if there is $n \in \mathbb{N}_{\geq 2}$ such that*

$$4(c_0)^2(c_1 - \epsilon\tilde{c}_1)^2 - 8(n-1)^2 c_0(c_1+\epsilon\tilde{c}_1) + (2n-1)(2n-3)(n-1)^2 = 0 \tag{86}$$

and, for any $2 \leq r \leq n-1$, $r \in \mathbb{N}$,

$$c_0(c_1+\epsilon\tilde{c}_1) \neq \frac{(2n-1)(2n-3)(n-1)^2 - (2r-1)(2r-3)(r-1)^2}{8(n-r)(n-2+r)}. \tag{87}$$

Proof. i) We consider a formal isomorphism $T := \sum_{n\geq 0} T^{(n)} z^n$ which maps ∇ to $\tilde{\nabla}$. We write $T^{(n)} = \tau_1^{(n)}C_1 + \tau_2^{(n)}C_2 + \tau_3^{(n)}D + \tau_4^{(n)}E$, where $\tau_i^{(n)} \in \mathbb{C}$. Relation (16) with $r = 0$, applied to ∇, $\tilde{\nabla}$ and T, gives $c = \tilde{c}$, $\tau_4^{(0)} = 0$ and

$$\tau_1^{(0)}(c_0 - \tilde{c}_0) + \tau_3^{(0)}(c_0 + \tilde{c}_0) = 0. \tag{88}$$

Using that $\tau_4^{(0)} = 0$, relation (16) with $r = 1$ becomes

$$\begin{aligned}
&\frac{\tau_4^{(1)}}{2}(c_0 - \tilde{c}_0) + (\alpha - \tilde{\alpha})\tau_1^{(0)} + (c_0 - \tilde{c}_0)\frac{\tau_2^{(0)}}{2} = 0\\
&\tau_1^{(1)}(c_0 - \tilde{c}_0) + \tau_3^{(1)}(c_0 + \tilde{c}_0) + (\alpha - \tilde{\alpha} + \frac{1}{2})\tau_2^{(0)} +\\
&(c_1 - \tilde{c}_1)\tau_1^{(0)} + (c_1 + \tilde{c}_1)\tau_3^{(0)} = 0\\
&\frac{\tau_4^{(1)}}{2}(c_0 + \tilde{c}_0) - (\alpha - \tilde{\alpha})\tau_3^{(0)} - \frac{\tau_2^{(0)}}{2}(c_0 + \tilde{c}_0) = 0\\
&(c_0 - \tilde{c}_0)\tau_1^{(0)} - (c_0 + \tilde{c}_0)\tau_3^{(0)} = 0.
\end{aligned} \tag{89}$$

Suppose that $\tilde{c}_0 \neq c_0$. Then, from (88), $\tau_1^{(0)} = -(\frac{c_0+\tilde{c}_0}{c_0-\tilde{c}_0})\tau_3^{(0)}$ and $\tau_3^{(0)} \neq 0$ (if $\tau_3^{(0)} = 0$ then $\tau_1^{(0)} = 0$; since $\tau_4^{(0)} = 0$ we obtain that $T^{(0)}$ is not invertible, which is a contradiction). The last relation (89) implies $c_0 = -\tilde{c}_0$. We proved that $c = \tilde{c}$ and $c_0 = \epsilon\tilde{c}_0$ where $\epsilon \in \{\pm 1\}$. The first and third relations (89) (together with $T^{(0)}$-invertible and (88)) imply that $\alpha = \tilde{\alpha}$. The first claim follows.

ii) The claim follows by considering an holomorphic isomorphism T between ∇ and $\tilde{\nabla}$, identifying coefficients in (16) with computations similar to those already done before, and using that T is a polynomial (see Exercise 3.10 of [22]). □

Corollary 32. *In the setting of Proposition 31, assume that* $\tilde{c}_1 = 0$. *Then* ∇ *is isomorphic to* $\tilde{\nabla}$ *if and only if* $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $(c_0)^2 = (\tilde{c}_0)^2$ *and there is* $n \in \mathbb{Z}_{\geq 2}$ *such that*

$$c_0 c_1 \in \{\frac{(n-1)(2n-1)}{2}, \frac{(n-1)(2n-3)}{2}\}. \tag{90}$$

Proof. When $\tilde{c}_1 = 0$, relation (86) is an equation in c_0c_1, with solutions given by the right hand side of (90). If $\tilde{c}_1 = 0$ then (90) implies (87). □

Remark 33. i) Lemmas 29 and 30, combined with Proposition 31, provide a criterion to decide when two non-elementary (TE)-structures are isomorphic, using their restriction at the origin of $\mathcal{N}_2$

ii) The only non-elementary formal normal forms from Theorem 9 are those from Theorem 9 i) with $c_0 \neq 0$. They represent the formal normal forms of non-elementary (TE)-structures. Their restriction at the origin are in Birkhoff normal form, with

$$B_0^o = cC_1 + c_0C_2,\ B_\infty = \alpha C_1 - \frac{1}{4}D + c_0E, \tag{91}$$

(where $c_0 \neq 0$).

iii) There are non-elementary (TE)-structures which are not (holomorphically) isomorphic to their formal normal form(s): from Corollary 32, they coincide (up to isomorphism) with the Malgrange universal deformations of connections $\nabla^{B_0^o, B_\infty}$ in Birkhoff normal form, with matrices B_0^o, B_∞ as in (85), such that c_0c_1 does not satisfy (90).

In order to obtain a list of holomorphic normal forms for non-elementary (TE)-structures we will express the Malgrange universal deformations of the meromorphic connections $\nabla^{B_0^o, B_\infty}$ in Birkhoff normal form (82), with matrices B_0^o, B_∞ given by (85), with $c_0B_{12}^\infty \neq 0$, in local coordinates (t_1, t_2) of $\mathcal{N}_2$. This will be done in the next sections.

4.2.2. Non-elementary $(\boldsymbol{TE})$-structures and Malgrange universal connections

Let B_0^o, $B_\infty \in M_{2\times 2}(\mathbb{C})$ be two matrices, where B_0^o is regular, with one Jordan block, i.e. $B_0^o = cC_1 + c_0C_2$, and $c_0 \neq 0$. We are interested in the case $B_{12}^\infty \neq 0$ but for the moment we don't make this assumption. We denote by $\nabla^{\mathrm{univ}} = \nabla^{\mathrm{univ}, B_0^o, B_\infty}$ the Malgrange universal deformation of the meromorphic connection $\nabla^{B_0^o, B_\infty}$ with connection form

$$\Omega^{B_0^o, B_\infty} = \frac{1}{z^2}(B_0^o + B_\infty z)dz.$$

We consider $(M^{\mathrm{univ}}, \circ_{\mathrm{univ}}, e_{\mathrm{univ}}, E_{\mathrm{univ}})$ the parameter space of ∇^{univ}. The germs $((M^{\mathrm{univ}}, 0), \circ_{\mathrm{univ}}, e_{\mathrm{univ}})$ and $\mathcal{N}_2$ are isomorphic.

For any $\Gamma \in M_{2\times 2}(\mathbb{C})$, we identify $T_\Gamma M_{2\times 2}(\mathbb{C})$ with $M_{2\times 2}(\mathbb{C})$ in the natural way. Therefore, vector fields on $M_{2\times 2}(\mathbb{C})$ or on the submanifold M^{univ} will be viewed as $M_{2\times 2}(\mathbb{C})$-valued functions (defined on $M_{2\times 2}(\mathbb{C})$ or M^{univ} respectively).

Let X_0, X_1 be vector fields on $M_{2\times 2}(\mathbb{C})$ defined by $(X_0)_\Gamma = C_1$ and $(X_1)_\Gamma = B_0^o - \Gamma + [B_\infty, \Gamma]$, for any $\Gamma \in M_{2\times 2}(\mathbb{C})$. In the standard coordinates (Γ_{ij}) of $M_{2\times 2}(\mathbb{C})$ (where $\Gamma_{ij} : M_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ is the function which assigns to $\Gamma \in M_{2\times 2}(\mathbb{C})$ its (i,j)-entry),

(92)
$$X_0 = \sum_{i,j=1}^{2} \delta_{ij} \frac{\partial}{\partial \Gamma_{ij}},\ X_1 = \sum_{i,j=1}^{2} ((B_0^o)_{ij} - \Gamma_{ij} + (B_\infty)_{ik}\Gamma_{kj} - \Gamma_{ik}(B_\infty)_{kj}) \frac{\partial}{\partial \Gamma_{ij}}.$$

(To simplify notation, we omitted the summation sign over $k \in \{1,2\}$). Let

$$\tilde{k} : M_{2\times 2}(\mathbb{C}) \to \mathbb{C},\ \tilde{k}(\Gamma) := -\frac{1}{2}\mathrm{trace}(X_1)_\Gamma = \frac{1}{2}\sum_{i=1}^{2} \Gamma_{ii} - c. \tag{93}$$

Viewing a vector field X on $M_{2\times 2}(\mathbb{C})$ as an $M_{2\times 2}(\mathbb{C})$-valued function, we can consider its derivative along any other vector field Y on $M_{2\times 2}(\mathbb{C})$. The result is a function $Y(X) : M_{2\times 2}(\mathbb{C}) \to M_{2\times 2}(\mathbb{C})$, whose (i,j)-entry is the function $Y(X_{ij})$. Various such derivatives are computed in the next lemma (below C_1 denotes the constant function on $M_{2\times 2}(\mathbb{C})$ equal to C_1).

Lemma 34. *The following relations hold:*

$$\begin{aligned} &X_0(X_1) = -C_1,\ X_0(\tilde{k}) = 1,\ X_0(\tilde{k}X_0 + X_1) = 0; \\ &X_1(X_1) = -X_1 + [B_\infty, X_1],\ X_1(\tilde{k}) = -\tilde{k}. \end{aligned} \tag{94}$$

Proof. As $X_0(\Gamma_{ij}) = \delta_{ij}$ for any $i, j \in \{1,2\}$ we obtain

$$X_0([B_\infty,\Gamma]_{ik}) = \sum_{j=1}^{2} X_0((B_\infty)_{ij}\Gamma_{jk} - \Gamma_{ij}(B_\infty)_{jk})$$
$$= \sum_{j=1}^{2}((B_\infty)_{ij}\delta_{jk} - (B_\infty)_{jk}\delta_{ij}) = (B_\infty)_{ik} - (B_\infty)_{ik} = 0,$$
$$X_0((B_0^o)_{ik} - \Gamma_{ik} + [B_\infty,\Gamma]_{ik}) = -\delta_{ik}.$$

We proved that $X_0(X_1) = -C_1$. Since $X_0(\Gamma_{ii}) = 1$ we obtain $X_0(\tilde{k}) = 1$. Obviously, $X_0(X_0) = 0$ (since $(X_0)_{ij} = \delta_{ij}$ are constants) and so

$$X_0(\tilde{k}X_0 + X_1) = X_0(\tilde{k})X_0 + X_0(X_1) = C_1 - C_1 = 0.$$

The first line of (94) follows. The second line can be proved similarly. □

Using the expression (92) of X_0 and X_1 we compute the Lie derivative of X_1 in the direction of X_0: $L_{X_0}X_1 = -X_0$. Using $X_0(\tilde{k}) = 1$ we obtain that the vector fields X_0 and $\tilde{k}X_0 + X_1$ commute. Their restriction to M^{univ} are the fundamental vector fields of a coordinate system (t_1, t_2) on M^{univ}, which we choose to be centred at the origin of M^{univ}. As $X_0(\tilde{k}) = 1$ and $(\tilde{k}X_0 + X_1)(\tilde{k}) = 0$ (from Lemma 34) and $\tilde{k}(0) = -c$ (from the definition (93) of $\tilde{k}$) we obtain that $\tilde{k}(t_1, t_2) = t_1 - c$.

Remark 35. For any $\Gamma \in M^{\mathrm{univ}}$, the matrix $(\tilde{k}X_0 + X_1)(\Gamma)$ has the following properties: it is trace-free (from the definition of $\tilde{k}$); it is regular (since $(X_1)(\Gamma)$ is regular, being regular at $\Gamma = 0$); it has only one Jordan block (since B_0^o has this property; see appendix). We obtain that $(\tilde{k}X_0 + X_1)(\Gamma)$ is conjugated to a matrix with all entries zero except the $(2,1)$-entry which is non-zero. In particular, $(\tilde{k}X_0 + X_1)(\Gamma)^2 = 0$. Also, $(\tilde{k}X_0 + X_1)_{21} = (X_1)_{21}$ at $\Gamma = 0$ is equal to $c_0 \neq 0$. The function $y := (X_1)_{21} : M^{\mathrm{univ}} \to \mathbb{C}$ is non-vanishing in a neighborhood of the origin in M^{univ} and the function $\tilde{k}X_0 + X_1 : M^{\mathrm{univ}} \to M_{2\times 2}(\mathbb{C})$ can be written as

(95) $$\tilde{k}X_0 + X_1 = y\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}, \text{ where}$$

(96) $$x = \frac{1}{(X_1)_{21}}(\tilde{k} + (X_1)_{11}) = \frac{1}{2(X_1)_{21}}((X_1)_{11} - (X_1)_{22}) : M^{\mathrm{univ}} \to \mathbb{C}.$$

PROPOSITION 36. *In the coordinate system (t_1, t_2), the Malgrange universal connection ∇^{univ} is given by the matrices A_1, A_2 and $B = B^{(0)} + B^{(1)}z$, where*

(97) $$A_1 = C_1,\ A_2 = y\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}$$

and

$$B^{(0)} = \begin{pmatrix} xy - t_1 + c & -x^2 y \\ y & -xy - t_1 + c \end{pmatrix}, \; B^{(1)} = B_\infty. \tag{98}$$

Proof. Recall the definition (154) of the Malgrange universal connection. The equality $A_1 = C_1$ follows from the fact that $A_1 = A_{\frac{\partial}{\partial t_1}}$ is the matrix valued function on M^{univ} identified with the vector field $\frac{\partial}{\partial t_1} = X_0$, which is C_1. The expression of A_2 follows similarly, from $A_2 = C_{\frac{\partial}{\partial t_2}}$, together with $\frac{\partial}{\partial t_2} = \tilde{k} X_0 + X_1$ and (95). The expression of $B^{(0)}$ is obtained as follows: from (153), $B^{(0)}(\Gamma) = (X_1)(\Gamma)$ for any $\Gamma \in M^{\mathrm{univ}}$. Therefore,

$$\begin{aligned} B^{(0)} &= (\tilde{k} X_0 + X_1) - \tilde{k} X_0 = y \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} - (t_1 - c)\mathrm{Id} \\ &= \begin{pmatrix} xy - t_1 + c & -x^2 y \\ y & -xy - t_1 + c \end{pmatrix} \end{aligned}$$

where we used (95) and $\tilde{k}(t_1, t_2) = t_1 - c$. □

To simplify notation, in the proof of the next lemma instead of the vector field X_1 we simply write X. The (i,j)-entry of X_1 (viewed as a $M_{2\times 2}(\mathbb{C})$-valued function) will be denoted by X_{ij}.

LEMMA 37. *The functions x and y are independent on t_1 and their derivatives with respect to t_2 are given by*

$$\begin{aligned} \dot{x} &= -B_{21}^{\infty} x^2 + (B_{11}^{\infty} - B_{22}^{\infty}) x + B_{12}^{\infty} \\ \dot{y} &= y(2B_{21}^{\infty} x + B_{22}^{\infty} - B_{11}^{\infty} - 1), \end{aligned} \tag{99}$$

where B_{ij}^{∞} denotes the (i,j)-entry of the matrix B_∞. They satisfy the initial conditions $x(0) = 0$ and $y(0) = c_0$.

Proof. From the first line of (94) and the definition of x and y, $X_0(x) = X_0(y) = 0$, i.e. x and y are independent on t_1. From the second line of (94),

$$X(X_{11}) = -X_{11} + [B_\infty, X]_{11} = -X_{11} + B_{12}^{\infty} X_{21} - B_{21}^{\infty} X_{12} \tag{100}$$

and similarly

$$X(X_{21}) = -X_{21} + B_{21}^{\infty}(X_{11} - X_{22}) + (B_{22}^{\infty} - B_{11}^{\infty}) X_{21}. \tag{101}$$

Using (100), (101), $X(\tilde{k}) = -\tilde{k}$ and $\det(X + \tilde{k}\mathrm{Id}) = 0$, we obtain

$$X\left(\frac{\tilde{k} + X_{11}}{X_{21}}\right) = B_{12}^{\infty} + (B_{11}^{\infty} - B_{22}^{\infty}) \left(\frac{X_{11} - X_{12}}{2X_{21}}\right) - B_{21}^{\infty} \left(\frac{X_{11} - X_{22}}{2X_{21}}\right)^2,$$

which implies the first relation (99) (we use the definition (96) of x and $\dot{x} = X(x)$, since $\frac{\partial}{\partial t_2} = \tilde{k}X_0 + X$ and $X_0(x) = 0$). The second relation (99) can be obtained similarly. $\square$

Remark 38. When $B_{12}^{\infty} = 0$, the system (99) is solved by $x = 0$ and $y(t_1, t_2) = c_0 e^{kt_2}$ where $c_0 \in \mathbb{C}$ and $k := B_{22}^{\infty} - B_{11}^{\infty} - 1$. Assume that $k \neq 0$. Replacing the expressions of x and y in (97), (98) we obtain that ∇^{univ} is the pull-back by $\mu(t_1, t_2) = (t_1, \frac{c_0}{k}(e^{kt_2} - 1))$ of the (TE)-structure $\tilde{\nabla}$ given by

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2,\ \tilde{B} = (-t_1 + c)C_1 + (kt_2 + c_0)C_2 + zB_{\infty}.$$

When $k = 0$ the same statement holds with $\mu(t_1, t_2) = (t_1, c_0 t_2)$. We obtain that ∇^{univ} is isomorphic to $\tilde{\nabla}$. Remark that $\tilde{\nabla}$ is of the third type in Theorem 9.

We now turn to the Malgrange universal deformations ∇^{univ} we are interested in, namely those which are non-elementary. Therefore, we assume that $B_{12}^{\infty} \neq 0$. From Lemma 30, we may (and will) assume, without loss of generality, that $B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2}$ and $B_{12}^{\infty} = c_0$. We distinguish two subcases, namely $B_{21}^{\infty} = 0$ and $B_{21}^{\infty} \neq 0$. In the first subcase, ∇^{univ} is isomorphic to a (TE)-structure of the first type in Theorem 9:

Corollary 39. *If $B_{12}^{\infty} = c_0$, $B_{21}^{\infty} = 0$ and $B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2}$, then ∇^{univ} is isomorphic to the (TE)-structure $\tilde{\nabla}$ given by*

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + zE$$

$$\tilde{B} = (-t_1 + c + \alpha z)C_1 + (-\frac{t_2}{2} + c_0)C_2 - \frac{z}{4}D + z(-\frac{t_2}{2} + c_0)E, \tag{102}$$

where $\alpha := \frac{1}{2}(B_{11}^{\infty} + B_{22}^{\infty})$.

Proof. The functions $x(t_1, t_2) = 2c_0(1 - e^{-\frac{t_2}{2}})$ and $y(t_1, t_2) = c_0 e^{-\frac{t_2}{2}}$ solve the system (99). From $y = \dot{x}$ we obtain that ∇^{univ} is the pull-back, by the function $\mu(t_1, t_2) = (t_1, x(t_1, t_2))$, of the (TE)-structure $\nabla^{[1]}$ with matrices $A_1^{[1]}$, $A_2^{[1]}$, $B^{[1]} = \sum_{k \geq 0} B^{[1],(k)} z^k$ given by

$$A_1^{[1]} = C_1,\ A_2^{[1]} = C_2 + t_2 D - t_2^2 E$$

$$B^{[1],(0)} = (-t_1 + c)C_1 + (-\frac{t_2}{2} + c_0)C_2 + t_2(-\frac{t_2}{2} + c_0)D + t_2^2(\frac{t_2}{2} - c_0)E$$

$$B^{[1],(1)} = \alpha C_1 - \frac{1}{4}D + c_0 E$$

$$B^{[1],(k)} = 0,\ k \geq 2. \tag{103}$$

The gauge isomorphism $T := C_1 + t_2 E$ maps $\nabla^{[1]}$ to $\tilde{\nabla}$. $\square$

It remains to consider the case when both B^∞_{12} and B^∞_{21} are non-zero.

Corollary 40. *Assume that* $B^\infty_{12} = c_0$*,* $B^\infty_{21} \neq 0$ *and* $B^\infty_{11} - B^\infty_{22} = -\frac{1}{2}$*. Define* $a, b \in \mathbb{C}$ *by* $a + b = -\frac{1}{2B^\infty_{21}}$ *and* $ab = -\frac{c_0}{B^\infty_{21}}$*.*

i) *When* $B^\infty_{12}B^\infty_{21} \neq -\frac{1}{16}$*, the system (99) is solved by*

$$x(t_1, t_2) = ab(1 - e^{(b-a)B^\infty_{21}t_2})(b - ae^{(b-a)B^\infty_{21}t_2})^{-1}$$

$$y(t_1, t_2) = \frac{c_0}{(b-a)^2}(b - ae^{(b-a)B^\infty_{21}t_2})^2 e^{(B^\infty_{21}(a-b)-1)t_2}. \tag{104}$$

ii) *When* $B^\infty_{12}B^\infty_{21} = -\frac{1}{16}$*, it is solved by*

$$x(t_1, t_2) = \frac{4c_0t_2}{t_2 + 4}, \ y(t_1, t_2) = \frac{c_0}{16}e^{-t_2}(t_2 + 4)^2. \tag{105}$$

Proof. We write equation (99) as $\dot{x} = -B^\infty_{21}(x - a)(x - b)$. Since $x(0) = 0$ this determines x as stated in the lemma. Then y is determined from the second equation (99). Remark that $B^\infty_{12}B^\infty_{21} = -\frac{1}{16}$ if and only if $a = b$. □

To simplify terminology we introduce the following definition.

Definition 41. A non-elementary Malgrange normal form of the first (respectively second) type is a Malgrange universal deformation ∇^{univ} as in Proposition 36, with functions x and y satisfying (99), $B^\infty_{12} = c_0 \neq 0$, $B^\infty_{11} - B^\infty_{22} = -\frac{1}{2}$ and $B^\infty_{21} = 0$ (respectively, $B^\infty_{21} \neq 0$).

Non-elementary (TE)-structures coincide (up to isomorphisms) to non-elementary Malgrange normal forms. From Corollary 39, those of the first type are isomorphic to the (TE)-structures (32) from Theorem 9, with $c_0 \neq 0$. In the next section we study the non-elementary Malgrange normal forms of the second type.

4.2.3. **Non-elementary Malgrange normal forms of second type**

We are looking for (holomorphic) isomorphisms T which map an arbitrary non-elementary Malgrange normal form of the second type $\nabla := \nabla^{\text{univ}, B^0_0, B_\infty}$ to a (TE)-structure which is 'as close as possible' to the formal normal forms from Theorem 9. Let A_1, A_2, $B = B^{(0)} + zB^{(1)}$ be the matrices of ∇, described in Proposition 36, in terms of functions x, y determined in Corollary 40. Recall that $B^o_0 = cC_1 + c_0C_2$ (with $c_0 \neq 0$), $B^\infty_{12} = c_0$ and $B^\infty_{11} - B^\infty_{22} = -\frac{1}{2}$. From Remark 35, $A_2 = A_2^{(0)}$ is conjugated to a matrix of the form FC_2, for a function $F = F(t_2)$. Therefore, there is a gauge isomorphism $T = T^{(0)}$, which depends only on t_2, such that the underling (T)-structure of $\nabla^{[1]} := T \cdot \nabla$ is

$$A^{[1]}_1 = C_1, \ A^{[1]}_2 = FC_2 + zT^{-1}\partial_2 T. \tag{106}$$

As T is independent on z, the matrix $B^{[1]}$ of $\nabla^{[1]}$ is given by

$$B^{[1]} = T^{-1}B^{(0)}T + zT^{-1}B^{(1)}T. \tag{107}$$

LEMMA 42. *The gauge isomorphisms T, which depend only on t_2, and map ∇ to a (TE)-structure $\nabla^{[1]} := T \cdot \nabla$, whose underlying (T)-structure satisfies*

$$A_1^{[1]} = C_1,\ A_2^{[1]} = FC_2 + zGE \tag{108}$$

for functions $F = F(t_2)$ and $G = G(t_2)$, are of the form

$$T = \begin{pmatrix} k_0 k & \frac{k_1 x}{k-x} \\ k_0 & \frac{k_1}{k-x} \end{pmatrix},\ T = \begin{pmatrix} k_1 & k_0 x \\ 0 & k_0 \end{pmatrix} \tag{109}$$

where $k_0, k_1, k \in \mathbb{C}^*$. If T is given by the first formula (109), then we have $F = \frac{k_0}{k_1}(k-x)^2 y$, $G = \frac{k_1 \dot{x}}{k_0(k-x)^2}$. If T is given by the second formula (109), then $F = \frac{k_1}{k_0}y$, $G = \frac{k_0}{k_1}\dot{x}$. In both cases, $F(0) \neq 0$.

Proof. By a straightforward computation, the matrices $T = T(t_2)$ which satisfy $T^{-1}A_2T = FC_2$ and $T^{-1}\partial_2 T = GE$ are of the form

$$T = \begin{pmatrix} qx + \tilde{q}\frac{F}{y} & x\tilde{q} \\ q & \tilde{q} \end{pmatrix}, \tag{110}$$

where $q, \tilde{q} \in \mathbb{C}\{t_2\}$ and

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t_2}(\frac{\tilde{q}F}{y}) + q\dot{x} = 0 \\ &\dot{\tilde{q}}\frac{F}{y} = q\dot{x} \\ &q^2\dot{x} + q\frac{\mathrm{d}}{\mathrm{d}t_2}(\frac{\tilde{q}F}{y}) - \dot{q}\frac{\tilde{q}F}{y} = 0. \end{aligned} \tag{111}$$

Moreover, if (111) are satisfied, then $G = \frac{\dot{x}y}{F}$.

If $q(0) \neq 0$, we divide the third relation (111) by q^2 and we obtain $\frac{\mathrm{d}}{\mathrm{d}t_2}(\frac{\tilde{q}F}{qy}) = -\dot{x}$, which implies that $\tilde{q}F = qy(k-x)$ for $k \in \mathbb{C}$. Using $\frac{\tilde{q}F}{y} = q(k-x)$, the first relation (111) implies that $q = k_0$ is constant. The second relation (111) determines $\tilde{q}$ as $\tilde{q} = \frac{k_1}{k-x}$, for $k_1 \in \mathbb{C}^*$. The expressions for T, F and G follow. As T is invertible and $x(0) = 0$, $k, k_0 \in \mathbb{C}^*$.

If $q(0) = 0$ then $q = 0$ (otherwise $q(z) = z^r \eta(z)$ for $r \in \mathbb{Z}_{\geq 1}$ and $\eta \in \mathbb{C}\{t_2\}$ non-trivial. But writing q in this way we obtain a contradiction in the third relation (111)). The case $q = 0$ can be treated similarly and leads to the second expression in (109) for T and to F, G as required.

Since $x(0) = 0$, $k_0, k_1, y(0) \in \mathbb{C}^*$, we obtain that $F(0) \neq 0$ (in both cases). □

Let T be a gauge isomorphism as in Lemma 42. The underling (T)-structure of $\nabla^{[1]} = T \cdot \nabla$ is of the form

$$A_1^{[1]} = C_1,\ A_2^{[1]} = F(C_2 + \frac{G}{F}E) = \dot{\mu}_2(C_2 + zfE)$$

where $\mu_2 \in \mathrm{Aut}(\mathbb{C}, 0)$ satisfies $\dot{\mu}_2 = F$ and in the second expression for $A_2^{[1]}$ the function $\frac{G}{F}$ is written in terms of μ_2, i.e. $\frac{G}{F} = f(\mu_2)$. (Remark that $\dot{\mu}_2(0) \neq 0$ since $F(0) \neq 0$). We obtain that $\nabla^{[1]}$ is the pull-back by $\mu(t_1, t_2) = (t_1, \mu_2(t_2))$ of the (T)-structure

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + zfE. \tag{112}$$

Therefore, the underling (T)-structure of ∇ is isomorphic to the (T)-structure (112). In the following we will make suitable choices in Lemma 42 which lead to 'simplest' expressions for the function f.

Proposition 43. i) *If $B_{12}^{\infty} B_{21}^{\infty} = -\frac{1}{16}$, then the underlying (T)-stucture of ∇ is isomorphic to the (T)-structure given by*

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + z\frac{(c_0)^2}{1 - t_2}E. \tag{113}$$

ii) *If $B_{12}^{\infty} B_{21}^{\infty} \neq -\frac{1}{16}$ and $B_{21}^{\infty}(b-a) \neq 1$ then the underlying (T)-structure of ∇ is isomorphic to the (T)-structure given by*

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + z(\frac{\lambda}{c_0}t_2 + 1)^{-2-\frac{1}{\lambda}}E, \tag{114}$$

where $\lambda := B_{21}^{\infty}(b - a) - 1$ and $a, b \in \mathbb{C}$ are defined in Lemma 40.

iii) *If $B_{12}^{\infty} B_{21}^{\infty} \neq -\frac{1}{16}$ and $B_{21}^{\infty}(b-a) = 1$ then the underlying (T)-structure of ∇ is isomorphic to the (T)-structure given by*

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + z(c_0)^2 e^{-t_2}E. \tag{115}$$

Proof. i) Let T be the gauge isomorphism given by the first expression (109), with $k := 4c_0$ (and $k_0, k_1 \in \mathbb{C}^*$ arbitrary). Define $\gamma := \frac{k_0}{k_1}$. Using $F = \gamma(k - x)^2 y$ (see Lemma 42) and the expressions of x, y from (105), we obtain that

$$F = \gamma(4c_0 - x)^2 y = 16\gamma(c_0)^3 e^{-t_2} = \dot{\mu}_2(t_2)$$

where $\mu_2 \in \mathrm{Aut}(\mathbb{C}, 0)$ is given by $\mu_2(t_2) = 16\gamma(c_0)^3(1 - e^{-t_2})$. Then $e^{t_2} = \frac{16(c_0)^3\gamma}{16(c_0)^3\gamma - \mu_2}$ and

$$\frac{G}{F} = \frac{\dot{x}}{\gamma^2(4c_0 - x)^4 y} = (\frac{1}{16\gamma(c_0)^2})^2 e^{t_2} = \frac{1}{16\gamma c_0\,(16\gamma(c_0)^3 - \mu_2)}.$$

We obtain that the underlying (T)-structure of ∇ is isomorphic to the (T)-structure $\tilde{\nabla}$ given by

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + \frac{z}{16\gamma c_0\,(16\gamma(c_0)^3 - t_2)}E.$$

For $\gamma = \frac{1}{16(c_0)^3}$ we obtain the (T)-structure (113).

ii) The claim follows by a similar argument, by taking the gauge isomorphism T given by the first formula (109) with $k := a$, $\frac{k_1}{k_0} := a^2$ and the automorphism $\mu_2 \in \mathrm{Aut}(\mathbb{C}, 0)$ given by

$$\mu_2(t_2) := \frac{c_0}{B_{21}^{\infty}(b-a) - 1}(e^{(B_{21}^{\infty}(b-a)-1)t_2} - 1).$$

iii) The claim follows by a similar argument, by taking T given by the first formula in (109), with $k := a$ and $\frac{k_1}{k_0} := a^2 c_0$ and $\mu_2 \in \mathrm{Aut}(\mathbb{C}, 0)$ the identity automorphism. □

We arrive at our main result from this section.

THEOREM 44. *Let $\nabla = \nabla^{\mathrm{univ}, B_0^o, B_\infty}$ be a non-elementary Malgrange normal form of the second type, with $B_0^o = cC_1 + c_0 C_2$ and $B_\infty = (B_{ij}^{\infty})$ (thus $B_{12}^{\infty} = c_0 \neq 0$ and $B_{11}^{\infty} - B_{22}^{\infty} = -\frac{1}{2}$). Let $\alpha := \frac{1}{2}(B_{11}^{\infty} + B_{22}^{\infty})$.*

i) If $B_{12}^{\infty} B_{21}^{\infty} = -\frac{1}{16}$, then ∇ is isomorphic to the (TE)-structure $\tilde{\nabla}$ given by

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + \frac{z(c_0)^2}{1-t_2}E,$$

$$\tilde{B} = (-t_1 + c + \alpha z)C_1 + (1-t_2)C_2 + z(c_0)^2 E. \tag{116}$$

ii) If $B_{12}^{\infty} B_{21}^{\infty} \neq -\frac{1}{16}$ and $B_{21}^{\infty}(b-a) \neq 1$ then ∇ is isomorphic to the (TE)-structure $\tilde{\nabla}$ given by

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + z(\frac{\lambda}{c_0}t_2 + 1)^{-2-\frac{1}{\lambda}}E$$

$$\tilde{B} = (-t_1 + c + \alpha z)C_1 + (\lambda t_2 + c_0)C_2 - \frac{z}{2}(\lambda + 1)D + zc_0(\frac{\lambda}{c_0}t_2 + 1)^{-1-\frac{1}{\lambda}}E, \tag{117}$$

where $\lambda := B_{21}^{\infty}(b-a) - 1$ and $a, b \in \mathbb{C}$ are defined in Lemma 40.

iii) If $B_{12}^{\infty} B_{21}^{\infty} \neq -\frac{1}{16}$ and $B_{21}^{\infty}(b-a) = 1$, then ∇ is isomorphic to the (TE)-structure $\tilde{\nabla}$ given by

$$\tilde{A}_1 = C_1,\ \tilde{A}_2 = C_2 + z(c_0)^2 e^{-t_2}E$$

$$\tilde{B} = (-t_1 + c + \alpha z)C_1 + C_2 - \frac{z}{2}D + z(c_0)^2 e^{-t_2}E. \tag{118}$$

Proof. The idea of the proof is common to the three cases. Recall that the matrix B of ∇ is given by $B = B^{(0)} + zB^{(1)}$ with $B^{(0)}$, $B^{(1)}$ given by (98) and functions x, y as in Lemma 40. Let T be the gauge isomorphism used in the proof of Proposition 43. Recall that T is given by the first formula in (109) (with various choices of constants k, k_0, k_1, according to the three cases of Proposition 43). The (TE)-structure $\nabla^{[1]} = T\cdot\nabla$ has matrix $B^{[1]}$ given by $B^{[1]} = T^{-1}BT$ and a straightforward computation shows that

$$\begin{aligned} B^{[1]} &= (-t_1 + c + \alpha z)C_1 + \frac{k_0}{k_1}\left(y(k-x)^2 + z(\frac{k}{2} + k^2 B^{\infty}_{21} - c_0)\right)C_2 \\ &+ \frac{z}{2(k-x)}\left((-\frac{k}{2} + 2c_0) - x(2kB^{\infty}_{21} + \frac{1}{2})\right)D \\ &+ \frac{zk_1}{k_0(k-x)^2}\left(c_0 - x(\frac{1}{2} + B^{\infty}_{21}x)\right)E. \end{aligned} \tag{119}$$

The choice of k in the proof of Proposition 43 (in all three cases), implies that $\frac{k}{2} + k^2B^{\infty}_{21} - c_0 = 0$. Therefore, the coefficient of C_2 in $B^{[1]}$ reduces to $\frac{k_0}{k_1}y(k-x)^2$. Define $\tilde{B}$ by $\tilde{B} := (\mu^{-1})^*B^{[1]}$, where $\mu(t_1,t_2) = (t_1, \mu_2(t_2))$ and μ_2 is the automorphism of $(\mathbb{C},0)$ constructed in the proof of Proposition 43 (for each case). The matrix $\tilde{B}$ is obtained by writing $B^{[1]}$ in terms of μ_2 (and t_1). Together with the (T)-structures from Proposition 43, the matrices $\tilde{B}$ form (TE)-structures $\tilde{\nabla}$ isomorphic to ∇ (in all three cases). Their associated functions $\tilde{b}_2$ are obtained by writing the coefficient $b_2^{[1]} = \frac{k_0}{k_1}y(k-x)^2$ of C_2 in the expression of $B^{[1]}$ in terms of μ_2. Making the computations explicit we obtain that $\tilde{\nabla}$ have the expressions stated in Theorem 44.

To ilustrate our argument we consider the case $B^{\infty}_{12}B^{\infty}_{21} = -\frac{1}{16}$. Then, from the proof of Proposition 40, $k = 4c_0$, $\frac{k_0}{k_1} = \frac{1}{16(c_0)^3}$, $\mu_2(t_2) = 1 - e^{-t_2}$ and the underlying (T)-structure of $\tilde{\nabla}$ is given by the first line of (116). Using the expressions for x, y given by (105), we obtain that

$$b_2^{[1]} = \frac{1}{16(c_0)^3}y(4c_0 - x)^2 = e^{-t_2} = 1 - \mu_2(t_2).$$

Thus, $\tilde{b}_2(t_2) = 1 - t_2$, which leads to the matrix $\tilde{B}$ given in (116). □

Definition 45. A holomorphic normal form for (TE)-structures over $\mathcal{N}_2$ is a (TE)-structure which belongs either to the list of (TE)-structures from Theorem 9 or to the list of (TE)-structures from Theorem 44.

The next corollary summarises our discussion on the holomorphic classification.

Corollary 46. *Any (TE)-structure over $\mathcal{N}_2$ is isomorphic to a holomorphic normal form.*

It remains to establish when two holomorphic normal forms ∇ and $\tilde{\nabla}$ are isomorphic. If ∇ and $\tilde{\nabla}$ are isomorphic (and distinct), then they are both elementary or both non-elementary. In the first case, ∇ and $\tilde{\nabla}$ are as in Theorem 9 i) with $c_0 = 0$, ii) or iii). They are isomorphic if and only if they are formally isomorphic and this happens if and only if the conditions from Theorem 10 ii) are satisfied. In the second case, ∇ and $\tilde{\nabla}$ are as in Theorem 9 i) with $c_0 \neq 0$ or as in Theorem 44. We shall associate to ∇ (and $\tilde{\nabla}$) a constant c_1 (respectively, $\tilde{c}_1$) which will be used to establish when ∇ and $\tilde{\nabla}$ are isomorphic. If ∇ is of the form (116), we define $c_1 := -\frac{1}{16c_0}$; if ∇ is of the form (117), we define $c_1 = \frac{1}{16c_0}(4\lambda^2 + 8\lambda + 3)$; if ∇ is of the form (118), we define $c_1 := \frac{3}{16c_0}$. Finally, if ∇ is as in Theorem 9 i) with $c_0 \neq 0$, we define $c_1 := 0$. In a similar way, we assign to $\tilde{\nabla}$ a constant $\tilde{c}_1$.

Corollary 47. *Let ∇, $\tilde{\nabla}$ be two holomorphic normal forms, as in Theorem 9 i) or Theorem 44 (and $c_0\tilde{c}_0 \neq 0$ when ∇ and $\tilde{\nabla}$ belong to Theorem 9 i)). Then ∇ and $\tilde{\nabla}$ are isomorphic if and only if the constants (c, α, c_0) and $(\tilde{c}, \tilde{\alpha}, \tilde{c}_0)$ involved in their expressions satisfy $c = \tilde{c}$, $\alpha = \tilde{\alpha}$, $c_0 = \epsilon\tilde{c}_0$ where $\epsilon \in \{\pm 1\}$ and relations (86) and (87), with c_1 and $\tilde{c}_1$ defined above, are satisfied as well.*

Proof. We claim that the constant c_1 associated to ∇ as above coincides with the $(2,1)$-entry B_{21}^{∞} of the matrix B_∞ from the Malgrange universal deformation $\nabla^{\mathrm{univ},B_0^o,B_\infty}$ isomorphic to ∇ (and similarly for $\tilde{c}_1$ and $\tilde{\nabla}$). This is obvious for the (TE)-structures from Theorem 9 i) with $c_0 \neq 0$ and for the (TE)-structures from Theorem 44 i). For the (TE)-structures from Theorem 44 ii) the claim follows from $B_{12}^{\infty} = c_0$ and $\lambda = B_{21}^{\infty}(b-a) - 1$, which imply

$$4\lambda^2 + 8\lambda + 3 = 4(\lambda+1)^2 - 1 = 16B_{12}^{\infty}B_{21}^{\infty}$$

where we used $a + b = -\frac{1}{2B_{21}^{\infty}}$ and $ab = -\frac{c_0}{B_{21}^{\infty}}$ (see Corollary 40). For the (TE)-structures from Theorem 44 iii) the claim follows from the same argument with $\lambda = 0$. We conclude the proof using Proposition 31. □

5. WHICH EULER FIELDS ON $\mathcal{N}_2$ COME FROM (TE)-STRUCTURES?

This section has two parts. In the first part, we determine normal forms for Euler fields on $\mathcal{N}_2$. It turns out that $\mathcal{N}_2$ has surprisingly many Euler fields. In the second part, we characterize the Euler fields on $\mathcal{N}_2$ which are induced by a (TE)-structure. At first sight, one might expect that all are induced by (TE)-structures (as in the case of the F-manifolds $I_2(m)$ [6]). We will prove that this is not the case.

5.1. **Normal forms for Euler fields on $\mathcal{N}_2$**

THEOREM 48. i) *Up to an automorphism, any Euler field on $\mathcal{N}_2$ is of the form*

$$E = (t_1 + c)\partial_1 + \partial_2 \tag{120}$$

$$E = (t_1 + c)\partial_1 \tag{121}$$

$$E = (t_1 + c)\partial_1 + c_0 t_2 \partial_2, \tag{122}$$

$$E = (t_1 + c)\partial_1 + t_2^r(1 + c_1 t_2^{r-1})\partial_2, \tag{123}$$

where $c, c_1 \in \mathbb{C}$, $c_0 \in \mathbb{C}^*$ and $r \in \mathbb{Z}_{\geq 2}$.

ii) *Any two (distinct) Euler fields from the above list belong to distinct orbits of the natural action of* $\mathrm{Aut}(\mathcal{N}_2)$ *on the space of Euler fields.*

We divide the proof into several steps.

LEMMA 49. i) *A vector field on $\mathcal{N}_2$ is an Euler field if and only if*

$$E = (t_1 + c)\partial_1 + g(t_2)\partial_2, \tag{124}$$

for $c \in \mathbb{C}$ and $g \in \mathbb{C}\{t_2\}$.

ii) *If $g \neq 0$, then $r := \mathrm{ord}_0(g) \geq 0$ is* $\mathrm{Aut}(\mathcal{N}_2)$*-invariant. If $g \neq 0$ is constant, then up to an automorphism, E is of the form (120).*

rm iii) If $g = 0$ then E is of the form (121) and the $\mathrm{Aut}(\mathcal{N}_2)$*-orbit of E reduces to E.*

Proof. i) Let $E := f\partial_1 + g\partial_2$ be a vector field on $\mathcal{N}_2$, where $f, g \in \mathbb{C}\{t_1, t_2\}$. A straightforward computation shows that $L_E(\circ) = \circ$ if and only if $f = t_1 + c$ (with $c \in \mathbb{C}$) and $g \in \mathbb{C}\{t_2\}$, i.e. E is of the form (124).

ii) Let $h(t_1, t_2) = (t_1, \lambda(t_2))$ be an automorphism of $\mathcal{N}_2$, where $\lambda \in \mathrm{Aut}(\mathbb{C}, 0)$ and E an Euler field given by (124). Then

$$(h_*E)_{(t_1,t_2)} = (t_1 + c)\partial_1 + (\dot{\lambda} g) \circ \lambda^{-1}\partial_2. \tag{125}$$

Assume that $g \neq 0$ and let $r := \mathrm{ord}_0(g) \in \mathbb{Z}_{\geq 0}$. Relation (125) and $\lambda \in \mathrm{Aut}(\mathbb{C}, 0)$ implies that r is an invariant of the $\mathrm{Aut}(\mathcal{N}_2)$-action on Euler fields. If $g = g_0$ is a (non-zero) constant, let $h(t_1, t_2) := (t_1, g_0^{-1}t_2)$. Then $h_*E = (t_1 + c)\partial_1 + \partial_2$ is of the form (120).

iii) Claim iii) is obvious from (125). □

The next lemma concludes the proof of Theorem 48 i).

LEMMA 50. *Let E be an Euler field given by (124), with g non-constant and $r = \mathrm{ord}_0(g) \geq 0$.*

i) *If $r = 0$ then, up to an automorphism of $\mathcal{N}_2$, E is of the form (120).*

ii) *If $r = 1$ then, up to an automorphism of $\mathcal{N}_2$, E is of the form (122).*

iii) *If $r \geq 2$ then, up to an automorphism of $\mathcal{N}_2$, E is of the form (123).*

Proof. i) From (125), we need to find $\lambda \in \operatorname{Aut}(\mathbb{C},0)$ such that $(\dot{\lambda}g)(\lambda^{-1}(t)) = 1$, or $\dot{\lambda}(t)g(t) = 1$. Writing $\lambda(t) = t\tilde{\lambda}(t)$ with $\tilde{\lambda} \in \mathbb{C}\{t\}$ a unit, the problem reduces to showing that the differential equation

$$t\dot{\tilde{\lambda}}(t) + \tilde{\lambda}(t) = \frac{1}{g(t)} \tag{126}$$

admits a holomorphic solution with $\tilde{\lambda}(0) \neq 0$. Equation (126) admits a (unique) formal solution $\tilde{\lambda}$, which is holomorphic from Lemma 53. Moreover, $\tilde{\lambda}(0) = \frac{1}{g(0)} \in \mathbb{C}^*$. This proves claim i).

ii) From (125), we need to show that there is $\lambda \in \operatorname{Aut}(\mathbb{C},0)$ such that $(\dot{\lambda}g)(\lambda^{-1}(t)) = c_0 t$ or

$$\dot{\lambda}(t)g(t) = c_0\lambda(t), \tag{127}$$

for a suitable $c_0 \in \mathbb{C}^*$. Writing as before $\lambda(t) = t\tilde{\lambda}(t)$ and $g(t) = \frac{t}{f(t)}$, with $f, \tilde{\lambda} \in \mathbb{C}\{t\}$ units, we obtain that equation (127) is equivalent to

$$t\dot{\tilde{\lambda}}(t) + (1 - c_0 f(t))\tilde{\lambda}(t) = 0. \tag{128}$$

It is easy to check that (128) has a formal solution $\tilde{\lambda} = \sum_{n\geq 0} \tilde{\lambda}_n t^n$ (unique, when $\tilde{\lambda}_0$ is given) if and only if $c_0 = \frac{1}{f(0)}$. Define $c_0 := \frac{1}{f(0)}$ and let $\tilde{\lambda}$ be a formal solution of (128) with $\tilde{\lambda}_0 \neq 0$. From Lemma 53, $\tilde{\lambda}$ is holomorphic. Let $h(t_1, t_2) := (t_1, t_2\tilde{\lambda}(t_2))$. Then h_*E is of the form (122).

iii) From (125), we need to find $\lambda \in \operatorname{Aut}(\mathbb{C},0)$ such that $(\dot{\lambda}g)(\lambda^{-1}(t)) = t^r(1 + c_1 t^{r-1})$ or

$$(\dot{\lambda}g)(t) = \lambda(t)^r(1 + c_1\lambda(t)^{r-1}), \tag{129}$$

for a suitably chosen $c_1 \in \mathbb{C}$. Writing $g(t) = t^r f(t)$, $\lambda(t) = t\tilde{\lambda}(t)$ with $f, \tilde{\lambda} \in \mathbb{C}\{t\}$ units, and $\tau(t) = (1-r)\tilde{\lambda}(t)^{r-1}$, equation (129) becomes

$$t\dot{\tau}(t) + (r-1)\tau(t) = -\frac{\tau(t)^2}{f}(1 + \frac{c_1}{1-r}t^{r-1}\tau(t)). \tag{130}$$

This is an equation, in the unknown function τ, of type (134). Lemma 55 concludes claim iii). □

It remains to prove Theorem 48 ii). From (125), the constant c from the Euler fields of Theorem 48 i) is $\operatorname{Aut}(\mathcal{N}_2)$-invariant. From Lemma 49 we deduce that any two distinct Euler fields E and $\tilde{E}$ from Theorem 48 i), which belong to the same orbit of $\operatorname{Aut}(\mathcal{N}_2)$, are necessarily either both of the form (122) (with the same constant c and distinct constants $c_0, \tilde{c}_0$) or both of the form (123) (with the same constant c and distinct constants c_1, $\tilde{c}_1$). But these cases cannot hold: assume, e.g. that E and $\tilde{E}$ are of the form (123). Then there is a solution τ, with $\tau_0 \neq 0$, of the equation (130) with $f := 1 + \tilde{c}_1 t^{r-1}$. From the uniqueness of the constant c in Lemma 55 we obtain that $c_1 = \tilde{c}_1$. The other case can be treated similarly.

5.2. Euler fields and $(\boldsymbol{TE})$-structures

For a (TE)-structure in pre-normal form, determined by (f, b_2, c, α), the induced Euler field on $\mathcal{N}_2$ is given by

$$E = (t_1 - c)\partial_1 - b_2^{(0)}\partial_2 \tag{131}$$

(see Theorem 6). Recall also that any isomorphism $f : (M_1, \circ_1, e_1, E_1) \to (M_2, \circ_2, e_2, E_2)$ between F-manifolds with Euler fields defines (by the pull-back $(\mathrm{Id} \times f)^*$) an isomorphism between the spaces of (TE)-structures over $(M_2, \circ_2, e_2)$ and $(M_1, \circ_1, e_1)$, which induce E_2 and E_1 respectively. Using these facts, we obtain:

Proposition 51. *An Euler field on $\mathcal{N}_2$ is induced by a (TE)-structure if and only if its normal form is of the type (120), (121), (122) or of the type (123) with $r = 2$ and $c_1 = 0$.*

Proof. The Euler fields induced by the holomorphic normal forms are given by $E = (t_1 - c)\partial_1 + g(t_2)\partial_2$, where $c \in \mathbb{C}$ and $g = 0$, $g = -1$, $g = -t_2^2$ or g is non-constant with $\mathrm{ord}_0(g) \in \{0, 1\}$ (from (131) and the explicit expression of the holomorphic normal forms). Such Euler fields have the normal forms (120), (121), (122) or (123) with $r = 2$ and $c_1 = 0$. □

Remark 52. The Euler fields in normal form (120), (121) and (122) are the Euler fields of a Frobenius manifold with underlying F-manifold $\mathcal{N}_2$ (we may choose a constant metric $g = (g_{ij})$ with $g_{11} = g_{22} = 0$ in the standard coordinate system (t_1, t_2) for $\mathcal{N}_2$, as a Frobenius metric). The Euler fields in normal form (123) with $r = 2$ and $c_1 = 0$ are not Euler fields of a Frobenius manifold. The reason is that in the case of a Frobenius manifold, there is a (TE)-structure which induces the F-manifold with Euler field and which extends to a trivial bundle on $\mathbb{P}^1 \times M$ with a logarithmic pole along $\{0\} \times M$, i.e. which can be brought into a Birkhoff normal form. But the only (TE)-structures over $\mathcal{N}_2$ with an Euler field in normal form (123) with $r = 2$ and $c_1 = 0$ are the (TE)-structures in the second normal form in Theorem 9 iii). At the point $t_1 = t_2 = 0$ and $c = 0$, the matrix B takes the form $B = \alpha z C_1 - z\frac{1}{2}D - z^2 E$. Example 5.5 in [6] shows that this connection cannot be brought into Birkhoff normal form.

6. APPENDIX

6.1. Differential equations

Along this section $t \in (\mathbb{C}, 0)$ is the standard coordinate. We shall use repeatedly the following well-known lemma (see e.g. [25] and the proof of Lemma 12 of [5]).

LEMMA 53. *Any formal solution $u \in \mathbb{C}[[t]]$ of a differential equation of the form*

$$t\dot{u}(t) + A(t)u(t) = b(t), \tag{132}$$

where $A : (\mathbb{C}, 0) \to M_n(\mathbb{C})$ and $b : (\mathbb{C}, 0) \to \mathbb{C}^n$ are holomorphic, is holomorphic.

The next class of inequalities will be used in Lemma 55 below.

LEMMA 54. *Let $C := 4\sum_{n=1}^{\infty}(\frac{1}{n})^2 = \frac{2}{3}\pi^2$. For any $b, l \in \mathbb{Z}_{\geq 2}$ with $b \geq l$,*

$$\sum_{a_i:(*)_{l,b}} (a_1 \cdots a_l)^{-2} \leq C^{l-1} b^{-2}, \tag{133}$$

where the condition $(*)_{l,b}$ on $(a_1, \cdots, a_l)$ means $a_i \in \mathbb{Z}_{\geq 1}$ (for any $1 \leq i \leq l$) and $\sum_{i=1}^{l} a_i = b$.

Proof. We prove (133) by induction on l. Consider first $l = 2$.

$$\sum_{a_1,a_2:(*)_{2,b}} (a_1 a_2)^{-2} = \sum_{a=1}^{b-1} \left(a^{-1}(b-a)^{-1}\right)^2 = \sum_{a=1}^{b-1} \left((a^{-1} + (b-a)^{-1})b^{-1}\right)^2$$

$$= b^{-2} \sum_{a=1}^{b-1} \left(a^{-1} + (b-a)^{-1}\right)^2 \leq 2b^{-2} \sum_{a=1}^{b-1} \left(a^{-2} + (b-a)^{-2}\right) \leq b^{-2} C.$$

Suppose that (133) holds for any $l \leq n-1$. Using (133) for $l = 2$ and $l = n-1$,

$$\sum_{a_1,\ldots,a_n:(*)_{n,b}} (a_1 \cdots a_n)^{-2} = \sum_{b_1,b_2:(*)_{2,b}} \sum_{a_1,\cdots,a_{n-1}:(*)_{n-1,b_1}} (a_1 \cdots a_{n-1} \cdot b_2)^{-2}$$

$$\leq \sum_{b_1,b_2:(*)_{2,b}} C^{n-2}(b_1 b_2)^{-2} \leq C^{n-1} b^{-2},$$

i.e. (133) holds for $l = n$ as well. $\square$

LEMMA 55. *Let $f \in \mathbb{C}\{t\}$ be a unit and $r \in \mathbb{Z}_{\geq 1}$. There is a unique $c \in \mathbb{C}$ such that the differential equation*

$$t\dot{\tau}(t) + r\tau(t) = \tau(t)^2 f(t)(1 + ct^r \tau(t)) \tag{134}$$

admits a formal solution $\tau = \sum_{n\geq 0} \tau_n t^n \in \mathbb{C}[[t]]$ *with* $\tau_0 \neq 0$*. Any such solution* τ *is holomorphic. It is uniquely determined by* $\tau_r \in \mathbb{C}$*, which can be chosen arbitrarily.*

Proof. We write $f = \sum_{n\geq 0} f_n t^n$. Identifying the coefficients in (134) (and using that $\tau_0 \neq 0$) we obtain that τ_n, for $n \in \{1, \cdots, r\}$, are determined inductively by

$$\tau_0 = \frac{r}{f_0},\ \tau_n = \frac{1}{n-r} \sum_{j+k+p=n;\ k,p\leq n-1} f_j \tau_k \tau_p,\ \forall n < r. \tag{135}$$

Identifying the coefficients of t^r in (134) we obtain that c is determined by

$$c = -\frac{1}{\tau_0^3 f_0} \sum_{j+k+p=r;\ k,p\leq r-1} f_j \tau_k \tau_p. \tag{136}$$

It follows that there is a unique $c \in \mathbb{C}$, namely the one defined by (136), for which (134) admits a formal solution τ with $\tau_0 \neq 0$: the coefficient τ_r of τ can be chosen arbitrarily and the remaining coefficients τ_n, for $n \geq r+1$, are determined inductively by

$$\tau_n = \frac{1}{n-r}\left(\sum_{j+k+p=n;\ k,p\leq n-1} f_j \tau_k \tau_p + c \sum_{j+k+p+s=n-r} \tau_j \tau_k \tau_p f_s \right). \tag{137}$$

It remains to prove that τ is holomorphic. Since f is holomorphic, there is $M > 0$ and $\tilde{r} > 0$ such that

$$|f_n| \leq \frac{M\tilde{r}^n}{(n+1)^2},\ \forall n \geq 0. \tag{138}$$

The above relation for $n = 0$ implies that $M \geq |f_0|$. We further assume that $M \geq 1$. We claim that for a suitable choice of M and $\tilde{r}$ satisfying relations (138), the coefficients τ_n of τ satisfy

$$|\tau_n| \leq \frac{M^{n+1}\tilde{r}^n}{(n+1)^2},\ \forall n \geq 0. \tag{139}$$

Remark that for $n = 0$ relation (139) is equivalent to $M \geq |\tau_0| = \frac{r}{|f_0|}$.

To prove the claim, let $n \geq r+1$ be fixed. We assume that (139) holds for all $\tau_0, \cdots, \tau_{n-1}$ and we study when it holds for τ_n. For this, we evaluate, using (137),

$$|\tau_n| \leq \frac{1}{n-r}\left(\sum_{j+k+p=n;\ k,p\leq n-1} |f_j||\tau_k||\tau_p| + |c| \sum_{j+k+p+s=n-r} |\tau_j||\tau_k||\tau_p||f_s| \right)$$

$$\leq \frac{\tilde{r}^n}{n-r} \sum_{j+k+p=n;\ k,p\leq n-1} M^{n-j+3}(j+1)^{-2}(k+1)^{-2}(p+1)^{-2}$$
$$+ \frac{|c|\tilde{r}^{n-r}}{n-r} \sum_{j+k+p+s=n-r} M^{n-r-s+4}(j+1)^{-2}(k+1)^{-2}(p+1)^{-2}(s+1)^{-2}.$$

Since $M \geq 1$, $M^{n-j+3} \leq M^{n+3}$ and $M^{n-r-s+4} \leq M^{n-r+4}$. From Lemma 54, we obtain that

$$|\tau_n| \leq \frac{1}{n-r}\left(\tilde{r}^n M^{n+3} C^2 (n+3)^{-2} + |c| M^{n-r+4} \tilde{r}^{n-r} C^3 (n-r+4)^{-2}\right). \tag{140}$$

We deduce that a sufficient condition for (139) to hold also for τ_n is that

$$M^2 + \frac{C|c|M^{3-r}}{\tilde{r}^r}\left(\frac{n+3}{n-r+4}\right)^2 \leq \frac{1}{C^2}(n-r)\left(\frac{n+3}{n+1}\right)^2. \tag{141}$$

Consider now $\epsilon_0 > 0$ small, $M_0 \geq \max\{\frac{r}{|f_0|}, M\}$ and $n_0 > r$ such that

$$M_0^2 \leq \frac{1}{C^2}(n-r)\left(\frac{n+3}{n+1}\right)^2 - \epsilon_0, \forall n \geq n_0. \tag{142}$$

(This is possible since the right hand side of (142) tends to $+\infty$ for $n \to +\infty$). With this choice of (M_0, n_0, ϵ_0), we choose $\tilde{r}_0 \geq \tilde{r}$ such that

$$\tilde{r}_0^r \geq \frac{C|c|M_0^{3-r}}{\epsilon_0}\left(\frac{n+3}{n-r+4}\right)^2, \ \forall n \geq n_0. \tag{143}$$

(This is possible since the right hand side of (143) is bounded when $n \to +\infty$). Relations (142) and (143) imply

$$M_0^2 + \frac{C|c|M_0^{3-r}}{\tilde{r}_0^r}\left(\frac{n+3}{n-r+4}\right)^2 \leq \frac{1}{C^2}(n-r)\left(\frac{n+3}{n+1}\right)^2, \ \forall n \geq n_0, \tag{144}$$

i.e. relation (141) (with M and $\tilde{r}$ replaced by M_0 and $\tilde{r}_0$ respectively) holds.

The above argument shows that the inequalities

$$|\tau_n| \leq \frac{M_0^{n+1}\tilde{r}_0^n}{(n+1)^2}, \ \forall n \geq 0 \tag{145}$$

hold if they hold for any $n \leq n_0 - 1$ and

$$|f_n| \leq \frac{M_0 \tilde{r}_0^n}{(n+1)^2}, \ \forall n \geq 0. \tag{146}$$

But (146) is obviously true from (138), since $M_0 \geq M$ and $\tilde{r}_0 \geq \tilde{r}$. Relations (145) for $n \leq n_0 - 1$ are satisfied as well, by imposing to $\tilde{r}_0$ (which can be chosen as large as needed) the additional conditions

$$\tilde{r}_0^n \geq \frac{(n+1)^2|\tau_n|}{M_0^{n+1}}, \ \forall 0 \leq n \leq n_0 - 1. \tag{147}$$

From (145), $\tau \in \mathbb{C}\{t\}$. $\square$

The following lemma is used in our formal classification of (TE)-structures. Its proof is straightforward and will be omitted.

Lemma 56. *Consider the system of equations*

$$mx + \dot{b}x - b\dot{x} = g,\ x^{(3)} = 0 \tag{148}$$

in the unknown function $x \in \mathbb{C}\{t\}$, *where* $m \in \mathbb{C}^*$ *and* $g = g_2t^2 + g_1t + g_0$, *for* $g_i \in \mathbb{C}$.

i) *Assume that* $b(t) = \lambda t$, *for* $\lambda \in \mathbb{C}^*$. *If* $m \notin \{\pm\lambda\}$ *then there is a unique solution* x *of (148). If* $m = \lambda$, *then there is a solution of (148) if and only if* $g_2 = 0$. *If* $m = -\lambda$, *then there is a solution of (148) if and only if* $g_0 = 0$.

ii) *Assume that* $b(t) = \lambda t + 1$, *for* $\lambda \in \mathbb{C}$. *If* $m \notin \{\pm\lambda\}$, *then there is a unique solution* x *of (148). If* $m = \lambda$, *then there is a solution of (148) if and only if* $g_2 = 0$. *If* $m = -\lambda$, *then there is a solution of (148) if and only if* $m^2g_0 + mg_1 + g_2 = 0$.

iii) *Assume that* $b(t) = t^2$. *Then there is a unique solution of (148).*

6.2. The Fuchs criterion

For the definition and properties of meromorphic connections with regular singularities, see e.g. [22], Chapter II. Consider a meromorphic connection ∇ on the germ $\mathcal{M} = \boldsymbol{k}^d$ of the meromorphic rank d trivial vector bundle over $(\mathbb{C}, 0)$, with pole at the origin only. Its sections are germs at the origin of vector valued functions $(f_1, \cdots, f_d)$, where each f_i is meromorphic, with pole at the origin only. The Fuchs criterion is an effective way to check if ∇ has a regular singularity at the origin. Namely, one considers a cyclic vector, i.e. a section v_0 such that $\{v_0, v_1 := \nabla_{\partial_z}(v_0), \cdots, v_{d-1} := \nabla^{d-1}_{\partial_z}(v_0)\}$ is a basis of $\mathcal{M}$ (such a vector always exists). In this basis, ∇ has the expression

$$\nabla_{\partial_z}(v_i) = v_{i+1},\ 0 \le i \le d-2 \tag{149}$$

$$\nabla_{\partial_z}(v_{d-1}) = a_0v_0 + \cdots + a_{d-1}v_{d-1}, \tag{150}$$

for some $a_i \in \boldsymbol{k}$. We denote by $v(f)$ the valuation of a function $f \in \boldsymbol{k}$, i.e. the unique integer such that $f(z) = z^{v(f)}h(z)$, where $h \in \mathbb{C}\{z\}$ and $h(0) \neq 0$. The Fuchs criterion is stated as follows (see [20]):

Theorem 57. *The connection* ∇ *has a regular singularity at the origin if and only if* $v(a_i) \ge i - d$, *for any* $0 \le i \le d-1$.

6.3. Irreducible bundles and Birkhoff normal form

Consider a meromorphic connection ∇ on the germ $E = (\mathcal{O}_{(\mathbb{C},0)})^d$ of the holomorphic rank d trivial vector bundle over $(\mathbb{C}, 0)$, with pole of order $r \ge 0$

at the origin. In the standard basis of E, the connection form of ∇ is given by $A(z)dz$, where $A \in M_{d\times d}(\boldsymbol{k})$ is such that $z^{r+1}A(z)$ is holomorphic. We say that (E,∇) can be put in Birkhoff normal form if there is a holomorphic isomorphism $T \in M_{d\times d}(\mathcal{O}_{(\mathbb{C},0)})$ such that the image $T\cdot\nabla$ of ∇ by T has connection form

$$\Omega := z^{-(r+1)}\left(B_0 z^0 + \cdots + z^r B_r\right) dz$$

where B_i are constant matrices.

The irreducibility criterion (see [1] and [22], Chapter IV) provides a sufficient condition for (E,∇) as above to be put in Birkhoff normal form. Let $(\mathcal{M} = \boldsymbol{k}^d, \nabla)$ be the germ of the meromorphic bundle with connection, with singularity at the origin only, for which (E,∇) is a lattice. From the Riemann-Hilbert correspondence (see e.g. [22], page 99), there is a unique (up to isomorphism) meromorphic bundle with connection $(\tilde{\mathcal{M}},\tilde{\nabla})$ on $\mathbb{P}^1$, with poles at 0 and ∞, whose germ at 0 is isomorphic to $(\mathcal{M},\nabla)$ and such that ∞ is a regular singularity for $\tilde{\nabla}$. We say that $(\tilde{\mathcal{M}},\tilde{\nabla})$ is irreducible if there is no proper meromorphic subbundle $\mathcal{N}\to\mathbb{P}^1$ of $\tilde{\mathcal{M}}$, which is preserved by $\tilde{\nabla}$, i.e. $\tilde{\nabla}(\mathcal{N}) \subset \Omega^1_{\mathbb{P}^1}\otimes\mathcal{N}$. The irreducibility criterion states that if $(\tilde{\mathcal{M}},\tilde{\nabla})$ is irreducible, then (E,∇) can be put in Birkhoff normal form: one extends the lattice E around the origin to a globally defined lattice $\tilde{E}$ of $(\tilde{\mathcal{M}},\tilde{\nabla})$, logarithmic at ∞, and applies Corollary 2.6 of [22] (page 154) to obtain a new lattice $\tilde{E}'$ of $\tilde{\mathcal{M}}$, which also extends E, is logarithmic at ∞ and is trivial as a holomorphic vector bundle. A base change between the standard basis of E and a basis of $\tilde{E}'$ in a neighborhood of $0\in\mathbb{C}$ gives the holomorphic isomorphism T above. In particular, for rank 2 bundles we can state:

Lemma 58. *Assume that E is of rank two and let $\{v_1,v_2\}$ be its standard basis. If there is no non-zero $w = gv_1 + fv_2$, with $f,g\in\boldsymbol{k}$, such that $\nabla_{\partial_z}(w) = hw$ for a function $h\in\boldsymbol{k}$, then ∇ can be put in Birkhoff normal form.*

Proof. In the above notation, we claim that $(\tilde{\mathcal{M}},\tilde{\nabla})$ is irreducible: if it were reducible, then a basis of $\mathcal{N}$ around the origin would provide a section w as in the statement of the lemma. We obtain a contradiction. □

6.4. Malgrange universal connections

Let $(H\to\mathbb{C}\times(M,0),\nabla)$ be a (TE)-structure with unfolding condition over a germ $((M,0),\circ,e,E)$ of an F-manifold with Euler field. Assume that the restriction ∇^0 of ∇ to the slice at the origin $\mathbb{C}\times\{0\}$ can be put in Birkhoff normal form. Let $\vec{v}_0$ be a basis of $H|_{(\mathbb{C},0)\times\{0\}}$ in which the connection form of ∇^0 is given by

$$\Omega^0 = \left(\frac{B_0^o}{z} + B_\infty\right)\frac{\mathrm{d}z}{z}, \tag{151}$$

where $B_0^o, B_\infty \in M_{n\times n}(\mathbb{C})$ (and $n = \mathrm{rank}(H) = \dim(M)$). If B_0^o is a regular matrix (i.e. distinct Jordan blocks in its Jordan normal form have distinct eigenvalues, or the vector space of matrices which commute with B_0^o has dimension n, with basis $\{\mathrm{Id}, B_0^o, \cdots, (B_0^o)^{n-1}\}$), then ∇^0 has a universal deformation $\nabla^{\mathrm{univ}} := \nabla^{\mathrm{univ}, B_0^o, B_\infty}$. In particular, ∇^{univ} is isomorphic to the given (TE)-structure ∇ and so are the parameter spaces of ∇^{univ} and ∇ (as F-manifolds with Euler fields). The universal deformation ∇^{univ} was constructed by Malgrange in [18, 19] (see also [22], Chapter VI, Section 3.a; see e.g. [22], page 199, for the definition of the universal deformation). We now recall its definition. Let $\mathcal{D} \subset TM_{n\times n}(\mathbb{C})$ be defined by

$$\mathcal{D}_\Gamma := \mathrm{Span}_{\mathbb{C}}\{\mathrm{Id}, (B_0)_\Gamma, \cdots, (B_0)_\Gamma^{n-1}\} \subset T_\Gamma M_{n\times n}(\mathbb{C}) = M_{n\times n}(\mathbb{C}), \tag{152}$$

where

$$(B_0)_\Gamma := B_0^o - \Gamma + [B_\infty, \Gamma]. \tag{153}$$

Because B_0^o is regular, so is $(B_0)_\Gamma$, for any $\Gamma \in W$, where W is a small open neighborhood of 0 in $M_n(\mathbb{C})$. For any $\Gamma \in W$, $\mathcal{D}_\Gamma$ is the (n-dimensional) vector space of polynomials in $(B_0)_\Gamma$ and the distribution $\mathcal{D} \to W$ is integrable. The parameter space M^{univ} of ∇^{univ} is the maximal integral submanifold of $\mathcal{D}|_W$, passing through $0 \in M_{n\times n}(\mathbb{C})$ (the trivial matrix). Let $\circ_{\mathrm{univ}}$ be the multiplication on TM^{univ}, which, on any tangent space $T_\Gamma M^{\mathrm{univ}} = \mathcal{D}_\Gamma$, is given by multiplication of matrices. It has unit field $e_{\mathrm{univ}} := \mathrm{Id}$ (i.e. $(e_{\mathrm{univ}})_\Gamma := \mathrm{Id} \in \mathcal{D}_\Gamma$, for any $\Gamma \in M^{\mathrm{univ}}$). Let E_{univ} be the vector field on M^{univ} defined by $E_{\mathrm{univ}} := -B_0$ (i.e. $E_\Gamma := -(B_0)_\Gamma$, for any $\Gamma \in M^{\mathrm{univ}}$). Then $(M^{\mathrm{univ}}, \circ_{\mathrm{univ}}, e_{\mathrm{univ}}, E_{\mathrm{univ}})$ is a regular F-manifold (see Definition 2 of [4]). The germ $((M^{\mathrm{univ}}, 0), \circ_{\mathrm{univ}}, e_{\mathrm{univ}}, E_{\mathrm{univ}})$ is universal in the following sense: it is the unique (up to isomorphism) germ of F-manifold with Euler field $((M, 0), \circ, e, E)$ for which the endomorphism $\mathcal{U}(X) := E \circ X$ of T_0M has the same conjugacy class as B_0^o (see [4]). Moreover, if B_0^o has a unique eigenvalue (or a unique Jordan block) then the matrix $(B_0)_\Gamma$, for any $\Gamma \in M^{\mathrm{univ}}$, has this property as well (see Proposition 15 of [4]).

The universal deformation ∇^{univ} of ∇^0 is defined on the trivial bundle $E = (\mathbb{C} \times M^{\mathrm{univ}}) \times \mathbb{C}^n \to \mathbb{C} \times M^{\mathrm{univ}}$. Its connection form in the standard trivialization of E is given by

$$\Omega^{\mathrm{univ}} = \left(\frac{B_0}{z} + B_\infty\right)\frac{\mathrm{d}z}{z} + \frac{\mathcal{C}}{z}. \tag{154}$$

Here $B_0 : M^{\mathrm{univ}} \to M_{n\times n}(\mathbb{C})$, $(B_0)(\Gamma) := (B_0)_\Gamma$ is given by (153) and $\mathcal{C}_X := X$ is the action of the matrix X on $\mathbb{C}^n$, for any $X \in T_\Gamma M^{\mathrm{can}} \subset M_{n\times n}(\mathbb{C})$.

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