Dedicated to Vasile Brînzănescu on his 75^{th} birthday

CONES OF POSITIVE VECTOR BUNDLES

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We introduce and study extensions of the nef cone of divisors to the full numerical ring of a projective manifold. These are convex cones generated by positive and in one case also semistable vector bundles of arbitrary rank. They share some of the properties of the nef cone. Drezet's log-Chern character motivates us to revisit Bogomolov's inequality, and we find a version for threefolds that involves asymptotic cohomological functions.

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1. INTRODUCTION

Let X be a projective variety of dimension n over an algebraically closed field $\mathbf{k}$. A line bundle on X is a vector bundle $\mathcal{L}$ of rank 1. The isomorphism classes of line bundles on X form the Picard group $\operatorname{Pic} X$. The group operation here is tensor product, and inverses are given by taking duals. If X is embedded in a projective space $\mathbb{P}^N$ of rank 1 quotients of $\mathbf{k}^{N+1}$, an example of a nontrivial line bundle is the restriction of the universal quotient bundle $\mathcal{O}_{\mathbb{P}^N}(1)|_X$. Such a line bundle is called very ample. If $\mathcal{L}^{\otimes m}$ is very ample for some $m > 0$, then $\mathcal{L}$ is ample. Ampleness is stable under tensor product. The set of ample elements of $\operatorname{Pic}(X)$ is similar to a convex cone. To get an actual cone, one considers their classes in the Néron–Severi space $N^1(X)$, the $\mathbb{R}$-span of numerical classes of line bundles. The space $N^1(X)$ is finite dimensional, and the ample classes span the open convex cone $\operatorname{Amp}(X)$. Its topological closure is the nef cone $\operatorname{Nef}^1(X)$. It is a fundamental invariant of X, controlling morphisms to other projective varieties. See for example [13, 14].

It is also interesting to consider vector bundles of higher rank. Modulo short exact sequences, they generate the K-theory ring $K(X)$ with operations given by direct sum and tensor product. A prototypical example of a positive bundle is the universal quotient bundle of a Grassmann variety. The standard positivity notions for bundles such as ampleness, global generation, or nefness are again stable under tensor product, but also direct sum. Inspired by the

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rank 1 case, we look for a suitable ambient space for a cone of positive bundles. If $\mathcal{V}$ is a positive vector bundle, we could look at $c_1(\mathcal{V})$, the class of $\det \mathcal{V}$ in $N^1(X)$, but this does not capture enough information about $\mathcal{V}$. Instead we look at all characteristic classes of $\mathcal{V}$ in the graded numerical $\mathbb{R}$-algebra $N^*(X) = \bigoplus_{k=0}^n N^k(X)$, which also verifies $\dim_{\mathbb{R}} N^*(X) < \infty$.

We highlight three ways of packaging the characteristic classes of a vector bundle $\mathcal{V}$ of rank r:

1. The total Chern class $c(\mathcal{V}) = 1 + c_1(\mathcal{V}) + c_2(\mathcal{V}) + \ldots + c_n(\mathcal{V})$.
2. The Chern character $\operatorname{ch} \mathcal{V} = r + \operatorname{ch}_1(\mathcal{V}) + \ldots + \operatorname{ch}_n(\mathcal{V}) = r + c_1(\mathcal{V}) + \frac{c_1^2(\mathcal{V}) - 2c_2(\mathcal{V})}{2} + \ldots$
3. The log-Chern character $\operatorname{lc} \mathcal{V} := \log \operatorname{ch} \mathcal{V} = \log r + \operatorname{lc}_1 \mathcal{V} + \ldots + \operatorname{lc}_n \mathcal{V} = \log r + \frac{c_1(\mathcal{V})}{r} + \frac{2r \operatorname{ch}_2(\mathcal{V}) - c_1^2(\mathcal{V})}{2r^2} + \ldots$

If $\mathcal{L}$ is a line bundle, then $c(\mathcal{L}) = 1 + c_1(\mathcal{L})$, while $\operatorname{ch} \mathcal{L} = 1 + c_1(\mathcal{L}) + \ldots + \frac{1}{n!} c_1^n(\mathcal{L}) = \exp(c_1(\mathcal{L}))$ and $\operatorname{lc} \mathcal{L} = c_1(\mathcal{L})$. The Chern classes, the graded components of the total Chern classes, are the classical way of looking at the characteristic classes of $\mathcal{V}$. However the Chern character and log-Chern character are better suited for convexity purposes. The Chern character determines a morphism of rings $\operatorname{ch} : K(X) \to N^*(X)$, and $\operatorname{lc}(\mathcal{V} \otimes \mathcal{W}) = \operatorname{lc} \mathcal{V} + \operatorname{lc} \mathcal{W}$. We will show (cf. 2.1) that the Chern character morphism is surjective after tensoring by $\mathbb{R}$. Motivated by the additive and multiplicative properties of positive bundles and of the Chern and log-Chern characters, we consider the following cones:

1. $\mathscr{C}(X)$ is the closure of the convex span of monomials of form $\prod_i (\operatorname{ch} \mathcal{V}_i)^{a_i} \in N^*(X)$, where the $\mathcal{V}_i$ are nef vector bundles, and $a_i \in \mathbb{R}_+$. The power $(\operatorname{ch} \mathcal{V}_i)^{a_i}$ is computed formally as a power series.
2. $\operatorname{Nef}(X)$ is the closure in $N^*(X)$ of the convex span of classes $\operatorname{lc} \mathcal{V}$ with $\mathcal{V}$ a nef vector bundle.

The presence of the coefficients a_i guarantees that the formal exponential $\exp : N^*(X) \to N^*(X)$ maps $\operatorname{Nef}(X)$ in $\mathscr{C}(X)$. Both $\mathscr{C}(X)$ and $\operatorname{Nef}(X)$ surject onto the nef cone of divisors $\operatorname{Nef}^1(X)$, therefore both deserve the nomenclature *cone of nef vector bundles* of X. We assign it to $\operatorname{Nef}(X)$, since we also have $\operatorname{Nef}^1(X) \subset \operatorname{Nef}(X)$ via $\operatorname{lc} \mathcal{L} = c_1(\mathcal{L})$ when $\operatorname{rk} \mathcal{L} = 1$. Furthermore, tensor products are the ones that seem to improve positivity (e.g., ample line bundles have globally generated and even very ample high tensor powers), while direct sums only preserve it. Of the two cones, $\operatorname{Nef}(X)$ is the one that focuses on tensor powers.

Example 1.1 (3.3, 4.6). Let X be a smooth projective curve. Then $N^*(X) \simeq \mathbb{R}[x]/(x^2)$ where x is the class of a point. We have $\mathscr{C}(X) = \operatorname{Nef}(X) = \langle 1, x\rangle$.

Example 1.2 (3.10, 4.9). Let X be a smooth projective surface. Then $N^*(X) \simeq \mathbb{R}\cdot 1 \oplus N^1(X) \oplus \mathbb{R}\cdot p$, where p is the class of any point on X. We have $\mathscr{C}(X) = \operatorname{Nef}(X) = \langle 1, \operatorname{Nef}^1(X), \pm p\rangle$. In particular the cones contain the line $\mathbb{R}\cdot p$ through the origin and so they are not pointed. We see in this example that usually the class lc_2 is useful for studying positivity only when seen together with the other lc_k.

Concerning the generators and shape of these cones, we prove extensions of known properties of the nef cone of divisors.

THEOREM 1.3 (3.12, 3.13, 4.10, 4.11). *Let X be a projective variety over an algebraically closed field. Then*

(i) *One can replace nef by ample for the bundles $\mathcal{V}_i$ and $\mathcal{V}$ in the definition of $\mathscr{C}(X)$ and* $\operatorname{Nef}(X)$ *respectively. If the characteristic of the base field is zero, then nef can also be replaced by globally generated.*

(ii) *The cones $\mathscr{C}(X)$ and* $\operatorname{Nef}(X)$ *are full-dimensional, i.e., they generate $N^*(X)$ as a vector space.*

Returning to the log-Chern character $\mathrm{lc}\,\mathcal{V}$, observe that $\mathrm{lc}_1\,\mathcal{V}$ is relevant for computing slopes. Recall that if H is an ample divisor class on X, then the slope of $\mathcal{V}$ with respect to H is $\mu_H(\mathcal{V}) := \frac{c_1(\mathcal{V})\cdot H^{n-1}}{\operatorname{rk}\mathcal{V}}$. Thus $\mu_H(\mathcal{V}) = (\mathrm{lc}_1\,\mathcal{V})\cdot H^{n-1}$. Furthermore, $\mathrm{lc}_2\,\mathcal{V} = -\frac{\Delta(\mathcal{V})}{2r^2}$, where $\Delta(\mathcal{V})$ is the discriminant of $\mathcal{V}$. Both the slope and the discriminant are important invariants for studying slope semistability. Bogomolov's famous inequality states that if $\mathcal{V}$ is slope semistable on a complex projective surface, then $\Delta(\mathcal{V}) \geqslant 0$. Generalizations to arbitrary dimension exist, but usually give information about $\Delta(\mathcal{V})$ only.

We suspect that the higher log-Chern classes play a role in the study or refinement of semistability. We give here an inequality for threefolds. If $\mathcal{V}$ is a vector bundle of rank r, set

$$\breve{h}^i(\mathcal{V}) := \liminf_{m\to\infty} \frac{h^i(X, \operatorname{Sym}^m \mathcal{V})}{m^{n+r-1}/(n+r-1)!}.$$

For $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$, these are conjecturally equal to the asymptotic cohomological functions $\widehat{h}^i(\mathcal{L})$ of [10] where lim sup was used.

THEOREM 1.4 (5.2). *Let (X, H) be a polarized smooth complex projective threefold. Let $\mathcal{V}$ be a μ_H-semistable vector bundle with* $\det \mathcal{V} = 0$. *Then*

$$-\breve{h}^1(\mathcal{V}) \leqslant c_3(\mathcal{V}) \leqslant \breve{h}^2(\mathcal{V}).$$

The proof is analogous to the proof of Bogomolov's inequality in [9, Theorem 3.4.1] or [12, Theorem 1.3.1], which follows [16]. It is also similar to Gieseker's argument by reduction to positive characteristic in [6].

The Bogomolov inequality suggests a fix for $\mathrm{Nef}(X)$ not always being pointed. Assume that (X, H) is a polarized projective manifold, and let $\mathrm{Stab}_H(X)$ be the closure of the convex cone generated in $N^*(X)$ by classes $\mathrm{lc}\,\mathcal{V}$ with $\mathcal{V}$ nef *and* μ_H-semistable. Clearly $\mathrm{Stab}_H(X) \subseteq \mathrm{Nef}(X)$. The cone $\mathrm{Stab}_H(X)$ also projects onto $\mathrm{Nef}^1(X)$ in $N^1(X)$.

When X is a surface, the Bogomolov inequality forces $\mathrm{Stab}_H(X) \subseteq \langle 1, \mathrm{Nef}^1(X), -p\rangle$. In fact equality holds (cf. 4.9). In particular $\mathrm{Stab}_H(X)$ is a pointed cone in this case and the inclusion $\mathrm{Stab}_H(X) \subset \mathrm{Nef}(X)$ is strict.

2. NOTATION, CONVENTIONS, BACKGROUND

Throughout, X denotes a projective variety of dimension n over an algebraically closed field.

2.1. Numerical ring, K-theory, Chern character

Let $A_k(X)$ denote the Chow group of k-cycles with integer coefficients. If $\mathcal{V}$ is a vector bundle on X, then $c_i(\mathcal{V})$ induces an additive map $c_i(\mathcal{V}) \cap \_ : A_k(X) \to A_{k-i}(X)$. The degree map $\deg : A_0(X) \to \mathbb{Z}$ is the additive map determined by $\deg x = 1$ for all points $x \in X$.

Say that $\alpha \in A_k(X)$ is *numerically trivial* if $\deg(P \cap \alpha) = 0$ for all polynomial expressions P of weight k in Chern classes of possibly distinct vector bundles on X. For example, if $\mathcal{V}$ and $\mathcal{W}$ are vector bundles on X, then $c_3(\mathcal{V}) - c_1(\mathcal{V})c_2(\mathcal{W})$ is such an expression of weight 3. The numerical space $N_k(X)$ is the tensor product with $\mathbb{R}$ of the group of numerical equivalence classes of $A_k(X)$. It is a finite dimensional vector space by [5, Example 19.1.4]. Denote $N^k(X) := N_k(X)^{\vee}$. It is naturally generated by polynomial Chern expressions P as above. For example $N^1(X)$ is the real Néron–Severi space. If $\mathcal{V}$ is a vector bundle, its Chern class $c_k(\mathcal{V})$ can be seen as an element of $N^k(X)$.

The graded group $N^*(X) := \bigoplus_{k=0}^{n} N^k(X)$ is naturally a finite dimensional $\mathbb{R}$-algebra, and $N^0(X) \simeq \mathbb{R}$. Multiplication is induced from the multiplication of polynomials. The multiplicative unit 1 is the class of $c_0(\mathcal{O}_X)$ which acts as the identity on all $A_k(X)$. The algebra $N^*(X)$ is functorial for morphisms of projective varieties.

By [5, Example 19.1.5], when X is also smooth, the intersection pairing $A_k(X) \times A_{n-k}(X) \to \mathbb{Z}$ descends to a perfect pairing $N_k(X) \times N_{n-k}(X) \to \mathbb{R}$, and in particular $N^k(X) \simeq N_{n-k}(X)$. In this case, multiplication in $N^*(X)$ is induced from the intersection pairing.

The *Chern character* of a vector bundle $\mathcal{V}$ is the unique natural (for pullbacks) characteristic class $\operatorname{ch}\mathcal{V} \in N^*(X)$ determined by the following properties:

- $\operatorname{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L})) = 1 + c_1(\mathcal{L}) + \frac{1}{2}c_1^2(\mathcal{L}) + \ldots + \frac{1}{n!}c_1^n(\mathcal{L})$ when L is a line bundle,

- $\operatorname{ch}(B) = \operatorname{ch}(A) + \operatorname{ch}(C)$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence.

We have $\operatorname{ch}\mathcal{V} = \operatorname{rk}\mathcal{V} + c_1(\mathcal{V}) + \frac{1}{2}\big(c_1^2(\mathcal{V}) - 2c_2(\mathcal{V})\big) + \ldots$ If $x_1, \ldots, x_r$ are the Chern roots of $\mathcal{V}$, then the degree k part of $\operatorname{ch}\mathcal{V}$ is $\operatorname{ch}_k \mathcal{V} = \frac{1}{k!}\sum_{i=1}^r x_i^k$. Furthermore, $\operatorname{ch}(\mathcal{V} \otimes \mathcal{V}') = \operatorname{ch}\mathcal{V} \cdot \operatorname{ch}\mathcal{V}'$ whenever $\mathcal{V},\ \mathcal{V}'$ are vector bundles.

The *K-theory ring* of X is denoted $K(X)$. It is the free abelian group generated by vector bundles on X modulo relations $[B] = [A] + [C]$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence of vector bundles. The ring structure is induced linearly from the tensor product of vector bundles. The Chern character descends to a morphism of rings $\operatorname{ch} : K(X) \to N^*(X)$.

PROPOSITION 2.1. *Let X be a projective variety over an algebraically closed field. Then the induced Chern character morphism* $\operatorname{ch} : K(X) \otimes_{\mathbb{Z}} \mathbb{R} \to N^*(X)$ *is surjective. In particular $N^*(X)$ is generated linearly by* $\operatorname{ch}\mathcal{V}$ *with $\mathcal{V}$ ranging though the vector bundles on X.*

Proof. Assume first that X is also smooth. In this case coherent sheaves on X admit finite resolutions by vector bundles. Consequently the $K(X)$ theory ring is isomorphic as a group with the K-theory group of coherent sheaves modulo short exact sequences. In the smooth case, the ring $N^*(X)$ is the usual space of cycles modulo numerical equivalence (cf. [5, Example 19.1.5]). A form of Grothendieck-Riemann-Roch ([5, Example 15.2.16]) says that the Chern character map $\operatorname{ch} : K(X) \to A^*(X)$ is an isomorphism after tensoring by $\mathbb{Q}$, where $A^*(X)$ is the Chow ring of X. Since the class map $A^*(X) \otimes_{\mathbb{Z}} \mathbb{R} \to N^*(X)$ is clearly onto, the conclusion follows.

Assume now that X is an arbitrary projective variety. We claim that there exists an embedding $\imath : X \hookrightarrow Y$ into a smooth projective variety Y such that $\imath^* : N^*(Y) \to N^*(X)$ is surjective.

Since $N^*(X)$ is finitely generated, there exist $\mathcal{V}_1, \ldots, \mathcal{V}_m$, finitely many bundles, such that $N^*(X)$ is generated by all monomials in all Chern classes of the $\mathcal{V}_i$. Let H be a very ample divisor on X such that $\mathcal{V}_i(H)$ is globally generated for all i. Each $\mathcal{V}_i(H)$ induces a Gauss map $\gamma_i : X \to \mathbb{G}_i$, where $\mathbb{G}_i$

is the Grassmann variety of rk $\mathcal{V}_i$-dimensional quotients of $H^0(X, \mathcal{V}_i(H))$. Let $f : X \hookrightarrow \mathbb{P}^N$ be the embedding induced by H. Let $Y := \mathbb{P}^N \times \mathbb{G}_1 \times \ldots \times \mathbb{G}_m$, and let $\imath := (f, \gamma_1, \ldots, \gamma_m)$. Let $\mathcal{Q}_i$ be the pullback to Y of the universal quotient bundle on $\mathbb{G}_i$. Let A be the pullback to Y of the hyperplane class on $\mathbb{P}^N$. We have $\mathcal{V}_i(H) = \imath^*\mathcal{Q}_i$ and $H = \imath^* A$. The monomials in Chern classes of the $\mathcal{V}_i$ are in the linear span of monomials in the Chern classes of $\mathcal{O}_X(H), \mathcal{V}_1(H), \ldots, \mathcal{V}_m(H)$. See [5, Example 3.2.2]. By naturality, these are all pulled back from Y. This proves the claim.

Since Y is smooth, by finite dimensionality there exist finitely many bundles $\mathcal{W}_j$ on Y such that $N^*(Y)$ is generated by $\operatorname{ch}\mathcal{W}_j$. Then $N^*(X)$ is generated by $\operatorname{ch}\imath^*\mathcal{W}_j$. $\square$

2.2. **Exponential and logarithm**

Let $S = \bigoplus_{k=0}^n S_k$ be a finite dimensional graded commutative $\mathbb{R}$-algebra with $S_0 = \mathbb{R}$. For $v \in S$ with graded components $v_k \in S_k$, denote by $v_+ := \sum_{k=1}^n v_k = v - v_0$ the part of positive degree. We have $v_+^{n+1} = 0$, so v_+ is nilpotent. In particular $1/v$ exists if and only if $v_0 \neq 0$.

We also call v_0 the rank $\operatorname{rk} v$, inspired by $\operatorname{ch}\mathcal{V} = \operatorname{rk}\mathcal{V} + c_1(\mathcal{V}) + \ldots$ if $\mathcal{V}$ is a vector bundle.

The *formal exponential* of v is $\exp v := e^{v_0} \cdot \sum_{k=0}^n \frac{1}{k!} v_+^k \in S$. It verifies $\exp(v + w) = \exp v \cdot \exp w$.

Denote $H := \{v \in S \mid v_0 > 0\}$. If $v \in H$, the *formal logarithm* of v is

$$\log v := \log v_0 + \sum_{k=1}^n (-1)^{k-1} \frac{1}{k} \left(\frac{v_+}{v_0}\right)^k.$$

It verifies $\exp \log v = v$ and $\log(v \cdot w) = \log v + \log w$, for all $v, w \in H$. Furthermore $\log \exp v = v$ for all $v \in S$.

By looking at coordinates with respect to a basis of S, one sees that the operations on S are polynomial. Then $\log : H \to S$ is a diffeomorphism with inverse exp. The algebraic/analytic structure on S is that of an affine space over $\mathbb{R}$, and H is an open subset with the induced structure.

If $v \in H$, and a is a real number, we can also define the a-th *formal power* of v as

$$v^a := \exp(a \log v).$$

2.3. **Positivity and twists**

The projectivization of a coherent sheaf $\mathcal{V}$ on X is $\mathbb{P}(\mathcal{V}) = \operatorname{Proj}\operatorname{Sym}^*\mathcal{V}$ with natural projection $\pi : \mathbb{P}(\mathcal{V}) \to X$. Recall that if $\mathcal{L}$ is a line bundle, then $\mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{V} \otimes \mathcal{L})$, and $\mathcal{O}_{\mathbb{P}(\mathcal{V}\otimes\mathcal{L})}(1) = \mathcal{O}_{\mathbb{P}(\mathcal{V})} \otimes \pi^*\mathcal{L}$.

A coherent sheaf $\mathcal{V}$ is called *ample (or nef)* if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is ample (respectively nef). If $\mathcal{V}$ is a vector bundle, we also have that $\mathcal{V}$ is globally generated if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$ is globally generated.

Let $L \in \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The *formal twist* of $\mathcal{V}$ by L is the pair $(\mathcal{V}, L)$, and it is denoted $V\langle L\rangle$. As in [14, Section 6.2], we generally do not attach a more concrete meaning to the formal twists. When $L \in \operatorname{Pic}(X)$ is Cartier, then $V\langle L\rangle$ is seen as $V \otimes L$. We can twist twists by $(\mathcal{V}\langle L\rangle)\langle L'\rangle := \mathcal{V}\langle L + L'\rangle$. With a similar formula, we tensor twisted bundles.

Define $\mathcal{O}_{\mathbb{P}(\mathcal{V}\langle L\rangle)}(1) := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)\langle \pi^* L\rangle$. With this, ampleness and nefness for sheaves extend formally to twists. If L is ample, then $\mathcal{V}$ is nef if and only if $\mathcal{V}\langle \epsilon L\rangle$ is ample for all $\epsilon > 0$.

If $\mathcal{V}$ is a vector bundle, define $\operatorname{ch}(\mathcal{V}\langle L\rangle) := \operatorname{ch}\mathcal{V} \cdot \exp L$, where by abuse we identify L with its class in $N^1(X)$.

2.4. Semistable bundles

Let X be a smooth projective variety of dimension n over an algebraically closed field of characteristic zero. Let H be an ample divisor on X. If $\mathcal{F}$ is a coherent torsion-free sheaf on X, its slope with respect to H is

$$\mu(\mathcal{F}) = \mu_H(\mathcal{F}) = \frac{\left(c_1(\mathcal{F}) \cdot H^{n-1}\right)}{\operatorname{rk}\mathcal{F}}.$$

The Chern classes of coherent sheaves $\mathcal{F}$ on projective manifolds can be defined by using the multiplicativity of the total Chern class in exact sequences, and the existence of finite resolutions of $\mathcal{F}$ by vector bundles. For example, if $D \subset X$ is an effective divisor, then $c(\mathcal{O}_D) = c(\mathcal{O}_X)/c(\mathcal{O}_X(-D)) = 1 + D + (D^2) + \dots$

A coherent torsion-free sheaf $\mathcal{V}$ on X is said to be *μ-semistable* if $\mu(\mathcal{V}) \geqslant \mu(\mathcal{F})$ for all nonzero coherent $\mathcal{F} \subseteq \mathcal{V}$. If $\mathcal{L}$ is a line bundle, then $\mathcal{V}$ is semistable iff $\mathcal{V} \otimes \mathcal{L}$ is semistable. Inspired by this, we say that a twisted bundle $\mathcal{V}\langle L\rangle$ is semistable iff $\mathcal{V}$ is.

The Mehta-Ramanathan theorem implies that a twisted bundle is semistable if and only if its restriction to a general complete intersection of high degree is semistable. On curves, semistability can be reduced by a suitable twist to twisted bundles of degree 0, where by Hartshorne's theorem [14, Theorem 6.4.15] semistability is equivalent to nefness. As a consequence, semistability is homogeneous for tensor products: If $\mathcal{V}$ is a vector bundle and $m \geqslant 1$, then $\mathcal{V}$ is semistable iff $\mathcal{V}^{\otimes m}$ is. Note that $\mu(\mathcal{V} \otimes \mathcal{W}) = \mu(\mathcal{V}) + \mu(\mathcal{W})$ for any twisted bundles $\mathcal{V}$ and $\mathcal{W}$.

The Bogomolov inequality states that if $\mathcal{V}$ is semistable on a smooth complex projective surface, then $2\operatorname{rk}\mathcal{V} \cdot \operatorname{ch}_2 \mathcal{V} - c_1^2(\mathcal{V}) \leqslant 0$. It is also valid for twisted bundles.

Versions of the results above also exist in positive characteristic at the cost of replacing semistability by *strong semistability*, a notion that also takes into account Frobenius pullbacks. See [11] for details.

3. THE CONE OF POSITIVE CHERN CHARACTERS

The nefness of vector bundles, and even of coherent sheaves, is preserved under direct sums and tensor product (e.g., by [Corollary 3.31, Lemma 3.32][4]). A direct sum of bundles is nef iff each summand is nef. If $\mathcal{V}$ is a bundle, then $\mathcal{V}^{\otimes m}$ is nef iff $\mathcal{V}$ is nef, as follos from [4, Lemma 3.25, Theorem 3.12]. It then natural to put structure on the set of isomorphism classes of nef bundles. However this is quite a large space. The classes of nef bundles in the K-theory ring $K(X)$ form a set that is closed under addition and multiplication. While $K(X)_{\mathbb{R}} := K(X) \otimes_{\mathbb{Z}} \mathbb{R}$ may also be infinitely dimensional, it has a natural quotient ring $\mathrm{ch} : K(X)_{\mathbb{R}} \to N^*(X)$ that is a finite dimensional $\mathbb{R}$-algebra. These motivate the consideration of the following convex cone.

Definition 3.1. Let X be a projective variety over an algebraically closed field. The cone of positive Chern characters $\mathscr{C}(X)$ of $N^*(X)$ is the closure of the convex span of expressions of form

$$\prod_{i=1}^{m} (\mathrm{ch}\, \mathcal{V}_i)^{a_i},$$

where $m \in \mathbb{N}^*$, where $\mathcal{V}_i$ are nef vector bundles on X, and $a_i \in \mathbb{R}_+$.

Remark 3.2. The projection of $N^*(X)$ onto $N^1(X)$ maps $\mathscr{C}(X)$ onto the usual nef cone $\mathrm{Nef}^1(X)$. (Indeed the degree 1 part of $\prod_i (\mathrm{ch}\, \mathcal{V}_i)^{a_i}$ is a positive combination of $c_1(\det \mathcal{V}_i)$. Conversely, if $\lim_{m\to\infty} a_m c_1(D_m) = \alpha \in \mathrm{Nef}^1(X)$ with $a_m \in \mathbb{Q}_+$, then $(\mathrm{ch}\, \mathcal{O}(D_m))^{a_m}$ limits to $\exp \alpha$ which projects onto α in $N^1(X)$.)

Example 3.3 (Curves). If X is a smooth projective curve, then $N^*(X) \simeq \mathbb{R}[x]/(x^2)$, where x is the class of any point. Identifying $N^*(X)$ as vector space with $\mathbb{R}^2$, we have $\mathscr{C}(X) = \mathbb{R}^2_{\geqslant 0}$. Generators are $1 = \mathrm{ch}(\mathcal{O}_X)$ and x seen as limit of $\frac{1}{m}\,\mathrm{ch}(\mathcal{O}_X(mx)) = \frac{1}{m} + x$.

Example 3.4 (Totally split bundles). If $L_1, \ldots, L_r$ are nef line bundles on the projective variety X, then $\mathcal{V}_m = \mathcal{L}_1^{\otimes m} \oplus \ldots \oplus \mathcal{L}_r^{\otimes m}$ is nef, hence $\mathrm{ch}\, \mathcal{V}_m = \exp(mc_1(\mathcal{L}_1)) + \ldots + \exp(mc_1(\mathcal{L}_r)) \in \mathscr{C}(X)$.

Example 3.5 (Duals of kernel bundles). Let H be a globally generated line bundle on X. Let N_H be the dual of the kernel of the evaluation map

$$H^0(X, \mathcal{O}_X(H)) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(H).$$

It is a globally generated bundle, and

$$\operatorname{ch} N_H = h^0(X, \mathcal{O}_X(H)) - \exp(-H) \in \mathscr{C}(X).$$

Example 3.6 (Generic cokernels). Let X have dimension n, and consider H an ample divisor on X. Let F be any vector bundle of rank $n + r$ on X, where $r \geqslant n$. By [14, Theorem 6.3.65], for all large d, the cokernel of a general map $\mathcal{O}_X(-dH)^{\oplus n} \to F$ is an ample bundle $\mathcal{V}$ of rank r. In particular

$$\operatorname{ch} F - n \exp(-dH) \in \mathscr{C}(X).$$

After scaling and taking limits, we find $(-1)^{n+1}(H^n) \in \mathscr{C}(X)$. $\square$

Example 3.7 (Bundles of large rank). Let $\mathcal{V}$ be a globally generated vector bundle of rank $r > n = \dim X$. Then

$$\operatorname{ch} \mathcal{V} - (r - n) \in \mathscr{C}(X).$$

(By a Bertini type argument (cf. [8, Exercise II.8.2]), if $W \subset H^0(X, \mathcal{V})$ is a general subspace of dimension $r - n$, then the cokernel of the evaluation map $W \otimes \mathcal{O}_X \to \mathcal{V}$ is a globally generated bundle of rank n. Its Chern character is $\operatorname{ch} \mathcal{V} - (r - n)$.) $\square$

Corollary 3.8. *If $\mathcal{V}$ is a globally generated vector bundle, then* $\operatorname{ch}_+ \mathcal{V} \in \mathscr{C}(X)$.

Proof. We have $\operatorname{ch}(\mathcal{V}^{\oplus m}) = m \operatorname{ch} \mathcal{V}$. From Example 3.7, $n + m \operatorname{ch}_+ \mathcal{V} \in \mathscr{C}(X)$ for large m. The conclusion follows after scaling by $\frac{1}{m}$ and taking limits. $\square$

At least when the dimension of the ambient space is even, the cone $\mathscr{C}(X)$ may not be the best extension of $\operatorname{Nef}^1(X)$ in $N^*(X)$.

Corollary 3.9. *If* $\dim X$ *is even, then* $\mathscr{C}(X)$ *is not a pointed cone, i.e., it contains lines through the origin.*

Proof. Let H be an ample divisor on X. Example 3.6 shows that $-(H^n) \in \mathscr{C}(X)$. After scaling by $d^n/n!$, the classes $\operatorname{ch}(\mathcal{O}_X(dH)) \in \mathscr{C}(X)$ approach (H^n) as d grows. $\square$

Example 3.10 (Surfaces). Let X be a smooth projective surface. Then $N^*(X) = \mathbb{R}\cdot 1 \oplus N^1(X) \oplus \mathbb{R}\cdot p$, where p is the class of any point. We show

$$\mathscr{C}(X) = \langle 1, \operatorname{Nef}^1(X), \pm p \rangle.$$

First, we have $1 = \operatorname{ch} \mathcal{O}_X \in \mathscr{C}(X)$. Let h be a very ample divisor class on X From the proof of Corollary 3.9, we obtain $\pm h^2 \in \mathscr{C}(X)$. From Corollary 3.8, we deduce $h + \frac{1}{2}h^2 \in \mathscr{C}(X)$. But $-h^2 \in \mathscr{C}(X)$ now implies $h \in \mathscr{C}(X)$.

The arguments above prove $\mathscr{C}(X) \supseteq \langle 1, \mathrm{Nef}^1(X), \pm p \rangle$. The reverse inclusion is clear. $\square$

The cone $\mathscr{C}(X)$ does share some of the other useful properties of $\mathrm{Nef}^1(X)$.

Remark 3.11. If $f : X \to Y$ is a morphism of projective varieties, then $f^*\mathscr{C}(Y) \subseteq \mathscr{C}(X)$. (Indeed pullbacks preserve nefness and commute with characteristic classes.)

PROPOSITION 3.12. *The cone $\mathscr{C}(X)$ is full dimensional, i.e., it generates $N^*(X)$ as a vector space.*

Proof. By Proposition 2.1, $N^*(X)$ is generated by $\mathrm{ch}\,\mathcal{V}$ for all vector bundles $\mathcal{V}$, not necessarily nef. Let $\mathcal{L}$ be an ample line bundle on X. There exists $t_0 \geqslant 0$ such that $\mathcal{V} \otimes \mathcal{L}^{\otimes t}$ is globally generated, hence nef, for all $t \geqslant t_0$. Since $\mathrm{ch}(\mathcal{V} \otimes \mathcal{L}^{\otimes t}) = \mathrm{ch}\,\mathcal{V} \cdot \exp(t \cdot c_1(\mathcal{L}))$, it is enough to prove that 1 is in the linear span of $\{\exp(t \cdot c_1(\mathcal{L})) \mid t \geqslant t_0\}$. From $\exp(t \cdot c_1(\mathcal{L})) = \sum_{k=0}^{n} \frac{t^k}{k!} c_1^k(\mathcal{L})$, using that the Vandermonde matrix $((t_0+i)^j)_{i,j \in \{0,1,\ldots,n\}}$ is invertible, we deduce the result. $\square$

PROPOSITION 3.13. *Let X be a projective variety. Up to closure, the monomials $\prod_{i=1}^{m} (\mathrm{ch}\,\mathcal{V}_i)^{a_i}$ generate $\mathscr{C}(X)$ where $m \in \mathbb{N}^*$, $a_i \in \mathbb{R}_+$, and the $\mathcal{V}_i$ are all*

(i) *nef vector bundles;*

(ii) *ample vector bundles;*

(iii) *(in characteristic zero) globally generated vector bundles;*

(iv) *twisted nef bundles;*

(v) *twisted ample bundles;*

Proof. (i) is the definition of $\mathscr{C}(X)$. For (ii), let $\prod_i (\mathrm{ch}\,\mathcal{V}_i)^{a_i}$ with $\mathcal{V}_i$ nef. Let H be an ample divisor. For all $t \in \mathbb{N}^*$, the bundle $\mathcal{V}_i^{\otimes t}(H)$ is ample. The class $\prod_i (\mathrm{ch}\,\mathcal{V}_i)^{a_i}$ is the limit of the classes $\prod_i (\mathrm{ch}(\mathcal{V}_i^{\otimes t}(H)))^{a_i/t}$.

In characteristic zero, if $\mathcal{V}$ is ample, then $\mathcal{V}^{\otimes m}$ is globally generated for all sufficiently large m. Indeed $\mathcal{V}^{\otimes m}$ is a direct sum of (co)Schur functors indexed by partitions of m with at most $r = \mathrm{rk}\,\mathcal{V}$ parts. Each such functor is a direct summand of $\mathrm{Sym}^m(\mathcal{V}^{\oplus r})$. Since $\mathcal{V}$ is ample, so is $\mathcal{V}^{\oplus r}$, hence $\mathrm{Sym}^m(\mathcal{V}^{\oplus r})$ is globally generated for all sufficiently large m. This settles (iii) using (ii) since $(\mathrm{ch}\,\mathcal{V}_i)^{a_i} = (\mathrm{ch}(\mathcal{V}_i^{\otimes m}))^{a_i/m}$.

Assume $\mathcal{V}\langle L \rangle$ is a nef twisted bundle, where L is an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. Let H be an ample divisor. Let $\epsilon > 0$ The twisted bundle $\mathcal{V}^{\otimes m}\langle m(L + \epsilon H) \rangle$

is ample by [4, Lemma 3.29, Corollary 3.31]. Write $L = \sum_i a_i L_i$ where $a_i \in \mathbb{R}$ and L_i are Cartier divisors. Let $\lfloor m(L + \epsilon H)\rfloor := \sum_i \lfloor m a_i\rfloor + \lfloor m\epsilon\rfloor$. The vector bundle $\mathcal{V}^{\otimes m} \otimes \mathcal{O}_X(\lfloor m(L+\epsilon H)\rfloor)$ is ample for large m, since ampleness is open (if $\mathcal{V}$ is a bundle, then the set of classes $[L] \in N^1(X)$ such that $\mathcal{V}\langle L\rangle$ is ample is open) and $\lim_{m\to\infty} \frac{1}{m}\lfloor m(L+\epsilon H)\rfloor = L + \epsilon H$. We also have $\lim_{m\to\infty} \operatorname{ch}(\mathcal{V}^{\otimes m} \otimes \mathcal{O}_X(\lfloor m(L+\epsilon H)\rfloor))^{1/m} = \operatorname{ch}(\mathcal{V}\langle L + \epsilon H\rangle)$. Part (iv) follows by shrinking ϵ to 0. Part (v) is similar. $\square$

In positive characteristic p, in order to involve global generation, it is sufficient to replace ampleness by the stronger cohomological Γ-ampleness, and nefness by the corresponding limiting condition.

Remark 3.14. In a weak sense, the cone $\mathscr{C}(X)$ is also preserved by pushforward. Let $f : X \to Y$ be a flat morphism of complex projective manifolds. Let $\mathcal{V}$ be an ample vector bundle on X. For large $m > 0$ we have:

$$f_*\big((\operatorname{ch}\mathcal{V})^m \operatorname{td} f\big) = \operatorname{ch}\big(f_*(\mathcal{V}^{\otimes m})\big) \in \mathscr{C}(Y),$$

where $\operatorname{td} f$ is the formal fraction $\operatorname{td} X / f^* \operatorname{td} Y$, which agrees with the Todd class of the relative tangent space when f is smooth. (The idea is to prove that for large enough m we have $R^i f_*(\mathcal{V}^{\otimes m}) = 0$ for $i > 0$, and $f_*(\mathcal{V}^{\otimes m})$ is an ample and globally generated vector bundle. The statement is then a consequence of the Grothendieck-Riemann-Roch theorem.)

An analogue of this result for the usual nef cone of divisors is the following:

Proposition 3.15. *If $f : X \to Y$ is a surjective equidimensional morphism of relative dimension d of smooth projective varieties, and if H is an ample divisor class on X, then $f_*((H^{d+1}))$ is an ample divisor class on Y.*

The proposition follows from the previous remark by setting $\mathcal{V} = \mathcal{O}_X(H)$ and letting m grow. We also include a direct proof for comparison.

Proof. Since f is equidimensional, a general hyperplane section of large enough degree of X is still equidimensional over Y by [3, Lemma 4.9]. Such a hyperplane is also smooth by Bertini. By induction we reduce to the case where $d = 0$. Then f is a finite covering between smooth varieties, hence it is flat. Flat pullbacks preserves effectivity. We conclude by the Moishezon–Nakai criterion and the projection formula. $\square$

Remark 3.16 (Semistability). We have seen (e.g., Corollary 3.9) that $\mathscr{C}(X)$ does not satisfy all the properties of $\operatorname{Nef}^1(X)$. One possible attempt to fix this could be to shrink it further by refining the generating set. Let

(X, H) be a polarized projective manifold. All line bundles on X are semistable with respect to any ample polarization. Instead of considering all nef bundles, we could focus on the semistable ones.

Denote by $\mathscr{C}_H(X)$ the closure in $N^*(X)$ of the cone generated by products $\prod_{i=1}^m \operatorname{ch}(\mathcal{V}_i)^{a_i}$ where $\mathcal{V}_i$ are μ_H-semistable and nef, and $a_i \in \mathbb{R}_+$. This is not a convex cone. Semistability is preserved by tensor products, but not by direct sums unless the summands have the same slope.

By [15], the cone $\mathscr{C}_H(X)$ generates $N^*(X)$ as a vector space.

We do get a convex cone if we fix the slope. When the slope is zero, or equivalently $c_1(\mathcal{V}) = 0$, the resulting convex cone $\mathscr{C}_H^0(X)$ is also closed under products.

Remark 3.17 (Why not positive coherent sheaves?). Tensor products of coherent sheaves preserve positivity. This is not observed by the Chern character, which is no longer multiplicative. In fact, if $\mathcal{V}$ and $\mathcal{W}$ are coherent sheaves on a projective manifold, then $\operatorname{ch}\mathcal{V}\cdot \operatorname{ch}\mathcal{W} = \sum_{i\geqslant 0}(-1)^i \operatorname{ch}\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{V}, \mathcal{W})$.

4. THE CONE OF POSITIVE VECTOR BUNDLES

Definition 4.1. Let $\mathcal{V}$ be a nonzero (twisted) vector bundle on X, so that $\operatorname{ch}_0(\mathcal{V}) = \operatorname{rk}\mathcal{V} > 0$. Following Drezet, define the log-Chern character of $\mathcal{V}$ as the formal logarithm of the Chern character

$$\operatorname{lc}\mathcal{V} := \log \operatorname{ch}\mathcal{V} = \log \operatorname{rk}\mathcal{V} + \sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\left(\frac{\operatorname{ch}_+\mathcal{V}}{\operatorname{rk}\mathcal{V}}\right)^k .$$

If $\mathcal{V}$ and $\mathcal{V}'$ are vector bundles, then $\operatorname{lc}(\mathcal{V}\otimes\mathcal{V}') = \operatorname{lc}\mathcal{V} + \operatorname{lc}\mathcal{V}$
prime.

Example 4.2. If $\mathcal{L}$ is a line bundle, then $\operatorname{lc}\mathcal{L} = \log\operatorname{ch}\mathcal{L} = \log\exp c_1(\mathcal{L}) = c_1(\mathcal{L})$.

Remark 4.3. Denote $r := \operatorname{rk}\mathcal{V}$, and let $\operatorname{lc}_k\mathcal{V}$ denote the degree k component of $\operatorname{lc}\mathcal{V}$. Then

$$\operatorname{lc}_0\mathcal{V} = \log r$$

$$\operatorname{lc}_1\mathcal{V} = \frac{c_1(\mathcal{V})}{r}$$

$$\operatorname{lc}_2\mathcal{V} = \frac{2\operatorname{ch}_0\mathcal{V}\operatorname{ch}_2\mathcal{V} - \operatorname{ch}_1^2\mathcal{V}}{2r^2} = -\frac{2r\cdot c_2(\mathcal{V}) - (r-1)\cdot c_1^2(\mathcal{V})}{2r^2} = -\frac{\Delta(\mathcal{V})}{2r^2},$$

where $\Delta(\mathcal{V})$ is the discriminant of $\mathcal{V}$. Furthermore

$$\operatorname{lc}_3\mathcal{V} = \frac{3r^2\cdot\operatorname{ch}_3\mathcal{V} - 3r\cdot\operatorname{ch}_1\mathcal{V}\operatorname{ch}_2\mathcal{V} + \operatorname{ch}_1^3\mathcal{V}}{3r^3}$$

$$= \frac{(r-1)(r-2)\cdot c_1^3(\mathcal{V}) - 3r(r-2)\cdot c_1(\mathcal{V})c_2(\mathcal{V}) + 3r^2\cdot c_3(\mathcal{V})}{6r^3}.$$

In particular $\operatorname{lc}_3 \mathcal{V} = 0$ when $r \in \{1, 2\}$.

Example 4.4. If $\mathcal{V}$ is a vector bundle, then $\operatorname{lc}(\mathcal{E}nd\,\mathcal{V}) = \operatorname{lc}(\mathcal{V} \otimes \mathcal{V}^\vee) = \operatorname{lc}(\mathcal{V}) + \operatorname{lc}(\mathcal{V}^\vee) = 2\sum_{k\geqslant 0} \operatorname{lc}_{2k} \mathcal{V}$.

If $\mathcal{V}$ and $\mathcal{V}'$ are ample/nef/globally generated vector bundles, then so is $\mathcal{V} \otimes \mathcal{V}'$. The same holds true for the ampleness and for the nefness of twisted bundles. Through lc, positivity becomes an additive property. In characteristic zero, μ-semistability of bundles is presevered by tensor product. Then μ-semistability is also additive for lc.

Definition 4.5. Let X be a projective variety over an algebraically closed field. The closure of the convex span in $N^*(X)$ of the classes $\operatorname{lc}\mathcal{V}$ for all *nef* vector bundles $\mathcal{V}$ on X is the *cone of positive vector bundles* $\operatorname{Nef}(X) \subset N^*(X)$.

Let now (X, H) be a polarized complex projective manifold. Consider the closure of the convex span in $N^*(X)$ of classes $\operatorname{lc}\mathcal{V}$, where $\mathcal{V}$ is an μ_H*-slope semistable* and *nef* vector bundle on X. Denote it $\operatorname{Stab}_H(X)$, or $\operatorname{Stab}(X)$ when no confusion may arise.

Example 4.6 (Curves). Let C be a smooth projective curve so $N^*(C) = \mathbb{R}[x]/(x^2)$, where x is the class of any point. By fixing $x \in C$ and considering the nef bundles $\mathcal{O}_C(x)^{\oplus r}$ whose log-Chern characters are $\log r + x$, we see that in fact $\operatorname{Nef}(C) = \operatorname{Stab}_H(C) = \mathbb{R}^2_{\geqslant 0}$ for any ample H.

Remark 4.7. If $\mathcal{L}$ is a line bundle, then $\operatorname{lc}\mathcal{L} = c_1(\mathcal{L})$. We deduce that $\operatorname{Nef}^1(X) \subset \operatorname{Nef}(X)$ and $\operatorname{Nef}^1(X) \subset \operatorname{Stab}_H(X)$. In the previous example we see that $\operatorname{Nef}^1(X)$ could be a face of $\operatorname{Nef}(X)$ and of $\operatorname{Stab}_H(X)$. In general, $\operatorname{Nef}^1(X)$ is contained in the supporting hyperplane $\{0\} \oplus N^1(X) \oplus \ldots \oplus N^n(X)$, hence ample line bundles $\mathcal{L}$ do not have the property that $\operatorname{lc}\mathcal{L}$ is in the relative interior of $\operatorname{Nef}(X)$. By contrast, the classes in the interior of $\operatorname{Nef}^1(X)$ are all ample.

Remark 4.8 (Invariance under pullback). If $f : X \to Y$ is a morphism of projective varieties, then

$$f^* \operatorname{Nef}(Y) \subseteq \operatorname{Nef}(X).$$

Indeed if $\mathcal{V}$ is nef on Y, then $f^*\mathcal{V}$ is nef on X, and $\operatorname{lc} f^*\mathcal{V} = f^* \operatorname{lc}\mathcal{V}$ by the naturality of Chern classes.

Example 4.9 (Surfaces). Let X be a smooth projective surface. With notation as in Example 3.10, we have

$$\operatorname{Nef}(X) = \big\langle 1, \operatorname{Nef}^1(X), \pm p \big\rangle.$$

In characteristic zero, we also have

$$\mathrm{Stab}_H(X) = \big\langle 1, \mathrm{Nef}^1(X), -p \big\rangle.$$

In particular, $\mathrm{Stab}_H(X)$ is strictly contained in $\mathrm{Nef}(X)$, and pointed.

Proof. Nef bundles have positive rank and nef determinant. We have $\mathrm{lc}(\mathcal{O}_X^{\oplus 2}) = \log 2$ and $\mathrm{lc}\,\mathcal{L} = c_1(\mathcal{L})$ for any line bundle. If H is ample on X, then $\mathrm{lc}(\mathcal{O}_X \oplus \mathcal{O}_X(mH)) = \log 2 + \frac{m}{2}H + \frac{m^2}{8}(H^2)p$. After scaling by m^2 and taking limits, we deduce $p \in \mathrm{Nef}(X)$. Assume H is very ample, and let $\mathcal{V}_m$ be a generic rank 2 quotient of $(\mathcal{O}_X \oplus \mathcal{O}_X(H))^{\otimes m}$ as in Example 3.7. Then $\mathrm{lc}\,\mathcal{V}_m = \log 2 + m2^{m-2}H + \big(\binom{m+1}{2}2^{m-3} - m^2 2^{2m-5}\big)(H^2)p$. After scaling and taking limits, we find $-(H^2)p \in \mathrm{Nef}(X)$ hence also $-p \in \mathrm{Nef}(X)$.

For the case of stability, nef semistable vector bundles $\mathcal{V}$ have positive rank, nef determinant, and negative $\mathrm{lc}_2\,\mathcal{V}$ by the Bogomolov inequality. These justify one inclusion. For the reverse, 1 is a multiple of $\mathrm{lc}(\mathcal{O}_X^{\oplus 2}) = \log 2$, and $\mathrm{Nef}^1(X) \subset \mathrm{Stab}_H(X)$ since line bundles are stable. It remains to prove that $-p \in \mathrm{Stab}_H(X)$.

Let H be a very ample divisor on X and let C be a general curve in $|H|$. For large m and for a general subspace $V_m \subset H^0(X, \mathcal{O}_X(mH))$ of dimension $h^0(C, \mathcal{O}_C(mH))$, let N_m denote the dual of the kernel of the (surjective) evaluation map $V_m \otimes \mathcal{O}_X \to \mathcal{O}_X(mH)$. By [15, Proposition 2.1], the sheaf N_m is a stable vector bundle. It is clearly also globally generated. Put $r_m := h^0(C, \mathcal{O}_C(mH)) - 1$. Then $\mathrm{lc}\,N_m = \log r_m + \frac{m}{r_m}H - \frac{m^2(r_m-1)}{r_m^2}(H^2)p$. By Riemann-Roch, we have $\lim_{m\to\infty} \frac{r_m}{m} = \deg H|_C = (H^2)$. We deduce that $\lim_{m\to\infty} \frac{1}{m}\,\mathrm{lc}\,N_m = -p$. $\square$

[2] prove the stability of the dual kernel bundles N_{mH} on surfaces for large m and ample H. Their ranks are too large for our asymptotic considerations. There is also a more involved proof for showing $-p \in \mathrm{Stab}_H(X)$ on surfaces. One observes that $\mathrm{Stab}_H(X)$ is preserved by finite pullback. Noether normalization reduces the question to $X = \mathbb{P}^2$. Results of [17] about the existence of stable vector bundles on surfaces and the geometry of moduli spaces, and of [7] about the Brill-Noether theory of stable bundles on $\mathbb{P}^2$, guarantee the existence of stable Castelnuovo-Mumford regular (hence globally generated) vector bundles with convenient log-Chern character. See also [1] for a more detailed list of such log-Chern characters on $\mathbb{P}^2$ and on Hirzebruch surfaces.

Proposition 4.10. *Let X be a projective variety. Then* $\mathrm{Nef}(X)$ *is full-dimensional in* $N^*(X)$.

Proof. Let $f \in N^*(X)^\vee$ such that $f(\operatorname{lc}\mathcal{V}) = 0$ for all nef vector bundles $\mathcal{V}$. Denote by r the rank of $\mathcal{V}$. Then

$$r^n \log r \cdot f(1) + \sum_{k=1}^{n} (-1)^{k-1} \frac{r^{n-k}}{k} f(\operatorname{ch}_+^k \mathcal{V}) = 0, \tag{1}$$

for all nef $\mathcal{V}$. If $\mathcal{V}$ is nef, then so is $\mathcal{V} \oplus \mathcal{O}_X^{\oplus s}$ for all $s \geqslant 0$. Note that $\operatorname{ch}_+ \mathcal{V} = \operatorname{ch}_+(\mathcal{V} \oplus \mathcal{O}_X^{\oplus s})$, while $\operatorname{rk}(\mathcal{V} \oplus \mathcal{O}_X^{\oplus s}) = r + s$. As s grows to infinity, by looking at the fastest growing terms in (1), and by induction, we find $f(1) = 0$ and $f(\operatorname{ch}_+^k \mathcal{V}) = 0$ for all $k \geqslant 1$. In particular, $f(\operatorname{ch}\mathcal{V}) = r \cdot f(1) + f(\operatorname{ch}_+ \mathcal{V}) = 0$ for all nef $\mathcal{V}$. Therefore f vanishes on the linear span of $\operatorname{ch}\mathcal{V}$ for all nef $\mathcal{V}$. This is $N^*(X)$ by the proof of Proposition 3.12. □

We do not know if the same holds true for $\operatorname{Stab}_H(X)$ when $\dim X \geqslant 3$.

PROPOSITION 4.11. *Let X be a projective variety. Each of the following is a set of generators (up to closure) of* $\operatorname{Nef}(X)$.

(i) $\{\operatorname{lc}\mathcal{V} \mid \mathcal{V}$ *is nef*$\}$;

(ii) $\{\operatorname{lc}\mathcal{V} \mid \mathcal{V}$ *is ample*$\}$;

(iii) *(in characteristic zero)* $\{\operatorname{lc}\mathcal{V} \mid \mathcal{V}$ *is globally generated*$\}$;

(iv) $\{\operatorname{lc}\mathcal{V}\langle L\rangle \mid \mathcal{V}\langle L\rangle$ *is nef*$\}$;

(v) $\{\operatorname{lc}\mathcal{V}\langle L\rangle \mid \mathcal{V}\langle L\rangle$ *is ample*$\}$.

Proof. Analogous to Proposition 3.13. □

Remark 4.12. The presence of the positive real exponents a_i in the monomials $\prod_i(\operatorname{ch}\mathcal{V}_i)^{a_i}$ in the definition of $\mathscr{C}(X)$ makes it so that the formal exponential $\exp : N^*(X) \to N^*(X)$ maps $\operatorname{Nef}(X)$ into $\mathscr{C}(X)$.

Note that the formal logarithm map does not necessarily return the favor, because it is unclear what the logarithm of a sum of monomials is.

5. BOGOMOLOV INEQUALITY FOR COMPLEX PROJECTIVE THREEFOLDS

The Bogomolov inequality states that $\Delta(\mathcal{V}) \geqslant 0$ (equivalently $\operatorname{lc}_2 \mathcal{V} \leqslant 0$) whenever $\mathcal{V}$ is a μ-semistable bundle on a complex projective surface. A more pleasant equivalent formulation is $c_2(\mathcal{V}) \geqslant 0$ whenever $\mathcal{V}$ is μ-semistable with $\det \mathcal{V} = 0$. Higher-dimensional generalizations were found by restriction to complete intersection surfaces, but the focus was still on the codimension 2 class $\Delta(\mathcal{V})$.

If $\mathcal{V}$ is a vector bundle of rank r on the projective variety X of dimension n, we denote

$$\widehat{h}^i(\mathcal{V}) := \limsup_{m\to\infty} \frac{h^i(X, \operatorname{Sym}^m \mathcal{V})}{m^{n+r-1}/(n+r-1)!} \quad \text{and}$$
$$\widecheck{h}^i(\mathcal{V}) := \liminf_{m\to\infty} \frac{h^i(X, \operatorname{Sym}^m \mathcal{V})}{m^{n+r-1}/(n+r-1)!}.$$

In fact $\widehat{h}^i(\mathcal{V}) = \widehat{h}^i(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$ and $\widecheck{h}^i(\mathcal{V}) = \widecheck{h}^i(\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1))$. These quantities are finite by [10].

PROPOSITION 5.1. *Let (X, H) be a polarized complex projective manifold of dimension n. Let $\mathcal{V}$ be a μ_H-slope semistable vector bundle of rank r on X. Assume furthermore $\mu_H(\mathcal{V}) = 0$ (e.g., when $\det \mathcal{V} = 0$). Then*

$$-\sum_{k=1}^{\lfloor n/2\rfloor} \widehat{h}^{2i-1}(\mathcal{V}) \leqslant s_n(\mathcal{V}^\vee) \leqslant \sum_{k=1}^{\lfloor (n-1)/2\rfloor} \widehat{h}^{2i}(\mathcal{V}),$$

where $s_n(\mathcal{V}^\vee)$ is the n-th Segre class of $\mathcal{V}^\vee$ ([5, Chapter 3.1]), which agrees with the self-intersection number of $\mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. When only one summand $\widehat{h}^i$ appears either on the right or on the left, it can be replaced by $\widecheck{h}^i$.

Proof. Let $Y := \mathbb{P}(\mathcal{V})$, and let $\mathcal{L} := \mathcal{O}_{\mathbb{P}(\mathcal{V})}(1)$. Asymptotic Riemann-Roch on Y gives

$$(\mathcal{L}^{n+r-1}) = \lim_{m\to\infty} \frac{\chi(Y, \mathcal{L}^{\otimes m})}{m^{n+r-1}/(n+r-1)!}.$$

We have $(\mathcal{L}^{n+r-1}) = s_n(\mathcal{V}^\vee)$ by the definition of the Segre classes. Clearly

$$-h^1(Y, \mathcal{L}^{\otimes m}) - h^3(Y, \mathcal{L}^{\otimes m}) - \ldots \leqslant \chi(Y, \mathcal{L}^{\otimes m}) \leqslant h^0(Y, \mathcal{L}^{\otimes m}) + h^2(Y, \mathcal{L}^{\otimes m}) + \ldots$$

Note that $h^i(Y, \mathcal{L}^{\otimes m}) = h^i(X, \operatorname{Sym}^m \mathcal{V})$ for all i and all $m \geqslant 0$ by the Leray spectral sequence. To conclude, given the summation bounds, it remains to show that

$$\widehat{h}^0(\mathcal{V}) = \widehat{h}^n(\mathcal{V}) = 0.$$

The bundles $\operatorname{Sym}^m \mathcal{V}$ and $\operatorname{Sym}^m \mathcal{V}^\vee$ are semistable of slope 0. If H is sufficiently ample on X so that $H - K_X$ is ample, then $\operatorname{Sym}^m \mathcal{V}(-H)$ and $\omega_X \otimes \operatorname{Sym}^m \mathcal{V}^\vee(-H)$ are semistable of negative slope, hence they cannot have sections. If $Z \subset X$ is some smooth divisor of class H, it follows from the restriction sequences that

$$h^0(X, \operatorname{Sym}^m \mathcal{V}) \leqslant h^0\big(Z, \operatorname{Sym}^m \mathcal{V}|_Z\big) =$$
$$= h^0\big(\mathbb{P}(\mathcal{V}|_Z), \mathcal{O}_{\mathbb{P}(\mathcal{V}|_Z)}(m)\big) = O(m^{\dim \mathbb{P}(\mathcal{V}|_Z)}) = O(m^{n+r-2})$$

and

$$h^n(X,\operatorname{Sym}^m\mathcal{V}) = h^0\big(X,\omega_X\otimes\operatorname{Sym}^m\mathcal{V}^\vee\big) \leqslant h^0\big(Z,\omega_X|_Z\otimes\operatorname{Sym}^m\mathcal{V}^\vee|_Z\big) =$$
$$= h^0\big(\mathbb{P}(\mathcal{V}^\vee|_Z),\mathcal{O}_{\mathbb{P}(\mathcal{V}^\vee|_Z)}(m)\otimes\pi^*\omega_{X|Z}\big) = O(m^{\dim\mathbb{P}(\mathcal{V}^\vee|_Z)}) = O(m^{n+r-2}),$$

where $\pi:\mathbb{P}(\mathcal{V}^\vee|_Z)\to Z$ denotes the bundle map. $\square$

It is conjectured that if $\mathcal{L}$ is a line bundle on the projective variety X of dimension n, then $\lim_{m\to\infty}\frac{h^i(X,\mathcal{L}^{\otimes m})}{m^n/n!}$ exists. This is known for $i=0$ and $i=n$. In general it is only know that $\limsup_{m\to\infty}\frac{h^i(X,\mathcal{L}^{\otimes m})}{m^n/n!}$ exists. The conjecture would imply that the asymptotic cohomological functions $\widehat{h}^i(\mathcal{L})$ of [10] and $\breve{h}^i(\mathcal{L})$ coincide.

Remark 5.2. When $n=3$, the conclusion of the proposition reads $-\breve{h}^1(\mathcal{V})\leqslant s_3(\mathcal{V}^\vee)\leqslant\breve{h}^2(\mathcal{V})$. When $c_1(\mathcal{V})=0$, we have $s_3(\mathcal{V}^\vee)=c_3(\mathcal{V})$. When $\det\mathcal{V}$ is arbitrary, we can give a version of the result by reducing to the case $\det\mathcal{V}=0$.

Let (X,H) be a polarized smooth complex projective threefold. Let $\mathcal{V}$ be a μ_H-semistable vector bundle of rank r on X.

There exists a finite covering $f:X'\to X$ such that X' is smooth and supports a line bundle $\mathcal{L}$ with $f^*\det\mathcal{V}=\mathcal{L}^{\otimes r}$. See [13, Theorem 4.1.10]. The bundle $\mathcal{V}':=(f^*\mathcal{V})\otimes\mathcal{L}^\vee$ is μ_{f^*H}-semistable and $\det\mathcal{V}'=0$. Then

$$c_3(\mathcal{V}') = 2r\cdot\operatorname{lc}_3(\mathcal{V}') = 2r\cdot\operatorname{lc}_3\big((f^*\mathcal{V})\otimes\mathcal{L}^\vee\big) = 2r\cdot\operatorname{lc}_3(f^*\mathcal{V}) = 2r\cdot(\deg f)\cdot\operatorname{lc}_3(\mathcal{V}),$$

since lc_k is invariant under tensoring by line bundles when $k>1$. We conclude

$$-\frac{\breve{h}^1(f^*\mathcal{V}\otimes\mathcal{L}^\vee)}{\deg f}\leqslant 2r\cdot\operatorname{lc}_3\mathcal{V}\leqslant\frac{\breve{h}^2(f^*\mathcal{V}\otimes\mathcal{L}^\vee)}{\deg f}.$$

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