Dedicated to Vasile Brînzănescu on his 75^{th} birthday

TOPOLOGICAL METHODS IN APPLIED ALGEBRAIC GEOMETRY AND ALGEBRAIC STATISTICS

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The aim of this note is to survey recent applications of topology and singularity theory in the study of the algebraic complexity of concrete optimization problems in applied algebraic geometry and algebraic statistics.

AMS 2010 Subject Classification: 13P25, 32S30, 57R20, 90C26.

Key words: Euclidean distance degree, defect of Euclidean distance degree, Euler characteristic, local Euler obstruction function, vanishing cycles, multiview variety, triangulation problem.

1. INTRODUCTION

This note surveys recent developments in the study of the algebraic complexity of concrete optimization problems in applied algebraic geometry and algebraic statistics. We will focus on the Euclidean distance degree of [7], which is an algebraic measure of the complexity of nearest point problems. For complete details, the interested reader may consult [20, 21, 19]. Similar methods apply to the computation of other important invariants in algebraic statistics (e.g., the maximum likelihood degree).

Without being particularly heavy on technical details, it is our hope that the results and techniques described in this note are of equal interest for pure mathematicians and applied scientists: besides acquainting applied scientists with a variety of tools from topology, algebraic geometry and singularity theory, the interdisciplinary nature of the work presented here should lead pure mathematicians to become more acquainted with a myriad of tools used in more applied research fields, such as computer vision.

1.1. **Nearest point problems. Euclidean distance degree**

Many models in data science or engineering are *algebraic models* (i.e., they can be realized as real algebraic varieties $X \subset \mathbb{R}^N$) for which one needs to solve a *nearest point problem.* Specifically, for such an algebraic model $X \subset \mathbb{R}^N$

and a generic *data point* $\underline{u} = (u_1, \ldots, u_N) \in \mathbb{R}^N$, one needs to find a nearest point $\underline{u}^* \in X_{\mathrm{reg}}$ to $\underline{u}$, i.e., a point $\underline{u}^*$ which minimizes the (squared) Euclidean distance from the given data point $u \in \mathbb{R}^N$. (Here, X_{reg} denotes the smooth locus of X.)

The standard approach for solving the nearest point problem for an algebraic model $X \subset \mathbb{R}^N$ and a generic data point $\underline{u} \in \mathbb{R}^N$ is to list and examine the critical points of the *squared distance function*

$$d_{\underline{u}}(x) = \sum_{i=1}^{N} (x_i - u_i)^2$$

on the smooth locus X_{reg}. In practice, algorithms (e.g., Gröbner bases, numerical algebraic geometry) detect all *complex critical points* of $d_{\underline{u}}$ (i.e., consider $X \subset \mathbb{C}^N$), and then sort out the real ones.

If X is an irreducible closed subvariety of $\mathbb{C}^N$ then, for a *generic* choice of data point $\underline{u}$, the function $d_{\underline{u}}|_{X_{\mathrm{reg}}}$ has finitely many critical points on the smooth locus X_{reg} of X. Moreover, this number of critical points is independent of the generic choice of $\underline{u}$, so it defines an invariant of the embedding of X in $\mathbb{C}^N$ called the *Euclidean distance (ED) degree of* X. It is denoted by $\mathrm{EDdeg}(X)$. Therefore, the ED degree of the complexified model variety gives an algebraic measure of the complexity of solving such an optimization problem, and it is a good indicator of the running time needed to solve the problem exactly.

The Euclidean distance degree was introduced in [7], and has since been extensively studied in areas like computer vision [3, 11, 20], biology [10], chemical reaction networks [1], engineering [6, 29], numerical algebraic geometry [12, 18], data science [14], etc. It is an additive analogue of another important invariant in algebraic statistics, namely the *maximum likelihood (ML) degree*, e.g., see [5, 8, 16, 15].

1.2. **Classical examples of nearest point problems**

Let us briefly indicate two main examples of nearest point problems. The interested reader may consult, e.g., [7, Section 3] and the references therein for more such examples.

Example 1.1 (Low-rank approximation). Fix positive integers $r \leq s \leq t$ and set $N = st$. Consider the following model of bordered-rank ($\leq r$) matrices:

$$X_r := \left\{ X = [x_{ij}] \in \mathbb{R}^{s \times t} \mid \mathrm{rank}(X) \leq r \right\} \subset \mathbb{R}^N.$$

As generic data point, we choose a general $s \times t$ matrix $U = [u_{ij}] \in \mathbb{R}^{s \times t} = \mathbb{R}^N$. The nearest point problem can be solved in this case by using the *singular value* *decomposition.* Indeed, the general matrix U admits a product decomposition

$$U = T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_s) \cdot T_2,$$

where $\sigma_1 > \cdots > \sigma_s$ are the *singular values* of the matrix U (all of which can be assumed non-zero since U is general), and T_1, T_2 are orthogonal matrices. Then the *Eckart-Young Theorem* (e.g., see [7, Example 2.3]) states that the matrix of rank $\leq r$ closest to U is:

$$U^* = T_1 \cdot \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0) \cdot T_2 \in X_r.$$

The other critical points of the squared distance function d_U are given by

$$T_1 \cdot \mathrm{diag}(0, \ldots, 0, \sigma_{i_1}, 0, \ldots, 0, \sigma_{i_r}, 0, \ldots, 0) \cdot T_2,$$

where $\{i_1 < \ldots < i_r\}$ runs over all r-element subsets of $\{1, \ldots, s\}$. In particular, there are $\binom{s}{r}$ critical points of the squared distance function d_U, all of which are real matrices of rank exactly r. (Note that the regular part of X_r consists exactly of rank-r matrices.)

Example 1.2 (Triangulation problem in computer vision). In computer vision, triangulation (or 3D-reconstruction) refers to the process of reconstructing a point in the three-dimensional (3D) space from its two-dimensional (2D) projections in $n \geq 2$ cameras in general position. The triangulation problem has many practical applications, e.g., in tourism, for reconstructing the 3D structure of a tourist attraction based on a large number of online pictures [2]; in robotics, for creating a virtual 3D space from multiple cameras mounted on an autonomous vehicle; in filmmaking, for adding animation and graphics to a movie scene after everything is already shot, etc. If the 2D projections are given with infinite precision, then two cameras suffice to determine the 3D point. In practice, however, various sources of "noise" (pixelation, lens distortion, etc.) lead to inaccuracies in the measured image coordinates. The problem, then, is to find a 3D point which optimally fits the measured image points.

The algebraic model fitting the triangulation problem is the space of all possible n-tuples of such 2D projections with infinite precision, called the *affine multiview variety* X_n; see [7, Example 3.3] and [20, Section 4] for more details. The above optimization problem translates into finding a point $\underline{u}^* \in X_n$ of minimum distance to a (generic) point $\underline{u} \in \mathbb{R}^{2n}$ obtained by collecting the 2D coordinates of n "noisy" images of the given 3D point. Once $\underline{u}^*$ is obtained, a 3D point is recovered by triangulating any two of its n projections. As already indicated in the previous section, in order to find such a minimizer $\underline{u}^*$ algebraically, one regards X_n as a complex algebraic variety and examines all complex critical points of the squared Euclidean distance function $d_{\underline{u}}$ on X_n. Under the assumption that $n \geq 3$, the complex algebraic variety X_n is smooth and 3-dimensional, and one is then interested in computing the Euclidean distance degree $\mathrm{EDdeg}(X_n)$ of the affine multiview variety X_n.

An explicit conjectural formula for the Euclidean distance degree $\mathrm{EDdeg}(X_n)$ was proposed in [7, Conjecture 3.4], based on numerical computations from [27] for configurations involving $n \leq 7$ cameras:

Conjecture 1.3 (Multiview conjecture). *The Euclidean distance degree of the affine multiview variety X_n is given by:*

$$\mathrm{EDdeg}(X_n) = \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4. \tag{1}$$

This conjecture was the main motivation for the introduction of the Euclidean distance degree in [7].

A proof of Conjecture 1.3 was obtained in [20] for $n \geq 3$ cameras in general position, by first giving a purely topological interpretation of the Euclidean distance degree of any complex affine variety as an "Euler-Mather characteristic" involving MacPherson's local Euler obstruction function. This approach will be explained in Section 2 below. In Section 3, we discuss topological formulae for the (projective) ED degree of complex projective varieties (cf. [21]), answering positively a conjecture of Aluffi-Harris. Section 4 deals with a computation of the ED degree of a smooth projective variety Y in terms of *generic* ED degrees associated to the singularities of a certain hypersurface on Y (cf. [19]).

2. ED DEGREES OF COMPLEX AFFINE VARIETIES AND THE MULTIVIEW CONJECTURE

In this section we explain how to compute the Euclidean distance degree of a complex affine variety as an Euler characteristic. We apply this computation to the resolution of the multiview conjecture (Conjecture 1.3).

2.1. Euclidean distance degree

Let us first recall the following definition from [7]:

Definition 2.1. The *Euclidean distance (ED) degree* $\mathrm{EDdeg}(X)$ of an irreducible closed variety $X \subset \mathbb{C}^N$ is the number of complex critical points of

$$d_{\underline{u}}(x) = \sum_{i=1}^{N} (x_i - u_i)^2$$

on the smooth locus X_{reg} of X (for general $\underline{u} = (u_1, \ldots, u_N) \in \mathbb{C}^N$).

Example 2.2. Every linear space X has ED degree 1.

Example 2.3. As already discussed in Example 1.1, if X_r denotes the variety of $s \times t$ real matrices (with $s \leq t$) of rank at most r, then $\mathrm{EDdeg}(X_r) = \binom{s}{r}$.

Remark 2.4. Let us explain the reason for the use of the term "degree" in Definition 2.1, see [7, Theorem 4.1] for complete details. For an irreducible closed variety $X \subset \mathbb{C}^N$ of codimension c, consider the *ED correspondence* $\mathcal{E}_X$ defined as the topological closure in $\mathbb{C}^N \times \mathbb{C}^N$ of the set of pairs (x, u) such that $x \in X_{\mathrm{reg}}$ is a critical point of $d_{\underline{u}}$. Note that $\mathcal{E}_X$ can be identified with the conormal space $T^*_X\mathbb{C}^N$ of X in $\mathbb{C}^N$. In particular, the first projection $\pi_1 : \mathcal{E}_X \to X$ is an affine vector bundle of rank c over X_{reg}, whereas for general data points $\underline{u} \in \mathbb{C}^N$ the second projection $\pi_2 : \mathcal{E}_X \to \mathbb{C}^N$ has finite fibers $\pi_2^{-1}(\underline{u})$ of cardinality equal to $\mathrm{EDdeg}(X)$.

2.2. **Topological interpretation of ED degrees**

Our approach to studying ED degrees in [20] makes use of Whitney stratifications and constructible functions. Let us recall here the main ingredients.

Let X be a complex algebraic variety. Then it is known that X admits a *Whitney stratification*, i.e., a partition $\mathcal{S}$ into locally closed nonsingular subvarieties (called *strata*), along which X is topologically equisingular. For example, the variety X_r of bordered-rank matrices is Whitney stratified (over $\mathbb{C}$) by the rank condition.

Definition 2.5. Given a complex algebraic variety X with a Whitney stratification $\mathcal{S}$, a function $\varphi : X \to \mathbb{Z}$ is *$\mathcal{S}$-constructible* if φ is constant along each stratum $S \in \mathcal{S}$.

Example 2.6. A constant function $\varphi = c \in \mathbb{Z}$ (e.g., $\varphi = 1_X$) is constructible with respect to any Whitney stratification of X.

Example 2.7 (MacPherson's local Euler obstruction). The *local Euler obstruction* function

$$\mathrm{Eu}_X : X \to \mathbb{Z}$$

is an essential ingredient in MacPherson's definition of Chern classes for singular varieties, cf. [17]. It satisfies the following properties:

(a) Eu_X is $\mathcal{S}$-constructible for any fixed Whitney stratification $\mathcal{S}$ of X.

(b) If $x \in X$ is a smooth point, then $\mathrm{Eu}_X(x) = 1$. In particular, if X is smooth then $\mathrm{Eu}_X = 1_X$.

(c) If X is a curve, then $\mathrm{Eu}_X(x)$ is the multiplicity of X at x.

(d) If (X, x) is an isolated singularity germ, then $\mathrm{Eu}_X(x) = \chi(CL(X, x))$, where $CL(X, x)$ denotes the complex link of x in X.

(e) The local Euler obstruction function is an analytic invariant. In particular, if U is a Zariski open set in X, then $\mathrm{Eu}_X|_U = \mathrm{Eu}_U$.

(f) The local Euler obstruction function is preserved under generic hyperplane sections.

Definition 2.8. Let X be a complex algebraic variety with a fixed Whitney stratification $\mathcal{S}$. The (weighted) *Euler characteristic* of an $\mathcal{S}$-constructible function φ is defined as:

$$\chi(\varphi) := \sum_{S \in \mathcal{S}} \chi(S) \cdot \varphi(S),$$

with $\varphi(S)$ denoting the (constant) value of φ on the stratum $S \in \mathcal{S}$.

Example 2.9. Using the additivity of the Euler-Poincaré characteristic in complex algebraic geometry, one has:

$$\chi(1_X) = \sum_{S \in \mathcal{S}} \chi(S) = \chi(X).$$

Definition 2.10. The Euler characteristic $\chi(\mathrm{Eu}_X)$ of the local Euler obstruction function is usually referred to as the *Euler-Mather characteristic* of X.

We can now state our main result from [20]:

Theorem 2.11. *Let $X \subset \mathbb{C}^N$ be an irreducible closed subvariety. Then, for general $\underline{u} = (u_0, \dots, u_N) \in \mathbb{C}^{N+1}$, we have:*

$$\mathrm{EDdeg}(X) = (-1)^{\dim_{\mathbb{C}} X} \chi(\mathrm{Eu}_{X \setminus Q_{\underline{u}}}), \tag{2}$$

where $Q_{\underline{u}} = \{\sum_{i=1}^N (x_i - u_i)^2 = u_0\} \subset \mathbb{C}^N$.
In particular, if X is smooth (e.g., the affine multiview variety), then

$$\mathrm{EDdeg}(X) = (-1)^{\dim_{\mathbb{C}} X} \chi(X \setminus Q_{\underline{u}}) \tag{3}$$

for general $\underline{u} = (u_0, \dots, u_N) \in \mathbb{C}^{N+1}$.

Example 2.12. If $X = \mathbb{C}$ is a complex line, then (2) yields:

$$\mathrm{EDdeg}(X) = -\chi(X \setminus Q_{\underline{u}}) = -\big(\chi(X) - \chi(X \cap Q_{\underline{u}})\big) = -(1-2) = 1.$$

In order to explain the proof of Theorem 2.11, we first *linearize* the optimization problem as follows. Consider the closed embedding

$$i : \mathbb{C}^N \hookrightarrow \mathbb{C}^{N+1}\ , \quad (x_1, \ldots, x_N) \mapsto (x_1^2 + \cdots + x_N^2, x_1, \ldots, x_N),$$

and let $w_0, \ldots, w_N$ be the coordinates of $\mathbb{C}^{N+1}$. Then the function $\sum_{1\leq i\leq N}(x_i - u_i)^2 - u_0$ on $\mathbb{C}^N$ is the pullback of the function

$$w_0 + \sum_{1\leq i\leq N} -2u_i w_i + \sum_{1\leq i\leq N} u_i^2 - u_0$$

on $\mathbb{C}^{N+1}$. The computation of the ED degree $\mathrm{EDdeg}(X)$ amounts now to counting the number of complex critical points of a generic linear function on the regular part of the affine variety $i(X) \subset \mathbb{C}^{N+1}$. Theorem 2.11 is then a consequence of the following more general result from stratified Morse theory (see [26, Equation (2)]):

Theorem 2.13. *Let $X \subset \mathbb{C}^N$ be an irreducible closed subvariety. Let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a general linear function, and let H_c be the hyperplane in $\mathbb{C}^N$ defined by the equation $\ell = c$ for a general $c \in \mathbb{C}$. Then the number of critical points of $\ell|_{X_{\mathrm{reg}}}$ equals*

$$(-1)^{\dim_{\mathbb{C}} X} \chi(\mathrm{Eu}_{X\setminus H_c}).$$

When X is smooth (e.g., the affine multiview variety), one can give a simpler proof of (3) by the following Lefschetz-type result (see [20, Theorem 3.1]) applied to the smooth affine variety $i(X)$:

Theorem 2.14. *Let $X \subset \mathbb{C}^N$ be a smooth closed subvariety of complex dimension d. Let $\ell : \mathbb{C}^N \to \mathbb{C}$ be a general linear function, and let H_c be the hyperplane in $\mathbb{C}^N$ defined by the equation $\ell = c$ for a general $c \in \mathbb{C}$. Then:*

(a) *X is homotopy equivalent to $X \cap H_c$ with finitely many d-cells attached.*

(b) *the numbers of d-cells attached equals the number of critical points of $\ell|_X$.*

(c) *the number of critical points of $\ell|_X$ is equal to $(-1)^d \cdot \chi(X \setminus H_c)$.*

Theorem 2.14 is proved in [20] by using Morse theory. Specifically, we consider real Morse functions of the form $\log|f|$, where f is a nonvanishing holomorphic Morse function on a complex manifold. Such a Morse function has the following key properties:

(i) The critical points of $\log|f|$ coincide with the critical points of f.

(ii) The index of every critical point of $\log|f|$ is equal to the complex dimension of the manifold on which f is defined.

However, as a real-valued Morse function, $\log|f|$ is almost never proper. So one needs to employ the non-proper Morse theory techniques developed by Palais-Smale [24].

2.3. **The multiview conjecture**

Our result from (3) can be used to confirm the multiview conjecture of [7] (Conjecture 1.3). Indeed, one has:

THEOREM 2.15. *The ED degree of the affine multiview variety $X_n \subset \mathbb{C}^{2n}$ corresponding to $n \geq 3$ cameras in general position satisfies:*

$$\mathrm{EDdeg}(X_n) = -\chi(X_n \setminus Q_{\underline{u}}) = \frac{9}{2}n^3 - \frac{21}{2}n^2 + 8n - 4.$$

The computation of $\chi(X_n \setminus Q_{\underline{u}})$ is quite involved and it relies on topological and algebraic techniques from Singularity theory, see [20, Section 4] for complete details. Let us only indicate here the key technical points. Even though both X_n and $Q_{\underline{u}}$ are smooth in $\mathbb{C}^{2n}$ and they intersect transversally, their intersection "at infinity" is very singular. We regard the affine multiview variety X_n as a Zariski open subset in its closure Y_n in $(\mathbb{CP}^2)^n$, with divisor at infinity $Y_n \setminus X_n = D_\infty$. * It can be easily seen that Y_n is isomorphic to the blowup of $\mathbb{CP}^3$ at n points. By using the additivity of the Euler-Poincaré characteristic, for the computation of $\chi(X_n \setminus Q_{\underline{u}})$ it suffices to calculate $\chi(Y_n)$, $\chi(D_\infty)$, $\chi(D_{\underline{u}})$, $\chi(D_\infty \cap D_{\underline{u}})$, where $D_{\underline{u}} := Y_n \cap \overline{Q}_{\underline{u}}$. The main difficulty arises in the calculation of $\chi(D_{\underline{u}})$, since $D_{\underline{u}}$ is an irreducible (hyper)surface in Y_n with a 1-dimensional singular locus. For the computation of Euler-Poincaré characteristics of complex projective hypersurfaces, we refer the reader to [25] or [22, Section 10.4]. Theorem 2.15 is then a direct consequence of the following formulae obtained in [20, Theorem 4.1]:

(i) $\chi(Y_n) = 2n + 4$.

(ii) $\chi(D_\infty) = \frac{n^3}{6} - \frac{3n^2}{2} + \frac{16n}{3}$.

(iii) $\chi(D_{\underline{u}}) = 4n^3 - 9n^2 + 9n$.

(iv) $\chi(D_\infty \cap D_{\underline{u}}) = -\frac{n^3}{3} + \frac{13n}{3}$.

3. **PROJECTIVE EUCLIDEAN DISTANCE DEGREE**

Many models in data science, engineering and other applied fields are realized as *affine cones* (defined by homogeneous polynomials), so it is natural to consider such models as *projective varieties*. Examples of such models

*A different compactification of X_n, in $\mathbb{CP}^{2n}$, was considered in [11], where the ED degree of the affine multiview variety X_n was studied via characteristic classes. This leads to an upper bound for the Euclidean distance degree of X_n given by: $\mathrm{EDdeg}(X_n) \leq 6n^3 - 15n^2 + 11n - 4$.

occur in (structured) low rank matrix approximation [23], low rank tensor approximation, formation shape control [4], and all across algebraic statistics [8, 28].

Example 3.1. The variety X_r of $s \times t$ matrices of rank $\leq r$ is an affine cone.

We make the following natural definition (cf. [7]):

Definition 3.2. If $Y \subset \mathbb{CP}^N$ is an irreducible complex projective variety, define the *projective Euclidean distance degree* of Y by

$$\mathrm{pEDdeg}(Y) := \mathrm{EDdeg}(C(Y)),$$

where $C(Y)$ is the affine cone of Y in $\mathbb{C}^{N+1}$.

The affine cone $C(Y)$ on a projective variety Y acquires a very complicated singularity at the cone point, so the computation of $\mathrm{pEDdeg}(Y)$ via formula (2) is in general very difficult. Instead, one aims in this case to describe $\mathrm{EDdeg}(C(Y))$ in terms of the topology of the projective variety Y itself. This problem has been addressed by Aluffi and Harris in [3] (building on preliminary results from [7]) in the special case when Y is a smooth projective variety. The main result of Aluffi-Harris can be formulated as follows (see [3, Theorem 8.1]):

THEOREM 3.3. *Let $Y \subset \mathbb{CP}^N$ be a smooth complex projective variety, and assume that $Y \not\subseteq Q$, where $Q = \{x_0^2 + \cdots + x_N^2 = 0\}$ is the isotropic quadric in $\mathbb{CP}^N$. Then*

$$\mathrm{pEDdeg}(Y) = (-1)^{\dim_{\mathbb{C}} Y} \chi(Y \setminus (Q \cup H)) \tag{4}$$

where $H \subset \mathbb{CP}^N$ is a general hyperplane.

Theorem 3.3 was proved in [3] by using the theory of characteristic classes for singular varieties, and it provides a generalization of [7, Theorem 5.8], where it was assumed that the smooth projective variety Y intersects the isotropic quadric Q transversally, i.e., that $Y \cap Q$ is a smooth hypersurface in Y. Aluffi and Harris also conjectured that formula (4) should admit a natural generalization to arbitrary (possibly singular) projective varieties by using the "Euler-Mather characteristic" defined in terms of the local Euler obstruction function. We addressed their conjecture in [21, Theorem 1.3], where we proved the following result:

THEOREM 3.4. *Let $Y \subset \mathbb{CP}^N$ be an irreducible complex projective variety. Then*

$$\mathrm{pEDdeg}(Y) = (-1)^{\dim_{\mathbb{C}} Y} \chi(\mathrm{Eu}_{Y \setminus (Q \cup H)}), \tag{5}$$

where Q is the isotropic quadric and H is a general hyperplane in $\mathbb{CP}^N$.

The proof of Theorem 3.4 is Morse-theoretic, and it employs ideas similar to those used to prove Theorem 2.11.

Note that in the case when $Y \subset \mathbb{CP}^N$ is smooth, Theorem 3.4 reduces to the statement of Theorem 3.3. Theorem 3.4 also generalizes [3, Proposition 3.1], where the ED degree of a possibly singular projective variety $Y \subset \mathbb{CP}^N$ is computed under the assumption that Y intersects the isotropic quadric Q transversally. In this case, one actually computes what is called the *generic* ED degree of Y. For more results concerning generic ED degrees, see also [3, 7, 13, 23], and Section 4 below.

Our topological interpretation of ED degrees reduces their calculation to the problem of computing MacPherson's local Euler obstruction function and the Euler-Poincaré characteristics of certain smooth algebraic varieties (strata). We present such computations in the following examples.

Example 3.5 (Nodal curve). Let $Y = \{x_0^2x_2 - x_1^2(x_1 + x_2) = 0\} \subset \mathbb{CP}^2$. It has only one singular point $p = [0 : 0 : 1]$. Therefore, the local Euler obstruction function Eu_Y equals 1 on the smooth locus Y_{reg} of Y, and $\mathrm{Eu}_Y(p) = 2$. Note that Y intersects the isotropic quadric Q transversally at 6 points, and it intersects a generic hyperplane H at 3 points. Moreover, Y_{reg} is isomorphic to $\mathbb{C}^*$. So by inclusion-exclusion, we get that $\chi(Y_{\mathrm{reg}} \setminus (Q \cup H)) = -9$. It then follows from (5) that $\mathrm{pEDdeg}(Y) = (-1) \cdot [(-9) + 2] = 7$.

Example 3.6 (Whitney umbrella). Consider the Whitney umbrella, i.e., the projective surface $Y = \{x_0^2x_1 - x_2x_3^2 = 0\} \subset \mathbb{CP}^3$. The singular locus of Y is defined by $x_0 = x_3 = 0$. Y has a Whitney stratification with strata: $S_3 := \{[0 : 1 : 0 : 0], [0 : 0 : 1 : 0]\}$, $S_2 = \{x_0 = x_3 = 0\} \setminus S_3$, and $S_1 = Y \setminus \{x_0 = x_3 = 0\}$. It is well known that Eu_Y takes the values 1, 2 and 1 along S_1, S_2 and S_3, respectively. Therefore, if we let $U := \mathbb{CP}^3 \setminus (Q \cup H)$ for a generic hyperplane $H \subset \mathbb{CP}^3$ and Q the isotropic quadric, then

$$\chi(\mathrm{Eu}_Y|_U) = \chi(Y \cap U) + \chi(S_2 \cap U).$$

The terms on the right-hand side of the above equality can be computed directly by using the inclusion-exclusion property of the Euler characteristic. One gets: $\chi(Y \cap U) = 13$ and $\chi(S_2 \cap U) = -3$ (see [21, Example 4.4] for complete details). Altogether, this yields that $\mathrm{pEDdeg}(Y) = \chi(\mathrm{Eu}_Y|_U) = 10$.

Remark 3.7. In view of recent computations of the local Euler obstruction function for determinantal varieties [9], it is an interesting exercise to check that (2) or (5) recovers the Euclidean distance degree of the variety of $s \times t$ matrices of rank $\leq r$, as discussed in Example 2.3.

4. DEFECT OF ED DEGREE

We begin this section by noting that the projective ED degree $\mathrm{pEDdeg}(Y)$ is difficult to compute even if $Y \subset \mathbb{CP}^N$ is smooth, since Y and Q may intersect non-transversally in $\mathbb{CP}^N$. The idea is then to perturb the objective (i.e., squared distance) function to create a transversal intersection. For this purpose, we make the following:

Definition 4.1. The *λ-Euclidean distance (ED) degree* $\mathrm{EDdeg}_\lambda(X)$ of a closed irreducible variety $X \subset \mathbb{C}^N$ is the number of complex critical points of

$$d_{\underline{u}}^{\lambda}(x) = \sum_{i=1}^{N} \lambda_i (x_i - u_i)^2 \ , \quad \lambda = (\lambda_1, \ldots, \lambda_N)$$

on the smooth locus X_{reg} of X (for general $\underline{u} \in \mathbb{C}^N$).

Similarly, If $Y \subset \mathbb{CP}^N$ is an irreducible complex projective variety, we define the *projective λ-Euclidean distance degree* of Y by

$$\mathrm{pEDdeg}_\lambda(Y) := \mathrm{EDdeg}_\lambda(C(Y)),$$

where $C(Y)$ is the affine cone of Y in $\mathbb{C}^{N+1}$.

If $\lambda = \mathbb{1}$, we get the *(unit) ED degree*, $\mathrm{EDdeg} := \mathrm{EDdeg}_{\mathbb{1}}$, resp., $\mathrm{pEDdeg} = \mathrm{pEDdeg}_{\mathbb{1}}$. If λ is generic, we get the corresponding *generic ED degrees*.

Theorem 3.4 can be easily adapted to the weighted context to obtain the following result:

THEOREM 4.2. *Let $Y \subset \mathbb{CP}^N$ be an irreducible complex projective variety. Then*

$$\mathrm{pEDdeg}_\lambda(Y) = (-1)^{\dim_{\mathbb{C}} Y} \chi(\mathrm{Eu}_{Y \setminus (Q_\lambda \cup H)}), \tag{6}$$

where $Q_\lambda := \{\lambda_0 x_0^2 + \cdots + \lambda_N x_N^2 = 0\}$ and H is a general hyperplane in $\mathbb{P}^N$. In particular, if Y is smooth, then

$$\mathrm{pEDdeg}_\lambda(Y) = (-1)^{\dim_{\mathbb{C}} Y} \chi(Y \setminus (Q_\lambda \cup H)). \tag{7}$$

For generic λ, the quadric Q_λ intersects Y transversally in $\mathbb{CP}^N$, and the computation of the generic projective ED degree $\mathrm{pEDdeg}(Y)$ is more manageable (e.g., see [7, 13, 3], etc). This motivates the following:

Definition 4.3 (Defect of ED degree). If $Y \subset \mathbb{CP}^N$ is an irreducible projective variety and λ is generic, the *defect of Euclidean distance degree* of Y is defined as:

$$\mathrm{EDdefect}(Y) := \mathrm{pEDdeg}_\lambda(Y) - \mathrm{pEDdeg}(Y).$$

It is known that $\mathrm{EDdefect}(Y)$ is non-negative, but for many varieties appearing in optimization, engineering, statistics, and data science, this defect is quite substantial. In [19], we give a new topological interpretation of this defect in terms of invariants of singularities of $Y \cap Q$ (i.e., the non-transversal intersection locus) when Y is a smooth irreducible complex projective variety in $\mathbb{CP}^N$. Specifically, we prove the following result (see [19, Theorem 1.5]):

THEOREM 4.4. *Let $Y \subset \mathbb{CP}^N$ be a smooth irreducible variety, with $Y \not\subseteq Q$, and let $Z = \mathrm{Sing}(Y \cap Q)$. Let $\mathcal{V}$ be the collection of strata of a Whitney stratification of $Y \cap Q$ which are contained in Z, and choose λ generic. Then:*

$$\mathrm{EDdefect}(Y) = \sum_{V \in \mathcal{V}} \alpha_V \cdot \mathrm{pEDdeg}_\lambda(\bar{V}), \tag{8}$$

where, for any stratum $V \in \mathcal{V}$,

$$\alpha_V = (-1)^{\mathrm{codim}_{Y\cap Q} V} \cdot \left(\mu_V - \sum_{\{S \mid V < S\}} \chi_c(L_{V,S}) \cdot \mu_S \right),$$

with $\mu_V = \chi(\widetilde{H}^*(F_V;\mathbb{Q}))$ the Euler characteristic of the reduced cohomology of the Milnor fiber F_V of the hypersurface $Y \cap Q \subset Y$ at some point in V, and $L_{V,S}$ the complex link of a pair of distinct strata (V,S) with $V \subset \bar{S}$.

The proof of Theorem 4.4 relies on the theory of vanishing cycles, adapted to the pencil of quadrics Q_λ on Y, see [19, Section 2] for complete details.

Note that computing the ED degree defect of $Y \subset \mathbb{CP}^N$ yields a formula for the projective ED degree $\mathrm{pEDdeg}(Y)$ only in terms of generic ED degrees (which, as already mentioned, are easier to compute). Also, computing the ED degree defect directly is generally much easier than the individual computations of $\mathrm{pEDdeg}(Y)$ and $\mathrm{pEDdeg}_\lambda(Y)$ for generic λ.

As an immediate consequence of Theorem 4.4, we get the following result from [3, Corollary 6.3]:

COROLLARY 4.5. *Under the notations of Theorem 4.4, assume that $Z = \mathrm{Sing}(Y \cap Q)$ has only isolated singularities. Then*

$$\mathrm{EDdefect}(Y) = \sum_{x \in Z} \mu_x, \tag{9}$$

where μ_x is the Milnor number of the isolated hypersurface singularity germ $(Y \cap Q, x)$ in Y.

Furthermore, if $Y \cap Q$ is equisingular along the non-transversal intersection locus Z, then Theorem 4.4 yields the following:

Corollary 4.6. *Under the notations of Theorem 4.4, assume that* $Z = \mathrm{Sing}(Y \cap Q)$ *is connected and* $Y \cap Q$ *is equisingular along* Z. *Then:*

$$\mathrm{EDdefect}(Y) = \mu \cdot \mathrm{pEDdeg}_{\lambda}(Z), \tag{10}$$

where μ *is the Milnor number of the isolated transversal singularity at some point* $x \in Z$ *(i.e., the Milnor number of the isolated hypersurface singularity in a normal slice to* Z *at* x*).*

Theorem 4.4 was motivated by the "duality conjecture" of [23, (3.5)] in structured low-rank approximation, which predicts a formula for the Euclidean distance degree defect of the restriction of (the dual variety of) Y to a linear space L. At this point let us note that, since intersecting Y with a general linear space L does not change the multiplicities α_V on the right-hand side of formula (8), Theorem 4.4 has the following immediate consequence:

Corollary 4.7. *With the notations of Theorem 4.4, and for L a general linear subspace of* $\mathbb{CP}^N$, *we have:*

$$\mathrm{EDdefect}(Y \cap L) = \sum_{V \in \mathcal{V}} \alpha_V \cdot \mathrm{pEDdeg}_{\lambda}(\bar{V} \cap L). \tag{11}$$

Let us conclude this section with the following example:

Example 4.8 (2×2 matrices of rank 1). Let $Y = \{x_0x_3 - x_1x_2 = 0\} \subset \mathbb{CP}^3$, with isotropic quadric $Q = \{\sum_{i=0}^{3} x_i^2 = 0\}$. Then $Y \cap Q$ consists of 4 lines, with 4 isolated double point singularities (hence, each having Milnor number 1). Corollary 4.5 yields that $\mathrm{EDdefect}(Y) = 4$. In fact, as shown in [7], one has in this case that $\mathrm{pEDdeg}(Y) = 2$ and $\mathrm{pEDdeg}_{\lambda}(Y) = 6$ for generic λ. For a higher-dimensional generalization of this example, see [19, Example 3.3].

Acknowledgments. The author thanks Jose Rodriguez and Botong Wang for their collaboration on the work surveyed in this note. The author is partially supported by the Simons Foundation Collaboration Grant #567077, and by a Van Vleck Professorship Research Award (University of Wisconsin).

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Received December 18, 2019

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