CHEN TYPE OF SOME CLASSES OF CR-SUBMANIFOLDS IN $\mathbb{C}P^m$ AND $\mathbb{C}H^m$

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We survey some of the fundamental classification results on low-type submanifolds of non-flat model complex space forms (complex projective and hyperbolic spaces) via the standard embeddings by projection operators. These results include classification of submanifolds of type 1 in these spaces, of CMC and Hopf hypersurfaces of type 2, and investigation of the Chen type of totally real and Kähler submanifolds. Some examples of submanifolds of type 3 are presented. We also give some nonexistence results for certain families of CR-submanifolds of complex space forms of Chen type two. For example, there exist no holomorphic submanifolds of the complex hyperbolic space which are of type 2 via the standard embedding by projectors. This is contrasted with the situation in the complex projective space, where there exist some parallel Einstein Kähler submanifolds of type 2.

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1. THE IMMERSIONS OF FINITE TYPE

In the study of isometric immersions of Riemannian manifolds into Euclidean or psedo-Euclidean spaces, one considers an isometric immersion $x : M^n \to \mathbb{E}^N_{(K)}$ of a Riemannian n-manifold, whose position vector allows spectral resolution into a sum of a constant vector x_0 (accounting for a translation) and a finite number of vector eigenfunctions of the Laplacian on M,

$$x = x_0 + x_{t_1} + x_{t_2} + \cdots + x_{t_k}, \tag{1.1}$$

where $x_0 = \text{const}$, $x_{t_i} \neq \text{const}$, and $\Delta x_{t_i} = \lambda_{t_i} x_{t_i}$, $i = 1, ..., k$, $\lambda_{t_i} \in \mathbb{R}$, and the Laplacian acts on an $\mathbb{E}^N$-valued function componentwise. If all λ_{t_i}, $i = 1, 2, \ldots, k$ are different, the submanifold M^n is said to be of *k-type* (also of *Chen type k*) in the ambient (pseudo) Euclidean space via the immersion x. If such finite decomposition is not possible, the immersion x is of infinite type. This concept was introduced and developed by Bang-Yen Chen (see [3]) and over the past four decades many differential geometers contributed to the theory of submanifolds of finite type, which, in some ways, is an extension of the study of minimal submanifolds of Euclidean spaces and round spheres, the class of submanifolds which are of Chen type 1 in this terminology. An example of a surface of type 2 in $\mathbb{E}^3$ is a right circular cylinder

$$x(\theta, t) = (\cos\theta, \sin\theta, t) = (0, 0, t) + (\cos\theta, \sin\theta, 0),$$

for which the (non-constant) eigenfunction $(0, 0, t)$ comes from the 0-eigenspace of the Laplacian. When that happens for some x_{t_i} in the decomposition (1.1), the immersion x, i.e. the submanifold M^n (which must be necessarily non-compact), is of *null k-type*. For a compact submanifold M^n, the constant term x_0 is the center of mass, i.e. $x_0 = \frac{1}{Vol(M)} \int_M x\, dV$. The set of indices $T(x) = \{t_1, t_2, \ldots, t_k\}$, indicating from which eigenspaces of the Laplacian the constituent functions x_{t_i} came from, is called the *order* of the immersion.

The Chen type of a submanifold is not an intrinsic invariant, but it depends on a particular immersion used - one and the same underlying manifold can be of different types if different immersions are considered. For example, the Cartan hypersurface $M^3 = SO(3)/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, which is one of the minimal isoparametric hypersurface of S^4, is of 1-type in the ambient $\mathbb{E}^5$, but of 3-type in $\mathbb{E}^{15}$ via the Veronese immersion of S^4 by means of the products of coordinate functions in $\mathbb{E}^5$ [7].

For any desired type k and any preassigned order, one can construct a k-type immersion of that particular order, using the diagonal immersion of the standard immersions of a compact homogeneous space, but the codimension of such submanifold is, generally speaking, high. For submanifolds of small codimension in a Euclidean space, there is a paucity of known examples of those that are of finite type, which prompted B. Y. Chen to formulate a conjecture (still open), stating that the round spheres are the only compact finite-type hypersurfaces of Euclidean space.

When the image $x(M^n)$ of a submanifold lies in a certain centered hyperquadric (e.g. a hypersphere) of the ambient (pseudo) Euclidean space, the immersion is said to be *mass-symmetric* in that hyperquadric, if the constant term x_0 in decomposition (1.1) coincides with the center of that hyperquadric. Submanifolds of null k-type are automatically mass-symmetric, since the constant term x_0 can be changed to match any desired value.

After studying finite-type submanifolds in Euclidean spaces and spheres, the researchers in this area turned their attention to submanifolds of more general spaces such as complex space forms, because these ambient spaces admit some of the simplest equivariant embeddings with parallel second fundamental form into (pseudo) Euclidean spaces, realized by means of the projection operators associated to the orthogonal projections onto the complex lines. Equivariancy and constant isotropy of the embedding are important, because these properties enable us to do computations locally, at a selected point, and produce a simple formula for the expression of the shape operator of the embedding in the direction of the second fundamental form (formula (3.8)).

While the standard embeddings of projective spaces and their hyperbolic duals by projection operators are constant isotropic, in the sense that $||\sigma(X,X)||$ is constant for every unit tangent vector X, that is no longer the case with a general Grassmannian of a higher rank, for which there is no convenient expression for the shape operator, akin to the one available for the projective and hyperbolic spaces. That is why the study of finite type is more promising for submanifolds of projective and hyperbolic spaces than for submanifolds of a general Grassmannian [6].

By applying the Laplacian to (1.1) in succession k times and eliminating x_{t_i}, $i = 1, 2, \ldots, k$, from the resulting $k+1$ equations, we see that a k-type submanifold satisfies a polynomial equation in the Laplacian, viz. $P(\Delta)(x - x_0) = 0$, where P is the monic polynomial of degree k whose coefficients are elementary symmetric functions of the eigenvalues λ_{t_i}. For example a 2-type immersion $x = x_0 + x_p + x_q$ satisfies

$$\Delta^2 x - a\Delta x + b(x - x_0) = 0, \quad a = \lambda_p + \lambda_q, \;\; b = \lambda_p\lambda_q \tag{1.2}$$

The converse is also true for compact submanifolds, in the sense that if $P(\Delta)(x - x_0) = 0$, where P is a monic polynomial of degree k, then the immersion x is of finite type $\le k$.

2. STANDARD EMBEDDINGS OF $\mathbb{C}P^m$ AND $\mathbb{C}H^m$

We use the Hopf fibration $\pi : S^{2m+1}(1) \to \mathbb{C}P^m(4)$ from an odd dimensional unit sphere in $\mathbb{C}^{m+1} = \mathbb{R}^{2m+2}$ to the complex projective space. Given a point $p \in \mathbb{C}P^m$ we select $z \in \pi^{-1}(p) \subset S^{2m+1} \subset \mathbb{C}^{m+1}$, where z is regarded as a column vector with $m+1$ components, and define $\Phi(p) = zz^*$, where $*$ denotes conjugate transpose. The matrix $\Phi(p)$ does not depend on the selection of z in the fiber $\pi^{-1}(p) = [z]$ and defines an embedding of $\mathbb{C}P^m$ into the set of Hermitian matrices of order $m+1$,

$$H(m+1) = \{A \in M_{m+1}(\mathbb{C}) \,|\, A^* = A\},$$

which is made into a Euclidean space E^N of dimension $N = (m+1)^2$ by introducing the metric $\tilde{g}(A,B) = \frac{1}{2}\mathrm{tr}\,(AB)$. $\Phi([z])$ is the matrix of the projection operator P onto the complex line $\mathbb{C}z$ (a 2-dimensional plane in $\mathbb{R}^{2m+2}$) given by

$$P(v) = g(z,v)z + g(iz,v)iz,$$

where $g = \mathrm{Re}\,\Psi$ and $\Psi(z,w) = \sum_{k=0}^{m} \bar{z}_k w_k,\ z = (z_0, z_1, \ldots, z_m),\ w = (w_0, w_1, \ldots, w_m)$. The image of $\mathbb{C}P^m(4)$ under Φ is given by

$$\Phi(\mathbb{C}P^m) = \{P \in H(m+1) |\, P^2 = P,\ \mathrm{tr}_{\mathbb{C}} P = 1\}$$

and it lies in the hypersphere of $\mathbb{E}^N = H(m+1)$ centered at $I/(m+1)$ of radius $\sqrt{m/2(m+1)}$ as a minimal submanifold. It also lies in the hyperplane $\{A \,|\, \mathrm{tr}\, A = 1\}$, whose normal vector is the identity matrix I of order $m+1$. The embedding Φ has parallel second fundamental form σ and is $U(m+1)$-equivariant, where the action of the unitary group in $\mathbb{E}^N = H(m+1)$ is by conjugation. In other words, $\Phi(A[z]) = A\Phi([z])A^{-1},\ A \in U(m+1)$. This embedding is realized, up to a translation, by a basis of eigenfunctions of the Laplacian on $\mathbb{C}P^m$ coming all from the first eigenspace that corresponds to the smallest non-zero eigenvalue $\lambda_1 = 2(m+1)$ of the Laplacian. Hence, Φ is an example of a mass-symmetric embedding of Chen type 1, whose order is $\{1\}$.

A similar construction is used to get the complex hyperbolic space $\mathbb{C}H^m(-4)$ by Hopf fibration from the anti - de Sitter space H_1^{2m+1}, which is a complete Lorentzian hypersurface of $\mathbb{C}_1^{m+1}$ with the standard Hermitian form Ψ now of signature $(m,1)$. By identifying a time-like complex line $L = [z]$ with the operator of the orthogonal projection P onto L we obtain an embedding of $\mathbb{C}H^m$ into the space of Ψ-Hermitian matrices $H^1(m+1)$, which is a pseudo-Euclidean space $\mathbb{E}_K^N$ of dimension $N = (m+1)^2$, index $K = m^2+1$, and the metric $\tilde{g}(A,B) = -\frac{1}{2}\mathrm{tr}\,(AB)$. We denote the complex projective and hyperbolic spaces jointly by $\mathbb{C}Q^m(4c)$, with $c = +1$ corresponding to the projective case and $c = -1$ to the hyperbolic case. Then the projection onto a (time-like) complex line $L = [z]$, where $\Psi(z,z) = c$, is given by

$$P(v) = c\Psi(z,v)z = cg(z,v)z + cg(iz,v)iz,$$

for any $v \in \mathbb{C}^{m+1}$. Its matrix is

$$P = \begin{pmatrix} |z_0|^2 & cz_0\bar{z}_1 & \cdots & cz_0\bar{z}_m \\ z_1\bar{z}_0 & c|z_1|^2 & \cdots & cz_1\bar{z}_m \\ \vdots & \vdots & \ddots & \vdots \\ z_m\bar{z}_0 & cz_m\bar{z}_1 & \cdots & c|z_m|^2 \end{pmatrix} \in H^{(1)}(m+1) = E_{(K)}^N.$$

The image $\Phi(\mathbb{C}Q^m)$ lies in the hyperquadric centered at $I/(m+1)$ defined by

$$\tilde{g}\left(A - \frac{I}{m+1}, A - \frac{I}{m+1}\right) = \frac{cm}{2(m+1)},$$

as a minimal submanifold. Although the dimension N of $H^{(1)}(m+1)$ is $(m+1)^2$, the image $\Phi(\mathbb{C}Q^m)$ of the complex space form lies fully in the hyperplane $\{A \,|\, \mathrm{tr}\, A = 1\}$, so there is actually reduction of codimension by 1.

The essence of the preceding discussion is that there is an equivariant embedding Φ of a complex space form with parallel second fundamental form, namely

$$\Phi : \mathbb{C}P^m(4) \longrightarrow \mathbb{E}^N = H(m+1), \quad \tilde{g}(A,B) = \frac{1}{2}\mathrm{tr}\,(AB)$$

for the complex projective space and

$$\Phi : \mathbb{C}H^m(-4) \longrightarrow \mathbb{E}^N_K = H^1(m+1), \quad \tilde{g}(A,B) = -\frac{1}{2}\mathrm{tr}\,(AB)$$

for the complex hyperbolic space, equivariant with respect to the action of the Ψ-unitary group, where $\Psi(z,w) = c\,\bar{z}_0 w_0 + \sum_{j=1}^{m} \bar{z}_j w_j,\ z,w \in \mathbb{C}^{m+1}$.

For more details on the embedding Φ see [23], [18], [11], [5].

Therefore, with any isometric immersion $x : M^n \to \mathbb{C}Q^m(4c)$, the submanifold M^n of $\mathbb{C}Q^m$ can be seen also as a submanifold of the (pseudo) Euclidean space $\mathbb{E}^N_{(K)}$ via the composite immersion $\tilde{x} = \Phi \circ x$, so it makes sense to study Chen type of submanifolds of non-flat complex space forms via the embedding Φ. In other words, we want to see when such immersion $\tilde{x}$ can be decomposed, up to a translation, into a sum of one, two, tree, etc. vector eigenfunctions of the Laplace operator Δ_M of M. Such an eigenfunction, we recall, is a field of Ψ-Hermitian matrices over M, seen as $\mathbb{E}^N_{(K)}$-valued vector field.

3. SOME BASIC FORMULAS AND NOTATION

Let $x : M^n \to \mathbb{C}Q^m(4c)$ be an isometric immersion of a Riemannian n-manifold into a non-flat model space form $\mathbb{C}Q^m(4c)$. Let $\Phi : \mathbb{C}Q^m \to H^{(1)}(m+1)$ be the standard embedding of $\mathbb{C}Q^m$ into the set of Hermitian matrices of order $m+1$, as in the preceding section. The composite immersion $\tilde{x} = \Phi \circ x$ realizes M as a submanifold of a (pseudo) Euclidean space $E^N_{(K)} = H^{(1)}(m+1)$, equipped with the usual trace metric $\langle A,B\rangle = \frac{c}{2}\mathrm{tr}\,(AB)$. Let $\overline{\nabla}, \overline{A}, \overline{D}$, denote respectively the Levi-Civita connection, the Weingarten endomorphism, and the metric connection in the normal bundle, related to $\mathbb{C}Q^m$ and the embedding Φ. Let the same letters without bar denote the respective objects for a submanifold M and the immersion x, whereas the same symbols with tilde will denote the corresponding objects related to the composite immersion $\tilde{x} = \Phi\circ x$ of M into the (pseudo) Euclidean space $H^{(1)}(m+1)$. As usual, we use σ for the second fundamental form of $\mathbb{C}Q^m$ in $E^N_{(K)}$ via Φ and symbol h for the second fundamental form of a submanifold in $\mathbb{C}Q^m$. An orthonormal basis of the tangent space T_pM at a general point will be denoted by $\{e_i\}$, $i = 1, 2, \cdots, n$, and a basis of the normal space $T_p^{\perp}M$ will be represented by $\{e_r\}$. In general, indices i, j, k will range from 1 to n and indices r, s from $n+1$ to $2m$. The tangent vector fields to M will be denoted by letters $X, Y, \cdots$, and vectors normal to M in $\mathbb{C}Q^m$ by $\xi, \eta, \cdots$. For a normal basis vector e_r, the Weingarten map A_{e_r} is abbreviated to A_r and is related to the second fundamental form by

$$\langle h(X,Y), e_r\rangle = \langle A_r X, Y\rangle.$$

The mean curvature vector H is defined by $H := \frac{1}{n}\sum_i h(e_i, e_i) = \frac{1}{n}\sum_r (\operatorname{tr} A_r)e_r$, and the squared norm of the second fundamental form by

$$||h||^2 := \sum_{i,j}\langle h(e_i, e_j), h(e_i, e_j)\rangle = \sum_r \operatorname{tr} A_r^2.$$

For a normal vector $\xi \in T^\perp M$ we define its *ancillary normal vector field* $\hat{\mathfrak{a}}(\xi)$ by

$$\hat{\mathfrak{a}}(\xi) := \sum_i h(e_i, A_\xi e_i) = \sum_r \operatorname{tr}(A_\xi A_r)\, e_r, \tag{3.1}$$

which is related to the allied vector field $\mathfrak{a}(\xi)$ of Chen [3], the latter being the component of $\hat{\mathfrak{a}}(\xi)$ in the direction perpendicular to ξ. Then $\hat{\mathfrak{a}} : \Gamma(T^\perp M) \to \Gamma(T^\perp M)$ is a symmetric endomorphism of the normal bundle satisfying

$$\langle \hat{\mathfrak{a}}(\xi), \eta\rangle = \langle \xi, \hat{\mathfrak{a}}(\eta)\rangle = \operatorname{tr}(A_\xi A_\eta), \qquad \operatorname{tr}\hat{\mathfrak{a}} = \|h\|^2. \tag{3.2}$$

The operator $\hat{\mathfrak{a}}$ is called the *Simons operator* of the immersion (after [21]) and figures prominently in many of our formulas.

We give next some important formulas, which are repeatedly used in subsequent expressions for the iterated Laplacians. For a general submanifold M, local tangent fields $X, Y \in \Gamma(TM)$ and a local normal field $\xi \in \Gamma(T^\perp M)$, the formulas of Gauss and Weingarten are

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \qquad \overline{\nabla}_X \xi = -A_\xi X + D_X \xi. \tag{3.3}$$

Let J be the standard Kähler almost complex structure of $\mathbb{C}Q^m$ and M an arbitrary submanifold. Define an endomorphism S of the tangent space and a normal bundle valued 1-form F by $SX = (JX)_T$, $FX = (JX)_N$, i.e. for $X \in \Gamma(TM)$, $JX = SX + FX$ is the decomposition of JX into tangential and normal parts. Similarly, for a normal vector ξ we set $J\xi = (J\xi)_T + (J\xi)_N = L\xi + K\xi$, K defining an endomorphism of the normal space and $L : T^\perp M \to TM$ a linear map from the normal to the tangent space. The curvature tensor of $\mathbb{C}Q^m(4c)$ is given by

(3.4)

$$\overline{R}(X,Y)Z = c\,[\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY - 2\langle JX,Y\rangle JZ\,],$$

and the equations of Gauss and Codazzi for a submanifold of $\mathbb{C}Q^m(4c)$ are respectively given by

(3.5)
$$R(X,Y,Z,W) = \overline{R}(X,Y,Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle,$$

(3.6)
$$(\overline{\nabla}_X h)(Y,Z) - (\overline{\nabla}_Y h)(X,Z) = c\,[\langle SY,Z\rangle FX - \langle SX,Z\rangle FY - 2\langle SX,Y\rangle FZ].$$

The Ricci (1,1)-tensor Q of M as an endomorphism of the tangent space of M is defined by $Q(X) = \sum_i R(X,e_i)\,e_i$. From (3.4) and (3.5) one gets

$$Q(X) = c(n-1)X - 3cS^2X + nA_HX - \sum_r A_r^2 X. \tag{3.7}$$

The following formula of A. Ros for the shape operator of Φ in the direction of $\sigma(X,Y)$ is also well known (see, for example, [18] and [5])

(3.8)
$$\bar{A}_{\sigma(X,Y)}V = c\,[2\langle X,Y\rangle V + \langle Y,V\rangle X + \langle X,V\rangle Y + \langle JY,V\rangle JX + \langle JX,V\rangle JY].$$

We also have

$$\sigma(JX,JY) = \sigma(X,Y), \quad \langle \sigma(X,Y), \tilde{x}\rangle = -\langle X,Y\rangle, \quad \langle \sigma(X,Y),\, I\rangle = 0. \tag{3.9}$$

Aurel Bejancu [1], [2], defined a CR-submanifold of a Kähler manifold $(\overline{M}, g, J)$ with almost complex structure J, as a submanifold M for which there exists a differentiable distribution $\mathcal{D} : p \to \mathcal{D}_p \subset T_pM$ on M so that at each of its points (1) $\mathcal{D}$ is holomorphic, i.e. $J\mathcal{D}_p = \mathcal{D}_p$ and (2) The complementary orthogonal distribution $\mathcal{D}^\perp : p \to \mathcal{D}_p^\perp \subset T_pM$ is anti-invariant, i.e. $J\mathcal{D}_p^\perp \subset T_p^\perp M$ for every point $p \in M$. So, its tangent space splits at each point $p \in M$ into a direct sum of two complementary orthogonal subspaces $\mathcal{D}_p$ and $\mathcal{D}_p^\perp$ of constant dimensions so that

$$T_pM = \mathcal{D}_p \oplus \mathcal{D}_p^\perp, \quad \text{with} \quad J\mathcal{D}_p \subset \mathcal{D}_p \quad \text{and} \quad J\mathcal{D}_p^\perp \subset T_p^\perp M.$$

When dim $\mathcal{D}_p^\perp = 0$, a submanifold is said to be *holomorphic* (or $J-$invariant), and when dim $\mathcal{D}_p = 0$, it is said to be *totally real* submanifold. Further, if dim $\mathcal{D}_p^\perp = 1$ at every point, a CR-submanifold is said to be of maximal CR-dimension [10]. At each point of a CR-submanifold M, the tangent space of the ambient manifold $\overline{M}$ is decomposed into an orthogonal direct sum of subspaces as follows

$$T_p\overline{M} = \mathcal{D}_p \oplus \mathcal{D}_p^\perp \oplus J\mathcal{D}_p^\perp \oplus \mathcal{W}_p,$$

where the first two subspaces span T_pM, the last two span $T_p^\perp M$, and $\mathcal{D}$ and $\mathcal{W}$ are $J-$invariant (holomorphic) subspaces. It can be shown that M is a CR-submanifold if and only if $FS = 0$ [29]. A totally real submanifold M^n is called a *Lagrangian submanifold* if $T_p^\perp M = J(T_pM)$ at each point $p \in M^n$.

The gradient of a smooth function f is a vector field given by $\nabla f = \sum_i (e_i f) e_i$ and the Laplacian of f is

$$\Delta f = \sum_{i=1}^{n} [(\nabla_{e_i} e_i) f - e_i e_i f].$$

The Laplace operator is then extended to act on a vector field V along $\tilde{x}(M)$ by

$$\Delta V = \sum_i [\widetilde{\nabla}_{\nabla_{e_i} e_i} V - \widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} V].$$

The product formula for the Laplacian, which is used in the ensuing computations, is

$$\Delta(f\, g) = (\Delta f)\, g + f(\Delta g) - 2 \sum_i (e_i f)(e_i g), \tag{3.10}$$

for smooth functions $f,\, g \in C^\infty(M)$ and it can be then extended to the scalar product of vector valued functions, so that the formula is valid also for matrix products.

The following computation of the second iterated Laplacian of $\tilde{x}$ is presented in detail in [9]. From the Beltrami's formula $\Delta \tilde{x} = -n\widetilde{H}$ we get

$$\Delta \tilde{x} = -n\widetilde{H} = -nH - \sum_{i=1}^{n} \sigma(e_i, e_i), \tag{3.11}$$

where, here and in the following computations, we understand the Laplacian to be applied to vector fields along M (viewed as $E^N_{(K)}-$valued functions, i.e. matrices) componentwise. Taking the Laplacian of (3.11) and using the preceding formulas we obtain, after a long computation, the expression for the second iterated Laplacian of the immersion $\tilde{x}$:

$$\begin{aligned}
\Delta^2 \tilde{x} \;\; = \;\; & - 2n \operatorname{tr} A_{DH} - \frac{n^2}{2} \nabla \alpha^2 - 3cnS(LH) - n\Delta^\perp H - cn(3n+4)H \\
& + cnJ(JH)_T - n\hat{\mathfrak{a}}(H) - 4c \sum_{i,j} \langle Jh(e_j, e_i), e_j \rangle Je_i \\
& + n^2 \sigma(H,H) - 2c(n+1) \sum_i \sigma(e_i, e_i) + 6c \sum_i \sigma(FSe_i, e_i) \\
& - 2n \sum_i \sigma(e_i, A_H e_i) + 4n \sum_i \sigma(e_i, D_{e_i} H) - 2c \sum_i \sigma(Se_i, Je_i) \\
& - 2 \sum_{i,r} \sigma(A_r e_i, A_r e_i) + 2 \sum_r \sigma(e_r, \hat{\mathfrak{a}}(e_r)),
\end{aligned} \tag{3.12}$$

where $\alpha^2 = \langle H, H \rangle$ is the squared mean curvature and

$$\operatorname{tr} A_{DH} := \sum_i A_{D_{e_i} H} e_i, \qquad \Delta^\perp H := \sum_i \Big(D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H \Big).$$

Formula (3.12) holds for any submanifold of $\mathbb{C}Q^m(4c)$. Note that for CR-submanifolds, $S \circ L = 0$ and $F \circ S = 0$, so that the above expression simplifies accordingly.

4. SUBMANIFOLDS OF $\mathbb{C}Q^m(4c)$ OF TYPE 1 OR 2

The study of 1-type submanifolds of $\mathbb{C}P^m$ was initiated in [18], where CR-minimal submanifolds of the complex projective space were classified. A parallel investigation for hypersurfaces of $\mathbb{C}H^m$ was later carried out in [11]. The complete classification of 1-type submanifolds of a non-flat complex space form $\mathbb{C}Q^m(4c)$, without any a priori assumptions, was achieved in our papers [4], [5] and is presented below. These submanifolds are of three kinds: canonically embedded complex space forms of lower dimensions, minimal Lagrangian submanifolds M^n of a canonically embedded $\mathbb{C}Q^n$, and, in the case of $\mathbb{C}P^m$ only, a geodesic sphere of radius arctan $\sqrt{n+2}$ of a canonical $\mathbb{C}P^n$.

Theorem 1 ([5]). *Let M^n be a complete connected Riemannian manifold and $x : M^n \to \mathbb{C}Q^m(4c)$ an isometric immersion into a non-Euclidean complex space form. Then $\tilde{x} = \Phi \circ x$ is of Chen type1 if and only if one of the following cases occurs*

(i) *n is even, M^n is congruent to a complex space form $\mathbb{C}Q^{n/2}(4c)$ and x embedds M^n as a complex totally geodesic $\mathbb{C}Q^{n/2}(4c) \subset \mathbb{C}Q^m(4c)$.*

(ii) *M^n is immersed as a totally real minimal submanifold of a complex totally geodesic $\mathbb{C}Q^n(4c) \subset \mathbb{C}Q^m(4c)$.*

(iii) *n is odd and M^n is embedded by x as a geodesic hypersphere of radius $\rho = \cot^{-1}(1/\sqrt{n+2})$ of a complex, totally geodesic $\mathbb{C}P^{(n+1)/2}(4c) \subset \mathbb{C}P^m(4c)$.*

There is also a local version of this result. Note that all three examples are CR-submanifolds of $\mathbb{C}Q^m$: $\mathbb{C}Q^{n/2}$ (n even) in part (*i*) is a holomorphic submanifold, any submanifold in part (*ii*) is a totally real (actually Lagrangian) one, and the geodesic sphere in (*iii*) is an example of CR-submanifold of maximal CR-dimension. This particular geodesic sphere is not minimal (the minimal one has radius $\rho = \cot^{-1}(1/\sqrt{n})$), but is distinguished by a certain stability property, namely it is the maximal stable geodesic hypersphere of $\mathbb{C}P^n(4)$ under variations that preserve the enclosed volume.

Submanifolds of complex space forms of Chen-type 2 have been studied by several authors. A non-totally geodesic compact Kähler submanifold M^n of type 2, lying fully in $\mathbb{C}P^m$, was characterized by A. Ros [19] by being Einstein-Kähler submanifold for which the Simons operator $\hat{\mathfrak{a}}$ is constant-homothetic. This condition is then shown in [25] to be equivalent to M^n being Einstein parallel submanifold. The known list of these parallel submanifolds from [17] was used to obtain the following classification

Theorem 2 ([26]). *Let $x : M^n \to \mathbb{C}P^m$ be a full isometric holomorphic immersion of a compact Kähler manifold, which is not totally geodesic. Then $\tilde{x} = \Phi \circ x$ is of type 2 if and only if M is an Einstein Kähler parallel submanifold of degree 2, i.e. one of the following:*

(i) $\mathbb{C}P^n(1/2)$ *with complex codimension* $n(n+1)/2$.

(ii) *A complex quadric* Q^n *with complex codimension 1.*

(iii) $\mathbb{C}P^n \times \mathbb{C}P^n$ *with complex codimension* n^2.

(iv) $U(s+2)/U(2) \times U(s)$, $s \geq 3$, *with complex codimension* $s(s-1)/2$.

(v) $SO(10)/U(5)$ *with complex dimension 10 and complex codimension 5.*

(vi) $E_6/Spin(10) \times T$ *with complex dimension 16 and complex codimension 10.*

It is noteworthy, as observed by Udagawa [26], that all of these submanifolds have the same order $\{1, 2\}$. With the addition of totally geodesic complex submanifolds and Segre embeddings of general $\mathbb{C}P^k \times \mathbb{C}P^l$, these are precisely the complex symmetric submanifolds (equivalently, those with parallel second fundamental form) of $\mathbb{C}P^m$, by a result of [17]. In particular, $\mathbb{C}P^n$ in part (*i*) is not a totally geodesic one, but immersed by the second canonical embedding F_2. Example (*ii*) is the Grassmannian $G_2^+(\mathbb{R}^{n+1})$ of oriented 2-planes, embedded as a complex quadric that is determined in homogeneous coordinates by the equation $z_0^2 + z_1^2 + \cdots + z_n^2 = 0$. Example (*iii*) is realized by the Segre embedding, explicitly given in homogeneous coordinate by

$$([z_0 : \cdots : z_n], [w_0 : \cdots : w_n]) \mapsto [z_0 w_0 : \cdots : z_\mu w_\nu : \ldots z_n w_n]$$

with all possible products of the coordinates, each coordinate of z multiplying each coordinate of w. The example in part (*iv*) is realized by the Plücker embedding of the complex 2-plane Grassmannian $G_2(\mathbb{C}^n)$.

In contrast to this situation, we have the following non-existence result for submanifolds of $\mathbb{C}H^m(-4)$.

Theorem 3 ([9]). *There are no holomorphic immersions of Kähler manifolds into $\mathbb{C}H^m$ which are of 2-type in $H^1(m+1)$.*

In the proofs of both of the preceding two theorems, the key step is to show that a submanifold is Einstein and that Simons operator $\hat{\mathfrak{a}}$ is constant-homothetic. Since the main result of Umehara [28] states that holomorphic Einstein submanifolds of $\mathbb{C}H^m$ must be totally geodesic (and these are actually of type 1), the result of Theorem 3 follows. Furthermore, it was shown in [25] that homothetic $\hat{\mathfrak{a}}$ for a Kähler Einstein submanifold M^n of $\mathbb{C}P^m$ is equivalent to the normal bundle $T^\perp M$ admitting Einstein Kähler metric, which is further shown to be equivalent to M^n being parallel Einstein Kähler submanifold, which led to the classification in Theorem 2.

On the opposite end from holomorphic submanifolds among all CR-submanifolds are the totally real (or anti-invariant) submanifolds, for which the tangent space of such submanifold is sent into the normal space by the almost complex structure J of $\mathbb{C}Q^m(4c)$. From $\bar{\nabla}_X(JY) = J\bar{\nabla}_X Y$ we get

$$A_{JY}X = A_{JX}Y = -[Jh(X,Y)]_T = -L(h(X,Y)), \tag{4.1}$$

so that

$$\sum_{i,j}\langle Jh(e_j,e_i),e_j\rangle Je_i = -\sum_j JA_{Je_j}e_j = nJ(LH). \tag{4.2}$$

This expression is then used in formula (3.12) in order to analyze the 2-type equation (1.2) for minimal or mass-symmetric totally real submanifolds of $\mathbb{C}Q^m$. Separating the components of that equation in all possible directions leads to the following characterization

Theorem 4 ([9]). *Let $x : M^n \to \mathbb{C}Q^m(4c)$ be a totally real isometric immersion of a Riemannian $n-$manifold into a non-flat complex space form of constant holomorphic sectional curvature $4c$, $c = \pm 1$, and let $T^\perp M = \mathcal{V}\oplus\mathcal{W}$ be the orthogonal decomposition of the normal bundle into the totally real and holomorphic subbundles. Then if $\tilde{x} = \Phi\circ x : M^n \to H^{(1)}(m+1)$ is a mass-symmetric immersion of type 2 satisfying*

$$\Delta^2\tilde{x} - p\Delta\tilde{x} + q(\tilde{x} - I/(m+1)) = 0,$$

we have

(i) *The mean curvature α is constant.*

(ii) $tr\, A_{DH} = 0$;

(iii) $\Delta^\perp H + \hat{\mathfrak{a}}(H) + [c(3n+4) - p]\,H + 3cJ(LH) = 0$;

(iv) $Q(X) - J(\hat{\mathfrak{a}}(JX)) - 2nA_H X - kX + \frac{n^2}{2}\langle LH, X\rangle LH = 0$, *for every $X \in \Gamma(TM)$, where $k = \frac{c}{4}[q + 2cf^2 + 4n(n+3) - 2c(n+1)p]$ is a constant;*

(v) $\langle JD_X H, Y, \rangle = \langle JD_Y H, X\rangle$, *for every* $X, Y \in \Gamma(TM)$;

(vi) $\frac{n^2}{2}\langle LH, X\rangle KH - nK^2(D_X H) - K(\hat{\mathfrak{a}}(JX)) = 0$;

(vii) $\frac{n^2}{2}\langle KH, \xi\rangle LH + n\sum_i\langle D_{e_i}H, \xi\rangle e_i - L(\hat{\mathfrak{a}}(J\xi)) = 0$, *for* $\xi \in \mathcal{W}$;

(viii)
$$\frac{n^2}{2}\langle H, \xi\rangle H + \frac{n^2}{2}\langle KH, \xi\rangle KH + k'\xi \\ - n\sum_i\langle D_{e_i}H, K\xi\rangle Je_i + \hat{\mathfrak{a}}(\xi) - K(\hat{\mathfrak{a}}(J\xi)) = 0,$$

for any $\xi \in \mathcal{W}$ *where* $k' = \frac{c}{4}[2cnp - 4n(n+1) - 2cf^2 - q]$.

Conversely, if (i) – (viii) hold, then the immersion is mass-symmetric and of type 1 or 2, provided that the polynomial $t^2 - pt + q = 0$ *has simple real roots or* M *is compact.*

Shen [20] attempted a study of minimal totally real submanifolds in $\mathbb{C}P^m$ which are of Chen type 2, obtaining some information on eigenvalue relations. Nonetheless, it seems difficult to obtain any concrete classification or information on totally real submanifolds of type 2 (minimal, mass-symmetric, etc.) beyond the above relations describing their geometry. While any classification of totally real submanifolds of type 2 in $\mathbb{C}Q^m$ remains elusive at present, one can try to determine the Chen type of some model examples of totally real symmetric submanifolds in $\mathbb{C}P^m$ given in [15] and [16], such as (in irreducible case) $SU(n)/SO(n), SU(n), SU(2n)/Sp(n)$ and E_6/F_4.

From this characterization we obtain for a Lagrangian immersion the following

Corollary 5 ([9]). *If* $x : M^n \to \mathbb{C}Q^n(4c)$ *is a Lagrangian immersion for which* $\tilde{x}$ *is mass-symmetric and of type 2 then*

(i) $f := n\alpha = const$;

(ii) $tr\,A_{DH} = 0$

(iii) $\Delta^\perp H + \hat{\mathfrak{a}}(H) + [c(3n+1) - p]\, H = 0$;

(iv) $Q(X) - J(\hat{\mathfrak{a}}(JX)) - 2nA_H X - kX + \frac{n^2}{2}\langle JH, X\rangle JH = 0$, *for every* $X \in \Gamma(TM)$

(v) $\langle JD_X H, Y\rangle = \langle JD_Y H, X\rangle$, *for every* $X, Y \in \Gamma(TM)$.

(vi) $D_{JH}H = 0, \quad \nabla_{JH}(JH) = 0, \quad$ *and* $\quad A_H(JH) = Jh(JH, JH)$.
In particular, the integral curves of JH are geodesics.

As we know from Theorem 1 (*ii*), minimal Lagrangian submanifolds in $\mathbb{C}Q^m(4c)$ are actually of type 1 (they are also mass-symmetric) and the only totally real submanifolds M^n of $\mathbb{C}Q^m$ which are of type 1 are minimal Lagrangian submanifolds in a canonically embedded $\mathbb{C}Q^n \subset \mathbb{C}Q^m$, [18] , [5]. It would be interesting to find out which totally real minimal submanifolds (which are not Lagrangian in some smaller-dimensional $\mathbb{C}Q^n$) are of type 2.

The classification of real hypersurfaces of Chen type 2 was undertaken in [14], [27] for hypersurfaces of $\mathbb{C}P^m$ with constant mean curvature and in [8] for Hopf hypersurfaces of $\mathbb{C}Q^m$, where Udagawa's classification from [27] was corrected and completed. A hypersurface $M^{2m-1} \subset \mathbb{C}Q^m$ is a Hopf hypersurface if the structure vector field $U := -J\xi$, where ξ is a unit normal to the hypersurface, is proper for the shape operator A, i.e $A(U) = \mu U$.

Theorem 6 ([8]). *Let M^{2m-1} be a Hopf hypersurface of $\mathbb{C}P^m(4)$, $(m \geq 2)$. Then M^{2m-1} is of type 2 in $H(m+1)$ via Φ if and only if it is an open portion of one of the following*

(i) *A geodesic hypersphere of any radius $r \in (0, \frac{\pi}{2})$, except $r = \cot^{-1}\sqrt{\frac{1}{2m+1}}$*

(ii) *The tube of radius $r = \cot^{-1}\sqrt{\frac{k+1}{m-k}}$ about a canonically embedded totally geodesic $\mathbb{C}P^k(4) \subset \mathbb{C}P^m(4)$, for any $k = 1, ..., m-2$*

(iii) *The tube of radius $r = \cot^{-1}\sqrt{\frac{2k+1}{2(m-k)+1}}$ about a canonically embedded $\mathbb{C}P^k(4) \subset \mathbb{C}P^m(4)$, for any $k = 1, ..., m-2$*

(iv) *The tube of radius $r = \cot^{-1}(\sqrt{m} + \sqrt{m+1})$ about the complex quadric $Q^{m-1} \subset \mathbb{C}P^m(4)$*

(v) *The tube of radius $r = \cot^{-1}\sqrt{\sqrt{2m^2-1} + \sqrt{2m^2-2}}$ about the complex quadric $Q^{m-1} \subset \mathbb{C}P^m(4)$.*

It turns out that under the 2-type condition, being a Hopf hypersurface is equivalent to having constant mean curvature (CMC). Therefore, the same classification holds when M is assumed to have constant mean curvature, instead of being Hopf. In that regard Theorems 1 and 2 in [27] are deficient and incomplete since Udagawa's list contains examples $(i)-(iii)$ only. The list of items $(i)-(v)$ is the correct and complete classification of CMC hypersurfaces of type 2 in $\mathbb{C}P^m(4)$. Likewise, a previous announcement of our theorem in [6] is incomplete, since it was based on Udagawa's result.

In the complex hyperbolic space we have the following

Theorem 7 ([8]). *Let M^{2m-1} be a real hypersurface of $\mathbb{C}H^m(-4)$, ($m \geq 2$) for which we assume that it is a Hopf hypersurface or has constant mean curvature. Then M^{2m-1} is of type 2 in $H^1(m+1)$ via Φ if and only if it is (an open portion of) either a geodesic hypersphere of arbitrary radius $r > 0$ or a tube of arbitrary radius $r > 0$ about a totally geodesic complex hyperbolic hyperplane $\mathbb{C}H^{m-1}(-4)$.*

Another well-studied class of hypersurfaces of $\mathbb{C}Q^m$ is the class of ruled real hypersurfaces [12], [13]. Such a hypersurface M^{2m-1} is obtained by moving a totally geodesic $\mathbb{C}Q^{m-1}(4c)$ along a curve in $\mathbb{C}Q^m(4c)$ so that M is foliated by complex hyperplanes $\mathbb{C}Q^{m-1}(4c)$. Ruled hypersurfaces are never Hopf hypersurfaces, as AU has components in both U and $U^\perp$ directions, where $U := -J\xi$ is the structure vector field. Thus, there is a unit tangent vector $W \perp U$ in the holomorphic subspace of TM so that $AW = \nu U$, $\nu \neq 0$. Then, $AU = \langle AU, U\rangle U + \nu W$. For these hypersurfaces we have

Theorem 8 ([9]). *There exist no real ruled hypersurface of $\mathbb{C}Q^m$ which is mass-symmetric and of type 2 in $E^N_{(K)}$.*

5. SOME EXAMPLES OF CR-SUBMANIFOLDS OF 3-TYPE

In his papers [24] and [26], S. Udagawa gave several examples of compact Kähler submanifolds in $\mathbb{C}P^m$ that are of type 3.

Example 1 ([26]). All compact irreducible Hermitian symmetric submanifolds of degree 3 in $\mathbb{C}P^m(1)$ are of type 3 and of order $\{1, 2, 3\}$. They have one of the following homogeneous representations

$$\mathbb{C}P^n(1/3),\quad SU(r+3)/S(U(r)\times U(3))\,(r\geq 3),\quad Sp(3)/U(3),$$
$$SO(12)/U(6),\quad SO(14)/U(7),\quad E_7/E_6\times T.$$

Example 2 ([24]). The Segre embedding $f : \mathbb{C}P^n \times \mathbb{C}P^m \longrightarrow \mathbb{C}P^{mn+n+m}$, where $m > n$, is of type 3 and of order (a) $\{1, 3, 4\}$, if $2n + 4 < m + 1$ (b) $\{1, 2, 4\}$, if $m + 1 < 2n + 4 < n + m + 2$ (c) $\{1, 2, 3\}$, if $2n + 4 = m + 1$ or $n + m + 2 \leq 2n + 4$. None of these submanifolds is an Einstein manifold.

Example 3 ([24]). The extended Segre embedding

$$\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n \longrightarrow \mathbb{C}P^{(n+1)^3-1} \quad (n \geq 2),$$

where homogeneous coordinates, one chosen in each of the three factors, are multiplied together for all possible selections of these coordinates to obtain the embedding. This embedding is of type 3 and of order $\{1, 2, 4\}$. Moreover, the embedded submanifold in this case is Einstein.

Hopf hypersurfaces of $\mathbb{C}P^m$ and $\mathbb{C}H^m$, $m \geq 2$, with constant principal curvatures are homogeneous and they are known. There are six classes of them in $\mathbb{C}P^m$ labeled as A_1, A_2, B, C, D, and E (forming what is known as Takagi's list) and five of them in $\mathbb{C}H^m$, labeled as A_0, A_1', A_1'', A_2 and B (forming the so-called Montiel's list). In particular, a class-B Hopf hypersurface in $\mathbb{C}P^m(4)$ is a tube of arbitrary radius $r \in (0, \pi/4)$ about a canonically embedded complex quadric $Q^{m-1} = SO(m+1)/SO(2) \times SO(m-1)$, whereas a class-$B$ Hopf hypersurface in $\mathbb{C}H^m(-4)$ is a tube of arbitrary radius $r \in \mathbb{R}_+$ about the canonically embedded (totally geodesic, totally real) $\mathbb{R}H^m$ in $\mathbb{C}H^m$. The two examples presented in Theorem 6 $(iv), (v)$, are class-B Hopf hypersurfaces with constant principal curvatures which are of Chen type 2. We can show that all the others in that class are of Chen type 3. Moreover, any Hopf hypersurface of class-B is mass-symmetric in a hyperquadric of $\mathbb{E}^N_{(K)}$ centered at $I/(m+1)$.

Example 4 ([8]). Every class-B Hopf hypersurface in $\mathbb{C}H^m(-4)$ is mass-symmetric and of type 3 via $\tilde{x}$.

Example 5 ([8]). Every class-B Hopf hypersurface in $\mathbb{C}P^m(4)$ is mass-symmetric and of type 3, with the exception of those two tubes about Q^{m-1} given in Theorem 2 $(iv), (v)$, which are mass-symmetric and of type 2.

Continuing the investigation, it is possible to compute the third iterated Laplacian of $\tilde{x}$ for holomorphic submanifolds. Namely, using the necessary formulas from Section 3, take the Laplacian of a simplified formula (3.12) for holomorphic submanifolds to get

$$\begin{aligned}
\Delta^3 \tilde{x} \;=\; & -8c\,\nabla\|h\|^2 - 4(n+2)(3n+8)\sum_i \sigma(e_i, e_i) - 4\sum_i \sigma(Q^2 e_i, e_i) \\
& + 4c(3n+8)\sum_i \sigma(Qe_i, e_i) - 8\sum_{i,r} \sigma((\nabla_{e_i} Q)(A_r e_i), e_r) \\
& + 2\sum_i \sigma((\Delta Q)e_i, e_i) + 4\sum_{i,j,r} \sigma(h(e_i, A_r e_j), h(e_i, A_r e_j)) \\
& + 4c(n+2)\sum_r \sigma(e_r, \hat{\mathfrak{a}}(e_r)) + 4\sum_r \sigma(\hat{\mathfrak{a}}(e_r), \hat{\mathfrak{a}}(e_r)) \qquad (5.1) \\
& - 4\sum_{i,r,s} \operatorname{tr}(A_r A_s)\sigma(A_r e_i, A_s e_i) + 8\sum_{i,r} \sigma(A_r e_i, (D_{e_i}\hat{\mathfrak{a}})e_r) \\
& + 2\sum_r \sigma(e_r, (\Delta^{\perp}\hat{\mathfrak{a}})e_r).
\end{aligned}$$

where Q is the Ricci tensor.

Since the only term tangent to a holomorphic submanifold M in these iterated Laplacians is $-8c\nabla\|h\|^2$, it follows that a type-3 full holomorphic immersion $x : M^n \to \mathbb{C}Q^m$ is mass-symmetric if and only if $\|h\|$ is constant, which is equivalent, by means of the Gauss equation, to M having constant scalar curvature and conversely. The formula (5.1) is essential in any investigation of holomorphic submanifolds of complex space forms of type 3, the study of which we hope to pursue in a subsequent paper.

In this survey we presented results on CR-submanifolds of low Chen type, dealing primarily with real hypersurfaces and complex and totally real submanifolds, the two classes that are on the opposite ends of the entire spectrum of CR-submanifolds. The case of Chen type of a general (proper) CR-submanifold, which is none of the mentioned kinds, largely remains *terra incognita* and deserves closer scrutiny in future research.

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