GENERALIZED DIRAC'S EQUATION AND ITS Z_3-GRADED SYMMETRIES

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The present article investigates the Lorentz-Poincaré covariance properties of the generalized Dirac equation for quarks endowed, besides the half-integer spin, with "color", a Z_3-graded discrete variable. Thus to the the $Z_2 \times Z_2$ symmetry of the Dirac equation, corresponding to the double-valued spin variable and the charge conjugation imposing the particle-antiparticle duality, we shall add an extra Z_3 symmetry describing the color variable. The difference with currently accepted QCD is that instead of three Dirac spinors carrying different colors, the generalized equation introduced in ([14], [6], [7], [9]) attributes color variables to double-valued Pauli spinors first, and the charge conjugation next, creating 12-component wave functions. The 3×3 generators of the $SU(3)$ color Lie algebra appear spontaneously.

The invariance under Z_3-graded spinorial representation of the Lorentz-Poincaré algebra is investigated. It leads to the enlargement of symmetry by introduction of new degrees of freedom, corresponding to extra $Z_2 \times Z_3$ symmetry, which we attribute to the existence of three families with two flavors in each ([12], [13]). The orbital representation of Z_3-graded Lorentz-Poincaré algebra is constructed using the enlarged Z_3-graded Minkowskian space-time $M_4 \times Z_3$. Matrix and differential operator representations are presented *in extenso*, along with the generalized Casimir operators.

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1. FROM PAULI'S TO DIRAC'S EQUATION

Let us start by recalling the principles of Quantum Mechanics in its version based on Schrödinger's representation.

The *quantum states* of an elementary object (usually an elementary particle) are elements of some Hilbert space $\mathcal{H}$, which is supposed to be a *module* on which certain operator algebra acts. In Dirac's notation the states are represented by "ket" vectors $\mid \psi >$. The scalar product in Hilbert space is represented by a product of a "ket" vector with its conjugate "bra" vector, denoted by $< \chi \mid$ giving a complex number:

$$< \chi \mid \psi > \in \mathbf{C}$$

Operators are denoted with a hat over their symbols, e.g. $\hat{E}$, $\hat{\mathbf{p}}$ for the energy and momentum operators, etc.

Observable quantities must be represented by *hermitian operators*, such that $\hat{S}^\dagger = \hat{S}$.

The system yields a definite measure of observable k when it is found in an eigenstate of the corresponding quantum operator $\hat{K}$:

$$\hat{K} \mid \psi >= k \mid \psi >, \quad k \in R^1 \text{ if } \hat{K}^\dagger = \hat{K}.$$

The *superposition principle* announces that for any two states $\mid \chi >$ and $\mid \psi >$ their linear superposition (with complex coefficients) is also an accessible state of the system:

$$\mid \chi >, \ \mid \psi > \in \mathcal{H} \to \alpha \ \mid \chi > + \beta \ \mid \psi >= \mid \varphi > \in \mathcal{H}.$$

In Schrödinger's realization the states are represented by complex functions of time and space, belonging to the functional Hilbert space L^2, i.e. square-integrable functions of space (but not necessarily over time!).

Then classical relationships between velocity, kinetic energy, momentum and potential energy of a point-like particle with a given mass m are transposed into the quantum realm as differential equations which must be satisfied by the wave function describing the probability for this particle to be found in a given state. The equations are obtained via the *correspondence principle*:

$$E \to -i\hbar \, \frac{\partial}{\partial t}, \ \ \mathbf{p} \to -i\hbar \, \boldsymbol{\nabla} = -i\hbar [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}].$$

The Schrödinger equation can be interpreted now as an expression for the energy of a point-like particle interacting with potential $V(\mathbf{r})$:

$$E = \frac{m\mathbf{v}^2}{2} + V(\mathbf{r}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

translated into the following differential equation for the unknown "wave function" $\psi(\mathbf{r}, t)$:

$$-i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \triangle \psi + V(\mathbf{r}) \, \psi. \tag{1}$$

After the discovery of spin of the electron (the Stern-Gerlach experiment), Pauli understood that a Schroedinger-like equation involving only one complex-valued wave function is not enough to take into account this new degree of freedom. He proposed then to describe the dichotomic spin variable by introducing a two-component function forming a column on which hermitian matrices can act as linear operators.

The basis of complex traceless 2×2 hermitian matrices contains just three elements since then known as the *Pauli matrices*:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3].$$

The three Pauli matrices multiplied by $\frac{i}{2}$ span the three dimensional Lie algebra: let $\tau_k = \frac{i}{2}\sigma_k$, then

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2.$$

On the other hand, the three Pauli matrices form the Clifford algebra related to the Euclidean 3-dimensional metric:

$$\sigma_i \sigma_k + \sigma_k \sigma_i = 2\delta_{ik} \mathbb{1}_2$$

The simplest linear relation between the operators of energy, mass and momentum acting on a column vector (called *a Pauli spinor*) would read then:

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & mc^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + c\, \boldsymbol{\sigma} \cdot \mathbf{p} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \tag{2}$$

where

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \sigma_1\, p^1 + \sigma_2\, p^2 + \sigma_3\, p^3 = \begin{pmatrix} p^3 & p^1 - i\, p^2 \\ p^1 + i\, p^2 & -p^3 \end{pmatrix}. \tag{3}$$

We can write (2) in a simplified manner, denoting the Pauli spinor by one letter ψ and treating the unit matrix symbolically like a number:

$$E\, \psi = mc^2\, \psi + c\, \boldsymbol{\sigma} \cdot \mathbf{p}\, \psi. \tag{4}$$

This equation is not invariant under Lorentz transformations. Indeed, by iterating, i.e. taking the square of this operator, we arrive at the following relation between the operators of energy and momentum and the mass of the particle:

$$E^2 = m^2 c^4 + 2\, mc^3\, \boldsymbol{\sigma} \cdot \mathbf{p} + c^2 \mathbf{p}^2, \tag{5}$$

instead of the relativistic relation

$$E^2 - c^2\, \mathbf{p}^2 = m^2 c^4. \tag{6}$$

The double product in the expression for the energy squared can be removed if one introduces a second Pauli spinor satisfying a smilar equation, but with negative mass term, and intertwining the two spinors. In 1924 Pauli was not ready to accept the idea of negative mass (or negative energy); this is why he did not continue in this direction, investigating instead a non-relativistic equation for the electron endowed with spin and magnetic moment, interacting with electric and magnetic fields. This equation bears his name:

$$E\, \psi = \left[\frac{1}{2m} \left(\boldsymbol{\sigma} \cdot (\mathbf{p} - e\mathbf{A}) \right)^2 + e\, V \right] \psi \tag{7}$$

where $\mathbf{A}$ is the electromagnetic vector potential, and V the scalar potential.

Let us introduce a second Pauli spinor with the negative sign for mass. Denote the first Pauli spinor by ψ_+ and the second one by ψ_-, and let them satisfy the following coupled system of equations:

$$E\,\psi_+ = mc^2\,\psi_+ + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,$$

$$E\,\psi_- = -mc^2\,\psi_- + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+, \tag{8}$$

which coincides with the relativistic equation for the electron found by Dirac a few years later. The relativistic invariance is now manifest: due to the negative mass term in the second equation, the iteration leads to the separation of variables, and all the components satisfy the desired relation

$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2c^4\,\psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2c^4\,\psi_-.$$

In a more appropriate basis the Dirac equation becomes manifestly relativistic: $[\gamma^\mu p_\mu - mc]\ \psi = 0$, with $p_0 = \frac{E}{c}$,

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \gamma^k = (i\sigma_2)\otimes\sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{9}$$

The choice of tensor products in (9) is dictated by the following considerations.

In order to fulfil the requirements imposed by the Minkowskian metric and to implement the corresponding Clifford algebra, we must keep the trace zero condition, implement the anti-commutation between its four generators, and make the result of the anti-commutators proportional to the Minkowskian metric tensor $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

Recalling the rules of matrix multiplication of tensor products,

$$(A\otimes a)\cdot(B\otimes b) = (A\cdot B)\otimes(a\cdot b), \tag{10}$$

we see that if we choose

$$\gamma^0 = a\otimes\mathbb{1}_2 \ \text{ and } \ \gamma^i = b\otimes\sigma^i, \tag{11}$$

with a, b two different 2×2 matrices, then, in order to fulfil the above conditions, the matrix a must be traceless, and as $\mathbb{1}_2$ commutes with all σ^i, a should anti-commute with b; finally, the square of b should give $-\mathbb{1}_2$. Therefore if we choose $a = \sigma_3$, then b should be either $\imath\sigma_1$ or $i\sigma_2$. In fact, six different choices are possible, but the most frequent one (the Dirac γ^μ matrices) is given by $a = \sigma_3$ and $b = i\sigma_2$.

The so defined Dirac's γ^μ-matrices span the Clifford algebra related to the Minkowskian metric tensor:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbb{1}_4$$

with $g^{\mu\nu} = diag[1, -1, -1, -1]$. while the six anti-symmetric matrices

$$\Sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu\right]$$

satisfy commutation relations characterizing the Lorentz group:

$$[M_{\mu\nu}, M_{\rho\lambda}] = i\ \left(\eta_{\mu\rho} M_{\nu\lambda} - \eta_{\nu\rho} M_{\mu\lambda} + \eta_{\nu\lambda} M_{\mu\rho} - \eta_{\mu\lambda} M_{\nu\rho}\right], \tag{12}$$

where $M_{\mu\nu}$ denote the Lorentz algebra generators satisfying the commutation relations (12) regardless of representation. The same commutation relations are often represented using separately 3-space rotations $J^i = \frac{1}{2}\epsilon^{ijk}\ M_{jk}$ and the Lorentz boosts $K^i = M^{oi}$; we have then

$$[J_i, J_k]{=}\epsilon_{ikl} J_l, \quad [J_i, K_k]{=}\epsilon_{ikm} K_m, \quad [K_i, K_k]{=}{-}\epsilon_{ikl} J_l. \tag{13}$$

The side effect of this modification was the presence of solutions with negative mass, which led Dirac to the conclusion that the "holes" in the sea of such solutions could be interpreted as electrons with the same mass as the usual ones, but with an opposite charge.

Now we have clearly the invariance under the product group $Z_2 \times Z_2$. The first Z_2 factor acknowledges the presence of half-integer *dichotomic spin variable*, the second Z_2 group is related to this new universal symmetry, called *charge conjugation*, stipulating the existence of anti-particles. In fact, the four-component complex function ψ is composed of two two-component spinors, ξ_α and $\chi_{\dot\beta}$,

$$\psi = \begin{pmatrix} \xi \\ \chi \end{pmatrix},$$

which are supposed to transform under two non-equivalent representations of the $SL(2, \mathbf{C})$ group:

$$\xi_{\alpha'} = S^{\alpha}_{\alpha'} \xi_\alpha, \quad \chi_{\dot\beta'} = S^{\dot\beta}_{\dot\beta'} \chi_{\dot\beta'}, \tag{14}$$

The electric charge conservation is equivalent to the vanishing of the four-divergence of j^μ:

$$\partial_\mu j^\mu = \left(\partial_\mu \psi^\dagger \gamma^\mu\right) \psi + \psi^\dagger \left(\gamma^\mu \partial_\mu \psi\right) = 0, \tag{15}$$

from which we infer that this condition will be satisfied if we have

$$\partial_\mu \psi^\dagger \gamma^\mu = -m\psi^\dagger \ \text{ and } \ \gamma^\mu \partial_\mu \psi = m\psi, \tag{16}$$

which again coincides with the Dirac equation.

2. GENERALIZED DIRAC EQUATION INCLUDING COLORS

The two coupled Pauli equations can be re-written using a matrix notation:

$$(17)\qquad \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} + \begin{pmatrix} 0 & c\,\boldsymbol{\sigma}\mathbf{p} \\ c\,\boldsymbol{\sigma}\mathbf{p} & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

where the entries in the energy operator and the mass matrix are in fact 2×2 unit matrices, as well as the σ-matrices appearing in the last matrix, so that in reality the above equation represents the 4×4 Dirac equation, only in a different basis.

According to the correspondence principle, E and $\mathbf{p}$ yield the following differential operators acting on Schrödinger's wave functions:

$$E \to -i\hbar \frac{\partial}{\partial t}, \;\; \mathbf{p} \to -i\hbar\, \boldsymbol{\nabla} = -i\hbar[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}].$$

The system of linear equations (17) displays two discrete Z_2 symmetries: the space reflection consisting in simultaneous change of directions of spin and momentum,

$$\boldsymbol{\sigma} \to -\boldsymbol{\sigma}, \quad \mathbf{p} \to -\mathbf{p},$$

and the particle-antiparticle symmetry realized by the transformation

$$m \to -m, \; \psi_+ \to \psi_-, \quad \psi_- \to \psi_+.$$

Our next aim is to enlarge the $Z_2 \times Z_2$ symmetry by including the Z_3 group which will mix not only the two spin states and particles with anti-particles, but also the three colors.

What is amazing here is that the requirement of Lorentz invariance on the equation for a two-component wave function acknowledging the two-valued spin degree of freedom led to an extra Z_2 symmetry corresponding to charge conjugation and introducing a new kind of particles entangling two Pauli spinors into one 4-component Dirac spinor.

We shall follow a similar logic while constructing a generalized equation for quarks, incorporating not only their half-integer spin and particle-antiparticle content (due to the charge conjugation, producing the anti-quark states), but also the new discrete degree of freedom, the *color*, taking on three possible values.

So let us describe three different two-component fields (Pauli spinors), which can be incidentally given the names of three colors, the "red" one φ_+, the "blue" one χ_+, and the "green" one ψ_+; more explicitly,

$$(18)\qquad \varphi_+ = \begin{pmatrix} \varphi_+^1 \\ \varphi_+^2 \end{pmatrix}, \; \chi_+ = \begin{pmatrix} \chi_+^1 \\ \chi_+^2 \end{pmatrix}, \; \psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix},$$

We follow the minimal scheme taking into account the existence of spin by using only Pauli spinors on which the 3-dimensional momentum operator acts through the scalar product $\boldsymbol{\sigma} \cdot \mathbf{p}$.

To acknowledge the existence of anti-particles, we should also introduce three "anti-colors", denoted by a "minus" underscript, corresponding to: "cyan" for φ_-, "yellow" for χ_- and "magenta" for ψ_-; here, too, we have to do with two-component columns:

$$\varphi_- = \begin{pmatrix} \varphi_-^1 \\ \varphi_-^2 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} \chi_-^1 \\ \chi_-^2 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix}, \tag{19}$$

altogether *twelve* components.

This reflects the overall $Z_2 \times Z_2 \times Z_3$ symmetry: one Z_2 group corresponding to the spin-like dichotomic degree of freedom, describing two (exclusively) accessible states; the second Z_2 required in order to account for the particle-anti-particle symmetry, and the Z_3 group corresponding to the color symmetry.

The "colored" Pauli spinors should satisfy first order equations conceived in such a way that neither can propagate by itself, just like in the case of $\mathbf{E}$ and $\mathbf{B}$ components of Maxwell's tensor in electrodynamics, or the couple of two-component Pauli spinors which cannot propagate alone, but constitute one single entity, the four-component Dirac spinor.

This leaves little space for the choice of the system of intertwined equations; here is the ternary generalization of Dirac's equation, intertwining not only particles with antiparticles, but also the three "colors", in such a way that the entire system becomes invariant under the action of the $Z_2 \times Z_2 \times Z_3$ group.

The set of linear equations for three Pauli spinors endowed with colors, and another three Pauli spinors corresponding to their anti-particles endowed with "anti-colors" involves altogether twelve complex functions. The twelve components could describe three independent Dirac particles, but here they are intertwined in a particular manner, mixing together not only spin states and particle-antiparticle states, but also the three colors.

Let us follow the logic that led from Pauli's to Dirac's equation extending it to the colors acted upon by the Z_3-group. In the expression for the energy operator (i.e. the Hamiltonian), the mass term is positive when acting on particles, and acquires a negative sign acting on anti-particles, i.e. it changes sign while intertwining particle-antiparticle components.

We shall now assume that mass terms should acquire the factor j when we switch from the red component φ to the blue component χ, and another j-factor when we switch from blue component χ to the green component ψ.

We remind that

$$j = e^{\frac{2\pi i}{3}}, \quad j^2 = e^{\frac{4\pi i}{3}}, \quad j^3 = 1, \quad \text{and } 1 + j + j^2 = 0.$$

The momentum operator will be non-diagonal, as in the Dirac equation, systematically intertwining not only particles with antiparticles, but also colors and anti-colors among themselves.

The system that satisfies all these assumptions is as follows: [14], [7], [10])

We shall choose the basis in which both particle and anti-particle with given color and its anti-color are grouped in pairs:

$$(\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-)^T . \tag{20}$$

where $\varphi_\pm$, $\chi_\pm$ and $\psi_\pm$ are two-component Pauli spinors defined above in (18 and (19), on which the 2×2 Pauli matrices act in natural manner.

In this basis our "colored Dirac equation" takes on the following form:

$$\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\chi_-, \\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\chi_+ \\
E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\psi_-, \\
E\,\chi_- &= -j\;mc^2\,\chi_- + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\psi_+ \\
E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\varphi_-, \\
E\,\psi_- &= -j^2\;mc^2\,\psi_- + c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,.\varphi_+
\end{aligned} \tag{21}$$

The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-color symmetry is realized by multiplication of mass m by j each time the color changes, i.e. more explicitly, Z_3 symmetry is realized as follows:

$$m \to jm, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm, \tag{22}$$

$$m \to j^2 m, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm, \tag{23}$$

The system of equations (21 can be written using 12×12 matrices acting on the 12-component vectors build up from six "colored" Pauli spinors. In this shortened form we can write

$$E\mathbb{1}_{12}\,\Psi = \left[mc^2\;M + cP\right]\;\Psi, \tag{24}$$

where $\mathbb{1}_{12}$ is the 12×12 unit matrix, and the matrices M and P are given explicitly below:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & jm & 0 & 0 & 0 \\ 0 & 0 & 0 & -jm & 0 & 0 \\ 0 & 0 & 0 & 0 & j^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & -j^2 m \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} \\ 0 & 0 & 0 & 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 \\ \boldsymbol{\sigma}\cdot\mathbf{p} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The dimension of the two matrices M and P displayed above is in reality 12×12: all the entries in the first one are proportional to the 2×2 unit matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\,m$, etc.

The entries in the second matrix P contain 2×2 Pauli's sigma-matrices, so that P is also a 12×12 matrix. The energy operator E is proportional to the 12×12 unit matrix.

Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of 3×3 circulant matrix that are multiples of 3 are proportional to $\mathbb{1}_3$.

This is why the diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,

$$E^6\,\varphi_+ = m^6 c^{12}\,\varphi_+ + c^6\,|\,\mathbf{p}\,|^6\;\varphi_+,$$

$$E^6\,\varphi_- = m^6 c^{12}\,\varphi_- + c^6\,|\,\mathbf{p}\,|^6\;\varphi_-. \tag{25}$$

and similarly for all other components.

Using a more rigorous approach the three operators can be expressed in terms of tensor products of matrices of lower dimensions. Let us introduce two following 3×3 matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{26}$$

whose products and powers generate the $U(3)$ Lie group algebra, or the $SU(3)$ algebra if we remove the unit matrix.

The 12×12 matrices M and P can be represented as the following tensor products:

$$M = m\,B \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad P = Q_3 \otimes \sigma_1 \otimes (\boldsymbol{\sigma}\cdot\mathbf{p}) \tag{27}$$

with as usual, $\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let us rewrite the matrix operator generating the system (21) when it acts on the column vector Ψ containing twelve components of three "color" fields, in the basis (20) $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:

with energy and momentum operators on the left hand side, and the mass operator on the right hand side:

(28) $$E\mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 - Q_3 \otimes \sigma_1 \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2\, B \otimes \sigma_3 \otimes \mathbb{1}_2$$

Like with the standard Dirac equation, let us transform this equation so that the mass operator would become proportional the the unit matrix. To do so, we multiply the equation (28) on the left by the matrix $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.

Now we get the following equation which enables us to interpret the energy and the momentum as the components of a Minkowskian four-vector $c\, p^\mu = [E,\ c\mathbf{p}]$:

(29) $$E\, B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2,$$

where we used the fact that under matrix multiplication, $\sigma_3 \sigma^3 = \mathbb{1}_2$, $B^\dagger B = \mathbb{1}_3$ and $B^\dagger Q_3 = Q_2$. The sixth power of this operator gives the same result as before,

(30) $$\left[E\, B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p}\right]^6 = \left[E^6 - c^6 \mathbf{p}^6\right] \mathbb{1}_{12} = m^6 c^{12} \mathbb{1}_{12}$$

It is also worthwhile to note that taking the the determinant on both sides yields the twelvth-order equation:

(31) $$\left(E^6 - c^6\ \mid \mathbf{p} \mid^6\right)^2 = m^{12} c^{24}.$$

The equation (29) can be written in a concise manner using Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

(32) $$\Gamma^\mu p_\mu\, \Psi = mc \mathbb{1}_{12}\, \Psi, \quad \text{with } p^0 = \frac{E}{c},\ \ p^k = [\, p^x, p^y, p^z\,].$$

with 12×12 matrices Γ^μ, $(\mu = 0, 1, 2, 3)$ defined as follows:

(33) $$\Gamma^0 = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k$$

where

(34) $$B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix},$$

The two traceless matrices B and Q_2 are both cubic roots of unit 3×3 matrix. They generate the entire Lie algebra of the $SU(3)$ group. The full basis of 3×3 traceless matrices generated by all possible powers and products of B and Q_2 is as follows:

(35) $$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad Q_2^\dagger = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{36}$$

and the two traceless diagonal matrices B and $B^\dagger$ defined in (34).

This special basis of the $SU(3)$ algebra has been studed by V.G. Kac in 1994 ([5]) The following multiplication table for B, $B^\dagger$, Q_A and $Q^\dagger$, $A, B, ... = 1, 2, 3$) defines the algebra :

$$\begin{gathered} BQ_A{=}j^2Q_AB{=}Q_{A+1}, \;\; B^\dagger Q_A{=}jQ_AB^\dagger{=}Q_{A-1}, \\ Q^\dagger B{=}j^2BQ_A^\dagger{=}Q_{A-1}^\dagger, \;\; Q_A^\dagger B^\dagger{=}B^\dagger Q_A^\dagger{=}Q_{A+1}^\dagger, \\ Q_AQ_{A-1}{=}jQ_{A+1}^\dagger, \;\; Q_{A-1}^\dagger Q_A^\dagger{=}j^2Q_{A+1}, \\ Q_AQ_{A+1}^\dagger{=}B^\dagger, \;\; Q_AQ_{A-1}^\dagger{=}B, \;\; Q_A^\dagger Q_{A-1}{=}jB^\dagger, \;\; Q_A^\dagger Q_{A+1}{=}j^2B. \end{gathered} \tag{37}$$

One has also $Q_AQ_A^\dagger = Q_a^\dagger Q_A = \mathbb{1}_3$. where the indices A, $A+1$, $A-1$ are always taken modulo 3, so that e.g. $3+1 \mid_{modulo\ 3} = 4 \mid_{modulo\ 3} = 1$, etc., and the cube of each of the eight matrices in (37) is the unit 3×3 matrix.

Quite obvously, the generalized Dirac equation (32) is invariant under an arbitrary similitude transformation: if we set

$$\tilde{\Gamma}^\mu = P\,\Gamma^\mu\,P^{-1} \quad \tilde{\Gamma}^\mu p_\mu\,\Psi = mc\,\Psi, \tag{38}$$

then obviously

$$\left[\tilde{\Gamma}^\mu p_\mu\right]^6 = (k_0^6 - \mid \mathbf{k} \mid^6)\mathbb{1}_{12}$$

as before, and the determinant of the transformed operator will remain the same.

A natural question that can be raised at this point is the following: is the form of the colored Dirac equation (21) unique, or are there other choices for generalized matrices Γ^μ, involving other generators of $SU(3)$ algebra? The problem is not trivial at all, because the 3×3 matrix factors in time and space components of 12×12 tensor products must be different for Γ^o (time) and for Γ^i (space) components. In the following section we shall show how twelve new realizations of the "color" generalization of the Dirac equation do naturally appear in the construction of faithful spinorial representation of the Z_3-graded Lorentz algebra. However, the question whether these realizations form a maximal set remains open.

3. SPINORIAL Z_3-GRADED LORENTZ ALGEBRA

The 12×12 matrices Γ^μ appearing in the colored Dirac equation (32) do not span 4-dimensional Clifford algebra, as the 4×4 Dirac matrices γ^μ. In fact, the $Z_3 \otimes Z_2$ structure of Γ^μ-matrices implies that only their sixth powers are proportional to the unit matrix $\mathbb{1}_{12}$ (see also (31)).

Thus, in order to obtain the realization of $D = 4$ Lorentz algebra generators one can not use just two standard commutators

$$J_i = \frac{i}{2}\,\epsilon_{ijk}\left[\Gamma^j, \Gamma^k\right], \quad K_l = \frac{1}{2}\,\left[\Gamma_l, \Gamma_0\right]. \tag{39}$$

However, the generators $\left(J_i^{(0)},\ K_l^{(0)}\right)$ satisfying the standard Lorentz algebra relations (see also (modulocomm) for $r = 0, s = 0$) can be defined by triple commutators:

$$\begin{aligned} [J_i, [J_j, J_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ J_l^{(0)}, \\ [K_i, [K_j, K_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ K_l^{(0)}\ . \end{aligned} \tag{40}$$

Indeed, substituting in (40) the explicit form of Γ^μ given in (33), we get

$$\begin{aligned} J_i &= -\frac{i}{2}\,Q_2^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l = -\frac{1}{2}\,Q_1 \otimes \sigma_1 \otimes \sigma_l, \\ J_i^{(0)} &= -\frac{i}{2}\mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_l^{(0)} = -\frac{1}{2}\mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_l. \end{aligned} \tag{41}$$

In order to close the generalized Lorentz algebra where $L^{(0)}=(J_i^{(0)}, K_j^{(0)})$, $L^{(1)} = (J_i^{(1)}, K_j^{(1)})$, $\quad L^{(2)} = (J_i^{(2)}, K_j^{(2)})$, one should supplement (40) by two missing triple commutators:

$$\begin{aligned} [J_i, [J_j, K_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ K_l^{(2)}, \\ [K_i, [K_j, J_k]] &= (\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl})\ J_l^{(1)}, \end{aligned} \tag{42}$$

where using the representation (41) we get

$$J_l^{(1)} = -\frac{i}{2}\,Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(2)} = -\frac{1}{2}\,Q_3^\dagger \otimes \sigma_1 \otimes \sigma_i. \tag{43}$$

The full set of Z_3-graded relations defining the Z_3-graded Lorentz algebra (with $r, s, = 0, 1, 2,\ r + s$ taken modulo 3) are:

$$\begin{aligned} &\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)}, \\ &\left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \end{aligned} \tag{44}$$

From the commutators $[K_i^{(1)}, K_m^{(1)}] \simeq J^{(2)}$ and $[J^{(1)}, J^{(1)}] \simeq J^{(2)}$ one gets the remaining generators of $\mathcal{L}$: the following extension of the formulae (43):

$$J_i^{(2)} = -\frac{i}{2}\,Q_3^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_m^{(1)} = -\frac{1}{2}\,Q_3 \otimes \sigma_1 \otimes \sigma_m. \tag{45}$$

The formulae (41), (43) and (45) describe the realization of $\mathcal{L}$ which follows from the choice (33) of matrices Γ^μ.

Following ([12]) we introduce a unified notation:

(46) $$\Gamma^{\mu}_{(A;\alpha)} = I_A \otimes \sigma_\alpha \otimes \sigma^\mu, \ A = 0,1,..,8; \ \alpha = 2,3; \ \mu = 0,1,2,3 \ .$$

Let the 3×3 "color matrices" I_A appearing as the first factor be defined as follows: $I_0 = \mathbb{1}_3, \ I_r = Q_r, \ I_{r+3} = Q_r^\dagger, \ I_7 = B, \ I_8 = B^\dagger$.

Then the original Γ-matrices given by (33) encoded as $\Gamma^0_{(8,3)} = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$ and $\Gamma^i_{(2;2)} = Q_2 \otimes (i\sigma_2) \otimes \sigma^i$. The eight matrices with $A = 1,2,...8$ with the multiplication rules given above in (37) span the ternary basis, generated by the cyclic Z_3-automorphism of the $SU(3)$ algebra. ([5], Sect. 8).

In order to get a closed formula for the action $\mathcal{S}^{(0)}\Gamma^\mu[\mathcal{S}^{(0)}]^{-1}$ of classical spinorial Lorentz symmetries generated by $L^{(0)}$, we should introduce the pairs of Γ^μ-matrices $\Gamma^\mu = (\Gamma^i_{(A;2)}, \ \Gamma^0_{(B;3)})$ and $\tilde{\Gamma}^\mu = (\Gamma^i_{(B;2)}, \ \Gamma^0_{(A;3)})$, $A \neq B$. For any choice of Γ^μ's in (46) we get:

(47) $$\left[J^{(0)}_i, \Gamma^j_{(A;\alpha)}\right] = \epsilon_{ijk}\Gamma^k_{(A;\alpha)}, \quad \left[J^{(0)}_i, \Gamma^0_{(A;\alpha)}\right] = 0,$$

The boosts $K^{(0)}_i$ act covariantly on doublets $\left(\Gamma^\mu, \tilde{\Gamma}^\mu\right)$ as follows:

$$\left[K^{(0)}_i, \Gamma^j_{(A;2)}\right] = \delta^j_i \ \Gamma^0_{(A;3)}, \quad \left[K^{(0)}_i, \Gamma^0_{(B;3)}\right] = \Gamma^i_{(B;2)},$$

(48) $$\left[K^{(0)}_i, \Gamma^j_{(B;2)}\right] = \delta^j_i \ \Gamma^0_{(B;3)}, \ \left[K^{(0)}_i, \Gamma^0_{(A;3)}\right] = \Gamma^i_{(A;2)},$$

with $A \neq B$, i.e. the standard Lorentz covariance requires the doublet of colored Dirac spinors;

In particular, the Γ^μ matrices (33) should be supplemented by:

(49) $$\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes (\sigma_3) \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^k_{(8;2)} = B^\dagger \otimes i\sigma_2 \otimes \sigma^k.$$

It is tempting to formulate the following conjecture: the pairs of Γ-matrices generated by the standard Lorentz covariance requirement can be used for the introduction of weak isospin doublets of the $SU(2) \times U(1)$ electroweak symmetry.

We conclude that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of the Standard Model follow from the imposition of Lorentz covariance on color Dirac multiplets.

It follows that in order to obtain the closure of the faithful action of generators $(J^{(s)}_k, \ K^{(s)}_m)$ $(s = 0,1,2)$ of the generalized spinorial transformations on matrices Γ^μ,one should introduce two sets $\Gamma^\mu_{(a)}, \Gamma^\mu_{\dot{a}} = (\Gamma^\mu_{(a)})^\dagger$ $(a = 1,2,...6)$ of colored 12×12 Dirac matrices supplemented by Lorentz doublet partners $(\tilde{\Gamma}^\mu_{(a)}, \tilde{\Gamma}^\mu_{(\dot{a})})$.

Choosing $(J^{(1)}_k, K^{(1)}_m)$ as given by Eqs. (43), (45), and assuming that $\Gamma^\mu_{(1)}$ are given by the same formula, (33) by calculating the multicommutators of

$\left(J_i^{(1)}, K_l^{(1)}\right)$ with the set $\Gamma^\mu_{(a)}$, $(a = 1, 2...6)$, we get the following six Γ-matrices closed under the action of $L^{(1)}$:

$$\begin{aligned}
\Gamma^\mu_{(1)} &= \left(\Gamma^0_{(8;3)},\ \Gamma^i_{(2;2)}\right);\ \Gamma^\mu_{(4)} = \left(\Gamma^0_{(8;2)},\ \Gamma^i_{(2;3)}\right);\\
\Gamma^\mu_{(2)} &= \left(\Gamma^0_{(2;2)},\ \Gamma^i_{(4;3)}\right);\ \Gamma^\mu_{(5)} = \left(\Gamma^0_{(2;3)},\ \Gamma^i_{(4;2)}\right);\\
\Gamma^\mu_{(3)} &= \left(\Gamma^0_{(4;3)},\ \Gamma^i_{(8;2)}\right);\ \Gamma^\mu_{(6)} = \left(\Gamma^0_{(4;2)},\ \Gamma^i_{(8;3)}\right).
\end{aligned} \tag{50}$$

The six matrices (50) form three pairs, each of which transforms in itself under the action of the 0-grade subalgebra $(\overset{(0)}{J}_i,\ \overset{(0)}{K}_l)$.

The Z_3-graded components of $\mathcal{L}_{Z_3}$, $\overset{(1)}{\mathcal{L}}$ and $\overset{(2)}{\mathcal{L}}$, act on these pairs transforming them into other pairs, with conjugate Q and B matrices. The realization of $L^{(2)}$ sector is obtained by introducing the Hermitean-conjugate sextet $\Gamma^\mu_{(\dot{a})} = (\Gamma^\mu_{(a)})^\dagger$; further one should add $\tilde{\Gamma}^\mu_{(\dot{a})} = (\tilde{\Gamma}^\mu_{(a)})^\dagger$ due to standard Lorentz covariance.

4. THE Z_3-GRADED POINCARÉ ALGEBRA

The Z_3-graded extension of full Poincaré algebra is quite obvious ([10]). Whatever the representation we choose (spinorial or "orbital"), the commutation relations for the Z_3-graded Lorentz subalgebra remain the same. Let us denote the generators of the extended Z_3-graded Poincaré algebra by $\overset{(r)}{\mathcal{K}_i}$ (generalized Lorentz boosts) and $\overset{(r)}{\mathcal{J}_i}$ (the generalized spatial rotations), where the superscript $r = 0, 1, 2$ refers to the Z_3-grade of one of the three components of Z_3-graded extended Lorentz algebra, and $i, k = 1, 2, 3$ are the 3-space indices. The commutation rules of the Lorentz algebra:

$$[\,\overset{(r)}{\mathcal{K}_i},\ \overset{(s)}{\mathcal{K}_k}\,] = -\epsilon_{ikl}\ \overset{(r+s)}{\mathcal{J}_l}\ ,\qquad [\,\overset{(r)}{\mathcal{J}_i},\ \overset{(s)}{\mathcal{K}_k}\,] = \epsilon_{ikl}\ \overset{(r+s)}{\mathcal{K}_l}\ , \tag{51}$$

$$[\,\overset{(r)}{\mathcal{J}_i},\ \overset{(s)}{\mathcal{J}_k}\,] = \epsilon_{ikl}\ \overset{(r+s)}{\mathcal{J}_l}\ . \tag{52}$$

now must be supplemented by set of commutation rules between Lorentz generators and the generators of 4-translations, which should also form a Z_3-graded extension of usual 4-dimensional Minkowskian translations P_μ.

Denoting them by

$$(\ \overset{(0)}{\mathcal{P}_\mu},\quad \overset{(1)}{\mathcal{P}_\mu},\quad \overset{(2)}{\mathcal{P}_\mu}\ ), \tag{53}$$

with $r = 0, 1, 2$ and $\mu, \nu = 0, 12, 3$, we impose the following Z_3-graded extra commutation relations:

$$[\overset{(r)}{\mathcal{P}_0}, \overset{(s)}{\mathcal{P}_k}] = 0; \quad [\overset{(r)}{\mathcal{P}_i}, \overset{(s)}{\mathcal{P}_j}] = 0, \tag{54}$$

$$[\overset{(r)}{\mathcal{J}_k}, \overset{(s)}{\mathcal{P}_0}] = 0; \quad [\overset{(r)}{\mathcal{J}_i}, \overset{(s)}{\mathcal{P}_k}] = \epsilon_{ikl} \overset{(r+s)}{\mathcal{P}_l}, \tag{55}$$

$$[\overset{(r)}{\mathcal{K}_i}, \overset{(s)}{\mathcal{P}_0}] = \overset{(r+s)}{\mathcal{P}_i}, \quad [\overset{(r)}{\mathcal{K}_i}, \overset{(s)}{\mathcal{P}_k}] = -\delta_{ik} \overset{(r+s)}{\mathcal{P}_0}. \tag{56}$$

In all the above relations the grades $r, s = 0, 1, 2$ add up modulo 3.

To realize these commutation relations in terms of differential operators, the ordinary 4-dimensional Minkowskian space cannot suffice; it must be extended so as to accomodate the Z_3-grading.

We denote by M_4 the standard 4-dimensional real vector space endowed with pseudo-Euclidean (Minkowskian) metric $\eta_{\mu\nu} = \text{diag}[+, -, -, -]$. A space-time vector is given by its coordinates in a chosen orthonormal frame:

$$k^\mu = [k^0, \mathbf{k}] = [k^0, k^x, k^y, k^z] \tag{57}$$

replaced by a more practical notation with small Greek indices running from 0 to 3:

$$k^\mu = [k^0, \mathbf{k}] = [k^0, k^1, k^2, k^3] \tag{58}$$

The three replicas of a 4-vector k^μ will be labeled with the superscripts relative to the elements of the Z_3-group as follows:

$$\overset{(0)}{k^\mu} = (\overset{(0)}{k^0}, \overset{(0)}{\mathbf{k}}), \quad \overset{(1)}{k^\mu} = (\overset{(1)}{k^0}, \overset{(1)}{\mathbf{k}}), \quad \overset{(2)}{k^\mu} = (\overset{(2)}{k^0}, \overset{(2)}{\mathbf{k}}). \tag{59}$$

In each of the three sectors the specific quadratic form is defined:

$$(\overset{(0)}{k^0})^2 - (\overset{(0)}{\mathbf{k}})^2 = m^2, \quad (\overset{(1)}{k^0})^2 - j(\overset{(1)}{\mathbf{k}})^2 = jm^2, \quad (\overset{(2)}{k^0})^2 - j^2(\overset{(2)}{\mathbf{k}})^2 = j^2m^2. \tag{60}$$

leading to the following explicit expressions of $(\overset{(r)}{k^0})$ as functions of $\overset{(r)}{\mathbf{k}}$ and m $(r = 0, 1, 2)$:

$$\overset{(0)}{k^0} = \pm\sqrt{\overset{(0)}{\mathbf{k}^2} + m^2}, \quad \overset{(1)}{k^0} = \pm j^2\sqrt{\overset{(1)}{\mathbf{k}^2} + m^2}, \quad \overset{(2)}{k^0} = \pm j\sqrt{\overset{(2)}{\mathbf{k}^2} + m^2}, \tag{61}$$

If we set $\overset{(0)}{\mathbf{k}} = \overset{(1)}{\mathbf{k}} = \overset{(2)}{\mathbf{k}} = \mathbf{k}$, then the product of the above three quadratic invariants is equal to the following *real* invariant of the sixth order

$$(k_0^2 - \mathbf{k}^2) \cdot (k_0^2 - j\mathbf{k}^2) \cdot (k_0^2 - j^2\mathbf{k}^2) = k_0^6 - |\mathbf{k}|^6 = m^6. \tag{62}$$

Let us denote the three quadratic forms, one real and two mutually complex conjugate ones, by the following three tensors

$$\overset{(0)}{\eta}_{\mu\nu} = \text{diag}[+1, -1, -1, -1], \quad \overset{(1)}{\eta}_{\mu\nu} = \text{diag}[+1, -j, -j, -j],$$

(63) $$\overset{(2)}{\eta}_{\mu\nu} = \text{diag}[+1, -j^2, -j^2, -j^2]$$

defined on each of the subspaces of the generalized Minkowskian space

(64) $$\overset{(Z_3)}{M}_{12} = \overset{(0)}{M}_4 \oplus \overset{(1)}{M}_4 \oplus \overset{(2)}{M}_4$$

The superscripts $(r) = (0),\ (1),\ (2)$ refer to the Z_3-grades attributed to each of the three subspaces. These grades will play an important role in defining the Z_3-graded extension of the Poincaré algebra acting on the extended Minkowskian space-time $\overset{(Z_3)}{M}_{12}$.

The three "replicas" are to be treated as really independent components of the resulting 12-dimensional manifold. For convenience, we shall use the same letters designing three types of space-time components, labeling them with an extra index as follows:

(65) $$x^\mu_r = (x^\mu_0,\ x^\mu_1,\ x^\mu_2) = [\tau_0, x_0, y_0, z_0;\ \tau_1, x_1, y_1, z_0;\ \tau_2, x_2, y_2, z_2].$$

Idempotent operators projecting on one of the three subspaces of the generalized Minkowskian space-time $\overset{(Z_3)}{M}_{12}$ can be constructed using the 3×3 matrices B and $B^\dagger$ as follows. Let us define two 12×12 matrices acting on $\overset{(Z_3)}{M}_{12}$:

$$\mathcal{B} = B \otimes \mathbb{1}_4, \quad \mathcal{B}^\dagger = B^\dagger \otimes \mathbb{1}_4,$$

Then the following three projection operators can be formed:

(66)
$$\overset{(0)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + \mathcal{B} + \mathcal{B}^\dagger), \quad \overset{(1)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + j^2\,\mathcal{B} + j\,\mathcal{B}^\dagger), \quad \overset{(2)}{\Pi} = \frac{1}{3}\,(\mathbb{1}_{12} + j\,\mathcal{B} + j^2\,\mathcal{B}^\dagger).$$

One checks easily that $[\overset{(r)}{\Pi}]^2 = \overset{(r)}{\Pi}$, $r = 0, 1, 2$ and $\overset{(r)}{\Pi}\overset{(s)}{\Pi} = 0$ for $r \neq s$.

Interesting higher-dimensional and complex extensions of Minkowskian space-time were investigated in [4], [2], albeit without the introduction of Z_3 grading.

5. VECTOR Z_3-GRADED REPRESENTATION

The quadratic Minkowskian square of the 4 vector k^μ, $(k^0)^2 - \mathbf{k}^2$ is invariant under the transformations of the Lorentz group. The space rotations touching only the 3-dimensional vector $\mathbf{k}$ leave all the three quadratic expressions invariant, because they depend only on its 3-dimensional Euclidean square $\mathbf{k}^2$; therefore we can fix our attention at the Lorentzian boosts.

As we can always align the relative velocity along one of the axes of chosen inertial frame, say $0x$, those boosts can be considered only between the time and the x coordinates. Here are the three 2×2 matrices representing the same Lorentz boost (with real parameter u equal to $\tanh \frac{v}{c}$) leaving invariant one of the three quadratic invariants given in (60):

$$\overset{(0)}{L}_{00} = \begin{pmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{pmatrix}, \quad \overset{(0)}{L}_{11} = \begin{pmatrix} \cosh u & j^2 \sinh u \\ j \sinh u & \cosh u \end{pmatrix}, \quad \overset{(0)}{L}_{22} = \begin{pmatrix} \cosh u & j \sinh u \\ j^2 \sinh u & \cosh u \end{pmatrix}. \tag{67}$$

The three matrices are self-adjoint:

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}, \quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}, \quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22}. \tag{68}$$

The above matrices transform each of the three sectors of the Z_3-Minkowski space into itself, which founds its reflection in the lower indices quite obviously: L_{00} transforms a vector belonging to the 0-th sector of the Z_3-graded Minkowskian space into a 4-vector belonging to the same sector, and similarly for the matrix operators L_{11} and L_{22}.

Each set is a representation of a one-parameter subgroup representing a particular Lorentz boost, here between the time variable (hereafter always represented by $\tau = ct$ and one cartesian coordinate, say x. For example, the product of two Lorentz boosts acting on the sector (1), is a boost of the same type:

$$\overset{(0)}{L}_{11}(u) \cdot \overset{(0)}{L}_{11}(v) = \overset{(0)}{L}_{11}(u + v), \tag{69}$$

and similarly for a product of two boosts acting on the sector (2),

$$\overset{(0)}{L}_{22}(u) \cdot \overset{(0)}{L}_{22}(v) = \overset{(0)}{L}_{22}(u + v). \tag{70}$$

The logic of the lower indices is quite transparent: L_{00} transforms a vector belonging to the 0-th sector of the Z_3-graded Minkowskian space into a 4-vector belonging to the same sector, and similarly for the matrix operators L_{11} and L_{22}. The full set of three independent "classical" (i.e. belonging to the subgroup denoted by $\overset{(0)}{L}$) Lorentz boosts is given by three 4×4 matrices, with independent parameters u, v, w:

(71)
$$\begin{pmatrix} \cosh u & \sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix}$$

To make the extension of the Lorentz boosts complete we need also two sets of complementary matrix operators transforming one sector into another. There are two types of such operators, one raising the Z_3 index of each subspace, another lowering the Z_3 index by 1. It is quite easy to find out their matrix representation.

The matrices lowering the Z_3 index by 1 are:

(72)
$$\overset{(1)}{L}_{01} = \begin{pmatrix} j\cosh u & \sinh u \\ j\sinh u & \cosh u \end{pmatrix}, \ \overset{(1)}{L}_{12} = \begin{pmatrix} j\cosh u & j^2\sinh u \\ j^2\sinh u & \cosh u \end{pmatrix},$$
$$\overset{(1)}{L}_{20} = \begin{pmatrix} j\cosh u & j\sinh u \\ \sinh u & \cosh u \end{pmatrix}.$$

The determinant of each of these matrices is equal to j.

The matrices raising the Z_3 index by 1 (or decreasing it by 2, which is equivalent from the point of view of the Z_3-grading are:

(73)
$$\overset{(2)}{L}_{10} = \begin{pmatrix} j^2\cosh u & j^2\sinh u \\ \sinh u & \cosh u \end{pmatrix}, \ \overset{(2)}{L}_{21} = \begin{pmatrix} j^2\cosh u & j\sinh u \\ j\sinh u & \cosh u \end{pmatrix},$$
$$\overset{(2)}{L}_{02} = \begin{pmatrix} j^2\cosh u & \sinh u \\ j^2\sinh u & \cosh u \end{pmatrix}.$$

The determinant of each of these matrices is equal to j^2.

The above sets of three matrices each, decreasing and raising the Z_3 index, are mutually hermitian adjoint:

(74)
$$\overset{(1)\dagger}{L}_{01} = \overset{(2)}{L}_{10}, \quad \overset{(1)\dagger}{L}_{12} = \overset{(2)}{L}_{21}, \quad \overset{(1)\dagger}{L}_{20} = \overset{(2)}{L}_{02}.$$

Here again, the logic of the lower indices is quite transparent: a matrix labeled L_{12} transforms a 4-vector belonging to the sector (2) into a 4-vector belonging to the sector (1), a matrix labeled L_{20} transforms a 4-vector belonging to the sector (0) into a 4-vector belonging to the sector (2) and so forth, e.g.:

(75)
$$L_{01} \overset{(1)}{k^\mu} = \overset{(0)}{k^{\mu'}}, \qquad L_{20} \overset{(0)}{k^\mu} = \overset{(2)}{k^{\mu'}}, \qquad L_{12} \overset{(2)}{k^\mu} = \overset{(1)}{k^{\mu'}}, \ \text{etc.}$$

The matrices raising or lowering the Z_3-grade of the particular type of the 4-vector they are acting on do not form a group, because most of the products of two such matrices produce new matrices not belonging to the set defined above. However, inside each of one-parameter families corresponding to a given choice of the single space direction concerned by the Lorentz boost, $0x$, $0y$ or $0z$ displays the group property if the products are taken according to the chain rule, with second index of the first factor equal to the first index of the second, like in the following examples:

$$\overset{(1)}{L}_{12}(\tau, x;\ u)\overset{(1)}{L}_{20}(\tau, x;\ v) = \overset{(2)}{L}_{10}(\tau, x;\ (u+v)),$$
$$\overset{(2)}{L}_{21}(\tau, y;\ u)\overset{(1)}{L}_{12}(\tau, y;\ v) = \overset{(0)}{L}_{22}(\tau, y;\ (u+v)), \text{ etc.} \tag{76}$$

The 2×2 matrices (68, 73, 74) represent a reduced version of Lorentz boosts with relative velocity aligned on the axis Ox.

As in the previous case, the full 4×4 versions are given by the following three matrices corresponding to the three independent Lorentz boosts.

The boosts of the increasing type, transforming 4-vectors from sector 2 to 0, from sector 1 to 2 and from sector 0 to 1, respectively, are given in the case of :

- the three boosts $\overset{(1)}{L}_{20}(\tau, x)$, $\overset{(1)}{L}_{20}(\tau, y)$, $\overset{(1)}{L}_{20}(\tau, z)$ are given by:

$$\begin{pmatrix} j\cosh u & j\sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & j\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j\cosh w & 0 & 0 & j\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{77}$$

- the three boosts $\overset{(1)}{L}_{12}(\tau, x)$, $\overset{(1)}{L}_{12}(\tau, y)$, $\overset{(1)}{L}_{12}(\tau, z)$ are given by:

$$\begin{pmatrix} j\cosh u & j^2\sinh u & 0 & 0 \\ j^2\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & j^2\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j^2\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j\cosh w & 0 & 0 & j^2\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j^2\sinh w & 0 & 0 & \cosh w \end{pmatrix} \tag{78}$$

and the three boosts $\overset{(1)}{L}_{01}(\tau, x)$, $\overset{(1)}{L}_{01}(\tau, y)$, $\overset{(1)}{L}_{01}(\tau, z)$ are given by:

(79)
$$\begin{pmatrix} j\cosh u & \sinh u & 0 & 0 \\ j\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j\cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j\cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j\sinh w & 0 & 0 & \cosh w \end{pmatrix}$$

The boosts of the decreasing type, transforming 4-vectors from sector 1 to 0, from sector 2 to 1 and from sector 0 to 2, respectively, are as follows:

- the three boosts $\overset{(2)}{L}_{10}(\tau, x)$, $\overset{(2)}{L}_{10}(\tau, y)$, $\overset{(2)}{L}_{10}(\tau, z)$ are given by:

(80)
$$\begin{pmatrix} j^2\cosh u & j^2\sinh u & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & j^2\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ \sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j^2\cosh w & 0 & 0 & j^2\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh w & 0 & 0 & \cosh w \end{pmatrix}$$

- the three boosts $\overset{(2)}{L}_{21}(\tau, x)$, $\overset{(2)}{L}_{21}(\tau, y)$, $\overset{(2)}{L}_{21}(\tau, z)$ are given by:

(81)
$$\begin{pmatrix} j^2\cosh u & j\sinh u & 0 & 0 \\ j\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & j\sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j^2\cosh w & 0 & 0 & j\sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j\sinh w & 0 & 0 & \cosh w \end{pmatrix}$$

and the three boosts $\overset{(2)}{L}_{02}(\tau,x)$, $\overset{(2)}{L}_{02}(\tau,y)$, $\overset{(2)}{L}_{02}(\tau,z)$ are given by:

(82)
$$\begin{pmatrix} j^2\cosh u & \sinh u & 0 & 0 \\ j^2\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} j^2\cosh v & 0 & \sinh v & 0 \\ 0 & 1 & 0 & 0 \\ j^2\sinh v & 0 & \cosh v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} j^2\cosh w & 0 & 0 & \sinh w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ j^2\sinh w & 0 & 0 & \cosh w \end{pmatrix}$$

The nine 4×4 matrices $\overset{(r)}{L}_{st}$, $r,s,t = 0,1,2$ act on the Z_3-extended Minkowskian vector in a specifically ordered way. Let us write a Z_3-extended vector as a column with 12 entries, composed of three 4-vectors belonging each to one of the Z_3-graded sectors

$$(\overset{(0)}{k^\mu}, \overset{(1)}{k^\mu}, \overset{(2)}{k^\mu})$$

(83)
$$\overset{(0)}{\Lambda} = \begin{pmatrix} \overset{(0)}{L}_{00} & 0 & 0 \\ 0 & \overset{(0)}{L}_{11} & 0 \\ 0 & 0 & \overset{(0)}{L}_{22} \end{pmatrix} \quad \overset{(1)}{\Lambda} = \begin{pmatrix} 0 & \overset{(1)}{L}_{01} & 0 \\ 0 & 0 & \overset{(1)}{L}_{12} \\ \overset{(1)}{L}_{20} & 0 & 0 \end{pmatrix} \quad \overset{(2)}{\Lambda} = \begin{pmatrix} 0 & 0 & \overset{(2)}{L}_{02} \\ \overset{(2)}{L}_{10} & 0 & 0 \\ 0 & \overset{(2)}{L}_{21} & 0 \end{pmatrix}$$

It is easy to see that the so defined matrices display not only the group property, but also the Z_3 grading in the following sense:

(84)
$$\overset{(0)}{\Lambda}\cdot\overset{(0)}{\Lambda} \subset \overset{(0)}{\Lambda}, \quad \overset{(0)}{\Lambda}\cdot\overset{(1)}{\Lambda} \subset \overset{(1)}{\Lambda}, \quad \overset{(0)}{\Lambda}\cdot\overset{(2)}{\Lambda} \subset \overset{(2)}{\Lambda},$$
$$\overset{(1)}{\Lambda}\cdot\overset{(1)}{\Lambda} \subset \overset{(2)}{\Lambda}, \quad \overset{(2)}{\Lambda}\cdot\overset{(2)}{\Lambda} \subset \overset{(1)}{\Lambda}, \quad \overset{(1)}{\Lambda}\cdot\overset{(2)}{\Lambda} = \overset{(2)}{\Lambda}\cdot\overset{(1)}{\Lambda} \subset \overset{(0)}{\Lambda}.$$

The elements of three subsets of the Z_3-graded group of boosts behave under associative matrix multiplication as folows:

(85) $$\overset{(r)}{\Lambda}\cdot\overset{(s)}{\Lambda} \subset \overset{(r+s)|_3}{\Lambda}, \quad \text{with } r,s,.. = 0,1,2, \quad (r+s)\,|_3 = (r+s) \text{ modulo } 3.$$

The three sets of matrices ordered in particular blocks (83) form a three-parameter family which can be considered as the extension of the set of three independent Lorentz boosts. To obtain the extension of the entire Lorentz group including the 3-parameter subgroup of space rotations we shall first investigate the Z_3-graded infinitesimal generators of the Lorentz boosts, then, taking their commutators, define the Z_3-graded extension of space rotations.

6. DIFFERENTIAL OPERATORS

The generators of the Lorentz algebra can be produced from the standard matrix representation by the following well-known procedure. Let us take for example the 4×4 matrix representation of 3-dimensional rotations given by formulae (39). The differential operators corresponding to J_x, J_y and J_z are obtained by taking formally the scalar product of the space-time 4-covector $[\tau, x, y, z]$ with the 4-gradient ∂_μ transformed by the corresponding matrix J_i. Take for example the matrix J_x:

$$(86) \qquad \begin{pmatrix} \tau, & x, & y, & z \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_\tau \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = z\partial_y - y\partial_z.$$

Similarly we get $J_y \to x\partial_z - z\partial_x, \quad J_z \to y\partial_x - x\partial_y$.

The construction of differential operators representing the Z_3-graded Poincaré algebra (56) follows the prescription given above by with 12×12 matrices introduced in previous section, and 12-component generalizations of Minkowskian 4-vectors and co-vectors. Let us introduce the following notation for generalized vectors in triple Minkowskian space-time:

$$(87) \qquad [\tau_0, x_0, y_0, z_0;\ \tau_1, x_1, y_1, z_1;\ \tau_2, x_2, y_2, z_2]\,,$$

The notations are obvious: the lower index "0" refers to standard Minkowskian component (graded 0), while the indices "1" and "2" refer to mutually conjugate complex extensions of Z_3 grades 1 and 2, respectively.

Partial derivatives with respect to these variables are represented by the following 12-component column vector (written here as a horizontal co-vector transposed, in order to spare the space):

$$(88) \qquad [\ \partial_{\tau_0}, \partial_{x_0}, \partial_{y_0}, \partial_{z_0};\ \partial_{\tau_1}, \partial_{x_1}, \partial_{y_1}, \partial_{z_1};\ \partial_{\tau_2}, \partial_{x_2}, \partial_{y_2}, \partial_{z_2};\ ]^T$$

Now we compute the results of contraction of the co-vector (87) with the 12-component generator of generalized translations (88) with one of the eighteen 12×12 matrices representing the generalized Lorentz algebra (13) sandwiched in between.

This will produce 18 generators of the Z_3-graded Poincaré algebra represented in form of linear differential operators. With twelve translations we shall get the 30-dimensional Z_3-graded extended algebra; its 10-dimensional subalgebra is the standard Poincaré algebra.

The results are a bit cumbersome, but their construction and symmetry properties are quite clear.

Let us start with the nine generalized Lorentz boosts $\overset{(r)}{\mathcal{K}}_i$. We have explicitly:

(89)
$$\overset{(0)}{\mathcal{K}}_x = (\tau_0 \partial_{x_0} + x_0 \partial_{\tau_0}) + (j^2\,\tau_1 \partial_{x_1} + j\,x_1 \partial_{\tau_1}) + (j\,\tau_2 \partial_{x_2} + j^2\,x_2 \partial_{\tau_2}),$$
$$\overset{(0)}{\mathcal{K}}_y = (\tau_0 \partial_{y_0} + y_0 \partial_{\tau_0}) + (j^2\,\tau_1 \partial_{y_1} + j\,y_1 \partial_{\tau_1}) + (j\,\tau_2 \partial_{y_2} + j^2\,y_2 \partial_{\tau_2}),$$
$$\overset{(0)}{\mathcal{K}}_z = (\tau_0 \partial_{z_0} + z_0 \partial_{\tau_0}) + (j^2\,\tau_1 \partial_{z_1} + j\,z_1 \partial_{\tau_1}) + (j\,\tau_2 \partial_{z_2} + j^2\,z_2 \partial_{\tau_2}).$$

(90)
$$\overset{(1)}{\mathcal{K}}_x = (\tau_0 \partial_{x_1} + j\,x_0 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{x_2} + j^2\,x_1 \partial_{\tau_2}) + (j\,\tau_2 \partial_{x_0} + x_2 \partial_{\tau_0}),$$
$$\overset{(1)}{\mathcal{K}}_y = (\tau_0 \partial_{y_1} + j\,y_0 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{y_2} + j^2\,y_1 \partial_{\tau_2}) + (j\,\tau_2 \partial_{y_0} + y_2 \partial_{\tau_0}),$$
$$\overset{(1)}{\mathcal{K}}_z = (\tau_0 \partial_{z_1} + j\,z_0 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{z_2} + j^2\,z_1 \partial_{\tau_2}) + (j\,\tau_2 \partial_{z_0} + z_2 \partial_{\tau_0}).$$

(91)
$$\overset{(2)}{\mathcal{K}}_x = (\tau_0 \partial_{x_2} + j^2\,x_0 \partial_{\tau_2}) + (j\,\tau_2 \partial_{x_1} + j\,x_2 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{x_0} + x_1 \partial_{\tau_0}),$$
$$\overset{(2)}{\mathcal{K}}_y = (\tau_0 \partial_{y_2} + j^2\,y_0 \partial_{\tau_2}) + (j\,\tau_2 \partial_{y_1} + j\,y_2 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{y_0} + y_1 \partial_{\tau_0}),$$
$$\overset{(2)}{\mathcal{K}}_z = (\tau_0 \partial_{z_2} + j^2\,z_0 \partial_{\tau_2}) + (j\,\tau_2 \partial_{z_1} + j\,z_2 \partial_{\tau_1}) + (j^2\,\tau_1 \partial_{z_0} + z_1 \partial_{\tau_0}).$$

The Z_3-graded generalized differential operators representing the Lorentz boosts display remarkable symmetry properties. The "diagonal" generators $\overset{(0)}{\mathcal{K}}_i$ are hermitian: they are invariant under the simultaneous complex conjugation, replacing j by j^2 and vice versa, and switching the indices $1 \to 2,\ 2 \to 1$. Under the same hermitian symmetry operation the Z_3-graded boosts $\overset{(1)}{\mathcal{K}}_i$ and $\overset{(2)}{\mathcal{K}}_i$ transform into each other, so that we have

$$\overset{(1)\dagger}{\mathcal{K}_i} = \overset{(2)}{\mathcal{K}}_i, \qquad \overset{(2)\dagger}{\mathcal{K}_i} = \overset{(1)}{\mathcal{K}}_i.$$

The commutation relations between the generalized Lorentz boosts given by (89, 90) and (91) define the differential representation of Z_3-graded extension of pure rotations, $\overset{(r)}{\mathcal{J}_k}$, with $r = 0, 1, 2$ and $i, j, .. = 1, 2, 3$.

By tedious (but not too sophisticated) calculation we can check that the commutation relations between the Z_3-graded Lorentz boosts imposed as hypothesis in (13):

$$[\,\overset{(r)}{\mathcal{K}_i},\ \overset{(s)}{\mathcal{K}_k}\,] = -\epsilon_{ikl}\,\overset{(r+s)}{\mathcal{J}_l}\,,$$

lead indeed to the following expressions for spatial rotations $\overset{(s)}{\mathcal{J}_i}$:

$$\begin{aligned}
\overset{(0)}{\mathcal{J}_x} &= (z_0\partial_{y_0} - y_0\partial_{z_0}) + (z_1\partial_{y_1} - y_1\partial_{z_1}) + (z_2\partial_{y_2} - y_2\partial_{z_2}),\\
\overset{(0)}{\mathcal{J}_y} &= (x_0\partial_{z_0} - z_0\partial_{x_0}) + (x_1\partial_{z_1} - z_1\partial_{x_1}) + (x_2\partial_{z_2} - z_2\partial_{x_2}),\\
\overset{(0)}{\mathcal{J}_z} &= (y_0\partial_{x_0} - x_0\partial_{y_0}) + (y_1\partial_{x_1} - x_1\partial_{y_1}) + (y_2\partial_{x_2} - x_2\partial_{y_2}).
\end{aligned} \tag{92}$$

Note that the above generators are sums of classical expressions for J_k, each of them acting in its own sector of the Z_3-graded extension of Minkowskian space-time.

The grade 1 generators of rotations $\overset{(1)}{\mathcal{J}_i}$ have the same form, but mix up coordinates with derivatives from different sectors, in cyclical order, symbolically $0 \to 1,\ 1 \to 2,\ 2 \to 0$:

$$\begin{aligned}
\overset{(1)}{\mathcal{J}_x} &= (z_0\partial_{y_1} - y_0\partial_{z_1}) + (z_1\partial_{y_2} - y_1\partial_{z_2}) + (z_2\partial_{y_0} - y_2\partial_{z_0}),\\
\overset{(1)}{\mathcal{J}_y} &= (x_0\partial_{z_1} - z_0\partial_{x_1}) + (x_1\partial_{z_2} - z_1\partial_{x_2}) + (x_2\partial_{z_0} - z_2\partial_{x_0}),\\
\overset{(1)}{\mathcal{J}_z} &= (y_0\partial_{x_1} - x_0\partial_{y_1}) + (y_1\partial_{x_2} - x_1\partial_{y_2}) + (y_2\partial_{x_0} - x_2\partial_{y_0}).
\end{aligned} \tag{93}$$

Finally, the grade 2 generators of spatial rotations, $\overset{(2)}{\mathcal{J}_i}$, repeat the same scheme, but in reverse (anti-cyclic) order, i.e. $0 \to 2,\ 1 \to 0,\ 2 \to 1$:

$$\begin{aligned}
\overset{(2)}{\mathcal{J}_x} &= (z_0\partial_{y_2} - y_0\partial_{z_2}) + (z_1\partial_{y_0} - y_1\partial_{z_0}) + (z_2\partial_{y_1} - y_2\partial_{z_1}),\\
\overset{(2)}{\mathcal{J}_y} &= (x_0\partial_{z_2} - z_0\partial_{x_2}) + (x_1\partial_{z_0} - z_1\partial_{x_0}) + (x_2\partial_{z_1} - z_2\partial_{x_1}),\\
\overset{(2)}{\mathcal{J}_z} &= (y_0\partial_{x_2} - x_0\partial_{y_2}) + (y_1\partial_{x_0} - x_1\partial_{y_0}) + (y_2\partial_{x_1} - x_2\partial_{y_1}).
\end{aligned} \tag{94}$$

These differential operators correspond to what we would get by direct construction using the matrix representation given in the previous section.

The 18 differential operators acting on the cartesian product of Z_3 with Minkowskian space-time – the 9 generalized Lorentz boosts $\overset{(r)}{\mathcal{K}_i}$ and the 9 generalized space rotations $\overset{(s)}{\mathcal{J}_k}$, with $r, s = 0, 1, 2$ and $i, j = 1, 2, 3$ – define the faithful representation of the Z_3-graded generalization of the Lorentz group.

In order to introduce the extension to full Poincaré group we have to add three 4-component generators of translations each one acting on its own sector of the generalized Z_3-graded Minkowskian space-time. It turns out that in order to satisfy the Z_3-graded set of standard commutation relations given by (56), the three differential operators $\overset{(0)}{\mathcal{P}}_\mu$, $\overset{(1)}{\mathcal{P}}_\mu$, $\overset{(2)}{\mathcal{P}}_\mu$ must be defined as follows:

(95) $$\overset{(0)}{\mathcal{P}}_\mu = [\, \partial_{\tau_0},\ -\partial_{x_0},\ -\partial_{y_0},\ -\partial_{z_0}\,]$$

(96) $$\overset{(1)}{\mathcal{P}}_\mu = [\, j\partial_{\tau_1},\ -\partial_{x_1},\ -\partial_{y_1},\ -\partial_{z_1}\,]$$

(97) $$\overset{(2)}{\mathcal{P}}_\mu = \left[\, j^2\partial_{\tau_2},\ -\partial_{x_2},\ -\partial_{y_2},\ -\partial_{z_2}\,\right]$$

The eighteen generators $\overset{(r)}{\mathcal{K}}_i$ and $\overset{(s)}{\mathcal{J}}_k$ together with the twelve generalized Z_3-graded translations defined above by (95, 96, 97) satisfy the full set of Z_3-graded extension of the Poincaré algebra.

Its total dimension is $3 \times 10 = 30$, corresponding to the three replicas of the classical Poincaré group, one "diagonal", acting on three components of the Z_3-graded Minkowskian space-time separately, and two other replicas acting on all three components transforming them into one another. The commutations relations are given by the set defined in (53, 55) and (56).

Classical Poincaré algebra admits two Casimir operators which commute with all generators. These are the 4-square of the translation 4-vector $P_\mu P^\mu$, and the 4-square of the Pauli-Lubanski 4-vector $W_\mu W^\mu$, where

(98) $$W^\mu = \frac{1}{2}\, \varepsilon^{\mu\nu\lambda\rho}\, J_{\nu\lambda} P_\rho, \quad J_{0i} = K_i, \quad J_{ik} = \epsilon_{ikl} J^l.$$

In terms of more familiar generators K_i and J_l the Pauli-Lubanski vector takes on the following form:

(99) $$W_0 = J_i P^i = \mathbf{J}\cdot\mathbf{P}, \qquad W_i = P_0\, J_i - \epsilon_{ijk} P^j K^k, \ \text{ or } \ \mathbf{W} = P^0\, \mathbf{J} - \mathbf{P}\wedge\mathbf{K}.$$

The following relations are easily verified:

(100) $$W_\mu P^\mu = 0, \quad \left[W^\mu, P^\lambda\right] = 0, \quad \left[J^{\mu\lambda}, W^\rho\right] = \eta^{\lambda\rho} W^\mu - \eta^{\mu\rho} W^\lambda.$$

Irreducible representations of the Poincaré algebra (and also the group, by exponentiation) are characterized by eigenvalues of its Casimir operators, the most important of which is the mass operator $M^2 = P_\mu P^{mu}$. In order to generalize the Casimir operator given by the square of four-momentum we must take into account similar contributions from all possible combination of Z_3 grades:

(101) $$\mathcal{P}^2 = \overset{(0)}{\mathcal{P}}_\mu \overset{(0)}{\mathcal{P}}{}^\mu + \overset{(1)}{\mathcal{P}}_\mu \overset{(1)}{\mathcal{P}}{}^\mu + \overset{(2)}{\mathcal{P}}_\mu \overset{(2)}{\mathcal{P}}{}^\mu + \overset{(0)}{\mathcal{P}}_\mu \overset{(1)}{\mathcal{P}}{}^\mu + \overset{(1)}{\mathcal{P}}_\mu \overset{(2)}{\mathcal{P}}{}^\mu + \overset{(2)}{\mathcal{P}}_\mu \overset{(0)}{\mathcal{P}}{}^\mu,$$

This operator commutes with the full set of generators of the Lorentz-Poincaré algebra by virtue of (54, 55 and 56).

The Pauli-Lubanski 4-vector also possesses its Z_3-graded extensions. They are of the following form:

$$
\begin{aligned}
\overset{(0)}{\mathcal{W}}_\mu &= \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho} + \overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho}),\\
\overset{(1)}{\mathcal{W}}_\mu &= \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho} + \overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho}),\\
\overset{(2)}{\mathcal{W}}_\mu &= \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}(\overset{(1)}{\mathcal{J}^{\nu\lambda}}\overset{(1)}{\mathcal{P}^\rho} + \overset{(2)}{\mathcal{J}^{\nu\lambda}}\overset{(0)}{\mathcal{P}^\rho} + \overset{(0)}{\mathcal{J}^{\nu\lambda}}\overset{(2)}{\mathcal{P}^\rho}).
\end{aligned}
\tag{102}
$$

With these three graded Pauli-Lubanski vectors we can produce a Z_3-invariant extended Casimir operator of orbital spin:

$$
\mathcal{W}^2 = \overset{(0)}{\mathcal{W}}_\mu\overset{(0)}{\mathcal{W}^\mu} + \overset{(1)}{\mathcal{W}}_\mu\overset{(1)}{\mathcal{W}^\mu} + \overset{(2)}{\mathcal{W}}_\mu\overset{(2)}{\mathcal{W}^\mu} + \overset{(0)}{\mathcal{W}}_\mu\overset{(1)}{\mathcal{W}^\mu} + \overset{(1)}{\mathcal{W}}_\mu\overset{(2)}{\mathcal{W}^\mu} + \overset{(2)}{\mathcal{W}}_\mu\overset{(0)}{\mathcal{W}^\mu}.
$$

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