EIGENVALUES AND APPROXIMATION THROUGH SIMPLE FUNCTIONS

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Communicated by Lucian Beznea

We consider two eigenvalue problems, associated with a continuous function and with his approximation through a simple (step) function. We prove that their corresponding eigenvalues have very different properties.

AMS 2010 Subject Classification: 34B09, 34D20, 35C09, 35J20, 76S05.

Key words: Hele-Shaw displacements, hydrodynamic stability, eigenvalue problems, simple functions approximations.

1. INTRODUCTION

Consider $\mu_W < \mu_O$ and an increasing piecewise - C^1 function such that

$$\mu : (0, L) \to \mathbf{R}, \quad \mu_W \leq \mu(x) \leq \mu_O. \tag{1}$$

We study the eigenvalue problem

$$-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} k^2 f \mu_x, \quad x \in (0, L), \quad k > 0; \tag{2}$$

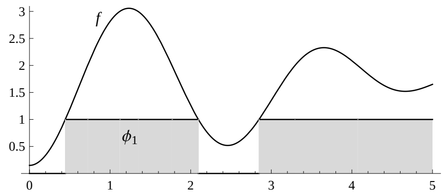
$$(\mu f_x f)^+(0) = \mu_W k f^2(0), \quad (\mu f_x f)^-(L) = -\mu_O k f^2(L); \tag{3}$$

where $^+$ and $^-$ are lateral limit values, k is a parameter (wave number) and σ, f are the eigenvalues and eigenfunctions.

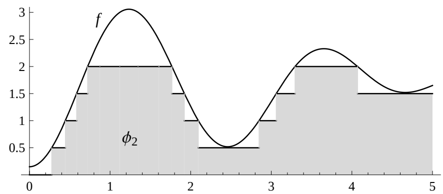
We consider the same problem (2)-(3) when μ is approximated through a simple (step) function μ_S, with jumps in some interior points $x_i \in (0, L)$; then $(\mu_S)_x$ is a Dirac distribution. The corresponding eigenvalues and eigenfunctions are denoted by f_S, σ_S. The functions μ, μ_S are "very close" if the number of interior points x_i is very large (see the Figure 1 below).

When μ is linear and continuous, we get an upper bound of σ which is not depending on k. Moreover, we prove that σ becomes arbitrary small (positive) with increasing L. On the contrary, we get f_S such that the corresponding σ_S become infinite with increasing k, independent of L.

The main results of this paper is following. Even if the step function μ_S is very close to μ, there exists a strong difference betwen the eigenvalues corresponding to μ_S and μ.



Graphs of a function f and the approximating simple function ϕ_1



Graphs of a function f and the approximating simple function ϕ_2

$$\phi_1(x) = \begin{cases} 0, & 0 \leq f(x) < 1, \\ 1, & f(x) \geq 1. \end{cases}$$

$$\phi_2(x) = \begin{cases} 0, & 0 \leq f(x) < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \leq f(x) < 1, \\ 1, & 1 \leq f(x) < \frac{3}{2}, \\ \frac{3}{2}, & \frac{3}{2} \leq f(x) < 2, \\ 2, & f(x) \geq 2. \end{cases}$$

Figure 1: Approximations of a continuous function by simple (step) functions.

The system (1)-(2) is related with the linear stability of the displacements in a Hele-Shaw cell (a technical device described in [1], [10]). The three-layer Hele-Shaw model is studied in [2], [8], [9]. The papers [3]-[6] are concerning the multi-layer Hele - Shaw model. In these models, the boundary conditions (3) contain also σ. Therefore, even if the system (1)-(2) is much more simpler, we have a "singular behavior" of the eigenvalues. Some contradictions concerning the multi-layer model are proved in [11].

The very strong dependence of the eigenvalues of an algebraic matrix, as functions of the coefficients, has been highlighted in the well-known book [12].

2. WHEN μ IS A CONTINUOUS FUNCTION

We consider the linear continuous function μ such that

$$\mu(x) = (\mu_O - \mu_W)x/L + \mu_W, \quad \forall x \in [0, L]. \tag{4}$$

We multiply with f in (2), we integrate on $[0, L]$ and get

$$(5) \qquad (\mu f_x f)^+(0) - (\mu f_x f)^-(L) + \int_0^L \mu f_x^2 + k^2 \int_0^L \mu f^2 = \frac{k^2}{\sigma} \int_0^L \mu_x f^2.$$

The relations (2)-(3) and (5) give us

$$(6) \qquad \sigma_L = \frac{k^2 \int_0^L \mu_x f^2}{\mu_W k f^2(0) + \mu_O k f^2(L) + \int_0^L (\mu f_x^2 + k^2 \mu f^2)}.$$

We neglect some positive terms in the denominator and for the linear μ we obtain

$$(7) \qquad \sigma_L \leq \frac{\int_0^L \mu_x f^2}{\int_0^L \mu f^2} \leq \frac{\mu_O - \mu_W}{L\mu_W}.$$

In the general case, when $\mu(x)$ is an arbitrary continuous function on $[0, L]$ (not related with μ_W and μ_O), from (6) we obtain the following estimate for the corresponding eigenvalue, denoted by σ_C:

$$(8) \qquad \sigma_C \leq \frac{Max_x\ (\mu_x)}{Min_x\ (\mu)}.$$

Both above estimates are not depending on k. Moreover, the relation (7) gives us

$$(9) \qquad L \to \infty \Rightarrow \sigma_L \to 0.$$

3. WHEN μ IS APPROXIMATED BY A SIMPLE (STEP) FUNCTION μ_S

We divide $(0, L)$ in N small intervals, separated by the interfaces x_i,

$$(10) \qquad x_i = iL/N, \quad i = 0, 1, ..., N; \quad x_0 = 0, \quad x_N = L.$$

For $i = 1, ..., N$ we introduce the step function

$$(11) \qquad \mu_S(x) = \mu_W + i[\mu], \quad x \in [x_{i-1}, x_i); \quad [\mu] = [\mu_O - \mu_W]/(N+1).$$

When $N = 3$ we have

- three "intermediate " values μ_i, such that $\mu_W < \mu_1 < \mu_2 < \mu_3 < \mu_O$;
- four equally spaced interfaces $x_0 = 0,\ x_1 = L/3,\ x_2 = 2L/3,\ x_3 = L$ such that

$$\mu_S(x) = \mu_W + \mu_i,\ x \in [x_{i-1}, x_i);$$

- four equal jumps $\mu_1 - \mu_W = \mu_2 - \mu_1 = \mu_3 - \mu_2 = \mu_O - \mu_3 = (\mu_O - \mu_W)/4$.

From now on, we omit the subscript $_S$.

The step function (11) is very "close" to the function (4) when N is large enough. The derivative μ_x is a Dirac distribution, therefore (see *Remark 2* below)

$$\int_0^L \mu_x f^2 = \sum_{i=1}^{i=N-1} f^2(x_i)[\mu]_i, \tag{12}$$

$$-\int_0^L (\mu f_x f)_x = (\mu^+ f_x^+ f)(x_0) - (\mu^- f_x^- f)(x_N) + \sum_{i=1}^{i=N-1} [\mu f_x]_i f(x_i), \tag{13}$$

where $[F]_i = F^+(x_i) - F^-(x_i)$. As μ_i is constant on each small interval, we get

$$-f_{xx} + k^2 f = 0, \quad x \in (x_{i-1}, x_i). \tag{14}$$

We consider the following particular solution of (14):

$$f(x) = \begin{cases} e^{kx} & , x \in [x_0, x_{N-1}]; \\ f(x_{N-1})e^{-k(x-x_{N-1})}, & x \in [x_{N-1}, x_N]. \end{cases} \tag{15}$$

The above function f is continuous, but f_x is not and (recall $x_N = L$)

$$f(x_{N-1}) = e^{kL(N-1)/N}, \quad f(x_N) = e^{k(2x_{N-1}-L)} = e^{kL(N-2)/N}. \tag{16}$$

We multiply (2) with$f$, then from (12), (13), (15) we get

$$\sigma = \frac{\sum_{i=1}^{i=N-1} k^2 [\mu]_i f_i^2}{DEN + \sum_{i=1}^{i=N} I_i}, \tag{17}$$

$$DEN = \mu_1 k f_0^2 + \mu_N k f_N^2 + \sum_{i=1}^{i=N-2} [\mu]_i k f_i^2 - (\mu_{N-1} + \mu_N) k f_{N-1}^2, \tag{18}$$

$$I_i = \int_{x_{i-1}}^{x_i} \mu_i (f_x^2 + k^2 f^2), \quad f_i = f(x_i). \tag{19}$$

Remark 1. For large N we have $DEN > 0$. For this, we need (see (18))

$$\mu_1 + \mu_N f_N^2 + \sum_{i=1}^{i=N-2} [\mu]_i f_i^2 > (\mu_{N-1} + \mu_N) f_{N-1}^2. \tag{20}$$

We recall (16) and use the well known relation

$$y + y^2 + ... + y^{N-2} = (y^{N-1} - 1)/(y - 1), \quad y = e^{2kL/N},$$

therefore (20) is equivalent with

$$\mu_1 + (\mu_O - \frac{\mu_O - \mu_W}{N+1}) e^{2kL(N-2)/N} + \frac{\mu_O - \mu_W}{N+1} \times \frac{y^{N-1}}{y-1} >$$

$$(\mu_O - 2\frac{\mu_O - \mu_W}{N+1})e^{2kL(N-1)/N}.$$

The last inequality holds for N large enough, when $(N-1)/N \to 1, \quad (N-2)/N \to 1$.

By direct calculation we get

LEMMA 1.

(21) $$f(x) = e^{\overset{+}{-}kx}, \quad J(a,c) = \int_a^c (f_x^2 + k^2 f^2) \Rightarrow J(a,c) \leq k\{f^2(a) + f^2(c)\}.$$

The main result of this section is set out in the following proposition.

PROPOSITION 1.

(22) $$\sigma \to \infty \quad when \quad k \to \infty.$$

Proof. From the relations (17), (20), (21), we get

(23) $$\sigma \geq k\frac{\sum_{i=1}^{i=N-1}[\mu]_i e^{2kx_i}}{a_0 + \sum_{i=1}^{i=N-1} a_i e^{2kx_i} + a_N e^{2kz}}, \quad a_i > 0.$$

We prove that the maximum value of the exponential is the same in the numerator and the denominator of the above ratio. Indeed, from the relations (15) - (16) it follows

$$f(x_N) = e^{kz}, \quad z = \frac{2(N-1)L}{N} - L = \frac{L(N-2)}{N}.$$

Therefore

$$2x_{N-1} = 2\frac{(N-1)L}{N} > 2\frac{L(N-2)}{N} = 2z.$$

It was very important that jumps $[\mu]_i$ were positive. □

Remark 2. We give a proof for the formula (12). Consider $\mu : [a,c] \to \mathbf{R}$, $b \in (a,c)$, $\mu(x) = A, x \in [a,b); \quad \mu(x) = B, x \in [b,c]$. We see that $\mu_x(x) = 0$ for almost every $x \in [a,c]$. Only in the point $x = b$, we have

$$\mu_x(b) = \lim_{\epsilon \to 0} \frac{\mu(b) - \mu(b-\epsilon)}{\epsilon} = \lim_{\epsilon \to 0} \frac{B-A}{\epsilon}.$$

The next property is verified by a sufficiently smooth function F: there exists a point $\chi \in (b-\epsilon, b)$ such that $\int_{b-\epsilon}^b F(x)dx = \epsilon F(\chi)$. Therefore, when $\epsilon \to 0$ we get

$$\int_a^c \mu_x(x)F(x)dx = \int_{b-\epsilon}^b \mu_x(x)F(x)dx \to [\mu]_b F(b), \quad [\mu]_b = \mu^+(b) - \mu^-(b) = B - A.$$

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Received January 15, 2020

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