RIGID LIE ALGEBRAS AND ALGEBRAICITY

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A finite dimensional complex Lie algebra $\mathfrak{g}$ is called rigid if any sufficiently close Lie algebra is isomorphic to it. We prove that this implies that the Lie algebra of inner derivations of $\mathfrak{g}$ is an algebraic Lie subalgebra of $gl(\mathfrak{g})$. We show that in general $\mathfrak{g}$ is not algebraic.

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1. INTRODUCTION

The notion of rigidity of Lie algebra is linked to the following problem: when does a Lie bracket μ on a vector space $\mathfrak{g}$ satisfy that every Lie bracket μ_1 sufficiently close to μ is of the form $\mu_1 = P \cdot \mu$ for some $P \in GL(\mathfrak{g})$ close to the identity? A Lie algebra which satisfies the above condition will be called rigid. The most famous example is the Lie algebra $sl(2, \mathbb{C})$ of square matrices of order 2 with vanishing trace. This Lie algebra is rigid, that is any close deformation of $sl(2, \mathbb{C})$ is isomorphic to it. Let us note that, for this Lie algebra, there exists a quantification of the universal algebra. This lead to the definition of the famous quantum group $SL(2)$. Another interest of studying the rigid Lie algebras is the fact that there exists, for a given dimension, only a finite number of isomorphic classes of rigid Lie algebras. So we are tempted to establish a classification using these classes. This problem has been solved up to the dimension 8. To progress in this direction, properties must be established on the structure of these algebras. One of the first results establishes an algebricity criterion [5]. However, the notion of algebricity which is used is not the classical notion and it includes non-algebraic Lie algebras in the usual sense. The aim of this work is to show that if a Lie algebra is rigid, then its algebra of inner derivations is algebraic.

2. RIGIDITY OF MULTIPLICATIONS ON A FINITE DIMENSIONAL VECTOR SPACE

Throughout the paper, $\mathbb{K}$ stands for an algebraically closed field of characteristic 0.

2.1. **Rigidity on the linear space of skew symmetric bilinear map**

Let E be a n-dimensional $\mathbb{K}$-vector space. We fix a basis $\mathcal{B} = \{e_1, \cdots, e_n\}$ of E. Let μ be a skew-symmetric bilinear map on E that is:

$$\mu : E \times E \to E$$

is bilinear and satisfies $\mu(X, Y) = -\mu(Y, X)$ for any $X, Y \in E$ or equivalently $\mu(X, X) = 0$ for any $X \in E$. The structure constants of μ related to the basis $\mathcal{B}$ are the scalars X_{ij}^k, $i, j, k \in [[1, n]]$, given by

$$\mu(e_i, e_j) = \sum_{k=1}^{n} X_{ij}^k e_k \tag{1}$$

and satisfying

$$X_{ij}^k = -X_{ji}^k.$$

We denote $\mathcal{V}_n$ the N-dimensional $\mathbb{K}$-vector space ($N = \frac{n^2(n-1)}{2}$) whose elements are the N-uples $\{X_{ij}^k, 1 \leq i < j \leq n, k = 1, \cdots, n\}$. It can be considered as an affine space of dimension N. If $\mathcal{SB}il(E)$ is the set of skew-symmetric bilinear maps

$$\mu : E \times E \to E,$$

then (1) shows that this vector space is isomorphic to $\mathcal{V}_n$ and we identify these two vector spaces. So, we shall often write μ for a point of $\mathcal{V}_n$.

Let $GL(E)$ be the algebraic group of linear isomorphisms of E. We have a natural action of $GL(E)$ on $\mathcal{SB}il(E)$ namely

$$\begin{array}{ccc} GL(E) \times \mathcal{SB}il(E) & \to & \mathcal{SB}il(E) \\ (f, \mu) & \mapsto & \mu_f \end{array}$$

where $\mu_f(X, Y) = f^{-1}(\mu(f(X), f(Y))$ for all $X, Y \in E$. This action is translated in an action of $GL(n, \mathbb{K})$ on $\mathcal{V}_n$ namely

$$(f = (a_{ij}), (X_{ij}^k)) \to (Y_{ij}^k)$$

with

$$\sum_{k=1}^{n} Y_{ij}^k a_{sk} = \sum_{l,r=1}^{n} a_{li} a_{rj} X_{lr}^s. \tag{2}$$

Since $f = (a_{ij}) \in GL(n, \mathbb{K})$, the vector (Y_{ij}^k) is completely determined by (2).

Definition 1. The element $\mu = (X_{ij}^k)$ of $\mathcal{V}_n$ is called rigid if its orbit $\mathcal{O}(\mu) = \{\mu_f, f \in GL(n, \mathbb{K})\}$ associated to the action of $GL(n, \mathbb{K})$ is open in the affine space $\mathcal{V}_n$.

This notion is a topological notion. Then the considered topology on the affine space is the Zariski-topology. Remember that in this case, the topology coincides with the finer metric topology. Then μ is rigid if any neighborhood of μ in $\mathcal{V}_n$ is contained in $\mathcal{O}(\mu)$.

Example. We consider $n = 2$ and the bilinear map

$$\mu(e_1, e_2) = X_{12}^1 e_1 + X_{12}^2 e_2$$

with nonzero X_{12}^1 or X_{12}^2. It corresponds to a point $\mu = (X_{12}^1, X_{12}^2) \neq (0,0)$ in $\mathcal{V}_2$. A direct computation gives

$$\mu_f = (Y_{12}^1, Y_{12}^2) \text{ with } \begin{pmatrix} Y_{12}^1 \\ Y_{12}^2 \end{pmatrix} = \Delta(f) f^{-1} \begin{pmatrix} X_{12}^1 \\ X_{12}^2 \end{pmatrix}$$

where $\Delta(f) = \det(f)$. Then $\mathcal{O}(\mu) = \mathcal{V}_2 \setminus \{(0,0)\}$ is open in $\mathcal{V}_2$ and the point μ is rigid.

Let μ be in $\mathcal{V}_n$ and G_μ be the isotropy group of μ,

$$G_\mu = \{f \in GL(n, \mathbb{C}) \,/\, \mu_f = \mu\}.$$

It is a closed subgroup of $GL(n, \mathbb{K})$ and the orbit $\mathcal{O}(\mu)$ is isomorphic to the homogeneous algebraic space

$$\mathcal{O}(\mu) = \frac{GL(n, \mathbb{K})}{G_\mu}.$$

In particular $\mathcal{O}(\mu)$ can be provided with a differentiable manifold structure and $\dim \mathcal{O}(\mu) = \operatorname{codim} G_\mu$. As a consequence, $\mu \in \mathcal{V}_n$ is rigid if and only if $\dim GL(n, \mathbb{K}) - \dim G_\mu = \dim \mathcal{V}_n = \frac{n^2(n-1)}{2}$. This implies $\dim G_\mu = \frac{n^2(3-n)}{2}$ and then $n \leq 3$. For $n = 2$ the point $\mu = (X_{12}^1, X_{12}^2) = (1, 0)$ is rigid.

Proposition 2. *If $n \geq 3$, no element $\mu \in \mathcal{V}_n$ is rigid.*

Proof. If $n > 3$ then for any $\mu \in \mathcal{V}_n$, $\dim \mathcal{O}(\mu) < \dim \mathcal{V}_n$ and $\mathcal{O}(\mu)$ is not rigid. If $n = 3$, from [13] $\dim G_\mu \geq 1$ and the orbit of μ is not rigid.

Remark that for $n \geq 4$, a μ in $\mathcal{V}_n$ with $\dim G_\mu = 0$ can be found (see [13]). But for such a μ we have $\dim \mathcal{O}(\mu) < \dim \mathcal{V}_n$ so $\mathcal{O}(\mu)$ is not an open set in $\mathcal{V}_n$.

2.2. Rigidity in stable subsets of $\mathcal{V}_n$

Let $\mathcal{W}$ be an algebraic subvariety of $\mathcal{V}_n$. It is defined by a finite polynomial system on $\mathcal{V}_n$. We assume that $\mathcal{W}$ is stable by the action of $GL(n, \mathbb{K})$ on $\mathcal{V}_n$, that is,

$$\forall \mu \in \mathcal{W}, \forall f \in GL(n, \mathbb{K}),\ \mu_f \in \mathcal{W}.$$

It is the case for example for

- $\mathcal{W} = \mathcal{L}ie_n$ the set of Lie algebra multiplications, that is,

$$\mu(\mu(X,Y),Z) + \mu(\mu(Y,Z),X) + \mu(\mu(Z,X),Y) = 0$$

for any $X, Y, Z \in E$, or equivalently

$$\sum_{l=1}^{n} X_{ij}^{l} X_{lk}^{s} + X_{jk}^{l} X_{li}^{s} + X_{ki}^{l} X_{lj}^{s} = 0$$

for any $1 \leq i < j \leq n, 1 \leq k \leq n$ and $1 \leq s \leq n$.

- $\mathcal{W} = \mathcal{SA}ss_n$ the set of skew-symmetric associative multiplications

$$\mu(\mu(X,Y),Z) - \mu(X,\mu(Y,Z)) = 0$$

for any $X, Y, Z \in E$ or equivalently

$$\sum_{l=1}^{n} (X_{ij}^{l} X_{lk}^{s} + X_{il}^{s} X_{jk}^{l})$$

for any $1 \leq i < j \leq j, 1 \leq k \leq n$ and $1 \leq s \leq n$. This set coincides with the set of multiplications satisfying

$$\mu(\mu(X,Y),Z) = 0,$$

that is, the subvariety of $\mathcal{L}_n$ constituted of 2-step nilpotent Lie algebras.

- $\mathcal{W} = \mathcal{N}il_n = \{\mu \in \mathcal{L}_n \ / \ \mu \text{ is nilpotent}\}$ or $\mathcal{S}ol_n = \{\mu \in \mathcal{L}_n \ / \ \mu \text{ is solvable}\}$. Recall that μ is nilpotent if the linear operators

$$ad_\mu X : Y \to \mu(X,Y)$$

are nilpotent. It is called k-step nilpotent if $(ad_\mu X)^k = 0$ for any X and if there exists Y such that $(ad_\mu Y)^{k-1} \neq 0$ (k is also called the nilindex of μ).

When $k = n-1$, the nipotent Lie algebra is called filiform. This case is studied in [14].

Definition 3. Let $\mathcal{W}$ be a stable algebraic subvariety of $\mathcal{V}_n$. An element $\mu \in \mathcal{W}$ is $\mathcal{W}$-rigid if the orbit $\mathcal{O}(\mu)$ is open in $\mathcal{W}$.

Since $\mathcal{W}$ is stable, $\mathcal{O}(\mu) \subset \mathcal{W}$ and

$$\dim \mathcal{O}(\mu) = n^2 - \dim G_\mu.$$

Remark. It is also interesting to consider stable but not necessarily closed subsets $\widetilde{\mathcal{W}}$ of $\mathcal{V}_n$ or of a stable subvariety $\mathcal{W}$ of $\mathcal{V}_n$ that is for every $\mu \in \widetilde{\mathcal{W}}$ then $\mathcal{O}(\mu) \subset \widetilde{\mathcal{W}}$. For example the subset $\mathcal{N}il_{n,k}$ of $\mathcal{N}il_n$ whose elements are k-step nilpotent, is stable for the action of $GL(n,\mathbb{K})$. For this stable subset, there exists another invariant up to an isomorphism which permits to describe it: the characteristic sequence of a nilpotent Lie algebra multiplication (see for example [12] for a detailled presentation of this notion). Let be $\mu \in \mathcal{N}il_n$. For any $X \in E$, let $c(X)$ be the ordered sequence, for the lexicographic order, of the dimensions of the Jordan blocks of the nilpotent operator $ad_\mu X$. The characteristic sequence of μ is the invariant, up to isomorphism,

$$c(\mu) = \max\{c(X),\ X \in E\}.$$

In particular, if $c(\mu) = (c_1, c_2, \cdots, 1)$, then μ is c_1-step nilpotent. A vector $X \in E$ such that $c(X) = c(\mu)$ is called a characteristic vector of μ. For a given sequence $(c_1, c_2, \cdots, 1)$ with $c_1 \geq c_2 \geq \cdots \geq c_s \geq 1$ and $c_1+c_2+\cdots+c_s+1 = n$, the subset $\mathcal{N}il_n^{(c_1,c_2,\cdots,1)}$ is stable in $\mathcal{N}il_n$.

Definition 4. Let $\widetilde{\mathcal{W}}$ be a stable subset of $\mathcal{V}_n$ or of a stable subvariety $\mathcal{W}$ of $\mathcal{V}_n$. A multiplication $\mu \in \widetilde{\mathcal{W}}$ is called $\widetilde{\mathcal{W}}$-rigid if for any neighborhood $V(\mu)$ of μ in $\mathcal{V}_n$ or in $\mathcal{W}$ then $V(\mu) \cap \mathcal{V}_n$ or $V(\mu) \cap \mathcal{W}$ is included in $\mathcal{O}(\mu)$.

This notion of $\widetilde{\mathcal{W}}$-rigidy has been introduced in [9] to study the set of k-step nilpotent Lie algebras.

2.3. **How to prove the rigidity**

We have two approaches

1. **A topological way**. By definition, μ is rigid if $\mathcal{O}(\mu)$ is open in $\mathcal{W}$ and $\mathcal{W} \backslash \mathcal{O}(\mu)$ is an algebraic subset of $\mathcal{W}$. But $\mathcal{O}(\mu)$ is provided with a differentiable homogeneous manifold contained in the affine space $\mathcal{V}_n$. So we can consider open neighbourhood of μ for the "metric" topology. Now, although $\mathcal{W}$ contains singular points, any rigid point μ in $\mathcal{W}$ is non singular since its orbit is open in $\mathcal{W}$. Thus in order to prove the rigidity we can consider an open neighbourhood of μ in $\mathcal{V}_n$. For this we can use a method inspired by the determination of the algebraic Lie algebra of an algebraic Lie group using the dual numbers and consider non archimedian extension of $\mathbb{K}$.
2. **A geometrical way**. Since $\mathcal{O}(\mu)$ is a differentiable manifold, its tangent space $T_\mu\mathcal{O}(\mu)$ to μ is well defined. It is isomorphic to the quotient space $\frac{gl(n,\mathbb{K})}{\mathcal{D}er_\mu}$ where $\mathcal{D}er_\mu = \{f \in gl(n,\mathbb{K}), \delta f(X,Y) = \mu(f(X),Y) + \mu(X,f(Y)) - f(\mu(X,Y)) = 0\}$ is the algebraic Lie algebra of G_μ. We deduce

$$\dim T_\mu\mathcal{O}(\mu) = n^2 - \dim \mathcal{D}er_\mu$$

and $T_\mu\mathcal{O}(\mu)$ is isomorphic to the subspace of bilinear maps whose elements are δf for any $f \in gl(n,\mathbb{C})$ generally denoted $B^2(\mu,\mu)$. Then

Proposition 5. *The application μ is rigid in $\mathcal{W}$ if*

$$\dim T_\mu \mathcal{O}(\mu) = \dim B^2(\mu,\mu) = \dim T_\mu \mathcal{W}.$$

The determination of $T_\mu \mathcal{W}$ is a little bit difficult. Since we assume that μ is rigid, necessarily $T_\mu \mathcal{W}$ exists. In a first time we can compute the Zariski tangent space $T^Z_\mu \mathcal{W}$ usually denoted $Z^2(\mu,\mu)$, defined by the linear system obtained by considering the polynomial system of definition of $\mathcal{W}$, translated to the point μ and taking its linear part. But $T_\mu \mathcal{W} \subset T^Z_\mu \mathcal{W} = Z^2(\mu,\mu)$ and this inclusion can be strict. This appears as soon as the affine schema which defines $\mathcal{W}$ is not reduced at the point μ. We can illustrate this in a simple example. Let us consider the algebraic variety M in $\mathbb{C}^3$ defined by the polynomial system

$$\begin{cases} X_1X_2 + X_1X_3 - 2X_2X_3 = 0, \\ 2X_1X_2 - 3X_2X_3 + X_3^2 = 0 \end{cases}$$

The only singular point is $(0,0,0)$. Let us compute $T^Z_\mu M$ at the point $(1,1,1)$. Linearizing the system we obtain

$$(\star)\begin{cases} 2X_1 - X_2 - X_3 + X_1X_2 + X_1X_3 - 2X_2X_3 = 0, \\ 2X_1 - X_2 - X_3 + 2X_1X_2 - 3X_2X_3 + X_3^2 = 0 \end{cases}$$

and $T^Z_\mu M = Ker\{\rho(X_1,X_2,X_3) = 2X_1 - X_2 - X_3\}$ is 2-dimensional although M is a one-dimensional curve. To compute $T_\mu M$ we come back to the definition of the tangent vector. We consider a point $(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ in M close to $(0,0,0)$ and satisfying $(\star)$. Thus

$$\begin{cases} 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 - 2\varepsilon_2\varepsilon_3 = 0, \\ 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + 2\varepsilon_1\varepsilon_2 - 3\varepsilon_2\varepsilon_3 + \varepsilon_3^2 = 0. \end{cases}$$

A similar approach with the dual number implies

$$2\varepsilon_1 - \varepsilon_2 - \varepsilon_3 = 0$$

and

$$\begin{cases} \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 - 2\varepsilon_2\varepsilon_3 = 0, \\ 2\varepsilon_1\varepsilon_2 - 3\varepsilon_2\varepsilon_3 + \varepsilon_3^2 = 0 \end{cases}$$

This is equivalent to

$$\varepsilon_1 = \frac{2\varepsilon_2\varepsilon_3}{\varepsilon_2 + \varepsilon_3} = \frac{\varepsilon_3(3\varepsilon_2 - \varepsilon_3)}{2\varepsilon_2}.$$

that is

$$\varepsilon_2 = \varepsilon_3 = \varepsilon_1.$$

We deduce that $T_\mu M = \{(x,x,x), x \in \mathbb{C}\} \neq T^Z_\mu M$.

Remarks. 1. We know many examples of rigid Lie algebras such that $T_\mu \mathcal{O}(\mu) \neq T^Z_\mu \mathcal{W}$. Since $T_\mu \mathcal{O}(\mu)$ and $T^Z_\mu \mathcal{W}$ coincide respectively with the space of coboundaries and the space of 2-cocycles associated with the Chevalley-Eilenberg cohomology $H^*(\mu,\mu)$ of μ (when μ is a Lie algebra) the classical Nijenhuis-Richardson's Theorem says that $\dim H^2(\mu,\mu)=0$ implies that μ is rigid. But the converse is not true because the determination of $T^Z_\mu\mathcal{W}$ is not suffisant to compute $\dim T_\mu\mathcal{W}$. There exists another approach of the rigidity using the deformation theory close to the cohomogical point of view. We consider a formal series $\mu_t=\mu+t\varphi_1+\cdots+t^n\varphi_n+\cdots$ and $\mu_t\in\mathcal{W}$ implies $\varphi_1\in T^Z_\mu\mathcal{W}$. But to compute $T_\mu\mathcal{W}$ it is necessary to look all the relations between $\varphi_1,\varphi_2,\cdots$

2. As the elements of $T^Z_\mu\mathcal{W}$ can be interpreted as cocycle associated with the Chevalley-Eilenberg cohomology, the elements of $T_\mu\mathcal{W}$ can be interpreted as particular cocycles. This will be the aim of a future work.

2.4. **Consequence of the reductivity of $GL(n,\mathbb{K})$**

The action of $G=GL(n,\mathbb{K})$ on $\mathcal{V}_n$ or on an algebraic subvariety $\mathcal{W}$ is an example of an action of a reductive group on an algebraic affine variety. Recall that any element $f\in GL(n,\mathbb{K})$ decomposes as $f=f_s\circ f_u$ where f_s is a semisimple and f_u is unipotent. So the fact that G is reductive implies that the maximal normal unipotent subgroup $R_u(G)$ of G is trivial.

Proposition 6. *Let μ be in $\mathcal{L}ie_n$ and G_U the maximal unipotent subgroup of G. Then the orbit $\mathcal{O}_{G_U}(\mu)=\{\mu_f\,/\,f\in G_U\}$ is closed in $\mathcal{L}ie_n$.*

Consequence. As we are interested in open orbits, we will be concerned with the action of the maximal torus (all the elements are semisimple and commuting). Let $\mu=(X^k_{ij})$ be in $\mathcal{L}ie_n$ and $f\in T$ where T is a maximal torus. We can assume that the basis $\{X_1,\cdots,X_n\}$ associated with the X^k_{ij}'s is a basis of eigenvectors of f. So

$$\mu_f(X_i,X_j)=\sum_k\frac{\lambda_i\lambda_j}{\lambda_k}X^k_{ij}X_k$$

and the structure constants Y^k_{ij} of μ_f satisfy

$$Y^k_{ij}=\frac{\lambda_i\lambda_j}{\lambda_k}X^k_{ij}.$$

If we assume now that μ is rigid, then the structure constant of a $\tilde{\mu}$ taken in a neighborhood of μ satisfy

$$Z^k_{ij}=X^k_{ij}(1+\rho^k_{ij})$$

with $\rho_{ij}^k \simeq 0$ and since $\tilde{\mu}$ is of type μ_f we have

$$\rho_{ij}^k = \frac{\lambda_i \lambda_j - \lambda_k}{\lambda_k}.$$

We will come back to this system in the Rank Theorem [3].

3. RIGID LIE ALGEBRAS AND ALGEBRAICITY

3.1. Rigid Lie algebras

Recall that a $\mathbb{K}$-Lie algebra is a pair $\mathfrak{g} = (E, \mu)$ where E is a $\mathbb{K}$-vector space and μ a Lie algebra multiplication on E. In this section we need to make a difference between $\mathfrak{g}$ and μ because the structure of $\mathfrak{g}$ depends also on the nature of the vector space E.

Definition 7. A n-dimensional $\mathbb{K}$-Lie algebra $\mathfrak{g} = (E, \mu)$ is called rigid if μ is $\mathcal{L}ie_n$-rigid.

From the previous discussion, we know that a Lie algebra is rigid (in $\mathcal{L}ie_n$) if and only if $\dim T_\mu \mathcal{O}(\mu) = \dim T_\mu \mathcal{L}ie_n$. In particular, since the Zariski tangent space $T_\mu^Z \mathcal{L}ie_n$ contains $T_\mu \mathcal{L}ie_n$, we have the classical Nijenhuis-Richardson Theorem:

THEOREM 8 (Nijenhuis-Richardson). *A $\mathbb{K}$-Lie algebra $\mathfrak{g} = (E, \mu)$ satisfying* $\dim H^2(\mu, \mu) = 0$ *is rigid.*

In fact $H^2(\mu, \mu) = \frac{Z^2(\mu,\mu)}{B^2(\mu,\mu)}$ and $Z^2(\mu, \mu)$ is isomorphic to $T_\mu^Z \mathcal{L}ie_n$ and $B^2(\mu, \mu)$ is isomorphic to $T_\mu \mathcal{O}(\mu)$.

Remarks. 1. We have discussed the converse in the previous section. The simplest example of rigid Lie algebra having a nonzero-dimensional $H^2(\mu, \mu)$ actually known is in dimension 13. It is given by the multiplication

$$\begin{cases} [X_1, X_i] = X_{i+1}, 2 \leq i \leq 11, \ [X_2, X_i] = X_{i+2}, \ 3 \leq i \leq 10, \\ [T, X_i] = i X_i, 1 \leq i \leq 12. \end{cases}$$

Here the dimension of the second cohomology group is 1. This means that the

$$\dim T_\mu^Z(\mathcal{L}ie_{13}) - \dim T_\mu(\mathcal{L}ie_{13}) = 1.$$

A direct computation considering a point of $\mathcal{L}ie_{13}$ close to μ shows that

$$T_\mu^Z(\mathcal{L}ie_{13}) = T_\mu(\mathcal{L}ie_{13}) \oplus \mathbb{C}\phi$$

where ϕ is the element of $\mathcal{V}_{13}$ given by

$$\begin{cases} \phi(X_2, X_i) = (4 - i) X_{2+i}, \ 5 \leq i \leq 11, \\ \phi(X_3, X_i) = X_{3+i}, \ 4 \leq i \leq 10. \end{cases}$$

This non tangent "cocycle" has been already defined in [10].

2. Assume that μ is rigid in $\mathcal{L}ie_n$. In this case $T_\mu(\mathcal{L}ie_n) = T_{\mu_1}(\mathcal{L}ie_n)$ for any $\mu_1 \in \overline{\mathcal{O}(\mu)}$. For example, if the rigid Lie algebra $\mathfrak{g} = (E, \mu)$ is a contact rigid Lie algebra, then $T_\mu(\mathcal{L}ie_n)$ can be computed considering the tangent space at μ_1 where μ_1 is the multiplication of the Heisenberg algebra (see [8, 6]).

3.2. **Algebraic Lie algebras**

Recall that $\mathbb{K}$ is an algebraically closed field of characteristic 0. (A study of rigid Lie algebra when $\mathbb{K} = \mathbb{R}$ is proposed in [1]).

Definition 9. A finite dimensional $\mathbb{K}$-Lie algebra is algebraic if it is the Lie algebra of an algebraic Lie group.

Examples:

1. Any complex semisimple Lie algebra is algebraic.
2. Any Lie algebra which coincides with its derived subalgebra is algebraic.
3. For any Lie algebra $\mathfrak{g}$, there exists an algebraic Lie algebra, containing $\mathfrak{g}$ and having the same derived subalgebra. It is called the algebraic Lie algebra generated by $\mathfrak{g}$.

Problem: A Lie algebra is not always algebraic so how to characterize an algebraic Lie algebra? There exists some criterium to study the algebraicity of a Lie algebra. We can always assume that an algebraic Lie algebra is linear that is it is a Lie subalgebra of some $gl(n, \mathbb{C})$. Let $\mathfrak{g}$ be a Lie subalgebra of $gl(n, \mathbb{K})$. A replica Y of an element $X \in \mathfrak{g}$ is an element of the smallest algebraic Lie subalgebra of $gl(n, \mathbb{K})$ containing X. A Lie subalgebra $\mathfrak{g}$ of $gl(n, \mathbb{K})$ is algebraic if and only if for any $X \in \mathfrak{g}$, all the replica of X are in $\mathfrak{g}$.

Recall also the structure of algebraic solvable or nilpotent Lie algebras.

PROPOSITION 10. *Let $\mathfrak{g}$ be an algebraic nilpotent Lie algebra, subalgebra of $gl(n, \mathbb{K})$. Let $\mathfrak{n}$ be the ideal of $\mathfrak{g}$ whose elements are the nilpotent elements of $\mathfrak{g}$. Then $\mathfrak{g}$ is the direct sum*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$$

where $\mathfrak{a}$ is an abelian algebraic subalgebra of $\mathfrak{g}$ contained in the center of $\mathfrak{g}$ whose all the elements are semisimple.

Let us note that $\mathfrak{a} \subset Z(\mathfrak{g})$. In fact if $X \in \mathfrak{a}$, then X semisimple implies that adX is also semisimple. But it is also nilpotent because $\mathfrak{g}$ is nilpotent, then $adX = 0$ and $X \in Z(\mathfrak{g})$.

Concerning the solvable case, we have

PROPOSITION 11. *Let $\mathfrak{g}$ be an algebraic solvable Lie algebra, subalgebra of $gl(n,\mathbb{K})$. Let $\mathfrak{n}$ be the ideal of $\mathfrak{g}$ whose elements are the nilpotent elements of $\mathfrak{g}$. Then $\mathfrak{g}$ is the direct sum*

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$$

where $\mathfrak{a}$ is an abelian algebraic subalgebra of $\mathfrak{g}$ with only semisimple elements.

Examples.

1. The one-dimensional abelian Lie algebra

$$\mathfrak{a}_1 = \left\{ \begin{pmatrix} x & x \\ 0 & x \end{pmatrix},\ x \in \mathbb{C} \right\}$$

is not algebraic. In fact the semisimple part of $\begin{pmatrix} x & x \\ 0 & x \end{pmatrix}$ is not in $\mathfrak{a}_1$ as soon as $x \neq 0$.

2. Let us consider the following 3-dimensional Lie algebras

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} x_1+x_2 & x_1+x_2 & 0 & x_1 \\ x_1+x_2 & x_1+x_2 & 0 & x_2 \\ x_1 & x_1+2x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_i \in \mathbb{C} \right\} \text{ and}$$

$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix},\ x_i \in \mathbb{C} \right\}.$$

They are isomorphic as Lie algebras but $\mathfrak{n}_2$ is algebraic and $\mathfrak{n}_1$ is not (see [15]).

We can even construct a family of 3-dimensional non algebraic Lie algebras which are isomorphic, as Lie algebra, to the algebraic Lie algebra $\mathfrak{n}_2$. We consider the 3-dimensional linear subspace $\mathfrak{h}_{\alpha,\beta}$ of $gl(4,\mathbb{C})$

$$\left\{ \begin{pmatrix} x_1+x_2 & x_1+x_2 & 0 & x_1 \\ x_1+x_2 & x_1+x_2 & 0 & x_2 \\ \alpha x_1+(\beta-1)x_2 & \beta x_1+(\alpha+1)x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ x_1,x_2,x_3 \in \mathbb{C} \right\},$$

where α, β are given elements of $\mathbb{C}$. A basis is given by

$$X_1 = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ X_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ \beta-1 & \alpha+1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We verify that

$$[X_1, X_2] = X_3, [X_1, X_3] = [X_2, X_3] = 0.$$

Then $\mathfrak{h}_{\alpha,\beta}$ is a 3-dimensional Lie subalgebra of $gl(4, \mathbb{C})$. It is isomorphic to the 3-dimensional Heisenberg Lie algebra, but this isomorphism is not the linear part of an isomorphism of algebraic groups. Let us consider X_1 with $\alpha + \beta \neq 0$. Its Chevalley-Jordan decomposition

$$X_1 = X_{1,s} + X_{1,n}$$

is given by

$$X_{1,s} = \begin{pmatrix} 1 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 \\ \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & 0 & \frac{\alpha+\beta}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ X_{1,n} = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & -1/2 \\ \frac{\alpha-\beta}{2} & -\frac{\alpha-\beta}{2} & 0 & -\frac{\alpha+\beta}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $X_{1,s} \notin \mathfrak{h}_{\alpha,\beta}$ and also $X_{1,n} \notin \mathfrak{h}_{\alpha,\beta}$, we deduce

Proposition 12. *The 3-dimensional nilpotent Lie subalgebra $\mathfrak{h}_{\alpha,\beta}$ of $gl(4, \mathbb{C})$ is not algebraic.*

A matrix $X \in \mathfrak{h}_{\alpha,\beta}$ is nilpotent if and only if $x_1 + x_2 = 0$. In fact the eigenvalues of X are 0 and $2(x_1 + x_2)$. We deduce that the set of nilpotent matrices of $\mathfrak{h}_{\alpha,\beta}$ is the subspace generated by $\{X_1 - X_2, X_3\}$ and it is a 2-dimensional abelian ideal.

We have recalled that any Lie algebra $\mathfrak{g}_0$ generates an algebraic Lie algebra $\mathfrak{g}_1$ which is the smallest algebraic algebra containing $\mathfrak{g}_0$ and these two algebras have the same derived Lie algebra. Let us determinate the algebraic Lie algebra generated by $\mathfrak{h}_{\alpha,\beta}$. This algebra contains the semisimple part X_s of any $X \in \mathfrak{h}_{\alpha,\beta}$. If $X = x_1 X_1 + x_2 X_2 + x_3 X_3$ then

$$X_s = \begin{pmatrix} x_1 + x_2 & x_1 + x_2 & 0 & \frac{x_1+x_2}{2} \\ x_1 + x_2 & x_1 + x_2 & 0 & \frac{x_1+x_2}{2} \\ (x_1 + x_2)\frac{\alpha+\beta}{2} & (x_1 + x_2)\frac{\alpha+\beta}{2} & 0 & (x_1 + x_2)\frac{\alpha+\beta}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider $X_4 = \begin{pmatrix} 1 & 1 & 0 & 1/2 \\ 1 & 1 & 0 & 1/2 \\ \frac{\alpha+\beta}{2} & \frac{\alpha+\beta}{2} & 0 & \frac{\alpha+\beta}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and the Lie algebra $\mathfrak{m} = \mathfrak{h}_{\alpha,\beta} \oplus$

$\mathbb{K}\{X_4\}$ that is

$$\mathfrak{m}=\left\{\begin{pmatrix} x_1+x_2+x_4 & x_1+x_2+x_4 & 0 & \frac{x_4}{2}+x_1 \\ x_1+x_2+x_4 & x_1+x_2+x_4 & 0 & \frac{x_4}{2}+x_2 \\ \alpha x_1+(\beta-1)x_2+\frac{\alpha+\beta}{2}x_4 & \beta x_1+(\alpha+1)x_2+\frac{\alpha+\beta}{2}x_4 & 0 & x_3+\frac{\alpha+\beta}{4}x_4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, x_i\in\mathbb{C}\right\}$$

Then we have $[X_i,X_4]=0$ for $i=1,2,3$ and $\mathfrak{m}$ is a 4-dimensional Lie algebra containing $\mathfrak{h}_{\alpha,\beta}$. Moreover, for any X in $\mathfrak{m}$, X_s and X_n the semisimple and nilpotent parts of the Chevalley-Jordan decomposition of X are in $\mathfrak{m}$. An element of $\mathfrak{m}$ is nilpotent if and only if $x_1+x_2+x_4=0$. Then the set $\mathfrak{n}_1$ of nilpotent elements of $\mathfrak{m}$ is the 3-dimensional linear subspace of $\mathfrak{m}$:

$$\mathfrak{n}_1=\left\{\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}(x_1-x_2) \\ 0 & 0 & 0 & -\frac{1}{2}(x_1-x_2) \\ \frac{\alpha-\beta}{2}x_1+\frac{-\alpha+\beta-2}{2}x_2 & \frac{-\alpha+\beta}{2}x_1+\frac{\alpha-\beta+1}{2}(\alpha+1)x_2 & 0 & x_3+\frac{\alpha+\beta}{4}(-x_1-x_2) \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}$$

The set of diagonalizable elements is the 1-dimensional subalgebra

$$\mathfrak{a}_1=\left\{\begin{pmatrix} y & y & 0 & \frac{1}{2}y \\ y & y & 0 & \frac{1}{2}y \\ \frac{\alpha+\beta}{2}y & \frac{\alpha+\beta}{2}y & 0 & \frac{\alpha+\beta}{4}y \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\},$$

with $y\in\mathbb{C}$ and $\mathfrak{m}$ is decomposable, that is

$$\mathfrak{m}=\mathfrak{n}_1\oplus\mathfrak{a}_1.$$

Moreover, for any $X\in\mathfrak{m}$ the components X_s and X_n are also in $\mathfrak{m}$. To study the algebraicity of $\mathfrak{m}$ we have to compute, for any $X\in\mathfrak{m}$, the algebraic Lie algebra $\mathfrak{g}(X)$ generated by X. Let X_s be its semisimple component. The eigenvalues are 0, which is a triple root, and $2(x_1+x_2+x_4)$. We assume that $x_1+x_2+x_4\neq 0$. The set Λ is constituted of 4-uples of integers $(p_1,p_2,p_3,0)$. If Y is a semisimple element of $\mathfrak{m}$ commuting with X_s, its eigenvalues μ_1,μ_2,μ_3,μ_4 satisfy $p_1\mu_1+p_2\mu_2+p_3\mu_3=0$ for any $p_1,p_2,p_3\in\mathbb{Z}$. Then $\mu_1=\mu_2=\mu_3=0$. We deduce that

$$Y=\begin{pmatrix} \frac{m}{2} & \frac{m}{2} & 0 & \frac{m}{4} \\ \frac{m}{2} & \frac{m}{2} & 0 & \frac{m}{4} \\ \frac{m(\alpha+\beta)}{4} & \frac{m(\alpha+\beta)}{4} & 0 & \frac{m(\alpha+\beta)}{8} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\mathfrak{g}(X_s)\subset\mathfrak{m}$. Let $X_n=X-X_s$ be the nilpotent component of X. Then

$$X_n=\begin{pmatrix} 0 & 0 & 0 & \frac{x_1-x_2}{2} \\ 0 & 0 & 0 & \frac{-x_1+x_2}{2} \\ \frac{\alpha-\beta}{2}x_1+(\frac{-\alpha+\beta}{2}-1)x_2 & \frac{-\alpha+\beta}{2}x_1+(\frac{\alpha-\beta}{2}+1)x_2 & 0 & x_3-(x_1+x_2)\frac{\alpha+\beta}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

These vectors belong to an algebraic 3-dimensional nilpotent Lie algebra whose elements are all nilpotent. Since $\mathfrak{g}(X_n)$ is contained in this algebra, it is also contained in $\mathfrak{m}$. Then we have

PROPOSITION 13. *The Lie algebra $\mathfrak{m}$ is algebraic. It is the algebraic Lie algebra generated by $\mathfrak{h}_{\alpha,\beta}$.*

3.3. **Rigidity and algebraicity**

Recall that in [5] we have the following result

Any complex rigid Lie algebra is algebraic.

In this form this result is not true. In fact, let us consider the following 2-dimensional Lie algebras

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix},\ x, y \in \mathbb{C} \right\}, \quad \mathfrak{g}_2 = \left\{ \begin{pmatrix} x & x & y \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix},\ x, y \in \mathbb{C} \right\}.$$

These Lie algebras are isomorphic since they have the same Lie multiplication

$$\mu(X, Y) = Y.$$

Computing the replica of any element of $\mathfrak{g}_1$, we can conclude that this Lie algebra is algebraic. Concerning $\mathfrak{g}_2$, the semisimple part of the element corresponding to $x = 1$ and $y = 0$ is not in $\mathfrak{g}_2$. This implies that $\mathfrak{g}_2$ is not algebraic.

PROPOSITION 14. *The two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are rigid. But $\mathfrak{g}_2$ is a non algebraic Lie algebra.*

The mistake in the Carles's result is not a consequence of a bad computation but to a strange definition of the algebraicity. In his paper, a Lie algebra is called algebraic if it is isomorphic as a Lie algebra to a Lie algebra of an algebraic group. If we consider the previous counter example, the Lie algebra $\mathfrak{g}_2$ is isomorphic to the algebraic Lie algebra $\mathfrak{g}_1$ and it is algebraic in the Carles's sense, but not algebraic in the classical sense. The isomorphism between $\mathfrak{g}_2$ and $\mathfrak{g}_1$ is a Lie algebras isomorphism but not an algebraic Lie algebra isomorphism. From the Carles's definition, we deduce that any nilpotent Lie algebra is algebraic which is wrong in the classical definition as we have already proved.

In the following we consider the classical definition of the algebraicity. In this classical context, we shall prove that if μ is a rigid Lie algebra multiplication, then the Lie algebra $\mathcal{A}d_\mu$ whose elements are the linear operators $ad_\mu X$ is algebraic.

3.4. The Lie algebra $\mathcal{A}d_\mu$

We have seen that the notion of rigidity refers to the Lie algebra multiplication and not with the Lie algebra. But a Lie multiplication on an n-dimensional vector space E defines a natural subalgebra of $gl(E)$ that is the Lie algebra $\mathcal{A}d_\mu$ whose elements are the operators $ad_\mu X$ for any $X \in E$. Let us recall some classical results. Let $\mathcal{D}er_\mu$ be the Lie algebra of derivations of $\mathfrak{g}$. It is an algebraic Lie subalgebra of $gl(E)$. The Lie algebra $\mathcal{A}d_\mu$ is an ideal of $\mathcal{D}er_\mu$ but it is not in general an algebraic Lie subalgebra of $\mathcal{D}er_\mu$. For example, let us consider the 3-dimensional Lie multiplication given by

$$[T, X_1] = eX_1, \ [T, X_2] = \pi X_2.$$

The element adT is semisimple with 0, e, and π as eigenvalues. Let (m_1, m_2, m_3) be in Z^3 with $m_1 0 + m_2 e + m_3 \pi = 0$. Then $m_2 = m_3 = 0$ and any replica of adT is semisimple with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = m_1\lambda_1 = 0$$

that is $\lambda_1 = 0$. Then the replica corresponding to $\lambda_1 = 0, \lambda_2 = \lambda_3 \neq 0$ is not in $\mathcal{A}d_\mu$ and the linear Lie algebra $\mathcal{A}d_\mu$ is not algebraic.

We know some general situations where $\mathcal{A}d_\mu$ is an algebraic linear Lie algebra. For example:

1. If μ is a nilpotent Lie multiplication on E, then from Engel's Theorem, any adX is nilpotent and the Lie algebra $\mathcal{A}d_\mu$ is unipotent. This implies that $\mathcal{A}d_\mu$ is algebraic.

2. If any derivation of μ is inner, that is $\mathcal{A}d_\mu = \mathcal{D}er_\mu$ then $\mathcal{A}d_\mu$ is algebraic.

3. We have the following result:

PROPOSITION 15. *Let $\mathfrak{g} \subset gl(E)$ be an algebraic linear Lie algebra. If μ is the Lie multiplication of $\mathfrak{g}$ corresponding to the bracket in $gl(E)$, then $\mathcal{A}d_\mu$ is also algebraic.*

Proof. To prove the algebraicity of $\mathcal{A}d_\mu$, we have to show that for any $adX \in \mathcal{A}d_\mu$ with $X \in \mathfrak{g}$, the set $\mathcal{A}d_\mu(adX)$ of replica of adX is contained in $\mathcal{A}d_\mu$. Let us determine this set. Assume for instance that $\mathfrak{g} \subset gl(E)$ is a linear Lie algebra (algebraic or not). For any $u \in gl(E)$ which satisfies

$$[u, \mathfrak{g}] \subset \mathfrak{g}$$

(where $[,]$ is the Lie bracket in $gl(E)$ corresponding to μ in $\mathfrak{g}$), we consider the endomorphism of $\mathfrak{g}$

$$\rho_u(X) = [u, X].$$

This is a derivation of $\mathfrak{g}$. The subalgebra $\mathfrak{h}$ of $gl(E)$ given by

$$\mathfrak{h} = \{u \in gl(E),\ \rho_u(\mathfrak{g}) \subset \mathfrak{g}\}$$

is an algebraic subalgebra of $gl(E)$ containing $\mathfrak{g}$. Let us denote also by $\widetilde{\mathfrak{g}}$ the algebraic Lie algebra generated by $\mathfrak{g}$. If $\mathfrak{g}$ is not algebraic, then $\mathfrak{g}$ is strictly contained in $\widetilde{\mathfrak{g}}$ but these two Lie algebras have the same derived subalgebra. Since $\mathfrak{h}$ is an algebraic Lie algebra containing $\mathfrak{g}$, then

$$\widetilde{\mathfrak{g}} \subset \mathfrak{h}.$$

We have

$$\mathcal{A}d_\mu(adX) = \{\rho_{\widetilde{X}},\ \widetilde{X} \in \widetilde{\mathfrak{g}}\}.$$

If we assume now that $\mathfrak{g}$ is algebraic, then $\mathfrak{g} = \widetilde{\mathfrak{g}}$ and

$$\mathcal{A}d_\mu(adX) = \{\rho_{\widetilde{X}},\ \widetilde{X} \in \widetilde{\mathfrak{g}}\} = \{\rho_X,\ X \in \widetilde{\mathfrak{g}}\} = \mathcal{A}d_\mu.$$

Remark. The converse of this proposition is not true. For example the 2-dimensional solvable algebra $\mathfrak{g}_2$ defined above is a not algebraic Lie algebra as proved. Let us compute for this Lie algebra $\mathcal{A}d_\mu$. If we consider the basis of this Lie algebra given by X corresponding to $x = 1, y = 0$ and Y corresponding to $x = 0, y = 1$, we have $\mu(X, Y) = Y$ and $\mathcal{A}d_\mu$ is the Lie algebra

$$\left\{ \begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix},\ x, y \in \mathbb{C} \right\}.$$

It is an algebraic abelian Lie algebra because all the replica of any element of this Lie algebra are contained in it.

THEOREM 16. *If the Lie multiplication μ is rigid in $\mathcal{L}ie_n$, then the Lie algebra $\mathcal{A}d_\mu$ is algebraic.*

Proof. A linear Lie algebra is algebraic if and only if its radical is algebraic. Thus we can assume $\mathcal{A}d_\mu$ to be solvable. Consider $X \in E$ and $U = adX$. Let $\widetilde{\mathcal{A}d_\mu}$ be the algebraic Lie algebra generated by $\mathcal{A}d_\mu$. Since $\mathcal{A}d_\mu$ is a Lie subalgebra of the algebraic Lie algebra $\mathcal{D}er_\mu$, then $\widetilde{\mathcal{A}d_\mu}$ is a Lie algebraic subalgebra of $\mathcal{D}er_\mu$, the replica $\widetilde{U}$ of U belongs to $\mathcal{D}er_\mu$ and it is a derivation of μ. We can assume that μ is not a nilpotent Lie multiplication as otherwise $\mathcal{A}d_\mu$ is trivially algebraic. This implies that the derivation $\widetilde{U}$ is singular. Moreover, the semisimple and nilpotent part of the Chevalley-Jordan decomposition of $\widetilde{U}$ are also derivations of μ. Let $\widetilde{U}_s$ be the semisimple part of $\widetilde{U}$. Since $\mathbb{K}$ is algebraically closed field, $\widetilde{U}_s$ is a diagonalizable endomorphism. If we consider the Chevalley-Jordan decomposition of U, $U = U_s + U_n$, then $\widetilde{U_s} = \widetilde{U}_s$. Assume now that μ is rigid. We shall show in a first step that U_s and U_n belong to $\mathcal{A}d_\mu$. Assume that $U_s \notin \mathcal{A}d_\mu$. Then U_s is a non inner semisimple derivation of μ. There exists $X_0 \in V$ such that $U = adX_0$. Let $\{X_0, X_1, \cdots, X_{n-1}\}$ be a basis of eigenvectors of U_s and $\{0, \lambda_1, \cdots, \lambda_{n-1}\}$ the corresponding eigenvalues. By hypothesis, we can also assume that these eigenvalues are non negative. Let us consider the deformation μ_ε of μ defined by

$$\mu_\varepsilon(X_0, X_i) = \mu(X_0, X_i) + \varepsilon U_s(X_i)$$

and

$$\mu_\varepsilon(X_i, X_j) = \mu(X_i, X_j)$$

for $1 \leq i < j \leq n-1$. This defines a non trivial deformation of μ. Since $\mathfrak{g}$ is rigid, we have a contradiction. Then $U_s \in \mathcal{A}d_\mu$ and $\mathcal{A}d_\mu$ is a split Lie algebra, that is, containing the semisimple and the nilpotent part of its elements. Now we can prove that $\mathcal{A}d_\mu$ is algebraic. Let us consider a semisimple element U of $\mathcal{A}d_\mu$. If $\{\lambda_0 = 0, \lambda_1, \cdots, \lambda_{n-1}\}$ is the set of eigenvalues, its replica $\widetilde{U}$ is a semisimple element of Der_μ whose eigenvalues $\{\rho_0, \rho_1, \cdots, \rho_{n-1}\}$ satisfy $p_0\rho_0 + p_1\rho_1 + \cdots + p_{n-1}\rho_{n-1} = 0$ with $(p_0, p_1, \cdots, p_{n-1}) \in \Lambda$ where Λ is the subset of $\mathbb{Z}^n$ whose elements satisfy $p_0\lambda_0 + p_1\lambda_1 + \cdots + p_{n-1}\lambda_{n-1} = 0$. As above, with such derivation we define an infinitesimal deformation of μ. Since μ is rigid, the replica $\widetilde{U}$ is in $\mathcal{A}d_\mu$ and this linear Lie algebra is algebraic.

4. STRUCTURE THEOREM OF RIGID LIE ALGEBRAS

In this section, we consider that $\mathbb{K}$ is the complex field $\mathbb{C}$.

Assume that $\mathfrak{g}$ is solvable with a trivial center. If $\mathfrak{g}$ is rigid, then $\mathcal{A}d_\mu$ is algebraic and we have the decomposition

$$\mathcal{A}d_\mu = \mathfrak{n} \oplus \mathfrak{a}$$

where all the elements of $\mathfrak{n}$ are nilpotent and $\mathfrak{a}$ is an abelian subalgebra whose elements are semisimple. There exists $X_1, \cdots, X_{n-r}, T_1, \cdots, T_r \in V$ such that $\{adX_1, \cdots, adX_{n-r}\}$ is a basis of $\mathfrak{n}$ and thus adX_i is nilpotent for $i = 1, \cdots, n-r$ and $\{adT_1, \cdots, adT_r\}$ is a basis of $\mathfrak{a}$ and thus adT_i is semisimple for $i = 1, \cdots, r$. Since $\mathfrak{a}$ is abelian, we can assume that the endomorphisms adT_i are diagonal. Let us denote by $\mathfrak{g}_\mathfrak{n}$ the subalgebra of $\mathfrak{g}$ generated by $\{X_1, \cdots, X_{n-r}\}$ and $\mathfrak{g}_\mathfrak{a}$ the subalgebra generated by $\{T_1, \cdots, T_r\}$. Then we have:

Proposition 17. *Let $\mathfrak{g}$ be a finite dimensional solvable rigid $\mathbb{K}$ Lie algebra with a trivial center. Then $\mathfrak{g}$ admits the decomposition*

$$\mathfrak{g} = \mathfrak{g}_\mathfrak{n} \oplus \mathfrak{g}_\mathfrak{a}$$

where $\mathfrak{g}_\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{g}_\mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{g}$ whose elements T are such that adT is semisimple.

Proof. Since the operators adX_i are nilpotent, $\mathfrak{g}_\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. From the decomposition of $\mathcal{A}d_\mu$, it is the maximal nilpotent ideal of $\mathfrak{g}$.

Let us note also that, sometimes, $\mathfrak{g}_{\mathfrak{a}}$ is called a maximal torus of $\mathfrak{g}$. This may lead to some confusion since $\mathfrak{g}$ is not necessarily algebraic. However, $\mathfrak{a}$ is a maxiamal algebraic torus of the algebraic Lie algebra $\mathcal{A}d_\mu$.

Example. Let us consider the 2-dimensional rigid Lie algebra

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} x & x & y \\ 0 & x & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; x, y \in \mathbb{C} \right\}.$$

If

$$X = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then $[X, Y] = Y$ and $\mathcal{A}d_\mu = \{U, V\}$ with $U = adX, V = adY$. We have the decomposition

$$\mathcal{A}d_\mu = \mathfrak{n} \oplus \mathfrak{a}$$

with $\mathfrak{n} = \mathbb{C}\{V\}$ and $\mathfrak{a} = \mathbb{C}\{U\}$. The corresponding decomposition of $\mathfrak{g}$ is

$$\mathfrak{g} = \mathbb{C}\{Y\} \oplus \mathbb{C}\{X\}.$$

We can see, in this example, that the elements of $\mathfrak{g}_{\mathfrak{a}}$ are not, in this case, semisimple.

From this decomposition of $\mathcal{A}d_\mu$ or of $\mathfrak{g}$, Ancochea and Goze established an interesting criterium of rigidity. Let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} \oplus \mathfrak{g}_{\mathfrak{n}}$ be the decomposition of the rigid Lie algebra $\mathfrak{g}$.

Definition 18. A vector $T_0 \in \mathfrak{g}_{\mathfrak{a}}$ is called regular if $\dim\ker(adT_0) \leq \dim\ker(adT)$ for any $T \in \mathfrak{g}_{\mathfrak{a}}$.

Moreover, since the elements of $\mathfrak{a}$ are semisimple and commuting, there exists a basis $\{T = X_1, X_2, \cdots, X_n\}$ of $\mathfrak{g}$ of eigenvectors for all the diagonal endomorphisms $adT \in \mathfrak{a}$, $T \in \mathfrak{g}_{\mathfrak{a}}$. Let T be in $\mathfrak{g}_{\mathfrak{a}}$. Then

$$[T, X_i] = \lambda_i(T) X_i,$$

and the linear function $\lambda_i \in \mathfrak{g}_{\mathfrak{a}}^*$ satisfy the relations

$$\lambda_i(T) + \lambda_j(T) = \lambda_k(T)$$

as soon as $[X_i, X_j]$ is an eigenvector corresponding to $\lambda_k(T)$. Let $S(T)$ be the linear system whose equations are

$$(x_i + x_j - x_k) = 0$$

when $C_{i,j}^k \neq 0$. In particular, the linear system associated with the roots

$$\lambda_i(T) + \lambda_j(T) = \lambda_k(T)$$

is a subsystem of $S(T)$.

Theorem 19 ([3]). *Let $\mathfrak{g}$ be a rigid solvable Lie algebra whose center is trivial. Then for any regular vector $T_0 \in \mathfrak{t}$, one has*

$$\operatorname{rank}(S(T_0)) = \dim \mathfrak{n} - 1.$$

An important example of such algebras are the Borel subalgebra of a complex semisimple Lie algebra. Theorem 19 permits also to construct rigid Lie algebras without cohomological criterium. See for example [2].

Proposition 20. *Let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} \oplus \mathfrak{g}_{\mathfrak{n}}$ be a rigid solvable Lie algebra with a trivial center. Then the nilradical $\mathfrak{g}_{\mathfrak{n}}$ is the nilradical of $\mathfrak{g}$ and it is an algebraic nilpotent Lie algebra.*

Proof. In fact, for any Lie algebra $\mathfrak{g}$, its derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is algebraic.
For example the "Heisenberg" Lie algebra

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} x_1 + x_2 & x_1 + x_2 & 0 & x_1 \\ x_1 + x_2 & x_1 + x_2 & 0 & x_2 \\ x_1 & x_1 + 2x_2 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ x_1, x_2, x_3 \in \mathbb{C} \right\}$$

cannot be the nilradical of a rigid Lie algebra. On the other hand, the (good) Heisenberg Lie algebra

$$\mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}, \ x_1, x_2, x_3 \in \mathbb{C} \right\}$$

is the nilradical of a 5-dimensional rigid Lie algebra whose multiplication is given by

$$\begin{cases} [T_1, X_i] = X_i, i = 1, 3, \quad [T_2, X_i] = X_i, i = 2, 3, \\ [X_1, X_2] = X_3 \end{cases}$$

Remark. Let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} \oplus \mathfrak{g}_{\mathfrak{n}}$ the decomposition of a solvable rigid Lie algebra. We call root of $\mathfrak{g}$ a non zero linear form $\alpha \in \mathfrak{g}_{\mathfrak{a}}^*$ such that the linear space

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathfrak{n}}, [T, X] = \alpha(T)X\}$$

is not $\{0\}$. If we denote by Δ the set of roots, we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_{\mathfrak{a}} \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

When $\mathfrak{g}$ is the Borel subalgebra of a semisimple Lie algebra, then Δ is the set of positive roots. As in this example, we can consider the subset of positive roots Π of Δ and the nilradical $\mathfrak{g}_{\mathfrak{n}}$ admit a Δ-grading. Recall that all the gradings of filiform Lie algebras are described in [4].

Examples.

1. Let us consider the 5-dimensional Lie algebra given by

$$\begin{cases} [T_1, X_i] = X_i, i = 1, 3, \quad [T_2, X_i] = X_i, i = 2, 3, \\ [X_1, X_2] = X_3. \end{cases}$$

The vector $T = T_1 + T_2$ is regular and $S(T)$ is the linear system

$$\begin{cases} t_2 + x_i = x_i, \ i = 2, 3, \\ x_1 + x_2 = x_3. \end{cases}$$

We have $\text{rank}(S(T)) = 2 = \dim \mathfrak{n} - 1$. Here $\mathfrak{n}$ is the 3-dimensional Heisenberg algebra and $\mathfrak{g}$ is a Borel subalgebra of $sl(3)$.

2. Let us consider the 8-dimensional Lie algebra given by

$$\begin{cases} [T_1, X_i] = X_i, i = 1, 2, 3, 4, & [T_1, X_5] = 2X_5 \\ [T_2, X_i] = X_i, i = 2, 3, 5, & [T_3, X_3] = X_3 \\ [T_3, X_4] = -X_4, & [X_1, X_2] = [X_3, X_4] = X_5. \end{cases}$$

The vector T_1 is regular and $S(T_1)$ is the linear system

$$\begin{cases} t_2 + x_i = x_i, \ i = 2, 4, 5, \\ t_3 + x_i = x_i, \ i = 3, 4, \\ x_1 + x_2 = x_5, \\ x_3 + x_4 = x_5. \end{cases}$$

We have $\text{rank}(S(T_1)) = 4 = \dim \mathfrak{n} - 1$. But $\mathfrak{g}$ is not a Borel Lie algebra.

REFERENCES

[1] J. M. Ancochéa Bermùdez, R. Campoamor-Stursberg, L. Garcia Vergnolle, and M. Goze, *Algèbres de Lie résolubles réelles algébriquement rigides*. Monatsh. Math. **152** (2007), *3*, 187–195.

[2] J. M. Ancochéa Bermùdez, R. Campoamor-Stursberg, and F. Oviano Garcìa, *New examples of rank one solvable real rigid Lie algebras possessing a nonvanishing Chevalley cohomology*. Appl. Math. Comput. **339** (2018), 431–440.

[3] J. M. Ancochéa-Bermùdez, and M. Goze, *Le rang du système linéaire des racines d'une algèbre de Lie rigide résoluble complexe*. Comm. Algebra **20** (1992), *3*, 875–887.

[4] Y Bahturin, M. Goze, and E. Remm, *Group gradings on filiform Lie algebras*. Comm. Algebra **44** (2016), *1*, 40–62.

[5] R. Carles, *Sur la structure des algèbres de Lie rigides*. Annales Institut Fourier, **34** (1984), *3*, 65–82.

[6] M. Goze, *Algèbres de Lie réelles et complexes*. In: *Algèbre dynamique et analyse pour la géométrie: aspects récents*. Edition Ellipses (2010)

[7] M. Goze and Y. Khakimdjanov, *Nilpotent Lie Algebras*. Mathematics and its Applications, Kluwer Academic Publisher.

[8] M. Goze and E. Remm, *Contact and Frobeniusian forms on Lie groups*. Differential Geom. Appl. **35** (2014), 74–94.

[9] M. Goze and E. Remm, *k-step nilpotent Lie algebras*. Georgian Math. J. **22** (2015), *2*, 219–234.

[10] N. Goze, *Poisson structures on* $\mathbb{C}[X_1, \cdots, X_n]$ *associated with rigid Lie algebras*. J. Gen. Lie Theory Appl. **4** (2010), Art. ID G090501.

[11] B. Khuhirun, K.C. Misra , and E. Stitzinger *On nilpotent Lie algebras of small breadth.* Journal of Algebra **444** (2015), 328–338.

[12] E. Remm, *Breadth and characteristic sequence of nilpotent Lie algebras*. Comm. Algebra **45** (2017), *7*, 2956–2966.

[13] E. Remm, 3-*dimensional skew-symmetric algebras and the variety of Hom-Lie algebras*. Algebra Colloq. **25** (2018), *4*, 547–566.

[14] E. Remm, *On filiform Lie algebras. Geometric and algebraic studies*. Rev. Roumaine Math. Pures Appl. **63** (2018), *2*, 179–209.

[15] E. Remm, Elisabeth, M. Goze, *Examples of non-algebraic nilpotent Lie algebra*. Preprint, arXiv:1802.00943 (2018).

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