# FUNDAMENTAL SOLUTIONS IN $\psi$-BESSEL POTENTIAL SPACES of certain generators of transient lévy processes 

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#### Abstract

We prove that generators of transient translation invariant Dirichlet forms have fundamental solutions belonging to certain $\psi$-Bessel potential spaces where $\psi$ is the symbol of the generator. This implies so far unknown regularity results for solutions of the equation $\psi(D) u=f$. Moreover, we can identify the fundamental solution with the kernel of the abstract potential operator and in the case where $\psi^{\frac{1}{2}}$ generates a metric on $\mathbb{R}^{n}$ (which is the generic case) the fundamental solution, hence the potential kernel often admits a representation with the help of $\psi^{\frac{1}{2}}$. This extends the well known case of rotational invariant symbols.


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## INTRODUCTION

Fundamental solutions of linear partial differential operators with constant coefficients are indeed fundamental tools to study such operators and corresponding partial differential equations. As a standard reference we refer to [14] and [15]. The analysis of the equation $P(D) E=\varepsilon_{0}$, where $\varepsilon_{0}$ denotes the Dirac measure at 0 , can make use of the fact that the symbol of $P(D)$ is a polynomial $P(\xi)$, hence an analytic function. As pointed out by Hörmander, in order to obtain optimal regularity results for solutions of $P(D) u=f$ given by $u=E * f$ it is desirable to find an $E$ that belongs to certain function spaces determined by $P(\xi)$, i.e. the symbol of $P(D)$.

The generator of a Lévy process is a pseudo-differential operator $-\psi(D)$ with a continuous negative definite symbol $\psi: \mathbb{R}^{n} \rightarrow \mathbb{C}$, the characteristic exponent of the process, and such a symbol is in general not even a $C^{k}$-function for some $k \geq 1$. Hence the study of the equation $\psi(D) E=\varepsilon_{0}$ cannot rely on all the tools available when $\psi(D)$ is replaced by a polynomial. The central part of the paper establishes, for a (real-valued) continuous negative definite symbol
$\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ under certain generic conditions, the existence of a fundamental solution to the operator $\psi(D)$ belonging to a certain $\psi$-Bessel potential space $H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)$, compare with [7] for this class of function spaces. The existence result has immediate consequences for solutions $u=E * f$ of the equation $\psi(D) u=f$, see Theorem 2.2 and its corollaries.

A key point in our investigations is the study of $F^{-1}\left(\frac{1}{\psi}\right)$ in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and we add the assumption $\frac{1}{\psi} \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ for some $p>1$ which implies that the symmetric Dirichlet form $\mathcal{E}$ and the corresponding symmetric sub-Markovian semigroup $\left(T_{t}\right)_{t \geq 0}$ are transient. Besides [9], the classical papers [2], [3] and [6] are still worthy of being consulted for our problems. It follows that the associated abstract potential operator, or the resolvent at 0 , is defined, and up to a constant it has to coincide with the fundamental solution $E$. This in turn allows us to study $E$ and the equation $\psi(D)=f$ from a second point of view, i.e. a potential theoretic one. This is done in Section 3, where the more obvious results are compiled, and in Section 4 where a new aspect is added. This new aspect takes into account that if $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a continuous negative definite function such that $\psi(\xi)=0$ if and only if $\xi=0$, then $d_{\psi}(\xi, \eta):=\psi^{\frac{1}{2}}(\xi-\eta)$ is a metric on $\mathbb{R}^{n}$ which under natural conditions generates the Euclidean topology in $\mathbb{R}^{n}$. In Section 4 we start to explore the consequences when $E$ (or the potential kernel $\kappa$ ) can be expressed in terms of this metric. For the rotational invariant case, i.e. $\psi(\xi)=\varphi(|\xi|)$, this is always possible, hence this extra assumption leads to a non-empty class. One question we discuss is that of further local regularity properties of $u * f$ provided $f$ satisfies a type of Hölder condition with respect to $d_{\psi}$. The final section provides examples and they are of some importance in order to demonstrate the non-triviality of the class of pseudo-differential operators under investigation.

We also would like to mention that using the existence results from [16] and [12], for further detail see [17]-[18], it is possible to construct a transient symmetric Dirichlet form generated by a pseudo-differential operator $-p(x, D)$ with a negative definite symbol $p(x, \xi)$ such that its extended Dirichlet space is comparable (with respect to the Dirichlet norm) with a fixed translation invariant Dirichlet form $\mathcal{E}^{\psi}$. By results of Tomisaki [24], see also [18], this leads to interesting comparison results for the corresponding transition densities. So far it is an open question as to whether these comparison results in turn will allow us to compare the fundamental solution $E^{\psi}$ related to $\psi(D)$ with the fundamental kernel or parametrix corresponding to $p(x, D)$. In cases where a symbolic calculus is available such results have already been obtained and we refer to [4].

In the first section we fix our notation and we collect auxiliary results from the theory of extended Dirichlet spaces and on $\psi$-Bessel potential spaces.

## 1. SETTING THE SCENE

Throughout this paper $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a continuous negative definite function. The associated symmetric $L^{2}$-sub-Markovian semigroup we denote by $\left(T_{t}^{\psi}\right)_{t \geq 0}$ or just by $\left(T_{t}\right)_{t \geq 0}$ and we note that this semigroup extends to all spaces $L^{p}\left(\mathbb{R}^{n}\right)$ as a symmetric $L^{p}$-sub-Markovian semigroup. Further, we assume the property that $\psi(\xi)=0$ if and only if $\xi=0$. Consequently, $d_{\psi}(\xi, \eta):=\psi^{\frac{1}{2}}(\xi-\eta)$ is a translation invariant metric on $\mathbb{R}^{n}$. Note $\psi$ satisfies the estimate

$$
\begin{equation*}
0 \leq \psi(\xi) \leq c_{\psi}\left(1+|\xi|^{2}\right) \quad \text { for all } \xi \in \mathbb{R}^{n} \tag{1.1}
\end{equation*}
$$

We add the assumption that the metric $d_{\psi}$ generates the Euclidean topology on $\mathbb{R}^{n}$ which is according to [19], Lemma 3.2, equivalent to the condition

$$
\begin{equation*}
\liminf _{|\xi| \rightarrow \infty} \psi(\xi)>0 \tag{1.2}
\end{equation*}
$$

Hence $\left(\mathbb{R}^{n}, d_{\psi}, \lambda^{(n)}\right)$ is a metric measure space, the underlying topology is the Euclidean one and $\lambda^{(n)}$ is the Lebesgue measure in $\mathbb{R}^{n}$. For certain considerations it is convenient to assume that $\left(\mathbb{R}^{n}, d_{\psi}, \lambda^{(n)}\right)$ has the doubling property, i.e.

$$
\begin{equation*}
\lambda^{(n)}\left(B_{2 r}^{d_{\psi}}(x)\right) \leq \gamma \lambda^{(n)}\left(B_{r}^{d_{\psi}}(x)\right) \tag{1.3}
\end{equation*}
$$

holds for all $x \in \mathbb{R}^{n}$ and $r>0$ where $\gamma>0$ is a constant independent of $x$ and $r$. Under this condition, the volume of the open ball $B_{r}^{d_{\psi}}(x):=\{y \in$ $\left.\mathbb{R}^{n} \mid d_{\psi}(x, y)<r\right\}$ has at most power growth as $r \rightarrow \infty$ and furthermore the doubling condition implies that $\left(\mathbb{R}^{n}, d_{\psi}, \lambda^{(n)}\right)$ is a homogeneous space in the sense of Coifman and Weiss [5]. It is convenient to introduce the volume function

$$
\begin{equation*}
V_{\psi}(r):=\lambda^{(n)}\left(B_{r}^{d_{\psi}}(0)\right) \tag{1.4}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\tilde{V}_{\psi}(r):=V_{\psi}(\sqrt{r}) \tag{1.5}
\end{equation*}
$$

In addition to conditions on $\psi$ or $d_{\psi}$ relating to properties of the metric measure space $\left(\mathbb{R}^{n}, d_{\psi}, \lambda^{(n)}\right)$, we assume that

$$
\begin{equation*}
\frac{1}{\psi} \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right) \tag{1.6}
\end{equation*}
$$

It is known, see [1], [9] or [17], that with a continuous negative definite function $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ we can associate a symmetric translation invariant Dirichlet space $\left(\mathcal{E}, D\left(\mathcal{E}^{\psi}\right)\right)$ where

$$
\begin{equation*}
\mathcal{E}^{\psi}(u, v)=\int_{\mathbb{R}^{n}} \psi(\xi) \hat{u}(\xi) \overline{\hat{v}(\xi)} \mathrm{d} \xi \tag{1.7}
\end{equation*}
$$

and

$$
\begin{equation*}
D\left(\mathcal{E}^{\psi}\right)=H^{\psi, 1}\left(\mathbb{R}^{n}\right) \tag{1.8}
\end{equation*}
$$

Here, $H^{\psi, s}\left(\mathbb{R}^{n}\right), s \in \mathbb{R}$, is the space

$$
\begin{equation*}
H^{\psi, s}\left(\mathbb{R}^{n}\right):=\left\{u \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right) \mid\|u\|_{\psi, s}<\infty\right\} \tag{1.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\|u\|_{\psi, s}=\|\left(1+\psi(\cdot)^{\frac{s}{2}} \hat{u} \|_{0} .\right. \tag{1.10}
\end{equation*}
$$

The space $H^{\psi, s}\left(\mathbb{R}^{n}\right)$ is a Hilbert space with scalar product

$$
\begin{equation*}
(u, v)_{s}:=\left((1+\psi(\cdot))^{\frac{s}{2}} \hat{u},(1+\psi(\cdot))^{\frac{s}{2}} \hat{v}\right)_{0} \tag{1.11}
\end{equation*}
$$

which takes on $\mathcal{S}\left(\mathbb{R}^{n}\right)$ the form

$$
\begin{equation*}
(u, v)_{s}=\int_{\mathbb{R}^{n}}(1+\psi(\xi))^{s} \hat{u}(\xi) \overline{\hat{v}(\xi)} \mathrm{d} \xi \tag{1.12}
\end{equation*}
$$

and $(\cdot, \cdot)_{0}$ is the scalar product in $L^{2}\left(\mathbb{R}^{n}\right)$. The condition

$$
\begin{equation*}
\psi(\xi) \geq c_{0}|\xi|^{\rho_{0}} \text { for } \xi \geq R \tag{1.13}
\end{equation*}
$$

with $c_{0}>0, \rho_{0}>0$ and $R_{0} \geq 0$ implies that if $s>\frac{n}{\rho_{0}}$ then $H^{\psi, s}\left(\mathbb{R}^{n}\right)$ is continuously embedded into the space $C_{\infty}\left(\mathbb{R}^{n}\right)$ and that the estimate

$$
\begin{equation*}
\|u\|_{\infty} \leq c_{\psi, s, n}\|u\|_{\psi, s} \tag{1.14}
\end{equation*}
$$

holds. The continuity of the embedding of $H^{\psi, t}\left(\mathbb{R}^{n}\right)$ into $H^{\psi, s}\left(\mathbb{R}^{n}\right)$ is for $t \geq s$ trivial. For the continuous negative definite function $\psi_{0}(\xi)=|\xi|^{2}$ we have

$$
\begin{equation*}
\|u\|_{\psi_{0}, s}=\left\|\left(1+|\cdot|^{2}\right)^{\frac{s}{2}} \hat{u}\right\|_{0} \tag{1.15}
\end{equation*}
$$

i.e. $H^{\psi_{0}, s}\left(\mathbb{R}^{n}\right)$ coincides with the classical Bessel potential space $H^{s}\left(\mathbb{R}^{n}\right)$.

For several reasons we have to extend the scale $H^{\psi, s}\left(\mathbb{R}^{n}\right), s \in \mathbb{R}$, and following [7] we introduce the scale of $\psi$-Bessel potential spaces $H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)$.

Definition 1.1. Let $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuous negative definite function. The $\psi$-Bessel potential space $H_{p}^{\psi, 2}\left(\mathbb{R}^{n}\right)$ of order 2 with respect to $L^{p}\left(\mathbb{R}^{n}\right)$, $1 \leq p<\infty$, is defined as

$$
\begin{equation*}
H_{p}^{\psi, 2}\left(\mathbb{R}^{n}\right):=\left\{u \in L^{p}\left(\mathbb{R}^{p}\right) \mid\|u\|_{H_{p}^{\psi, 2}}<\infty\right\} \tag{1.16}
\end{equation*}
$$

where

$$
\begin{equation*}
\|u\|_{H_{p}^{\psi, 2}}:=\|(\mathrm{id}+\psi(D)) u\|_{L^{p}} . \tag{1.17}
\end{equation*}
$$

Remark 1.2. A. The meaning of 1.17 is discussed in detail in [7]. For our purposes it is sufficient to interpret 1.17 in the obvious way, say for $u \in \mathcal{S}\left(\mathbb{R}^{n}\right)$. B. For $p=2$ we have of course $H_{2}^{\psi, 2}\left(\mathbb{R}^{n}\right)=H^{\psi, 2}\left(\mathbb{R}^{n}\right)$.

Denoting for a moment the generator of the $L^{p}$-semigroup $\left(T_{t}^{\psi}\right)_{t \geq 0}$ by

$$
\left(A^{(p)}, D\left(A^{(p)}\right)\right)
$$

we find

$$
\begin{equation*}
D\left(A^{(p)}\right)=H_{p}^{\psi, 2}\left(\mathbb{R}^{n}\right) \tag{1.18}
\end{equation*}
$$

and $\left(A^{(p)}, H_{p}^{\psi, 2}\left(\mathbb{R}^{n}\right)\right)$ is an extension of $\left(-\psi(D), \mathcal{S}\left(\mathbb{R}^{n}\right)\right)$, indeed the closure of $\left(-\psi(D), \mathcal{S}\left(\mathbb{R}^{n}\right)\right)$ in $L^{p}\left(\mathbb{R}^{n}\right)$.
Next we introduce for $s>0$ and $1 \leq p<\infty$

$$
\begin{equation*}
H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right):=\left(\mathrm{id}-A^{(p)}\right)^{-\frac{s}{2}}\left(L^{p}\left(\mathbb{R}^{n}\right)\right) \tag{1.19}
\end{equation*}
$$

With some effort, see [7], it can be proved that

$$
\begin{equation*}
H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)=\left\{u \in L^{p}\left(\mathbb{R}^{n}\right) \left\lvert\,\left\|F^{-1}\left((1+\psi)^{\frac{s}{2}} \hat{u}\right)\right\|_{L^{p}}<\infty\right.\right\} \tag{1.20}
\end{equation*}
$$

and we set

$$
\begin{equation*}
\|u\|_{H_{p}^{\psi, s}}:=\left\|F^{-1}\left((1+\psi)^{\frac{s}{2}} \hat{u}\right)\right\|_{L^{p}}=\left\|\left(\mathrm{id}-A^{(p)}\right)^{\frac{s}{2}} u\right\|_{L^{p}} . \tag{1.21}
\end{equation*}
$$

Now certain embedding results follow along standard arguments, e.g.

$$
\begin{equation*}
H_{p}^{\psi, s+t}\left(\mathbb{R}^{n}\right) \hookrightarrow H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right), t, s \geq 0 \tag{1.22}
\end{equation*}
$$

or the fact that $\mathcal{S}\left(\mathbb{R}^{n}\right)$ is dense in $H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)$. It is further possible to extend the scale $H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right), s \geq 0$, to $s \in \mathbb{R}$ and for $s \geq 0$ the space $H_{p^{\prime}}^{\psi,-s}\left(\mathbb{R}^{n}\right)$ we can identify with $\left(H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)\right)^{*}, \frac{1}{p}+\frac{1}{p^{\prime}}=1$, see [7]. In particular, we have for $u \in H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right)$ and $v \in H_{p^{\prime}}^{\psi,-s}\left(\mathbb{R}^{n}\right)$ the estimate

$$
\begin{equation*}
|\langle v, u\rangle| \leq\|u\|_{H_{p}^{\psi, s}}\|v\|_{H_{p^{\prime}}^{\psi,-s}} . \tag{1.23}
\end{equation*}
$$

To formulate the most general embedding theorem for $H_{p}^{\psi, s}$-spaces we need
Definition 1.3. A tempered distribution $m \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ is called a Fourier multiplier of type $(p, q)$ if

$$
\begin{equation*}
\|m\|_{(p, q)}:=\sup \left\{\left.\frac{\left\|F^{-1}(m \hat{\varphi})\right\|_{L^{q}\left(\mathbb{R}^{n}\right)}}{\|\varphi\|_{L^{p}\left(\mathbb{R}^{n}\right)}} \right\rvert\, 0 \neq \varphi \in \mathcal{S}\left(\mathbb{R}^{n}\right)\right\}<\infty \tag{1.24}
\end{equation*}
$$

By $\mathcal{M}_{p, q}$ we denote the space of all Fourier multipliers of type $(p, q)$.

We refer to [13] for the main properties of $\mathcal{M}_{p, q}$.
Here we only mention

$$
\begin{equation*}
F\left(L^{1}\left(\mathbb{R}^{n}\right)\right) \subset \mathcal{M}_{p, p} \subset M_{2,2}=L^{\infty}\left(\mathbb{R}^{n}\right), \quad 1<p<\infty \tag{1.25}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\mathcal{M}_{1, q} \subset \mathcal{M}_{q^{\prime}, \infty}=F\left(L^{q}\left(\mathbb{R}^{n}\right)\right), 1 \leq q \leq \infty, \frac{1}{q}+\frac{1}{q^{\prime}}=1 \tag{1.26}
\end{equation*}
$$

Moreover, we have $\mathcal{M}_{p, q} \subset L_{\text {loc }}^{1}\left(\mathbb{R}^{n}\right)$ for $q \leq 2$. Now we state the embedding result.

ThEOREM 1.4. Let $\psi_{1}, \psi_{2}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be two continuous negative definite functions. Furthermore, let $r, s \in \mathbb{R}$ and $1 \leq p, q<\infty$. In this case we have a continuous embedding

$$
\begin{equation*}
H_{p}^{\psi_{1}, s}\left(\mathbb{R}^{n}\right) \hookrightarrow H_{q}^{\psi_{2}, r}\left(\mathbb{R}^{n}\right) \tag{1.27}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
m:=\left(1+\psi_{2}\right)^{\frac{r}{2}}\left(1+\psi_{1}\right)^{-\frac{s}{2}} \in \mathcal{M}_{p, q} . \tag{1.28}
\end{equation*}
$$

In addition, a type of Sobolev embedding theorem holds, compare with Theorem 2.3.4 in [7].

Theorem 1.5. The condition

$$
\begin{equation*}
F^{-1}\left((1+\psi)^{-\frac{s}{2}}\right) \in L^{p^{\prime}}\left(\mathbb{R}^{n}\right), 1<p<\infty, s \in \mathbb{R} \tag{1.29}
\end{equation*}
$$

is necessary and sufficient for the continuous embedding

$$
\begin{equation*}
H_{p}^{\psi, s}\left(\mathbb{R}^{n}\right) \hookrightarrow C_{\infty}\left(\mathbb{R}^{n}\right), \frac{1}{p}+\frac{1}{p^{\prime}}=1 \tag{1.30}
\end{equation*}
$$

Now we return to our discussion of Dirichlet space and we note that (1.13) implies $e^{-t \psi(\cdot)} \in L^{1}\left(\mathbb{R}^{n}\right), t>0$, see [21], and therefore $(2 \pi)^{-\frac{n}{2}} e^{-t \psi(\cdot)}$ has the inverse Fourier transform

$$
\begin{equation*}
p_{t}(x):=(2 \pi)^{-n} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} e^{-t \psi(\xi)} \mathrm{d} \xi \tag{1.31}
\end{equation*}
$$

which belongs to $C_{\infty}\left(\mathbb{R}^{n}\right)$.
The condition (1.6) is necessary and sufficient for $\left(\mathcal{E}^{\psi}, H^{\psi, 1}\left(\mathbb{R}^{n}\right)\right)$ to be a transient Dirichlet form. Recall that a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ is transient if and only if for a strictly positive and bounded $L^{1}$-function $g$ we have the estimate

$$
\begin{equation*}
\int|u| g \mathrm{~d} x \leq \mathcal{E}^{\frac{1}{2}}(u, u) \tag{1.32}
\end{equation*}
$$

for all $u \in \mathcal{F}$. In this case we can define the corresponding extended Dirichlet space $\left(\mathcal{E}_{e}, \mathcal{F}_{e}\right)$ as follows:
$\mathcal{F}_{e}$ consists of all measurable functions $u: \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that $|u|<\infty$ a.e. and for $u$ there exists a sequence $\left(u_{k}\right)_{k \in \mathbb{N}}, u_{k} \in \mathcal{F}$, which converges a.e. to $u$ and $\left(u_{k}\right)_{k \in \mathbb{N}}$ is a Cauchy sequence with respect to $\mathcal{E}$, i.e. $\lim _{k, l \rightarrow \infty} \mathcal{E}\left(u_{k}-u_{l}, u_{k}-u_{l}\right)=0$.

It is well known that $\mathcal{F}=\mathcal{F}_{e} \cap L^{2}$ and

$$
\begin{equation*}
\mathcal{E}_{e}(u, u)=\lim _{k \rightarrow \infty} \mathcal{E}\left(u_{k}, u_{k}\right) \tag{1.33}
\end{equation*}
$$

for every sequence satisfying the condition in the above definition. As we have already mentioned under condition (1.6), the extended Dirichlet space $\left(\mathcal{E}^{\psi}, H_{e}^{\psi, 1}\left(\mathbb{R}^{n}\right)\right)$ exists where we also use $\mathcal{E}^{\psi}$ as a symbol for $\mathcal{E}_{e}^{\psi}$ for convenience, which is justified by (1.33).

The transience of a Dirichlet space is closely related to that of the associated operator semigroup $\left(T_{t}\right)_{t \geq 0}$. For $u \in L^{2}$ we set

$$
\begin{equation*}
S_{t} u:=\int_{0}^{t} T_{s} u \mathrm{~d} s, \quad t>0 \tag{1.34}
\end{equation*}
$$

and for $u \in L^{1}, u \geq 0$ a.e., the potential generator $G$ is defined by

$$
\begin{equation*}
G u(x):=\lim _{N \rightarrow \infty} S_{N} u(x)=\sup _{N \in \mathbb{N}} S_{N} u(x) \leq \infty \tag{1.35}
\end{equation*}
$$

where the non-negativity of $u$ entails the existence of the limit by monotone convergence. The semigroup $\left(T_{t}\right)_{t \geq 0}$ is called transient if and only if $G u(x)<\infty$ a.e. for all $u \in L^{1}, u \geq 0$ a.e. It is known that $(\mathcal{E}, D(\mathcal{E}))$ is transient if and only if $\left(T_{t}\right)_{t \geq 0}$ is transient. Note, and this is of importance, that $G$ is not a linear operator since its domain is not a vector space. Following K. Yosida we introduce in the $L^{2}$-context the abstract potential operator $\left(R_{0}, D\left(R_{0}\right)\right)$ associated with $\left(T_{t}\right)_{t \geq 0}$ or equivalently with $(\mathcal{E}, D(\mathcal{E}))$ by defining

$$
\begin{equation*}
D\left(R_{0}\right):=\left\{u \in L^{2} \mid \lim _{\lambda \rightarrow 0} R_{\lambda} u \text { exists in } L^{2}\right\} \tag{1.36}
\end{equation*}
$$

and

$$
\begin{equation*}
R_{0} u:=\lim _{\lambda \rightarrow 0} R_{\lambda} u, \quad u \in D\left(R_{0}\right), \tag{1.37}
\end{equation*}
$$

where the limit in (1.37) is the $L^{2}$-limit. Here, $\left(R_{\lambda}\right)_{\lambda>0}$ denotes the resolvent associated with $\left(T_{t}\right)_{t \geq 0}$ or $(\mathcal{E}, D(\mathcal{E})$ ), i.e.

$$
\begin{equation*}
R_{\lambda} u:=\int_{0}^{\infty} e^{-\lambda t} T_{t} u \mathrm{~d} t, u \in L^{2}, \lambda>0 \tag{1.38}
\end{equation*}
$$

Of importance is the following result, see [9].

Proposition 1.6. Let $\left(T_{t}\right)_{t \geq 0}$ be a symmetric transient sub-Markovian semigroup on $L^{2}$ with generator $(A, D(A))$. In this case, the corresponding potential operator and the abstract potential operator coincide as $L^{2}$-operators on the intersection of their domains and as an $L^{2}$-operator we have for $R_{0}$ the relations

$$
\begin{equation*}
A=-R_{0}^{-1} \quad \text { and } \quad R_{0}=-A^{-1} \tag{1.39}
\end{equation*}
$$

We remark that $R_{0}$ is often called the resolvent at 0 and if we extend $G$ to $L^{2}$ by $R_{0}$ we can read (1.39) as

$$
\begin{equation*}
A=-G^{-1} \text { and } G=-A^{-1} \tag{1.40}
\end{equation*}
$$

In both lines we have to consider the first equations to hold on $D(A)$ whereas the second equations hold on $L^{2}$.
We now return to the concrete case $\left(\mathcal{E}, H^{\psi, 1}\left(\mathbb{R}^{n}\right)\right)$ under the assumption (1.6). A formal calculation yields

$$
\begin{aligned}
\left(R_{0} u\right)(x) & =\int_{0}^{\infty}\left(T_{t} u\right)(x) \mathrm{d} x \\
& =\int_{0}^{\infty}(2 \pi)^{-\frac{n}{2}}\left(\int_{\mathbb{R}^{n}} e^{i x \cdot \xi} e^{-t \psi(\xi)} \hat{u}(\xi) \mathrm{d} \xi\right) \mathrm{d} t \\
& =(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} \int_{0}^{\infty} e^{-t \psi(\xi)} \mathrm{d} t \hat{u}(\xi) \mathrm{d} \xi \\
& =(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} \frac{1}{\psi(\xi)} \hat{u}(\xi) \mathrm{d} \xi
\end{aligned}
$$

i.e.

$$
\begin{equation*}
\left.R_{0} u=F^{-1}\left(\frac{1}{\psi}\right) \hat{u}\right)=\left(F^{-1}\left(\frac{1}{\psi}\right)\right) * u \tag{1.41}
\end{equation*}
$$

which we relate to

$$
\begin{equation*}
A u=-F^{-1}(\psi \hat{u}) . \tag{1.42}
\end{equation*}
$$

We introduce, at the moment still on a formal level,

$$
\begin{equation*}
\kappa:=E:=F^{-1}\left(\frac{1}{\psi}\right) \tag{1.43}
\end{equation*}
$$

In (1.43) we want to give $F^{-1}\left(\frac{1}{\psi}\right)$ two different interpretations. We want to interpret $\kappa$ as the potential kernel associated with $G\left(=R_{0}\right)$ or $\left(T_{t}\right)_{t \geq 0}$ where $\left(T_{t}\right)_{t \geq 0}$ is of course associated with the convolution semigroup $\left(\mu_{t}\right)_{t \geq 0}$, i.e.

$$
\begin{equation*}
T_{t} u=\mu_{t} * u \tag{1.44}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{\mu}_{t}(\xi)=(2 \pi)^{-\frac{n}{2}} e^{-t \psi(\xi)} \tag{1.45}
\end{equation*}
$$

However, we want to look at $E$ as a fundamental solution to the pseudodifferential operator $\psi(D)$ with symbol $\psi$, i.e.

$$
\begin{equation*}
\psi(D) u(x)=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \cdot \xi} \psi(\xi) \hat{u}(\xi) \mathrm{d} \xi \tag{1.46}
\end{equation*}
$$

or $\psi(D)=-A$. The goal of this paper is firstly to give a precise mathematical setting for these ideas and then to use the metric measure space $\left(\mathbb{R}^{n}, d_{\psi}, \lambda^{(n)}\right)$ if possible, to investigate $\kappa=E$.

## 2. FUNDAMENTAL SOLUTIONS FOR $\psi(D)$

By definition, a fundamental solution for a linear partial differential operator $P(D)=\sum_{|\alpha| \leq m} a_{\alpha} D^{\alpha}, a_{\alpha} \in \mathbb{C}$ and $D^{\alpha}=(-i)^{|\alpha|} \partial^{\alpha}$, is a distribution $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ satisfying

$$
\begin{equation*}
P(D) E=\varepsilon_{0}, \tag{2.1}
\end{equation*}
$$

where $\varepsilon_{0}$ denotes the Dirac measure at 0 . The precise meaning of (2.1) is

$$
\begin{equation*}
\left\langle E, P^{t}(D) \varphi\right\rangle=\left\langle\varepsilon_{0}, \varphi\right\rangle=\varphi(0) \tag{2.2}
\end{equation*}
$$

for all $\varphi \in \mathscr{D}\left(\mathbb{R}^{n}\right)=C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ where $P^{t}(D)=\sum_{|\alpha| \leq m}(-1)^{|\alpha|} a_{\alpha} D^{\alpha}$ is the formal transposed differential operator to $P(D)$. By the Malgrange-Ehrenpreis theorem every linear partial differential operator with constant coefficients admits a fundamental solution. If $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ and $P(D) u=0$ then $E+u$ is a further fundamental solution, thus a fundamental solution of $P(D)$ is not unique. Suppose that $E$ is a fundamental solution of $P(D)$ and $f \in \mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$ is a distribution with compact support then $u:=E * f$ is well defined and solves in $\mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ the equation $P(D) u=f$. Moreover, since

$$
\sin g \operatorname{supp}(E * f) \subset \operatorname{sing} \operatorname{supp} E+\sin g \operatorname{supp} f
$$

where sing supp $u$ denotes the singular support of $u \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$, we can use properties of a fundamental solution to obtain (local) regularity results for solutions of $P(D) u=f$. For example, if $\operatorname{sing} \operatorname{supp} E=\{0\}$ then

$$
\operatorname{sing} \operatorname{supp} u=\operatorname{sing} \operatorname{supp}(E * f) \subset \operatorname{sing} \operatorname{supp} f
$$

where $u:=E * f, f \in \mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$, solves $P(D) u=f$. In particular, if $\left.f\right|_{G} \in C^{\infty}(G)$, $G \subset \mathbb{R}^{n}$ open, then $\left.u\right|_{G} \in C^{\infty}(G)$. Thus fundamental solutions are rather useful tools to (locally) study linear partial differential operators with constant coefficients. Differential operators are local and pseudo-local operators, i.e. they satisfy

$$
\operatorname{supp} P(D) u \subset \operatorname{supp} u
$$

and
$\operatorname{sing} \operatorname{supp} P(D) u \subset \operatorname{sing} \operatorname{supp} u$.
The first property we used already in (2.2): for every $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ the function $P^{t}(D) \varphi$ belongs to $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ too and we can form the term $\left\langle E, P^{t}(D) \varphi\right\rangle$. Now let $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuous negative definite function and define

$$
\psi(D) u(x)=(2 \pi)^{-\frac{n}{2}} \int_{\mathbb{R}^{n}} e^{i x \xi} \psi(\xi) \hat{u}(\xi) d \xi
$$

which is due to (1.1) well defined on $\mathcal{S}\left(\mathbb{R}^{n}\right)$, hence on $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$. We may ask how to extend $\psi(D)$ to $\mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$, if possible, and whether we can introduce the notion of a fundamental solution for $\psi(D)$ analogously to the case of differential operators. Since we are dealing with real-valued symbols $\psi$, let us restrict for a moment to real-valued functions $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and then we need to define the term $\left\langle E, \psi^{t}(D) \varphi\right\rangle$ for $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$. But for $\psi$ real-valued we must have $\psi^{t}(D)=\psi(D)$ and the problem is to define $\langle E, \psi(D) \varphi\rangle$ for some $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$, $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$. However, the operator $\psi(D)$ is in general not local and it preserves $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ if and only if the Lévy measure $\nu$ of $\psi$ in its Lévy-Khinchine representation

$$
\psi(\xi)=\sum_{k, l=1}^{n} q_{k l} \xi_{k} \xi_{l}+\int_{\mathbb{R}^{n} \backslash\{0\}}(1-\cos y \xi) \nu(d y)
$$

has a bounded support. This condition is too restrictive for our purposes. Thus in general $\psi(D) \varphi$ does not belong to $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and therefore $\langle E, \psi(D) \varphi\rangle$ is for $E \in \mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ not defined. We want to find conditions under which $E=$ $F^{-1}\left(\frac{1}{\psi}\right)$ is defined and gives a fundamental solution of $\psi(D)$. (Note in the following that $\psi$ being real-valued implies that $F\left(\frac{1}{\psi}\right)=F^{-1}\left(\frac{1}{\psi}\right)$.
The problem to handle the potential operator within the theory of distributions was studied already in the 1970s and 1980s, with some first considerations even in the 1960s. The main contributors were C. Herz, J-P. Kahane and F. Hirsch who developed the theory of the "opérateur de Laplace généralisée" and used their result to handle the problem of spectral synthesis with the Newton kernel replaced by $F^{-1}\left(\frac{1}{\psi}\right)$. Eventually the opérateur de Laplace généralisée is the inverse to $u \mapsto F^{-1}\left(\frac{1}{\psi}\right) * u$, however the crucial question is about domains, i.e. regularity. We refer to the important paper [11] by F. Hirsch and the references given there. The main difference in our approach is that as in the theory of linear partial differential operator we aim at precise regularity conditions for
$F^{-1}\left(\frac{1}{\psi}\right)$, compare with the comments in [14]. In addition, we make a first attempt to use geometric information in the form of the metric $d_{\psi}$ to study solutions to $\psi(D) u=f$.
First we note the Hausdorff-Young theorem
Theorem 2.1. For $1<p \leq 2$ the Fourier transform $F: \mathcal{S}\left(\mathbb{R}^{n}\right) \rightarrow \mathcal{S}\left(\mathbb{R}^{n}\right)$ extends to a linear continuous mapping from $L^{p}\left(\mathbb{R}^{n}\right)$ to $L^{p^{\prime}}\left(\mathbb{R}^{n}\right)$ with operator norm $(2 \pi)^{\frac{(p-2) n}{2 p}}$, i.e. we have

$$
\|F u\|_{L^{p^{\prime}}} \leq(2 \pi)^{\frac{(p-2) n}{2 p}}\|u\|_{L^{p}}, \frac{1}{p^{\prime}}+\frac{1}{p}=1 .
$$

A consequence of Theorem 2.1 is the following estimate for $v \in \mathcal{S}\left(\mathbb{R}^{n}\right)$

$$
\begin{aligned}
\int_{\mathbb{R}^{n}}|\hat{v}(\xi)| d \xi & =\int_{\mathbb{R}^{n}}(1+\psi(\xi))^{-\frac{s_{0}}{2 p^{\prime}}}\left|(1+\psi(\xi))^{\frac{s_{0}}{2 p^{\prime}}} \hat{v}(\xi)\right| d \xi \\
& \left.=\int_{\mathbb{R}^{n}}(1+\psi(\xi))^{-\frac{s_{0}}{2 p^{\prime}}} \right\rvert\, F\left(\left.\left(1+\psi(D)^{\frac{s_{0}}{2 p^{\prime}}} v\right)(\xi) \right\rvert\, d \xi\right. \\
& \leq\left\|(1+\psi)^{-\frac{s_{0}}{2 p^{p}}}\right\|_{L^{p}}\left\|F\left((1+\psi(D))^{\frac{s_{0}}{2 p^{\prime}}} v\right)\right\|_{L^{p^{\prime}}} \\
& \leq(2 \pi)^{\frac{(p-2) n}{2 p}}\left\|(1+\psi)^{-\frac{s_{0}}{2 p^{\prime}}}\right\|_{L^{p}}\left\|(1+\psi(D))^{\frac{s_{0}}{2 p^{\prime}}} v\right\|_{L^{p}},
\end{aligned}
$$

or with the help of (1.21)

$$
\int_{\mathbb{R}^{n}}|\hat{v}(\xi)| d \xi \leq(2 \pi)^{\frac{(p-2) n}{2 p}}\left\|(1+\psi)^{-\frac{s_{0}}{2 p^{\prime}}}\right\|_{L^{p}}\|v\|_{H_{p}, \frac{s_{0}}{p^{\prime}}},
$$

provided $(1+\psi)^{-\frac{s_{0}}{2 p^{\prime}}} \in L^{p}\left(\mathbb{R}^{n}\right)$, i.e. $(1+\psi)^{-\frac{s_{0} p}{2 p^{\prime}}}=(1+\psi)^{-\frac{s_{0}(p-1)}{2}} \in L^{1}\left(\mathbb{R}^{n}\right)$. Now we assume that $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$ for some $1<p \leq 2$ and for $v \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ we find

$$
\begin{aligned}
\left|\left\langle\frac{1}{\psi}, \hat{v}\right\rangle\right| & \leq\left|\int_{B_{1}(0)} \frac{1}{\psi} \hat{v} d \xi\right|+\left|\int_{B_{1}^{c}(0)} \frac{1}{\psi} \hat{v} d \xi\right| \\
& \leq\left\|\frac{1}{\psi}\right\|_{L^{p}\left(B_{1}(0)\right)}\|\hat{v}\|_{L^{p^{\prime}\left(B_{1}(0)\right)}}+\tilde{c}_{\psi} \int_{B_{1}^{c}(0)}|\hat{v}(\xi)| d \xi
\end{aligned}
$$

where $\tilde{c}_{\psi}:=\sup _{\xi \in B_{1}^{c}(0)} \frac{1}{\psi(\xi)}$ which we assume to be finite. By the HausdorffYoung theorem we have

$$
\|\hat{v}\|_{L^{p^{\prime}}\left(B_{1}(0)\right)} \leq(2 \pi)^{\frac{(p-2) n}{2 p}}\|v\|_{L^{p}}
$$

If $(1+\psi)^{-\frac{s_{0}(p-1)}{2}} \in L^{1}\left(\mathbb{R}^{n}\right)$ then our previous calculation yields

$$
\tilde{c}_{\psi} \int_{\mathbb{R}^{n}}|\hat{v}| d \xi \leq(2 \pi)^{\frac{(p-2) n}{2 p}} \tilde{c}_{\psi}\left\|(1+\psi)^{-\frac{s_{0}(p-1)}{2}}\right\|_{L^{1}}^{\frac{1}{p}}\|v\|_{H_{p}, \frac{s_{0}}{p^{\prime}}} .
$$

Since for $\frac{s_{0}}{p^{\prime}}>0$ we have $\|v\|_{L^{p}} \leq\|v\|_{H_{p}^{\psi,}, \frac{s_{0}}{p^{\prime}}}$, we arrive at

$$
\left|\left\langle\frac{1}{\psi}, \hat{v}\right\rangle\right| \leq c_{s_{0}, p, n, \psi}\|v\|_{H_{p}, \frac{s_{0}}{p^{\prime}}},
$$

and we have proved
THEOREM 2.2. Let $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuous negative definite function such that (1.13) holds and such that $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$ for some $1<p \leq 2$. For any $s_{0}>\frac{2 n}{\rho_{0}(p-1)}$ it follows that we can identify with $\frac{1}{\psi}$ a distribution in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$, the Fourier transform of which belongs to $H_{p^{\prime}}^{\psi, \frac{-s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right)$.
From $F\left((1+\psi(D))^{t}(u * v)\right)=(2 \pi)^{\frac{n}{2}}(1+\psi(\xi))^{t} \hat{u}(\xi) \hat{v}(\xi)$ we deduce that

$$
(1+\psi(D))^{\frac{t}{2}}(u * v)=(1+\psi(D))^{\frac{t_{1}}{2}} u *(1+\psi(D))^{\frac{t-t_{1}}{2}} v
$$

and from Young's inequality

$$
\begin{equation*}
\|u * v\|_{L^{r}} \leq\|v\|_{L^{\tilde{q}}}\|u\|_{L^{\tilde{p}}} \tag{2.3}
\end{equation*}
$$

for $\frac{1}{r}+1=\frac{1}{\tilde{p}}+\frac{1}{\tilde{q}}, 1 \leq \tilde{p}, \tilde{q}, r<\infty$, we obtain now

$$
\left\|(1+\psi(D))^{\frac{t}{2}}(u * v)\right\|_{L^{r}} \leq\left\|(1+\psi(D))^{-\frac{s_{0}}{2 p^{\prime}}} v\right\|_{L^{\tilde{q}}}\left\|(1+\psi(D))^{\frac{1}{2}\left(t+\frac{s_{0}}{p^{\prime}}\right)} u\right\|_{L^{\tilde{p}}}
$$

or

$$
\begin{equation*}
\|u * v\|_{H_{r}^{\psi, t}} \leq\|v\|_{H_{\tilde{q}}^{\psi,-}}{\frac{s}{p_{0}}}^{p_{H_{\tilde{p}}}^{\psi, t+\frac{s_{0}}{p^{\prime}}},} \tag{2.4}
\end{equation*}
$$

which holds for $u \in H_{\tilde{p}}^{t+\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right)$ and $v \in H_{\tilde{q}}^{\psi,-\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right)$. In particular for $r=\infty$ inequality (2.3) reads as

$$
\|u * v\|_{\infty} \leq\|v\|_{L^{p^{\prime}}}\|u\|_{L^{\tilde{p}}}
$$

with $u * v \in C_{b}\left(\mathbb{R}^{n}\right)$, however we cannot work with the space $H_{\infty}^{\psi, s}\left(\mathbb{R}^{n}\right)$ using the result from [7].
We note that $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$ implies $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{q}\left(B_{1}(0)\right)$ for $1 \leq q \leq p$ which implies

Corollary 2.3. Suppose that $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a continuous negative definite function such that (1.13) holds and such that for some $p, 1<p \leq 2$, fixed $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$. Then $\frac{1}{\psi} \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ and for some $\tilde{E} \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ we have $\tilde{E}=F^{-1}\left(\frac{1}{\psi}\right)$. For every $1<p_{0} \leq p$ it follows that $s_{0}>\frac{2 n}{\rho_{0}\left(p_{0}-1\right)}$ implies

$$
\begin{equation*}
\tilde{E} \in \bigcap_{p_{0} \leq q \leq p} H_{q^{\prime}}^{\psi,-\frac{s_{0}}{q^{\prime}}}\left(\mathbb{R}^{n}\right)=\bigcap_{\frac{p}{p-1} \leq q^{\prime} \leq \frac{p_{0}}{p_{0}-1}} H_{q^{\prime}}^{\psi,-\frac{s_{0}}{q^{\prime}}}\left(\mathbb{R}^{n}\right), \frac{1}{q}+\frac{1}{q^{\prime}}=1 \tag{2.5}
\end{equation*}
$$

Moreover, $E:=(2 \pi)^{-\frac{n}{2}} \tilde{E}$ is a fundamental solution to $\psi(D)$ in the sense that in $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ we have $\psi(D) E=\varepsilon_{0}$.

Proof. Since $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$ implies $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{q}\left(B_{1}(0)\right)$ for $1 \leq$ $q \leq p$, we immediately deduce (2.5). Moreover, since we are allowed to take in $\psi(D) E=\varepsilon_{0}$ the Fourier transform on both sides and under our conditions we have $(\psi(D) E)^{\wedge}=\psi(\cdot) \hat{E}$ the result follows.

Corollary 2.4. Under the assumptions of Corollary 2.3 let $E:=(2 \pi)^{-\frac{n}{2}} F^{-1}\left(\frac{1}{\psi}\right)$ be a fundamental solution to $\psi(D)$, let $1<q<p$, $\frac{1}{q^{\prime}}+\frac{1}{q}=1$, and consider $E$ as an element in $H_{q^{\prime}}^{\psi,-\frac{s_{0}}{q^{\prime}}}\left(\mathbb{R}^{n}\right)$. Further let $f \in$ $H_{\tilde{p}}^{\psi, t+\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right), t>0$. Then by $w:=E * f$ a distributional solution to $\psi(D) u=f$ is given which belongs to $H_{r}^{\psi, t}\left(\mathbb{R}^{n}\right)$, where $r=\frac{\tilde{p} \cdot q}{q-\tilde{p}}$ provided $q>\tilde{p}$.

Proof. We may apply (2.4) with $v=E \in H_{q^{\prime}}^{\psi,-\frac{s_{0}}{q^{\prime}}}\left(\mathbb{R}^{n}\right)$ and
$u=f \in H_{p}^{\psi, t+\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right)$. It remains to check the conditions on $r$. Recall that we must have $\frac{1}{r}+1=\frac{1}{q^{\prime}}+\frac{1}{\tilde{p}}$ and $r \geq 1$. The identity is satisfied by our definition of $r$. The condition $r \geq 1$ holds if and only if $\frac{\tilde{p} q}{q-\tilde{p}} \geq 1$ and since $q>\tilde{p}$ this is equivalent to $1 \geq \frac{1}{\tilde{\tilde{p}}}-\frac{1}{q}$ or $\frac{q+1}{q} \geq \frac{1}{\tilde{p}}$ which means $\tilde{p} \geq \frac{q}{q+1}$ and by assumption we have $\tilde{p}>1$.

Of course we can combine Corollary 2.4 and (2.3) in order to obtain further (local) regularity results for solutions of the equation $\psi(D) u=f$. However our next step is to make use of the fact that for $\frac{1}{\psi} \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ the Dirichlet space $\left(\mathcal{E}^{\psi}, H^{\psi, 1}\left(\mathbb{R}^{n}\right)\right)$ is transient and $F^{-1}\left(\frac{1}{\psi}\right)$ is its potential kernel.

## 3. FUNDAMENTAL SOLUTIONS AS POTENTIAL KERNELS

Let $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a continuous negative definite function such that $\psi(\xi)=0$ if and only if $\xi=0$, (1.13) holds and $\left.\frac{1}{\psi}\right|_{B_{1}(0)}=L^{p}\left(B_{1}(0)\right)$ for some $1<p \leq 2$. It follows that the corresponding Dirichlet space $\left(\mathcal{E}^{\psi}, H^{\psi, 1}\left(\mathbb{R}^{n}\right)\right)$ is transient and its extended Dirichlet space exists. Denote by $\left(\mu_{t}\right)_{t \geq 0}$ the symmetric convolution semigroup corresponding to $\psi$, i.e.

$$
\hat{\mu}_{t}(\xi)=(2 \pi)^{-\frac{n}{2}} e^{-t \psi(\xi)}
$$

It follows that for every compact set $K \subset \mathbb{R}^{n}$ we have

$$
\kappa(K):=\int_{0}^{\infty} \mu_{t}(K) d t<\infty
$$

and by

$$
\kappa:=\int_{0}^{\infty} \mu_{t} d t
$$

a measure is defined. With

$$
\begin{equation*}
\rho_{\lambda}:=\int_{0}^{\infty} e^{-\lambda t} \mu_{t} d t \tag{3.1}
\end{equation*}
$$

we get for all $u \in C_{0}\left(\mathbb{R}^{n}\right)$

$$
\langle\kappa, u\rangle=\int_{\mathbb{R}^{n}} u d \kappa=\lim _{\lambda \rightarrow \infty} \int_{\mathbb{R}^{n}} u d \rho_{\lambda}=\lim _{\lambda \rightarrow \infty}\left\langle\rho_{\lambda}, u\right\rangle .
$$

Moreover, $\left(\mu_{t}\right)_{t \geq 0}$ is integrable and

$$
\begin{equation*}
\kappa=F^{-1}\left(\frac{1}{\psi}\right) \tag{3.2}
\end{equation*}
$$

We refer to [1] where these results are discussed in great detail. For our purpose the equality (3.2) is of importance since it states that in the case under consideration we have

$$
\begin{equation*}
\tilde{E}=\kappa=F^{-1}\left(\frac{1}{\psi}\right) \tag{3.3}
\end{equation*}
$$

which implies that properties of $\kappa$ are those of $E=(2 \pi)^{-\frac{n}{2}} \tilde{E}$.
From (3.3) it follows immediately that $E$ is even and that if $\psi$ is rotational invariant then $E$ is rotational invariant too. For reference purposes let us summarize all conditions on $\psi$ needed in the following.

Assumption 3.1. The function $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a continuous negative definite function satisfying
i)

$$
\begin{equation*}
\psi(\xi)=0 \text { if and only if } \xi=0 \tag{3.4}
\end{equation*}
$$

ii)

$$
\begin{equation*}
\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right) \text { for some } p>1 ; \tag{3.5}
\end{equation*}
$$

iii) for some $c_{0}>0, \gamma_{0}>0$, and $R_{0} \geq 0$ we have

$$
\begin{equation*}
c_{0}|\xi|^{\gamma_{0}} \leq \psi(\xi) \text { for }|\xi| \geq R_{0} \tag{3.6}
\end{equation*}
$$

iv) for some $c_{1} \geq 0, \gamma_{1}>0$, and $R_{1} \geq 0$ we have

$$
\begin{equation*}
\psi(\xi) \leq c_{1}|\xi|^{\gamma_{1}} \text { for }|\xi| \geq R_{1} \tag{3.7}
\end{equation*}
$$

Clearly we may choose $\gamma_{1} \leq 2$ and in (3.5) we can restrict ourselves always to the cases $1<p \leq 2$.

Lemma 3.2. Under (3.4) - (3.7) and $\gamma_{1} \leq n$ we have

$$
\kappa(0)=\lim _{r \rightarrow \infty} \int_{B_{r}(0)} \frac{1}{\psi(\xi)} d \xi=+\infty
$$

Proof. For $r>R_{1}$ we have

$$
\begin{aligned}
\int_{\overline{B_{r}(0)}} \frac{1}{\psi(\xi)} d \xi & \geq \int_{\overline{B_{r}(0) \backslash B_{R_{1}}}(0)} \frac{1}{\psi(\xi)} d \xi \geq \frac{1}{c_{1}} \int_{\overline{B_{r}(0)} \backslash B_{R_{1}}(0)} \frac{1}{|\xi|^{\gamma_{1}}} d \xi \\
& =\frac{\omega_{n}}{c_{1}} \int_{R^{1}}^{r} \rho^{n-1-\gamma_{1}} d \rho \\
& =\frac{\omega_{n}}{c_{1}}\left\{\left.\frac{1}{n-\gamma_{1}} \rho^{n-\gamma_{1}}\right|_{R^{1}} ^{r},\right. \\
\left.\ln \rho\right|_{R^{1}} ^{r}, & n \neq \gamma_{1},
\end{aligned}
$$

where $w_{n}$ is the volume of the unit ball. For $\gamma_{1} \leq n$ the result follows.
Since we always have $\gamma_{1} \leq 2$, only in the one-dimensional case we might not apply Lemma 3.2.
Since $\xi \mapsto \frac{1}{\varepsilon+\psi(\xi)}$ is a positive definite function, it follows that $\frac{1}{\psi} \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right) \subset$ $\mathscr{D}^{\prime}\left(\mathbb{R}^{n}\right)$ is a positive definite distribution and by the Bochner-Schwartz theorem $E=(2 \pi)^{\frac{n}{2}} F^{-1}\left(\frac{1}{\psi}\right)$ is a positive measure of at most polynomial growth (which of course can be identified with an element belonging to $\mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ ). From (3.6) it follows that the operator semigroup $\left(T_{t}\right)_{t \geq 0}$ associated with $\left(\mu_{t}\right)_{t \geq 0}$ admits a density given by

$$
p_{t}(x)=(2 \pi)^{-n} \int_{\mathbb{R}^{n}} e^{i x \xi} e^{-t \psi(\xi)} d \xi
$$

i.e.

$$
\left(T_{t} u\right)(x)=\left(\mu_{t} * u\right)(x)=\int_{\mathbb{R}^{n}} p_{t}(x-y) u(y) d y
$$

for all $u \in \mathcal{S}\left(\mathbb{R}^{n}\right)$ (and for all extensions to $L^{q}\left(\mathbb{R}^{n}\right), 1 \leq q \leq \infty$, or $C_{\infty}\left(\mathbb{R}^{n}\right)$ ). Since

$$
\kappa=\int_{0}^{\infty} \mu_{t} d t
$$

we deduce now that $\kappa$ has a density $k$ given by

$$
k(x)=\int_{0}^{\infty} p_{t}(x) d t, x \neq 0
$$

and

$$
k(0)=+\infty .
$$

In general however we cannot expect $k$ to be integrable, i.e. to belong to $L^{1}\left(\mathbb{R}^{n}\right)$.
Let us summarize where we are: Suppose that Assumption 3.1 holds and consider the pseudo-differential operator $\psi(D)$. This pseudo-differential operator admits a fundamental solution $E \in \mathcal{S}^{\prime}\left(\mathbb{R}^{n}\right)$ which even belongs to the space $H_{p^{\prime}}^{\psi,-\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right), s_{0}>\frac{2 n}{\rho_{0}(n-1)}$, and whenever $E * f$ is defined in the sense of distributions, e.g. $f \in \mathcal{E}^{\prime}\left(\mathbb{R}^{n}\right)$ or $f \in H_{\tilde{p}}^{\psi, t+\frac{s_{0}}{p^{\prime}}}\left(\mathbb{R}^{n}\right)$ as in Corollary 2.4, then the equation $\psi(D) u=f$ has the distributional solution $u=E * f$. In particular, we have the information

$$
\operatorname{sing} \operatorname{supp} u \subset \operatorname{sing} \operatorname{supp} E+\operatorname{sing} \operatorname{supp} f
$$

e.g. if $f \in C^{\infty}\left(\mathbb{R}^{n}\right)$ and $\operatorname{sing} \operatorname{supp} E=\{0\}$ then $u \in C^{\infty}\left(\mathbb{R}^{n}\right)$. In addition, we know that $E=(2 \pi)^{\frac{n}{2}} k, k \geq 0$, and $k \lambda^{(n)}$ is a polynomially bounded measure. Hence if $E * f$ is defined (as a distribution) and $f \geq 0$ then the solution $u=E * f$ to $\psi(D) u=f$ is non-negative too. Thus when considering $\psi(D)$ in certain spaces of distributions such that $f \mapsto E * f$ becomes its inverse operator, this inverse operator is positivity preserving, hence monotone.
In a next step we want to discuss whether the fact that $d_{\psi}(\xi, \eta):=\psi^{\frac{1}{2}}(\xi-\eta)$ is a metric in $\mathbb{R}^{n}$ will provided us with further information about solutions to $\psi(D) u=f$.

## 4. METRIC AND POTENTIAL

It is a natural question whether we can "recover" the metric $d_{\psi}$ or at least some of its properties when studying the fundamental solution $E$ or the potential kernel $\kappa$. Of course properties of the scalar product $x \cdot \xi$ in relation to $d_{\psi}$ must play a crucial role and we are far away from a detailed understanding of the general situation.
The best understood case is of course the rotational invariant case. The Fourier transform of a rotational invariant distribution is itself rotational invariant, hence if $\psi$ is a rotational invariant continuous negative definite function satisfying our standard conditions then $E$ is a rotational invariant distribution. If in addition $\psi$ is a homogeneous function, $E$ must be homogeneous too. Moreover, we note that generic rotational invariant continuous negative definite functions are of the type $\psi(\xi)=f\left(|\xi|^{2}\right.$ ) where $f$ is a Bernstein function (and the corresponding process or Dirichlet form is associated with a subordinate Brownian
motion). Thus in this case we must have for $E$ a representation

$$
E(x-y)=h(|x-y|)=g\left(\frac{1}{d_{\psi}(x, y)}\right)
$$

or

$$
E(x-y)=g\left(\frac{1}{\psi^{\frac{1}{2}}(x-y)}\right) .
$$

This implies in turn that if $f$ is a rotational invariant function such that $E * f$ is properly defined then the solution $u=E * f$ of $\psi(D) u=f$ is rotational invariant too. More interesting is that we can also deduce regularity properties of $u$ provided that $f$ satisfies further Hölder conditions. The result and the proof are obviously a modification of that for the Newton (or the Riesz) potentials and we refer to the considerations in [10].
In the general case we still might encounter situations where

$$
\begin{equation*}
E(x-y)=g\left(\frac{1}{d_{\psi}(x, y)}\right) \tag{4.1}
\end{equation*}
$$

and then a Hölder condition with respect to $d_{\psi}$, i.e. a condition such as

$$
\begin{equation*}
|f(x)-f(y)| \leq\left(\omega\left(d_{\psi}(x, y)\right)\right)^{\beta} \tag{4.2}
\end{equation*}
$$

will yield local Hölder continuity of $u=E * f$ with respect to $d_{\psi}$. More precisely, if we can find a non-negative continuous increasing function $\omega:[0, \infty) \rightarrow \mathbb{R}$ such that $\omega(0)=0$ and $\omega(t)>0$ for $t \neq 0$, and for some $\alpha>0$ we have for $\delta \leq r$ that

$$
\omega(\delta) \int_{\frac{\delta}{2} \leq d_{\psi}(z, 0)<r} g\left(\frac{1}{d_{\psi}(z, 0)}\right) d z \leq M(r)<\infty
$$

then (4.2) implies for $0<\gamma<\beta-\alpha$ that for $u=E * f$ we have for $x$ in a compact set

$$
\begin{equation*}
\frac{u(x)-u(x+\eta)}{\left(\omega\left(d_{\psi}(\eta, 0)\right)\right)^{\gamma}} \rightarrow 0 \text { as } \omega\left(d_{\psi}(\eta, 0)\right) \rightarrow 0 \tag{4.3}
\end{equation*}
$$

where $0<\gamma<\beta-\alpha$. The detailed proof of this result is given in [8] and as already mentioned it follows along the standard arguments as given in [10]. A remark to (4.3) is now in order. Since our general assumptions imply (locally) for $\psi(\xi)$ lower and upper bounds against some powers of $|\xi|$, the result (4.3) implies local Hölder continuity in the classical sense, but (4.3) gives more precise information for the module of continuity of $E * f$.
In the following paragraph we will provide examples for (4.1) to hold for $\psi$ not being rotational invariant.

## 5. SOME EXAMPLES

In this paragraph we want to provide some examples. The first class of examples shows that the set of continuous negative definite functions satisfying our basic assumptions is quite rich. The only non-trivial assumption to check will be $\left.\frac{1}{\psi}\right|_{B_{1}(0)} \in L^{p}\left(B_{1}(0)\right)$ for some $p>1$. The second class of examples is intended to assure the reader that for much more than just rotational invariant continuous negative definite function we can express the fundamental solutions, i.e. the potential kernel, as a function of $d_{\psi}$.

Example 5.1. The function $\xi \rightarrow|\xi|^{\alpha}, 0<\alpha \leq 2$, and $\xi \mapsto \psi(\xi)=$ $|\xi|^{\frac{\alpha}{2}} \arctan |\xi|^{\frac{\alpha}{2}}$ are both continuous negative definite functions with $\xi=0$ being their only zero and lower as well as upper bounds for $|\xi|$ large are obvious. Clearly, $\frac{1}{1 \cdot{ }^{\alpha}} \in L_{\text {loc }}^{p}\left(\mathbb{R}^{n}\right)$ if and only if $\alpha<\frac{n}{p}$. Since

$$
s^{\frac{1}{2}} \arctan ^{\frac{1}{2}}=s^{\frac{1}{2}} \int_{0}^{s^{\frac{1}{2}}} \frac{1}{1+x^{2}} d x \geq \frac{s}{2} \text { for } s \leq 1
$$

it follows that $\frac{1}{|\cdot|^{\frac{\alpha}{2}} \arctan |\cdot|^{\frac{\alpha}{2}}} \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ if and only if $\alpha<\frac{n}{p}$.
Example 5.2. This example gives the prototype of anisotropic continuous negative definite functions with mixed homogeneity. We restrict ourselves to two summands and a simple case, i.e. we do not consider all possible cases. Let $\mathbb{R}^{n}=\mathbb{R}^{n_{1}+n_{2}}=\mathbb{R}^{n_{1}} \times \mathbb{R}^{n_{2}}$ and $\xi \in \mathbb{R}^{n_{1}}, \eta \in \mathbb{R}^{n_{2}}$. Further let $0<\alpha<1$ and $0<\beta<\alpha$. The function $(\xi, \eta) \mapsto|\xi|^{\alpha}+|\eta|^{\beta}$ is on $\mathbb{R}^{n}$ a continuous negative definite function satisfying the required bounds for $|\xi|^{2}+|\eta|^{2}$ large. For $|\xi|^{2}+|\eta|^{2}<1$ we find

$$
|\xi|^{\alpha}+|\eta|^{\beta} \geq\left(|\xi|^{2}+|\eta|^{2}\right)^{\frac{\alpha}{2}}
$$

and consequently we have $\frac{1}{|\xi|^{\alpha}+|\eta|^{\beta}} \in L_{l o c}^{p}\left(\mathbb{R}^{n}\right)$ for $\alpha<\frac{2}{p}\left(n_{1}+n_{2}\right)=\frac{2 n}{p}$. This calculation also implies for the continuous negative definite function given on $\mathbb{R}^{n_{1}+n_{2}}$ by $\psi(\xi, \eta)=\left(|\xi|^{\alpha}+|\eta|^{\beta}\right) \arctan \left(|\xi|^{\alpha}+|\eta|^{\beta}\right)^{\frac{1}{2}}$ that $\frac{1}{\psi} \in L_{l o c}^{1}\left(\mathbb{R}^{n}\right)$ for $\alpha<\frac{2 n}{p}$. This part of our example extends when $s \mapsto s^{\frac{1}{2}} \arctan s^{\frac{1}{2}}$ is replaced by a Bernstein function $f$ satisfying $f(s) \geq s^{\rho_{0}}$ for $\rho_{0}>0$ and $s<1$. In this case we have $\frac{1}{f(\psi)} \in L_{l o c}^{1}\left(\mathbb{R}^{n_{1}+n_{2}}\right)$ if $\alpha \rho_{0}<\frac{2}{p}\left(n_{1}+n_{2}\right)$. For large collection of concrete Bernstein functions we refer to [23].

Example 5.3. The following symbol was introduced in [22], see also [20], as an example of a negative definite symbol not admitting a principal part. Let $1<\alpha_{1}, \alpha_{2}<2,1<\beta_{1}, \beta_{2}<2,0<\gamma_{1}, \gamma_{2}<1, \alpha_{1} \gamma_{1}=\beta_{2} \gamma_{2}, \alpha_{2} \gamma_{2}=\beta_{1} \gamma_{1}$, $\alpha_{1} \gamma_{1}>\alpha_{2} \gamma_{2}$ and $\alpha_{j} \gamma_{j}>1, \beta_{j} \gamma_{j}>1$ for $j=1,2$. Then we have for $\xi \in \mathbb{R}^{n_{1}}$ and $\eta \in \mathbb{R}^{n_{2}}$ for the continuous negative definite function

$$
\psi_{E R}(\xi, \eta)=\left(|\xi|^{\alpha_{1}}+|\eta|^{\beta_{1}}\right)^{\gamma_{1}}+\left(|\xi|^{\alpha_{2}}+|\eta|^{\beta_{2}}\right)^{\gamma_{2}}
$$

the estimate
and $\frac{1}{\psi_{E R}} \in L_{l o c}^{p}\left(\mathbb{R}^{\kappa_{1}+n_{2}}\left(|\xi|^{2}+|\eta|^{2}\right)^{\frac{\alpha_{1} \gamma_{1}}{2}} \leq \psi_{E R}\left(\xi, \eta \alpha_{1} \gamma_{1}<\frac{2}{p}\left(n_{1}+n_{2}\right)\right.\right.$.
We now want to investigate the fundamental solution related to $\psi(\xi, \eta)=$ $|\xi|+|\eta|,(\xi, \eta) \in \mathbb{R}^{2}$, or certain derived symbols such as $(|\xi|+|\eta|)^{\alpha}$. Since the potential kernel $\kappa$ corresponding to $\psi$ is given by

$$
\kappa(x, y)=\int_{0}^{\infty} p_{t}(x, y) d t=\frac{1}{\pi^{2}} \int_{0}^{\infty} \frac{t^{2}}{\left(x^{2}+t^{2}\right)\left(y^{2}+t^{2}\right)} d t
$$

with some efforts, see [8] for full details, we find

$$
\kappa(x, y)=\frac{1}{2 \pi} \frac{1}{|x|+|y|}, \quad x^{2} \neq y^{2}
$$

and for $x^{2}=y^{2}$ a separate calculation shows

$$
\kappa(x, x)=\frac{1}{4 \pi} \frac{1}{|x|}=\frac{1}{2 \pi} \frac{1}{|x|+|x|} .
$$

Thus we find
or

$$
E(x, y)=c \kappa(x, y)=\frac{c}{2 \pi} \frac{1}{\psi(x, y)}
$$

$$
E(x, y)=\tilde{c} \frac{1}{d_{\psi}^{2}((x, y),(0,0))}
$$

Note that

$$
\operatorname{sing} \operatorname{supp} E=\{(0,0)\} \cup \operatorname{sing} \operatorname{supp} \psi=\operatorname{sing} \operatorname{supp} \psi
$$

and

$$
\operatorname{sing} \operatorname{supp} \psi=(\{0\} \times \mathbb{R}) \cup(\mathbb{R} \times\{0\})
$$

Example 5.4. For $\psi_{\alpha}(\xi, \eta)=(|\xi|+|\eta|)^{\alpha},(\xi, \eta) \in \mathbb{R}^{2}, 0<\alpha<1$, the potential kernel $\kappa_{\alpha}$ is given by

$$
\kappa_{\alpha}(x, y)=\frac{2 \Gamma(1-\alpha)}{\pi} \sin \left(\frac{1-\alpha}{2} \pi\right) \frac{1}{|x|+|y|}\left(\frac{|x|^{\alpha}-|y|^{\alpha}}{|x|-|y|}\right),|x| \neq|y| .
$$

and in the limit $|y| \rightarrow|x|$ it follows that

$$
\kappa_{\alpha}(x, x)=\frac{\alpha \Gamma(1-\alpha)}{\pi} \sin \left(\frac{1-\alpha}{2} \pi\right) \frac{1}{|x|^{2-\alpha}} .
$$

Remark 5.5. For Example 5.4 we have so far not established a general relation of $\kappa(x, y)$ to $d_{\psi}(x, y)$. However, for $\alpha=\frac{1}{2}$ we can establish the estimate, see [8],

$$
\frac{1}{\sqrt{2 \pi}} \frac{1}{d_{\psi}^{6}((x, y),(0,0))} \leq k_{\frac{1}{2}}(x, y) \leq \sqrt{\frac{2}{\pi}} \frac{1}{d_{\psi}^{6}((x, y)(0,0))}
$$

Analogous results we can obtain for $\alpha=\frac{1}{2^{m}}, m \in \mathbb{N}$. It is open as to whether we can establish such a type of two-sided estimates for a general $\alpha \in(0,1)$.

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