Dedicated to the memory of Nicu Boboc

EQUICONTINUITY OF HARMONIC FUNCTIONS AND COMPACTNESS OF POTENTIAL KERNELS

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Within the framework of balayage spaces (the analytical equivalent of nice Hunt processes), we prove equicontinuity of bounded families of harmonic functions and apply it to obtain criteria for compactness of potential kernels.

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1. INTRODUCTION

The main purpose of this paper is to provide simple criteria for compactness of potential kernels (Theorem 4.3 and Corollary 4.5) in the general framework of balayage spaces.

These criteria shall be essential in a forthcoming paper [2]. They are based on [7, Lemma 3.1], where compactness of potential kernels for continuous real potentials with compact superharmonic support has been stated. Its proof used local equicontinuity of bounded families of harmonic functions without providing any details or references. Therefore we shall first prove such an equicontinuity before getting to compactness of potential kernels.

In classical potential theory this equicontinuity can be immediately obtained by looking at the Poisson kernel for balls (similarly for the theory of Riesz potentials). It has been proven with increasing generality for harmonic spaces by G. Mokobodzki (unpublished) and in [12], [3], [11].

In the following let X be a locally compact space with countable base. For every open set U in X, let $\mathcal{B}(U)$ ($\mathcal{C}(U)$, respectively) denote the set of all Borel measurable numerical functions (continuous real functions, respectively) on U. Further, let $\mathcal{C}_0(U)$ be the set of all functions in $\mathcal{C}(U)$ which vanish at

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infinity with respect to U. Given any set $\mathcal{F}$ of functions, let $\mathcal{F}_b$ ($\mathcal{F}^+$ resp.) denote the set of bounded (positive resp.) functions in $\mathcal{F}$.

We recall that $(X, \mathcal{W})$ is a *balayage space*, if $\mathcal{W}$ is a convex cone of positive numerical functions on X (they will be the positive hyperharmonic functions on X) such that (B_0) – (B_3) hold:

(B$_0$) $\mathcal{W}$ has the following continuity, separation and transience properties:

- (C) Every $w \in \mathcal{W}$ is the supremum of its minorants in $\mathcal{W} \cap \mathcal{C}(X)$.
- (S) For all $x \neq y$ and $\gamma > 0$, there is a function $v \in \mathcal{W}$ such that $v(x) \neq \gamma v(y)$.
- (T) There are strictly positive functions $u, v \in \mathcal{W} \cap \mathcal{C}(X)$ such that $u/v \in \mathcal{C}_0(X)$.

(B$_1$) If $v_n \in \mathcal{W}$, $v_n \uparrow v$, then $v \in \mathcal{W}$.

(B$_2$) If $\mathcal{V} \subset \mathcal{W}$, then $\widehat{\inf \mathcal{V}}^f \in \mathcal{W}$.[1]

(B$_3$) If $u, v', v'' \in \mathcal{W}$, $u \leq v' + v''$, then there exist $u', u'' \in \mathcal{W}$ such that $u = u' + u''$ and $u' \leq v'$, $u'' \leq v''$.

Remark 1.1. If $\mathbb{P} = (P_t)_{t>0}$ is a sub-Markov semigroup on X (for example, the transition semigroup of a Hunt process $\mathfrak{X}$) such that its convex cone

$$\mathcal{E}_{\mathbb{P}} := \{u \in \mathcal{B}^+(X) \colon \sup_{t>0} P_t u = u\}$$

of excessive functions satisfies (B_0), then $(X, \mathcal{E}_{\mathbb{P}})$ is a balayage space; see [1, II.4.9] or [8, Corollary 2.3.8]. We might note that the essential part of (B_0), the continuity property (C), holds, if the resolvent kernels $V_\lambda := \int_0^\infty e^{-\lambda t} P_t \, dt$, $\lambda > 0$, are strong Feller, that is, $V_\lambda(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$. – For a converse, see Remark 1.3.

For this and an exposition of the theory of balayage spaces in detail, see [1, 8]; for a description, which is more expanded than the one given here and includes a discussion of examples, we mention [6] and [10, Appendix 8.1].

In the following, let $(X, \mathcal{W})$ be a balayage space. The set $\mathcal{P}(X)$ of *continuous real potentials on* X (with respect to $(X, \mathcal{W})$) is defined by

$$\mathcal{P}(X) := \{p \in \mathcal{W} \cap \mathcal{C}(X) \colon \exists\, w \in \mathcal{W} \cap \mathcal{C}(X),\ w > 0, \text{ with } p/w \in \mathcal{C}_0(X)\}.$$

Of course, $\mathcal{P}(X)$ is a convex cone. Moreover,

$$S(\mathcal{P}(X)) := \{\sup p_n \colon (p_n) \subset \mathcal{P}(X) \text{ increasing}\} = \mathcal{W} \tag{1.1}$$

[1] Here $\hat{g}^f$ is the greatest finely lower semicontinuous minorant of g, where the ($\mathcal{W}$-)*fine topology* on X is the coarsest topology such that functions in $\mathcal{W}$ are continuous.

and, for every $p \in \mathcal{P}(X)$, there exists $q \in \mathcal{P}(X)$, $q > 0$, such that $p/q \in \mathcal{C}_0(X)$ (see [1, II.4.6] or [8, Proposition 1.2.1]).

For every open set V in X, we have the *harmonic kernel* H_V given by

$$H_V p = R_p^{X \setminus V} := \inf\{w \in \mathcal{W} \colon w \ge p \text{ on } X \setminus V\} \tag{1.2}$$

for every $p \in \mathcal{P}(X)$, and the *harmonic measures* $H_V(x, \cdot)$, $x \in V$, are supported by the complement $X \setminus V$ (see [1, p. 98 and II.5.4] or [8, Section 4.2]).

For the moment, let us fix $p \in \mathcal{P}(X)$ and define $\mathcal{F} := \mathcal{W} + \mathbb{R}p$. For $x \in X$, let $\mathcal{M}_x(\mathcal{F})$ be the set of all positive Radon measures μ on X such that $\int f\, d\mu \le f(x)$ for all $f \in \mathcal{F}$. The *fine support* $\delta(p)$ is the *Choquet boundary* of X with respect to $\mathcal{F}$, that is, the set of all $x \in X$ such that $\mathcal{M}_x(\mathcal{F})$ consists only of the Dirac measure ε_x at x (see [1, p. 70]; by (1.1), it does not matter if we take $\mathcal{W}$ or $\mathcal{P}(X)$ in the definition of $\mathcal{F}$).

By a general minimum principle, $f \ge 0$ on X, whenever $f \in \mathcal{F}$ such that $f \ge 0$ on $\delta(f)$ (see [1, I.2.2] or [8, 6.3.2]). Consequently, the closure $C(p)$ of $\delta(p)$, called the *carrier* of p, is the smallest closed set A in X such that

$$p = \inf\{w \in \mathcal{W} \colon w \ge p \text{ on } A\}, \qquad \text{that is,} \qquad p = H_{X \setminus A} p \tag{1.3}$$

(if (1.3) holds and $x \in V := X \setminus A$, then $x \notin \delta(p)$, since $H_V(x, \cdot) \in \mathcal{M}_x(\mathcal{F})$ and $H_V(x, \cdot) \ne \varepsilon_x$).

By [1, II.6.17]), there exists a unique kernel K_p on X, called *associated potential kernel* or *potential kernel for* p, such that

- $K_p 1 = p$,
- for every $f \in \mathcal{B}_b^+(X)$, $K_p f \in \mathcal{P}(X)$ and $C(K_p f) \subset \operatorname{supp}(f)$.

Remark 1.2. If there is a Green function G for $(X, \mathcal{W})$ and $p \in \mathcal{P}(X)$ such that

$$p = G\mu := \int G(\cdot, y)\, d\mu(y)$$

for some measure $\mu \ge 0$ on X (see [9] for such a representation), then, for every $f \in \mathcal{B}^+(X)$,

$$K_p f = G(f\mu) \qquad \text{and} \qquad C(K_p f) = \operatorname{supp}(f\mu).$$

We might note that $p \in \mathcal{P}(X)$ is *strict*, that is, $\delta(p) = X$, if and only if $K_p 1_W \ne 0$ for every finely open Borel set $W \ne \emptyset$ (see [1, VI.8.2]).

Remark 1.3. The following holds (see [1, II.8.6, proof of IV.8.1 and VI.3.14]): If $1 \in \mathcal{W}$, then, for every strict $p \in \mathcal{P}_b(X)$, there exists a Hunt process $\mathfrak{X}$ on X such that its transition semigroup $\mathbb{P} = (P_t)_{t>0}$ satisfies

$$\mathcal{E}_{\mathbb{P}} = \mathcal{W} \qquad \text{and} \qquad \int_0^\infty P_t\, dt = K_p$$

(so that, in particular, the resolvent kernels are strong Feller).

If τ_V is the *exit time* of an open set V, that is, $\tau_V := \inf\{t \ge 0 \colon X_t \notin V\}$, then

$$\mathbb{E}^x(f \circ X_{\tau_V}) = H_V f(x), \qquad f \in \mathcal{B}^+(X),\ x \in X.$$

Let us now turn to hyperharmonic functions and harmonic functions. Given an open set U in X, let $\mathcal{U}(U)$ be the set of all open $V \subset X$ with compact closure in U. Let ${}^*\mathcal{H}(U)$ denote the set of functions $u \in \mathcal{B}(X)$ which are *hyperharmonic on* U, that is, are lower semicontinuous on U and satisfy

$$-\infty < H_V u(x) \le u(x) \quad \text{ for all } x \in V \in \mathcal{U}(U).$$

We note that ${}^*\mathcal{H}^+(X) = \mathcal{W}$ (see [1, II.5.5] or [8, Proposition 4.1.7]). The set $\mathcal{H}(U) := {}^*\mathcal{H}(U) \cap (-{}^*\mathcal{H}(U))$ is the set of functions in $\mathcal{B}(X)$ which are *harmonic on* U, that is,

$$\mathcal{H}(U) = \{h \in \mathcal{B}(X) \colon h|_U \in \mathcal{C}(U),\ H_V h = h \text{ for every } V \in \mathcal{U}(U)\}.$$

It is easily seen that the carrier $C(p)$ for $p \in \mathcal{P}(X)$ is the smallest closed set such that p is harmonic on its complement. Hence $C(p)$ is also called the *superharmonic support* of p, and $C(K_p f) \subset \operatorname{supp}(f)$ for $f \in \mathcal{B}_b^+(X)$ amounts to $K_p f \in \mathcal{H}(X \setminus \operatorname{supp}(f))$.

By [1, III.2.8 and III.1.2],

$$\mathcal{P}(X) = \{p \in \mathcal{W} \cap C(X) \colon \text{If } h \in \mathcal{H}^+(X) \text{ and } h \le p, \text{ then } h = 0\}, \tag{1.4}$$

and

$$H_U f \in \mathcal{H}(U), \qquad \text{whenever } f \in \mathcal{B}(X),\ |f| \le s \in \mathcal{W} \cap \mathcal{C}(X). \tag{1.5}$$

Further, we note that a function $h \in \mathcal{B}^+(X)$ satisfying $h|_U \in \mathcal{C}(U)$ is harmonic on U provided that, for every $x \in U$, there is a fundamental system $\mathcal{V}(x) \subset \mathcal{U}(U)$ of neighborhoods of x with $H_V h(x) = h(x)$ for every $V \in \mathcal{V}(x)$ (see [1, III.4.4] or [8, Corollary 5.2.8]). Analogously for hyperharmonic functions. In particular, for positive functions, being harmonic (hyperharmonic resp.) on an open set is a local property in the following sense: If $(U_i)_{i\in I}$ is a family of open sets in X, then

$$\bigcap_{i\in I} \mathcal{H}^+(U_i) = \mathcal{H}^+\Big(\bigcup_{i\in I} U_i\Big) \quad \text{and} \quad \bigcap_{i\in I} {}^*\mathcal{H}^+(U_i) = {}^*\mathcal{H}^+\Big(\bigcup_{i\in I} U_i\Big). \tag{1.6}$$

Let us close this section with a simple, but useful observation on transforms of our balayage space $(X, \mathcal{W})$.

Remark 1.4. Let $s \in \mathcal{W} \cap \mathcal{C}(X)$ and $\widetilde{\mathcal{W}} := (1/s)\mathcal{W}$. It is immediately verified that $(X, \widetilde{\mathcal{W}})$ is a balayage space with $1 \in \widetilde{\mathcal{W}}$. Adding $\widetilde{\ }$ to the corresponding notations we see that $\widetilde{\mathcal{P}}(X) = (1/s)\mathcal{P}(X)$, $\widetilde{H}_V f = (1/s) H_V(sf)$,

$V \subset X$ open and $f \in \mathcal{B}^+(X)$. Clearly, this implies that, for all open sets U in X and $p \in \mathcal{P}(X)$,

$$^*\widetilde{\mathcal{H}}(U) = (1/s)^*\mathcal{H}(U), \quad \widetilde{\mathcal{H}}(U) = (1/s)\mathcal{H}(U) \quad \text{and} \quad \widetilde{K}_{(1/s)p} = (1/s)K_p.$$

2. EQUICONTINUITY OF SETS OF HARMONIC FUNCTIONS

Let U be an open set in X, $s \in \mathcal{W} \cap \mathcal{C}(X)$ and

$$\mathcal{H}_s(U) := \{h \in \mathcal{H}(U) \colon |h| \le s\}.$$

The main result of this section is the following.

THEOREM 2.1. *The set $\mathcal{H}_s(U)$ is locally equicontinuous on U.*

To prove it we may, by Remark 1.4, assume that $s = 1$. Moreover, it will be sufficient to prove the equicontinuity for $\mathcal{H}_1^+(U)$. Indeed, if $h \in \mathcal{H}_1(U)$ and $V \in \mathcal{U}(U)$, then $h = H_V h = H_V h^+ - H_V h^-$, where $H_V h^\pm \in \mathcal{H}_1^+(V)$, by (1.5).

Our proof of the equicontinuity at points in

$$X_0 := \{x \in X \colon \lim_{V \downarrow x} H_V(x, W) = 1 \text{ for every open neighborhood } W \text{ of } x\}$$

is inspired by the the work of G. Mokobodzki [13] on the composition of two strong Feller kernels on separable metric spaces.

Remark 2.2. In many cases, for example for harmonic spaces and for the balayage space given by Riesz potentials (symmetric α-stable processes) on $\mathbb{R}^d$, we have $X_0 = X$. We may note (but shall not use it) that in our general case the set $X \setminus X_0$ is (at most) countable and consists of all finely isolated points in X (see [1, III.7.2]). In [5] it is shown that, for $X = (0,1)$, the set $X \setminus X_0$ can be *any* given countable subset of X.

We start with two lemmas which are immediate consequences of [13, Lemmas 1 and 2] (cf. also the approach in [3, 11]). For the convenience of the reader we include their short proofs.

LEMMA 2.3. *Let $V \in \mathcal{U}(X)$ and let (f_n) be a bounded sequence in $\mathcal{B}_b(X)$. Then there exists a subsequence (f_n') of (f_n) such that the sequence $(H_V f_n')$ is pointwise convergent on V.*

Proof. Without loss of generality it should be assumed that $0 \le f_n \le 1$ for all n. Let $\{x_m \colon m \in \mathbb{N}\}$ be a dense sequence in V and $\sigma := \sum_{m=1}^\infty 2^{-m} H_V(x_m, \cdot)$. Then $\sigma(X) \le 1$. Since $L^\infty(\sigma)$ is the dual of $L^1(\sigma)$, by the Theorem of Banach-Alaoglu, there exists a subsequence (f_n') of (f_n) and $f \in \mathcal{B}_b(X)$ such that $0 \le f \le 1$ and

$$\lim_{n \to \infty} \int f_n' g \, d\sigma = \int f g \, d\sigma \qquad \text{for every } g \in \mathcal{L}^1(\sigma). \tag{2.1}$$

Let $x \in V$ and let A be a Borel set in V such that $\sigma(A) = 0$. Then $H_V 1_A(x_m) = 0$ for every $m \in \mathbb{N}$, and hence $H_V(x, A) = 0$, since the function $H_V 1_A$ is continuous on V, by (1.5). So, by the theorem of Radon-Nikodym, there exists $g \in \mathcal{L}^1(\sigma)$ such that $H_V(x, \cdot) = g\sigma$. By (2.1), we conclude that $\lim_{n\to\infty} H_V f'_n(x) = H_V f(x)$. $\square$

Lemma 2.4. *Let $W \in \mathcal{U}(X)$ and let (g_n) be a bounded sequence in $\mathcal{B}_b(X)$ which converges pointwise to a function g. Then the sequence $(H_W g_n)$ converges locally uniformly on W to $H_W g$.*

Proof. Without loss of generality $g = 0$. Then $g'_n := \sup_{k\geq n} |g_k| \downarrow 0$ and hence $H_W g'_n \downarrow 0$ as $n \to \infty$. By (1.5), the functions $H_W g'_n$ are harmonic (and hence continuous) on W. So, by Dini's theorem, the convergence of $(H_W g'_n)$ is locally uniform on W. The proof is completed observing that $0 \leq |H_W g_n| \leq H_W g'_n$ for every $n \in \mathbb{N}$. $\square$

Remark 2.5. Let us suppose for a moment that $(X, \mathcal{W})$ is a harmonic space and let $W, V \in \mathcal{U}(U)$ be such that $\overline{W} \subset V$. Then $H_W(1_V h) = h$ on W for every $h \in \mathcal{H}(U)$, since the measures $H_W(y, \cdot)$, $y \in W$, are supported by the boundary ∂W of W. Hence Lemmas 2.3 and 2.4 immediately yield that every bounded sequence (h_n) in $\mathcal{H}(U)$ contains a subsequence (h'_n) which converges locally uniformly on W.

For our general balayage space we obtain the following.

Proposition 2.6. *If $x \in U \cap X_0$, then $\mathcal{H}_1^+(U)$ is equicontinuous at x.*

Proof. Let us suppose that $\mathcal{H}_1^+(U)$ is not equicontinuous at a point x in $U \cap X_0$. We will show that this leads to a contradiction. To this end let (A_n) be a sequence of compact neighborhoods of x in U such that $A_n \downarrow \{x\}$ and A_{n+1} is contained in the interior of A_n, $n \in \mathbb{N}$. Then there exists $\delta \in (0,1)$ such that, for every $n \in \mathbb{N}$, there are $h_n \in \mathcal{H}_1^+(U)$ and $y_n \in A_n$ satisfying

$$|h_n(y_n) - h_n(x)| \geq 5\delta. \tag{2.2}$$

Let $V \in \mathcal{U}(U)$ be such that $x \in V$. Clearly, $H_V h_n = h_n$ for every $n \in \mathbb{N}$. Passing to a subsequence we may assume, by Lemma 2.3, that (h_n) converges pointwise on V.

Since $x \in X_0$, there exists a neighborhood $W \in \mathcal{U}(V)$ of x such that $H_W 1_V(x) > 1 - \delta$. By continuity of $H_W 1_U$ on W, there exists $n_0 \in \mathbb{N}$ such that $A := A_{n_0} \subset W$ and $H_W 1_U > 1 - \delta$ on A. Since $H_W 1 \leq 1$, we obtain that

$$H_W 1_{X\setminus U} < \delta \quad \text{on } A. \tag{2.3}$$

Let

$$g_n := 1_W h_n \text{ for every } n \in \mathbb{N} \quad \text{and} \quad g := \lim_{n\to\infty} g_n.$$

By (2.3), for every $n \in \mathbb{N}$,

$$(2.4)\quad |h_n - H_W g_n| = |H_W(h_n - g_n)| = H_W(1_{X\setminus W} h_n) \le H_W 1_{X\setminus W} < \delta \quad \text{on } A.$$

By Lemma 2.4, the sequence $(H_W g_n)$ converges locally uniformly on W to $H_W g$. For every $n \ge n_1$,

$$(2.5)\qquad |H_W g_n - H_W g| < \delta \quad \text{on } A.$$

Further, by continuity of $H_W g$ on W, there exists $n \ge n_1$ such that

$$(2.6)\qquad |H_W g - H_W g(x)| < \delta \quad \text{on } A_n.$$

Finally, combining the estimates (2.4), (2.5) and (2.6) we obtain that

$$|h_n - h_n(x)| < 5\delta \quad \text{on } A_n$$

contradicting (2.2). Thus $\mathcal{H}_1^+(U)$ is equicontinuous at x. $\square$

To continue our proof of Theorem 2.1 (and for later use) we define

$$\mathcal{W}_U := {}^*\mathcal{H}^+(U)|_U$$

and observe that $(U, \mathcal{W}_U)$ is a balayage space (see [1, V.1.1]).

Let us now consider a point $x \in X \setminus X_0$. By [1, III.2.7], it is finely isolated. Since $\mathcal{W}|_U \subset \mathcal{W}_U$, it is also finely isolated with respect to $(U, \mathcal{W}_U)$. Therefore

$$q_x := \inf\{w \in \mathcal{W}_U \colon w(x) \ge 1\}$$

is a continuous real potential for $(U, \mathcal{W}_U)$ with $C(q_x) = \{x\}$ (see [1, p. 94 and III.2.8] or [8, Lemma 4.2.13]).

Lemma 2.7. *The set $\mathcal{H}_1^+(U)$ is equicontinuous at every point $x \in U \setminus X_0$.*

Proof. Given $\delta > 0$, there exists a neighborhood V of x in U such that

$$(2.7)\qquad q_x > 1 - \delta \quad \text{on } V.$$

We now fix $h \in \mathcal{H}_1^+(U)$. Then $v := h|_U \in \mathcal{W}_U$ and $w := (1-h)|_U \in \mathcal{W}_U$. Applying (1.3) to the balayage space $(U, \mathcal{W}_U)$ we get that

$$v \ge h(x) q_x \quad \text{and} \quad w \ge (1 - h(x)) q_x.$$

Since $0 \le h(x) \le 1$, this implies that, for every $y \in V$, by (2.7),

$$h(y) > h(x) - \delta \quad \text{and} \quad 1 - h(y) > 1 - h(x) - \delta,$$

that is, $\delta > h(x) - h(y) > -\delta$. $\square$

Having Proposition 2.6 and Lemma 2.7 the proof of Theorem 2.1 is completed.

3. EQUICONTINUITY OF SPECIFIC MINORANTS OF $p\in\mathcal{P}(X)$

Let $\prec$ denote the *specific order* on $\mathcal{W}$, that is, if $u, v \in \mathcal{W}$, then $u \prec v$ if there exists $w \in \mathcal{W}$ such that $u + w = v$. If $q \in \mathcal{P}(X)$ and $f \in \mathcal{B}_b(X)$ such that $0 \le f \le 1$, then $K_q f \prec q$, since $K_q(1-f) \in \mathcal{P}(X)$. If $q, q' \in \mathcal{P}(X)$, then obviously $K_{q+q'} = K_q + K_{q'}$, and hence $K_q f \prec K_{q+q'} f$ for every $f \in \mathcal{B}^+(X)$. For $q \in \mathcal{P}(X)$ and Borel sets A in X, let

$$q_A := K_q 1_A.$$

Having Theorem 2.1 the proof given in [4] for the following result is complete. However, for the convenience of the reader we add a quick presentation.

PROPOSITION 3.1. *For every $p \in \mathcal{P}(X)$, $\mathcal{M}_p := \{q \in \mathcal{P}(X) \colon q \prec p\}$ is locally equicontinuous on X.*

Proof. Let $x \in X$ and $\delta > 0$. There exists an open neighborhood U of x such that $p_{U\setminus\{x\}}(x) < \delta$, and hence $p_{U\setminus\{x\}} < \delta$ on some neighborhood V of x. Moreover, we may assume that $|p_{\{x\}} - p_{\{x\}}(x)| < \delta$ on V (if $\{x\}$ is totally thin, then $p_{\{x\}} = 0$). By Theorem 2.1, there exists a neighborhood W of x in V such that, for every $q \in \mathcal{M}_p$, $|q_{X\setminus U} - q_{X\setminus U}(x)| < \delta$ on W.

Now let us fix $q \in \mathcal{M}_p$. Then $q_{U\setminus\{x\}} \prec p_{U\setminus\{x\}}$ and $q_{\{x\}} \prec p_{\{x\}}$. By (1.3), $q_{\{x\}} = \alpha p_{\{x\}}$ with $\alpha \in [0,1]$. Thus $|q - q(x)| < 3\delta$ on W. $\square$

COROLLARY 3.2. *For every $p \in \mathcal{P}(X)$, $\{K_p f \colon f \in \mathcal{B}(X),\, 0 \le f \le 1\}$ is locally equicontinuous on X.*

At first sight, Proposition 3.1 may look stronger than Corollary 3.2. However, it is not, since, for every $q \prec p$, there exists a function $f \in \mathcal{B}(X)$ such that $0 \le f \le 1$ and $K_p f = q$; see [1, II.7.11].

4. COMPACTNESS OF POTENTIAL KERNELS

Let us introduce the following boundedness property for $(X, \mathcal{W})$ (cf. Remark 4.4):

(B) There is a strictly positive bounded function $w_0 \in \mathcal{W}$.

We first recall the statement of [7, Lemma 3.1] and prove it using Corollary 3.2.

PROPOSITION 4.1. *Suppose* (B) *and let $p \in \mathcal{P}(X)$ be such that $C(p)$ is compact. Then K_p is a compact operator on $(\mathcal{B}_b(X), \|\cdot\|_\infty)$.*

Proof. Let (f_n) be a bounded sequence in $\mathcal{B}_b(X)$. To show that $(K_p f_n)$ contains a uniformly convergent subsequence, we may assume that $0 \le f_n \le 1$. Then, by Corollary 3.2 and the theorem of Arzelà-Ascoli, there is a subsequence (q_n) of $(K f_n)$ which is uniformly convergent on $C(p)$.

Let $\delta > 0$, $a := \inf w_0(C(p))$ and $b := \sup w_0(C(p))$. There exists $k \in \mathbb{N}$ such that, for all $m, n \ge k$,

$$q_m < q_n + (\delta/b)a \quad \text{ on } C(p),$$

where $C(q_m) \subset C(p)$, and therefore $q_m \le q_n + (\delta/b)w_0 \le q_n + \delta$ on X, by (1.3). So the sequence (q_n) is uniformly convergent. □

Given $g \in \mathcal{B}_b(X)$, let us denote the operator $f \mapsto fg$ on $\mathcal{B}_b(X)$ by M_g. Clearly, for all $g \in \mathcal{B}_b^+(X)$ and $p \in \mathcal{P}(X)$, the potential kernel for $K_p g$ is $K_p M_g$.

Corollary 4.2. *Suppose* (B) *and let* $p \in \mathcal{P}(X)$. *Then there exists a function* $\varphi_0 \in \mathcal{C}(X)$, $0 < \varphi_0 \le 1$, *such that the potential kernel of* $K_p\varphi_0$ *is a compact operator on* $(\mathcal{B}_b(X), \|\cdot\|_\infty)$.

Proof. Let us choose $\varphi_n \in \mathcal{C}(X)$ with compact support, $0 \le \varphi_n \le 1$, such that $\bigcup_{n\in\mathbb{N}}\{\varphi_n > 0\} = X$. For every $n \in \mathbb{N}$, $p_n := K_p\varphi_n \in \mathcal{P}(X)$ with $C(p_n) \subset \operatorname{supp}(\varphi_n)$, and hence K_{p_n} is a compact operator on $(\mathcal{B}_b(X), \|\cdot\|_\infty)$, by Proposition 4.1. Let $0 < \alpha_n \le 2^{-n}$, $n \in \mathbb{N}$, such that $\alpha_n p_n \le 2^{-n}$. Then $\varphi_0 := \sum_{n=1}^\infty \alpha_n\varphi_n \in \mathcal{C}(X)$, $0 < \varphi_0 \le 1$, $p_0 := K_p\varphi_0 = \sum_{n=1}^\infty \alpha_n p_n \in \mathcal{P}_b(X)$ and $K_{p_0} = \sum_{n=1}^\infty \alpha_n K_{p_n}$ is a compact operator on $(\mathcal{B}_b(X), \|\cdot\|_\infty)$. □

Let us fix an exhaustion of X by relatively compact open sets U_n, $n \in \mathbb{N}$.

Theorem 4.3. 1. *Assuming* (B) *the following are equivalent for every* $p \in \mathcal{P}(X)$:

(a) K_p *is a compact operator on* $(\mathcal{B}_b(X), \|\cdot\|_\infty)$ (*and* $K_p(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$).

(b) $\lim_{n\to\infty} \|K_p 1_{X\setminus U_n}\|_\infty = 0$.

(c) $\lim_{n\to\infty} \|1_{X\setminus U_n} K_p 1_{X\setminus U_n}\|_\infty = 0$.

2. *Suppose that there exists a strictly positive* $s_0 \in \mathcal{W} \cap \mathcal{C}_0(X)$, *and let* $p \in \mathcal{P}(X)$. *Then* K_p *is a compact operator on* $\mathcal{B}_b(X)$ *if and only if* $p \in \mathcal{C}_0(X)$.

Proof. 1. We note that $q_n := K_p 1_{X\setminus U_n} \in \mathcal{P}(X)$ and $q_n \downarrow 0$ pointwise as $n \to \infty$, hence $q_n \downarrow 0$ locally uniformly on X.

(a) ⇒ (b): If K_p is a compact operator on $\mathcal{B}_b(X)$, there is a uniformly convergent subsequence of (q_n). Thus (b) holds.

(b) ⇒ (c): Trivial.

(c)⇒(b): Let $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $1_{X\setminus U_n} q_n < \varepsilon$. There exists $m \geq n$ such that $q_m < \varepsilon$ on $\overline{U}_n$. Since $q_m \leq q_n$, we see that $q_m < \varepsilon$ on X.

(b)⇒(a): For every $n \in \mathbb{N}$, $p_n := K_p 1_{U_n} \in \mathcal{P}(X)$ and $C(p_n) \subset \overline{U}_n$. Hence K_{p_n} is a compact operator on $\mathcal{B}_b(X)$, by Proposition 4.1, and $K_{p_n}(\mathcal{B}_b(X))$ is contained in $\mathcal{C}_b(X)$. Since

$$K_p = K_p M_{1_{U_n}} + K_p M_{1_{X\setminus U_n}}, \qquad n \in \mathbb{N},$$

where $K_p M_{1_{U_n}} = K_{p_n}$ and $\|K_p M_{1_{X\setminus U_n}}\|_\infty = \|K_p 1_{X\setminus U_n}\|_\infty$, we obtain that K_p is a compact operator on $\mathcal{B}_b(X)$ and $K_p(\mathcal{B}_b(X)) \subset \mathcal{C}_b(X)$.

2. If $p \in \mathcal{C}_0(X)$, then (c) holds, since $1_{X\setminus U_n} K_p 1_{X\setminus U_n} \leq 1_{X\setminus U_n} p$, and hence K_p is a compact operator on $\mathcal{B}_b(X)$.

Finally, assume conversely that K_p is a compact operator on $\mathcal{B}_b(X)$. Then (b) holds, by (1). Further for every $n \in \mathbb{N}$, there is $a_n \geq 0$ such that $p_n := K_p 1_{U_n} \leq a_n p_0$ on $\overline{U}_n$ and hence on X, by (1.3). So (p_n) is a sequence in $\mathcal{C}_0(X)$. Having (b) we see that $p \in \mathcal{C}_0(X)$. □

Remark 4.4. Using Remark 1.4 we may apply Theorem 4.3 to general balayage spaces, that is, without assuming that $1 \in \mathcal{W}$. Indeed, let $p \in \mathcal{P}(X)$, $s \in \mathcal{W} \cap \mathcal{C}(X)$, $s > 0$, and $\widetilde{\mathcal{W}} := (1/s)\mathcal{W}$. Then $\tilde{p} := p/s$ is a continuous real potential with respect to the balayage space $(X, \widetilde{\mathcal{W}})$ and the corresponding potential kernel $\widetilde{K}_{\tilde{p}}$ is given by

$$\widetilde{K}_{\tilde{p}} f = (1/s) K_p f, \qquad f \in \mathcal{B}^+(X).$$

Then compactness of $\widetilde{K}_{\tilde{p}}$ on $(\mathcal{B}_b(X), \|\cdot\|_\infty)$ implies compactness of K_p on the space of all s-bounded functions equipped with the norm

$$\|f\| := \inf\{a \geq 0 \colon |f| \leq as\}.$$

In the following let U be an open set in X. We recall from the preceding section that taking $\mathcal{W}_U := {}^*\mathcal{H}^+(U)|_U$ we obtain a balayage space $(U, \mathcal{W}_U)$. By [1, V.1.1], we know, in addition, that $q - H_U q \in \mathcal{P}(U)$ for every $q \in \mathcal{P}(X)$.

Moreover, we recall that U is called *regular* if $\lim_{x\to z} H_U q(x) = q(z)$ for all $q \in \mathcal{P}(X)$ and $z \in \partial U$ or, equivalently, if for an (every) associated Hunt process (see Remark 1.3) $\tau_U = 0$ P^z-almost surely for every $z \in \partial U$.

Corollary 4.5. *Let U be relatively compact. Then the following hold:*

(a) *If $q \in \mathcal{P}(X)$ and $p := q - H_U q$, then $K_p = K_q - H_U K_q$, and K_p is a compact operator on $\mathcal{B}_b(U)$.*

(b) *If U is regular and $p \in \mathcal{P}(U)$, then K_p is a compact operator on $\mathcal{B}_b(U)$ if and only if $p \in \mathcal{C}_0(U)$, and then $K_p(\mathcal{B}_b(U)) \subset \mathcal{C}_0(U)$.*

Proof. (a) Obviously, $K_p = K_q - H_U K_q$. Let (V_n) be an exhaustion of U. We have $q_n := K_q 1_{U\setminus V_n} \downarrow 0$, where $q_n \in \mathcal{C}(X)$, and hence $q_n \downarrow 0$ uniformly on $\overline{U}$ as $n \to \infty$. Since $K_p 1_{U\setminus V_n} \le q_n$ for every $n \in \mathbb{N}$, we obtain that $\lim_{n\to\infty} \|K_p 1_{U\setminus V_n}\|_\infty = 0$. Choosing any $w \in \mathcal{W} \cap \mathcal{C}(X)$, $w > 0$, we have $w_0 := w|_U \in \mathcal{W}_U \cap \mathcal{C}_b(U)$. Thus (B) holds, and K_p is a compact operator on $\mathcal{B}_b(U)$, by Theorem 4.3.

(b) Taking a strict potential $q_0 \in \mathcal{P}(X)$, the function $p_0 := q_0 - H_U q_0$ is contained in $\mathcal{P}_b(U) \cap \mathcal{C}_0(U)$ and $p_0 > 0$. By Theorem 4.3, the proof is finished. □

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