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VIBRATIONS OF A FINITE STRING UNDER A FRACTIONAL GAUSSIAN RANDOM NOISE

ZEINA MAHDI KHALIL and CIPRIAN A. TUDOR

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We study the solution to the stochastic wave equation with Dirichlet boundary conditions driven by a Gaussian noise which behaves as a fractional Brownian motion in time and as a Wiener process in space. We obtain the existence of the solution as well as various other properties (pathwise regularity, scaling properties, or the behavior with respect to the Hurst parameter).

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1. INTRODUCTION

The wave equation represents a mathematical model for the vibrations of a perfectly flexible string. The stochastic wave equation models the vibrations of a string perturbed by a random force. Traditionally, the random noise is considered to be white in time, i.e. it behaves as a standard Brownian motion with respect to its time variable. There exists a vast literature on stochastic partial differential equations in general, and on the wave equation in particular (see e.g. (see e.g. [14], [11], [8], [10], [12], [17], [18] and the references therein). More recently, due to the development of the stochastic calculus with respect to the fractional Brownian motion (fBm in the sequel) and related processes, several authors considered the stochastic wave equation with fractional noise (in time and/or in space), i. e. which behaves as a fBm both in time and space. Among others, we mention the works [4], [9], [13], [15], [19], [22]. The model considered in these references assume that the string is infinite, i.e. the spatial variable belongs to the whole real line or to an interval with infinite Lebesgue measure. As far as we know, the case of a finite string, i.e. when the spatial variable belongs to a finite interval $[0, L]$, with Dirichlet boundary conditions at the endpoints of the interval has not been yet treated.

Our purpose is to analyzed the vibrations of a finite string forced by a random process that behaves as a fBm in time and which is white in space. We will analyze the existence of the mild solution to the wave equation, its relation with the weak solution and other properties of the solution, including the pathwise regularity, the scaling properties or the behavior with respect to the Hurst parameter. The main difference with respect to the case of the infinite string is the fact the Green kernel associated to the wave equation is different, it can be written as a trigonometric series. This makes the calculation different and lead to a different behavior of the solution. In particular, the solution is not anymore self-similar in time or stationary in space and its pathwise regularity is not the same as for the infinite random string.

We organized the paper as follows. Section 2 contains some preliminaries on the wave equation and on the calculus related to fBm. In Section 3 we discuss the existence and various distributional and trajectorial properties of the solution to (1) while in Section 4 we analyze the behavior of the solution when the Hurst parameter approaches its critical values.

2. PRELIMINARIES

We consider the boundary-value problem

$$
(1) \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) &= c^2 \Delta u(t,x) + \dot{W}^H(t,x), \ t \in [0,T], \ x \in [0,L], \\ u(0,x) &= 0, \quad x \in [0,L], \\ \frac{\partial u}{\partial t}(0,x) &= 0, \quad x \in [0,L], \\ u(t,0) &= u(t,L) = 0, \quad t \in [0,T]. \end{cases}
$$

with $c, L > 0$. The constant L represents the length of the string while c is related to the tension. The random perturbation W^H is a *fractional-white Gaussian noise* which is defined as a real valued centered Gaussian field $W^H = \{W_t^H(A); t \in [0,T], A \in B_b([0,L])\}$, over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with covariance function given by

$$
(2) \qquad \mathbf{E}\left(W_t^H(A) W_s^H(B)\right) = R_H(t,s)\lambda(A \cap B), \forall A, B \in \mathcal{B}_b([0,L]),
$$

where R_H is the covariance of the fractional Brownian motion

$$
(3) \qquad R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad s,t \in [0,T]
$$

We denoted by $B_b(I)$ the class of bounded Borel subsets of $I \subset \mathbb{R}$ and by λ the Lebesgue measure on $\mathbb{R}$. We will assume throughout this work $H \in \left(\frac{1}{2}, 1\right)$.

We will consider the mild formulation for the solution to the wave equation (1). That is, the mild solution to (1) is defined as the Green kernel (or the

fundamental solution) associated to the wave equation on $(t,x) \in [0,T] \times [0,L]$ integrated in the Wiener sense with respect to the centered Gaussian process with covariance (23), i.e.

$$u(t,x) = \int_0^t \int_0^L G_{t,x}(s,y) W^H(ds,dy) \text{ for every } t \in [0,T], x \in [0,L] \tag{4}$$

where the Green kernel $G_{t,x}$ is given by, for $0 \le s \le t \le T$ and $x, y \in [0,L]$

$$G_{t,x}(s,y) = \sum_{n=1}^{\infty} \frac{2}{Lw_n} \sin(w_n(t-s)) \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \text{ with } w_n = \frac{n\pi c}{L}. \tag{5}$$

We will say that the solution to (1) exists if the Wiener integral in the righ-hand side of (4) is well-defined and

$$\sup_{t\in[0,T]} \mathbf{E} u(t,x)^2 < \infty. \tag{6}$$

In order to check the square integrability of (4), let us denote by $\mathcal{P}$ the Hilbert space associated with the Gaussian field with covariance (23). The inner product in the space $\mathcal{P}$ is given by (see e.g. [3])

$$\begin{aligned} \langle f,g\rangle_{\mathcal{P}} &= \mathbf{E} \int_0^T \int_0^L f(s,y) W^H(ds,dy) \int_0^T \int_0^L g(s,y) W^H(ds,dy) \\ &= \alpha_H \int_0^T \int_0^T dudv |u-v|^{2H-2} \int_0^L dy f(u,y) g(v,y) \end{aligned} \tag{7}$$

for any measurable functions $f, g : [0,T] \times [0,L] \to \mathbb{R}$ such that

$$\int_0^T \int_0^T dudv |u-v|^{2H-2} \int_0^L dy |f(u,y)g(v,y)| < \infty.$$

We will also need to introduce the space of integrands with respect to the fractional Brownian motion B^H with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. Denote by $\mathcal{H}$ the Hilbert space associated with the fBm (see e.g. [16]) and recall that

$$\mathbf{E} \int_0^T f(u) dB_u^H \int_0^T g(u) dB_u^H = \alpha_H \int_0^T \int_0^T f(u) g(u) |u-v|^{2H-2} dudv := \langle f,g\rangle_{\mathcal{H}} \tag{8}$$

for any $f, g \in |\mathcal{H}|$ where $|\mathcal{H}|$ the space of measurable function $f : [0,T] \to \mathbb{R}$ such that

$$\int_0^T \int_0^T |f(u)g(u)||u-v|^{2H-2} dudv < \infty.$$

3. EXISTENCE AND BASIC PROPERTIES OF THE SOLUTION

We start by showing the existence of the solution to the wave equation (1), i.e. we prove that the process $(u(t,x), t \in [0,T], x \in [0,L])$ given by (4) is well-defined and (6) holds true. Then we deduce other properties related to the law of the solution to (1).

3.1. Existence of the solution

We will use the series representation of the Green kernel (5). Notice that (see e.g. [17]) the Wiener integral (4) can be also written as

$$u(t,x) = \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi x}{L}\right) \tag{9}$$

with

$$T_n(t) = \frac{2}{Lw_n} \int_0^t \int_0^L \sin(w_n(t-u)) \sin\left(\frac{n\pi y}{L}\right) W^H(du,dy). \tag{10}$$

We will shart by giving some useful properties of the family $(T_n, n \geq 1)$.

Lemma 1. *For every $n \geq 1$, $(T_n(t))_{t\in[0,T]}$ is a centered Gaussian process and its covariance function satisfies*

$$|\mathbf{E}T_n(t)T_n(s)| \leq \frac{2L}{\pi^2 c^2} R_H(t,s) \frac{1}{n^2} \text{ for every } s,t \in [0,T]$$

where R_H is given by (3).

If $n \neq m$, then $T_n(t)$ and $T_m(s)$ are independent random variables, for every $s,t \in [0,T]$.

Proof: For every $s,t \in [0,T]$, we have, with $\alpha_H = H(2H-1)$, by (7)

$$\begin{aligned}
&\mathbf{E}T_n(t)T_m(s) = \\
&\quad = \frac{4}{L^2 w_n w_m} \alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u)) \sin(w_m(s-v)) \\
&\quad \times \int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right)
\end{aligned}$$

and since

$$\int_0^L dy \sin\left(\frac{n\pi y}{L}\right) \sin\left(\frac{m\pi y}{L}\right) = \begin{cases} 0, & \text{if } n \neq m \\ \frac{L}{2}, & \text{if } m = n \end{cases}$$

we get

$$\mathbf{E}T_n(t)T_m(s) = 0 \text{ if } n \neq m \tag{11}$$

and for $n = m$, by (8),

$$\begin{aligned}
&\mathbf{E}T_n(t)T_n(s) = \\
&\quad = \frac{4}{L^2 w_n^2}\frac{L}{2}\alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u))\sin(w_m(s-v)) \qquad (12)\\
&\quad = \frac{2}{L w_n}\langle \sin(w_n(t-\cdot))1_{[0,t]})(\cdot), \sin(w_n(s-\cdot)1_{[0,s]}(\cdot))\rangle_{\mathcal{H}}.
\end{aligned}$$

so

$$\begin{aligned}
&|\mathbf{E}\ T_n(t)T_n(s)| \\
&\quad \le \frac{4}{L^2 w_n^2}\frac{L}{2}\alpha_H \left| \int_0^t du \int_0^s dv |u-v|^{2H-2} \sin(w_n(t-u))\sin(w_m(s-v)) \right| \\
&\quad \le \frac{4}{L^2 w_n^2}\frac{L}{2}\alpha_H \int_0^t du \int_0^s dv |u-v|^{2H-2} = \frac{2L}{\pi^2 c^2} R_H(t,s)\frac{1}{n^2}.
\end{aligned}$$

Relation (11) gives the independence of the Gaussian random variables $T_m(t)$ and $T_n(s)$ for every $n \neq m$ and for every $s, t \in [0,T]$. ■

An easy and useful consequence of the covariance expression of T_n is the following.

Lemma 2. *For every $n \ge 1$, the process $(T_n(t))_{t\in[0,T]}$ has the same finite dimensional distributions as the process*

$$\left(\frac{\sqrt{2}}{\sqrt{L} w_n} \int_0^t \sin(w_n(t-u)) dB_u^H \right), \qquad (13)$$

with $(B_t^H)_{t\in[0,T]}$ a fractional Brownian motion with Hurst parameter.

Proof: Both processes are centered Gaussian processes with covariance given by (12). ■

Let us show that the mild solution (9) is well-defined. By C we denote a generic strictly positive constant that may change from line to line.

Proposition 1. *For every $H \in \left(\frac{1}{2}, 1\right)$, the stochastic integral in (9) is well-defined and it holds that*

$$\sup_{t\in[0,T], x\in[0,L]} \mathbf{E}u(t,x)^2 < \infty.$$

Proof: We have from Lemma 1 and (12), for every $t \in [0,T]$ and $x \in [0,L]$

$$\mathbf{E}u(t,x)^2 \quad = \quad \sum_{n=1}^{\infty} \mathbf{E}T_n(t)^2 \left(\sin\left(\frac{n\pi x}{L}\right)\right)^2 \le C t^{2H} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

so

$$\sup_{t\in[0,T], x\in[0,L]} \mathbf{E}u(t,x)^2 \le C.$$

■

Notice that the solution existence for every $H \in \left(\frac{1}{2}, 1\right)$. The same holds in the case of the wave equation with fractional-white noise with space variable in $\mathbb{R}$ (see e.g. [4]).

Remark 1. It is possible to give an alternative representation of the Green kernel G (5) and implicitely of the solution (9) by calculating the sum of the series in (5). Recall that the Green kernel G is given by (5). By applying the trigonometric identities

$$\sin(x)\sin(y) = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right) \text{ and}$$
$$\sin(x)\cos(y) = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

we obtain

$$\begin{aligned} G_{t,x}(u,z) = &-\sum_{n\geq 1} \frac{1}{2Lw_n} \sin\left(\frac{n\pi}{L}(c(t-u)+z-x)\right) \\ &+\sum_{n\geq 1} \frac{1}{2Lw_n} \sin\left(\frac{n\pi}{L}(c(t-u)-(z-x))\right) \\ &-\sum_{n\geq 1} \frac{1}{2Lw_n} \sin\left(\frac{n\pi}{L}(c(t-u)+x+z)\right) \\ &+\sum_{n\geq 1} \frac{1}{2Lw_n} \sin\left(\frac{n\pi}{L}(c(t-u)-(z+x)\right) \\ :=&S_1 + S_2 + S_3 + S_4. \end{aligned}$$

and by using the formula

$$f(x) = \sum_{n\geq 1} \frac{\sin(nx)}{n} = \begin{cases} \frac{\pi - x}{2} \text{ if } x \in (0, 2\pi) \\ f(x+2\pi), \text{ if } x \in \mathbb{R} \end{cases} \tag{14}$$

we can calculate the sum of the four series above. By assuming $cT < L$, we find tht G is not zero only on the sets $(x - z \leq c(t-u) < x + z) \cap (x + z \leq L)$ (and its value is $\frac{L}{2\pi}$) and on the set $(x - z \leq c(t-u) < 2L - (x+z)) \cap (x + z \geq L)$ (and its value is $-\frac{L}{2\pi}$). This corresponds with the formula given in [10]. In this work, we will not use this expression of the fundamental solution G.

3.2. **Scaling property**

The process $(u(t,x), t \geq 0)$ is not self-similar. On the other hand, it verifies some scaling properties that depend on the parameter c in (1). In the

following result (and only here), let us use the notation $u(t,x) = u_c(t,x)$ in order to express the dependence on the parameter c. We will denote by "$\equiv^{(d)}$" the equivalence of finite-dimensional distributions. We have:

PROPOSITION 2. *Fix $x \in (0,L)$. For every $a > 0$, the process $(u_c(at,x), t \geq 0)$ has the same finite-dimensional distributions as the process $\left(a^{H+1}u_{ac}(t,x), t \geq 0\right)$.*

Proof: We can write, for every $a > 0$,

$$\begin{aligned} u_c(at,x) =& \sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{at} \int_0^L \sin\left(\frac{n\pi c}{L}(ta-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \\ & \times \sin\left(\frac{n\pi x}{L}\right) \\ =& \sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{t} \int_0^L \sin\left(\frac{n\pi ca}{L}(t-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(d(au),dz) \\ & \times \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

and by using the scaling property is time of the random noise W^H,

$$\begin{aligned} u_c(at,x) \quad \equiv^{(d)} & \sum_{n\geq 1} \frac{2}{n\pi c} a^H \int_0^{t} \int_0^L \sin\left(\frac{n\pi ca}{L}(t-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \\ & \times \sin\left(\frac{n\pi x}{L}\right) \\ =& \; a^{H+1} \sum_{n\geq 1} \frac{2}{n\pi ca} \int_0^{t} \int_0^L \sin\left(\frac{n\pi ca}{L}(t-u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du,dz) \\ & \times \sin\left(\frac{n\pi x}{L}\right) \\ =& \; a^{H+1} u_{ac}(t,x). \end{aligned}$$

■

Recall that (see e.g. [22]), when $x \in \mathbb{R}$, the process given by (4) is self-similar in time and stationary in space. When we work on a finite interval in space, these properties are not true. Instead, we have the scaling property from Proposition 2, which say that, the dilation of time is similar, modulo scaling, with a dilation of the tension of the string.

3.3. **Behavior at nodal times**

Let $T_k = \frac{k\pi}{w_1} = \frac{kL}{c}$, $k = 1, 2, ...$ be a sequence of times. These times are usually called *nodal times of vibrations.*

We have, for $k \geq 1$ integer and for every $x \in [0, L]$

$$\begin{aligned} u(T_k, x) &= \sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{T_k} \int_0^L \sin\left(\frac{n\pi c}{L}(T_k - u)\right) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \\ &\quad \times \sin\left(\frac{n\pi x}{L}\right). \end{aligned}$$

Since

$$\sin\left(\frac{n\pi c}{L}(T_k - u)\right) = \sin(nk\pi - w_n u) = (-1)^{kn+1} \sin(w_n u) \tag{15}$$

we obtain

$$\begin{aligned} u(T_k, x) = &\sum_{n\geq 1} \frac{2}{n\pi c} \int_0^{T_k} \int_0^L (-1)^{kn+1} \sin(w_n u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \\ &\times \sin\left(\frac{n\pi x}{L}\right). \end{aligned} \tag{16}$$

The increment of the solution between two nodal times satisfies the following interesting property. We denote by "$=^{(d)}$" the equality in distribution of two random variables.

Proposition 3. *Let T_k, T_l be two nodal times with $k > l$ and assume that k, l have the same parity. Then*

$$u(T_k, x) - u(T_l, x) =^{(d)} u(T_k - T_l, x).$$

Proof: For every k, l integers with $k > l$ and with the same parity (both are even or both are odd), we have from (16)

$$\begin{aligned} &u(T_k, x) - u(T_l, x) \\ = &\sum_{n\geq 1} \frac{2}{Lw_n} \int_{T_l}^{T_k} (-1)^{kn+1} \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \times \sin\left(\frac{n\pi x}{L}\right) \\ &+ \sum_{n\geq 1} \frac{2}{Lw_n} \int_0^{T_l} \left((-1)^{kn+1} - (-1)^{ln+1}\right) \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \\ &\times \sin\left(\frac{n\pi x}{L}\right) \\ = &\frac{2}{Lw_n} \sin\left(\frac{n\pi x}{L}\right) \int_{T_l}^{T_k} (-1)^{kn+1} \sin(w_u u) \sin\left(\frac{n\pi z}{L}\right) W^H(du, dz) \end{aligned}$$

because $(-1)^{kn+1} - (-1)^{ln+1} = 0$ when k and l have the same parity. Thus $u(T_k, x) - u(T_l, x)$ is a centered Gaussian random variable and, from (7) and (15), its variance can be calculated as follows

$$
\begin{aligned}
&\mathbf{E}\left(u(T_k,x)-u(T_l,x)\right)^2\\
&=\sum_{n\geq 1}\frac{4}{L^2w_n^2}\frac{L}{2}\left(\sin\left(\frac{n\pi x}{L}\right)\right)^2\alpha_H\int_{T_l}^{T_k}\int_{T_l}^{T_k}\sin(w_nu)\sin(w_nv)|u-v|^{2H-2}dudv\\
&=\sum_{n\geq 1}\frac{4}{Lw_n^2}\frac{L}{2}\left(\sin\left(\frac{n\pi x}{L}\right)\right)^2\alpha_H\int_{0}^{T_k-T_l}\int_{0}^{T_k-T_l}\sin(w_n(T_l-u))\\
&\qquad\qquad\times\sin(w_n(T_l-v))|u-v|^{2H-2}dudv\\
&=\sum_{n\geq 1}\frac{4}{Lw_n^2}\frac{L}{2}\left(\sin\left(\frac{n\pi x}{L}\right)\right)^2\alpha_H\int_{0}^{T_k-T_l}\int_{0}^{T_k-T_l}\sin(w_nu)\\
&\qquad\qquad\times\sin(w_nv)|u-v|^{2H-2}dudv\\
&=\mathbf{E}\left(u(T_k-T_l,x)\right)^2.
\end{aligned}
$$

■

This suggests that the position of the point x on the random string at time T_k is obtained by adding the position of the same point at times T_l and T_k-T_l.

3.4. **Relation with the weak solution**

Another concept of solution to the boundary value problem (1) is the weak solution. We will say that a stochastic process $(u(t,x),t\in[0,T],x\in[0,L])$ is a weak-solution to (1) if for every test function $\varphi\in C^\infty([0,T]\times[0,L])$ with $\varphi(T,x)=\frac{\partial\varphi}{\partial t}(T,x)=0$ for every $x\in[0,L]$ and $\varphi(t,0)=\varphi(t,L)=0$ for every $t\in[0,T]$, we have

$$
\int_0^T dt\int_0^L dx\quad u(t,x)\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}\right)=\int_0^T\int_0^L\varphi(s,y)W^H(ds,dy). \tag{17}
$$

We can show that our solution (4) also satisfies (1) in the weak sense.

Proposition 4. *The mild solution (4) is also a weak solution for (1),*

Proof: Let $\varphi\in C^\infty([0,T]\times[0,L])$ as above. Then

$$
\begin{aligned}
&\int_0^T dt\int_0^L dxu(t,x)\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}\right)\\
=&\int_0^T\int_0^L\left(\int_u^T dt\int_0^L dxG_{t,x}(u,y)\left(\frac{\partial^2\varphi}{\partial t^2}-\frac{\partial^2\varphi}{\partial x^2}\right)\right)W^H(du,dy).
\end{aligned}
$$

By integrating twice by parts and using the assumptions on φ,

$$\int_u^T dt \int_0^L dx G_{t,x}(u,y) \frac{\partial^2 \varphi}{\partial t^2} = - \int_u^T dt \int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t} \frac{\partial \varphi}{\partial t}(t,x)$$
$$= \int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t}\Big|_{t=u} \varphi(u,y) + \int_u^T dt \int_0^L dx \frac{\partial^2 G_{t,x}(u,y)}{\partial t^2} \varphi(t,x).$$

and

$$\int_u^T dt \int_0^L dx G_{t,x}(u,y) \frac{\partial^2 \varphi}{\partial x^2} = \int_u^T dt \int_0^L dx \frac{\partial^2 G_{t,x}(u,y)}{\partial x^2} \varphi(t,x).$$

We used the fact that G satisfies (1) when there is no noise. Notice that

$$\begin{aligned} \int_0^L dx \frac{\partial G_{t,x}(u,y)}{\partial t}\Big|_{t=u} &= \frac{2}{L} \sum_{n \geq 1} \left(\int_0^L dx \sin\left(\frac{n\pi x}{L}\right)\right) \sin\left(\frac{n\pi y}{L}\right) \\ &= \frac{2}{\pi} \sum_{n \geq 1} \frac{1}{n} \left[1 - (-1)^n\right] \sin\left(\frac{n\pi y}{L}\right) \end{aligned}$$

we obtain the conclusion since by (14)

$$\sum_{n \geq 1} \frac{1}{n} \sin\left(\frac{n\pi z}{L}\right) = \frac{\pi}{2} - \frac{\pi z}{2L}$$

and for a similar formula for the sawtooth wave function (see e.g. [23])

$$\sum_{n \geq 1} \frac{1}{n} (-1)^n \sin\left(\frac{n\pi z}{L}\right) = -\frac{\pi z}{2L}.$$

■

4. BEHAVIOR OF THE INCREMENTS OF THE SOLUTION

In this part, we study the regularity of the sample paths of the solution (9) with respect to its time and space variables.

4.1. The temporal increment

Let us fix the space variable $x \in [0, L]$ and study the behavior of the process $(u(t,x), t \in [0,T])$. We will need the following auxiliary lemma from [13]. Recall that $\mathcal{H}$ is the Hilbert space associated to the fractional Brownian motion.

LEMMA 3. *Let $f(x) = \cos(x)$ for $x \in \mathbb{R}$. Then for every $a, b \in \mathbb{R}$ with $a < b$ we have*

$$\|f1_{[a,b]}(\cdot)\|_{\mathcal{H}}^2 \leq 2\alpha_H \int_0^{b-a} dv \cos(v) v^{2H-2}(b-a-v).$$

We have the following result.

PROPOSITION 5. *Let $x \in [0, L]$. Then*

$$\mathbf{E}\,|u(t,x) - u(s,x)|^2 \leq C|t-s|^{2H-\varepsilon}$$

for every $\varepsilon \in (0, 2H)$ and for every $0 \leq s \leq t \leq T$.

Proof: Let $s, t \in [0, T]$ with $s \leq t$. Recall that C denotes a generic strictly positive constant (that may change from line to line). We have

$$\begin{aligned}
& \mathbf{E}\,|u(t,x) - u(s,x)|^2 \\
= \; & C \sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{n\pi x}{L}\right) \int_s^t \int_s^t dudv \sin(w_n(t-u)) \sin(w_n(t-v)) |u-v|^{2H-2} \\
& + C \sum_{n\geq 1} \frac{1}{n^2} \int_0^s \int_0^s dudv |u-v|^{2H-2} \\
& \times (\sin(w_n(t-u)) - \sin(w_n(s-u)))\,(\sin(w_n(t-v)) - \sin(w_n(s-v))) \\
:= \; & C(T_1 + T_2).
\end{aligned}$$

We show that $T_i \leq C|t-s|^{2H}$ for $i = 1, 2$. For T_1, this is trivial since, by majorizing the sinus function by 1,

$$T_1 \leq \sum_{n\geq 1} \frac{1}{n^2} \int_s^t \int_s^t dudv |u-v|^{2H-2} = C|t-s|^{2H} \sum_{n\geq 1} \frac{1}{n^2} = C|t-s|^{2H}.$$

Let us focus on the term T_2. Using the trigonometric identity

$$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

we can write

$$\begin{aligned}
T_2 \; \leq \; & C \sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{w_n(t-s)}{2}\right) \\
& \times \int_0^s \int_0^s dudv \cos\left(\frac{w_n(t+s-2u)}{2}\right) \cos\left(\frac{w_n(t+s-2v)}{2}\right) |u-v|^{2H-2} \\
= \; & C \sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{w_n(t-s)}{2}\right)
\end{aligned}$$

$$\times \int_{\frac{t-s}{2}}^{\frac{t+s}{2}} \int_{\frac{t-s}{2}}^{\frac{t+s}{2}} dudv \cos(w_n u)\cos(w_n v)|u-v|^{2H-2}$$

$$= C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_{w_n\frac{t-s}{2}}^{w_n\frac{t+s}{2}}\int_{w_n\frac{t-s}{2}}^{w_n\frac{t+s}{2}} dudv\cos(u)$$

$$\times \cos(v)|u-v|^{2H-2}$$

$$= C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\|\cos(\cdot)1_{[w_n\frac{t-s}{2},w_n\frac{t+s}{2}]}\|^2_{\mathcal{H}}$$

by using (8). Now, via Lemma 3,

$$T_2 \leq C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_0^{w_n s}\cos(v)v^{2H-2}(w_n s-v)dv$$

$$= Cs\sum_{n\geq 1}\frac{1}{n^{1+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_0^{w_n s}\cos(v)v^{2H-2}dv$$

$$-C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_0^{w_n s}\cos(v)v^{2H-1}dv.$$

Via an integration by parts

$$\int_0^{w_n s}\cos(v)v^{2H-1}dv = \sin(w_n s)(w_n s)^{2H-1} - (2H-1)\int_0^{w_n s}\sin(v)v^{2H-2}dv$$

and this implies

$$T_2 \leq Cs\sum_{n\geq 1}\frac{1}{n^{1+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_0^{w_n s}v^{2H-2}\cos(v)dv$$

$$-C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\sin(w_n s)(w_n s)^{2H-1}$$

$$-C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\int_0^{w_n s}\sin(v)v^{2H-2}dv$$

$$:= T_{2,1}+T_{2,2}+T_{2,3}. \tag{18}$$

We will bound separately the three summands $T_{2,1}, T_{2,2}$ and $T_{2,3}$. First, since the integral $\int_0^\infty v^{2H-2}\cos(v)dv$ is convergent,

$$T_{2,1} \leq C\sum_{n\geq 1}\frac{1}{n^{1+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)\left|\int_0^{w_n s}v^{2H-2}\cos(v)dv\right|$$

$$\leq C\sum_{n\geq 1}\frac{1}{n^{1+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right)$$

and writting, for every $\varepsilon \in (0, 2H)$

$$\sin^2\left(\frac{w_n(t-s)}{2}\right) = \left|\sin\left(\frac{w_n(t-s)}{2}\right)\right|^{2H-\varepsilon}\left|\sin\left(\frac{w_n(t-s)}{2}\right)\right|^{2-2H+\varepsilon}$$
$$\leq C(n(t-s))^{2H-\varepsilon}$$

we get

$$T_{2,1} \leq C|t-s|^{2H-\varepsilon}\sum_{n\geq 1}\frac{1}{n^{1+\varepsilon}} \leq C|t-s|^{2H-\varepsilon}. \tag{19}$$

For $T_{2,2}$, we have

$$T_{2,2} \leq C\sum_{n\geq 1}\frac{1}{n^3}\sin^2\left(\frac{w_n(t-s)}{2}\right) \leq C|t-s|^{2H-\varepsilon}\sum_{n\geq 1}\frac{1}{n^{3-2H+\varepsilon}}$$
$$\leq C|t-s|^{2H-\varepsilon}. \tag{20}$$

Finally, using the convergence of the integral $\int_0^\infty v^{2H-2}\sin(v)dv$

$$T_{2,3} \leq C\sum_{n\geq 1}\frac{1}{n^{2+2H}}\sin^2\left(\frac{w_n(t-s)}{2}\right) \leq C|t-s|^{2H-\varepsilon}. \tag{21}$$

By replacing (19), (20) and (21) in (18), we obtain the conclusion. ■

An immediate consequence is the following.

Corollary 1. *For every $x \in [0, L]$, the process $(u(t,x), t \in [0,T])$ is Hölder continuous of order δ, for every $\delta \in (0, H)$.*

Proof: This follows from the above Proposition 5 and the Kolmogorov continuity criterion. ■

4.2. **Spatial increment**

For the study of the spatial increment, we recall the following formula (see e.g. [23]): for every $x \in (0, \frac{\pi}{2})$,

$$\sum_{n\geq 1}\frac{\sin^2(nx)}{n^2} = \frac{\pi}{2}x - \frac{1}{2}x^2. \tag{22}$$

The spatial regularity of the process (4) states as follows.

Proposition 6. *For every $t \in [0,T]$ and for every $x, y \in (0, L)$ with $|x-y|$ small enough,*

$$\mathbf{E}(u(t,x) - u(t,y))^2 \leq C|x-y|.$$

Proof: First, for $t \in [0,T]$ and $x, y \in (0,L)$,

$$\begin{aligned} u(t,x) - u(t,y) &= \sum_{n\geq 1} T_n(t) \left(\sin\left(\frac{n\pi x}{L}\right) - \sin\left(\frac{n\pi y}{L}\right)\right) \\ &= 2\sum_{n\geq 1} T_n(t) \sin\left(\frac{n\pi(x-y)}{2L}\right) \cos\left(\frac{n\pi(x+y)}{2L}\right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{E}\left(u(t,x) - u(t,y)\right)^2 &= 4\sum_{n\geq 1} \mathbf{E} T_n(t)^2 \sin^2\left(\frac{n\pi(x-y)}{2L}\right) \cos^2\left(\frac{n\pi(x+y)}{2L}\right) \\ &\leq C\sum_{n\geq 1} \frac{1}{n^2} \sin^2\left(\frac{n\pi(x-y)}{2L}\right). \end{aligned}$$

By using the above formula (22), for $x, y \in (0,L)$,

$$\mathbf{E}\left(u(t,x) - u(t,y)\right)^2 \leq C\left(\frac{\pi}{2}(x-y) - \frac{1}{2}|x-y|^2\right) \leq C|x-y|$$

when $|x-y|$ is small enough. ■

The Hölder regularity in space is obtained via Proposition 6 and the Kolmogorov criterion.

COROLLARY 2. *For every $t \in [0,T]$, the process $(u(t,x), x \in [0,L])$ is Hölder continuous of order δ, for every $\delta \in \left(0, \frac{1}{2}\right)$.*

Notice that the regularity of the solution (9), both in time and in space, is different with respect to the case of the wave equation with fractional-colored noise with space variable in the whole real line. Recall (see [13], see also [12] for the white-noise case), that when $x \in \mathbb{R}$, then the corresponding solution is Hölder continuous of order $\delta \in (0, \frac{H}{2})$ in time and of order $\delta \in (0, H)$ in space. In the case of the finite string, while some regularity is gained in time, the solution seems to be less regular in space since $H > \frac{1}{2}$.

5. BEHAVIOR WITH RESPECT TO THE HURST PARAMETER

Now, we analyze the behavior of the solution to (1) with respect to the Hurst parameter. Recall that $H \in \left(\frac{1}{2}, 1\right)$ and the covariance of the solution is not defined for $H = \frac{1}{2}$ and $H = 1$ (see the covariance formula (12)). We will see what happens when H converges to its extreme values, i.e. when $H \to \frac{1}{2}$ and $H \to 1$. The behavior of several fractional processes with respect to the Hurst parameter has been studied in [1], [2], [5] while the particular case of solutions to SPDEs can be found in [20], [21].

Recall that a space -time white noise is a real valued centered Gaussian field
$W = \{W_t(A); t \in [0,T], A \in B_b([0,L])\}$, over a given complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, with covariance function given by

$$\mathbf{E}\left(W_t(A)W_s(B)\right) = (t \wedge s)\lambda(A \cap B), \forall A, B \in \mathcal{B}_b([0,L]). \tag{23}$$

By $C([0,T])$ we denote the set of continuous functions on $[0,T]$.

Proposition 7. *Let $(u(t,x), t \in [0,T], x \in [0,L])$ be given by (9). Then, as $H \to \frac{1}{2}$, for every $x \in [0,L]$, the process $(u(t,x), t \in [0,T])$ converges in the the space $C[0,T]$ to the process $(u_0(t,x), t \in [0,T])$ defined by*

$$u_0(t,x) = \int_0^t \int_0^L G_{t,x}(s,y)W(ds,dy) \tag{24}$$

where W is a space-time white noise and G is given by (5).

As $H \to 1$, for every $x \in [0,L]$, the process $(u(t,x), t \in [0,T])$ converges in the space $C[0,T]$ to the process $(u_1(t,x), t \in [0,T])$ defined by

$$u_1(t,x) = \sum_{n\geq 1} \frac{\sqrt{2}}{\sqrt{L}w_n} \left(\int_0^t \sin(w_n(t-u))du\right) Z_n \tag{25}$$

with $(Z_n)_{n\geq 1}$ independent standard normal random variables.

Proof: Recall that for every $s, t \in [0,T]$,

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) = &\sum_{n\geq 1} \frac{4}{L^2 w_n^2} \frac{L}{2} H(2H-1) \int_0^t \int_0^s dudv \sin(w_n(t-u)) \\ &\times \sin(w_n(s-v))|u-v|^{2H-2}.\end{aligned} \tag{26}$$

Let us look to the limit as $H \to \frac{1}{2}$ and $H \to 1$ of the quantity

$$I(H) = H(2H-1)\int_0^t du \int_0^s dv \sin(w_n(t-u))\sin(w_n(s-v))|u-v|^{2H-2}.$$

Assume $s \leq t$. We can express $I(H)$ as follows

$$\begin{aligned} I(H) &= H(2H-1)\int_s^t du \int_0^s dv \sin(w_n(t-u))\sin(w_n(s-v))(u-v)^{2H-2} \\ &\quad + H(2H-1)\int_0^s du \int_0^u dv \sin(w_n(t-u))\sin(w_n(s-v))(u-v)^{2H-2} \\ &\quad + H(2H-1)\int_0^s du \int_u^s dv \sin(w_n(t-u))\sin(w_n(s-v))(v-u)^{2H-2} \\ &:= I_1(H) + I_2(H) + I_3(H). \end{aligned}$$

Let us calculate first $I_1(H)$. By integrating by parts in the integral dv,

$$\begin{aligned} I_1(H) &= H\int_s^t du \sin(w_n(t-u))\left[-\sin(w_n(s-v))(u-v)^{2H-1}\Big|_{v=0}^{v=s}\right. \\ &\quad \left. -\int_0^s w_n\cos(w_n(s-v))(u-v)^{2H-1}\right] \\ &= H\int_s^t du\sin(w_n(t-u))\left[\sin(w_n s)u^{2H-1}\right. \\ &\quad \left. -\int_0^s dv w_n\cos(w_n(s-v))(u-v)^{2H-1}\right]. \end{aligned}$$

We take the limit of the above quantity when $H\to\frac{1}{2}$. Clearly

$$H\sin(w_n(t-u))\sin(w_n s)u^{2H-1} \to_{H\to\frac{1}{2}} \frac{1}{2}\sin(w_n(t-u))\sin(w_n s)$$

and

$$H\sin(w_n(t-u))\cos(w_n(s-v))(u-v)^{2H-1} \to_{H\to\frac{1}{2}} \frac{1}{2}\sin(w_n(t-u))\cos(w_n(s-v)).$$

By applying the dominated convergence theorem,

$$I_1(H)\to_{H\to\frac{1}{2}} \frac{1}{2}\int_s^t du\sin(w_n(t-u))\left(\sin(w_n(s)-\int_0^s dv w_n\cos(w_n(s-u))\right)=0.$$

For $I_2(H)$, we similarly get

$$\begin{aligned} I_2(H) &= H\int_0^s du\sin(w_n(t-u))\left[\sin(w_n s)u^{2H-1}\right. \\ &\quad \left. -\int_0^u dv \quad w_n\cos(w_n(s-v))(u-v)^{2H-1}\right] \\ &\to_{H\to\frac{1}{2}} \frac{1}{2}\int_0^s du\sin(w_n(t-u))\left[\sin(w_n s)+\sin(w_n(s-v))\Big|_{v=0}^{v=u}\right] \\ &= \frac{1}{2}\int_0^s du\sin(w_n(t-u))\sin(w_n(s-u)). \end{aligned}$$

Finally,

$$\begin{aligned} I_3(H) &= H\int_0^s du\sin(w_n(t-u))\int_u^s dv \quad w_n\cos(w_n(s-v))(v-u)^{2H-1} \\ &\to \frac{1}{2}\int_0^s du\sin(w_n(t-u))\sin(w_n(s-u)). \end{aligned}$$

We obtained

$$I(H)\to_{H\to\frac{1}{2}} \int_0^s du\sin(w_n(t-u))\sin(w_n(s-u)). \tag{27}$$

From (27) and (26),

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) &\to_{H\to\frac{1}{2}} \sum_{n\geq 1}\frac{4}{L^2w_n^2}\frac{L}{2}\int_0^s du\sin(w_n(t-u))\sin(w_n(s-u))\\ &= \mathbf{E}u_0(t,x)u_0(s,x)\end{aligned}$$

with u_0 given by (24). This gives the convergence of finite-dimensional distributions of u to those of u_0, since both processes are Gaussian. The tightness is obtained from Proposition 5 and the Billingsley criterion (see [6, Theorem 12.3] or [7]).

If $H \to 1$, it is easy to see that

$$I(H) \to \int_0^t \sin(w_n(t-u))du \int_0^s \sin(w_n(s-v))dv$$

and thus

$$\begin{aligned}\mathbf{E}u(t,x)u(s,x) &\to_{H\to 1} \sum_{n\geq 1}\frac{4}{L^2w_n^2}\frac{2}{L}\\ &\int_0^t \sin(w_n(t-u))du \int_0^s \sin(w_n(s-v))dv = \mathbf{E}u_1(t,x)u_1(s,x)\end{aligned}$$

with u_1 given by (25). So we have the convergence of finite dimensional distributions of u to those of u_1 and the tightness is obtained as above. ■

Let us remark that the above result shows that the solution (4) converges, when H approaches its extreme values, to the solution to the wave equation driven by the "limit of the noise". Indeed, when $H \to \frac{1}{2}$, it is clear that the fractional-white noise (2) converges to the white noise (23) while when H is close to 1 the solution (4) as the same law as the process (13) and we use that fact that $(B_t^1)_{t\in[0,T]} =^{(d)} (tZ, t \in [0,1])$ with Z a standard normal random variable.

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Université de Lille, CNRS, UMR 8524
Laboratoire Paul Painlevé
F-59655 Villeneuve d'Ascq, France
`zeina_kh@outlook.fr`
`ciprian.tudor@univ-lille.fr`