Dedicated to the memory of Nicu Boboc

ON THE MAXIMAL MONOTONICITY OF SUBDIFFERENTIAL OPERATORS

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The maximal monotonicity of the subdifferential operator in general Banach spaces is proved via Ekeland's variational principle. Using the Fitzpatrick function the Rockafellar surjectivity theorem follows as a corollary.

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1. INTRODUCTION

In this note, we give a very simple proof of Rockafellar's maximal monotonicity theorem based on Ekeland's variational principle. The paper is the result of particular discussions and remarks of Prof. C. Zalinescu on the simplified proofs of Rockafellar's results: maximal monotonicity theorem and surjectivity theorem. The proof for maximal monotonicity that we present here comes from a note on the maximal monotonicity of the subdifferential operator of the convex l.s.c. function $\Phi : C([0,T];\mathbb{R}^d) \to]-\infty,+\infty]$

$$\Phi(x) = \begin{cases} \displaystyle\int_0^T \varphi(x(t)dt, & \text{if } \varphi(x) \in L^1(0,T) \\ +\infty, & \text{otherwise} \end{cases}$$

given by Asiminoaei and Răşcanu in [1].

Remark that the first proof of maximal monotonicity theorem was given by Rockafellar in [7]. Other different and simplified proofs are given by S. Simons in [8] and M. Marques Alves and B. F. Svaiter in [6].

Let $(\mathbb{X}, \|.\|)$ be a real Banach space and $(\mathbb{X}^*, \|.\|_*)$ be its dual. For $x^* \in \mathbb{X}^*$ and $x \in \mathbb{X}$ we denote $x^*(x)$ (the value of x^* in x) by $\langle x, x^*\rangle$ or $\langle x^*, x\rangle$. The space $\mathbb{X} \times \mathbb{X}^*$ is also a Banach space with the norm $\|(x,x^*)\|_{\mathbb{X}\times\mathbb{X}^*} = \left(\|x\|^2 + \|x^*\|_*^2\right)^{1/2}$.

If $A:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is a point-to-set operator (from $\mathbb{X}$ to the family of subsets of $\mathbb{X}^*$), then $Dom\,(A)\overset{\text{def}}{=}\{x\in\mathbb{X}:A\,(x)\neq\emptyset\}$ and $R\,(A)\overset{\text{def}}{=}\{x^*\in\mathbb{X}^*:\exists\ x\in Dom\,(A)\text{ s.t. }x^*\in A\,(x)\}$. We say that A is proper if $Dom\,(A)\neq\emptyset$.

We shall use the notation $(x,x^*)\in A$ for $x\in Dom\,(A)$ and $x^*\in A\,(x)$; this means that the operator A is identified with its graph

$$gr\,(A)=\{(x,x^*)\in\mathbb{X}\times\mathbb{X}^*:x\in Dom\,(A)\,,\ x^*\in A\,(x)\}.$$

The operator $A:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is monotone ($A\subset\mathbb{X}\times\mathbb{X}^*$ is a monotone set) if

$$\langle x-y,x^*-y^*\rangle\geq 0,\ \forall\,(x,x^*)\,,(y,y^*)\in A.$$

A monotone operator (set) is maximal monotone if its graph is not properly contained in the graph of any other monotone operator (set). Hence a monotone operator is maximal monotone if and only if

$$\inf\{\langle x-u,x^*-u^*\rangle:(u,u^*)\in A\}\geq 0\quad\Longrightarrow\quad(x,x^*)\in A.$$

Given a function $\varphi:\mathbb{X}\to]-\infty,+\infty]$, we denote $Dom\,(\varphi)\overset{\text{def}}{=}\{x\in\mathbb{X}:\varphi\,(x)<\infty\}$. We say that φ is proper if $Dom\,(\varphi)\neq\emptyset$. The subdifferential $\partial\varphi:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is defined by

$$(x,x^*)\in\partial\varphi\quad\text{if}\quad\langle y-x,x^*\rangle+\varphi\,(x)\leq\varphi\,(y)\,,\ \forall\ y\in\mathbb{X}.$$

Clearly if φ is proper and $(x,x^*)\in\partial\varphi$ then $\varphi\,(x)\in\mathbb{R}$, that is $x\in Dom\,(\varphi)$.

Remark that if $\varphi:\mathbb{X}\to]-\infty,+\infty]$ is a proper convex l.s.c. function and $\psi:\mathbb{X}\to\mathbb{R}$ is a continuous convex function then $\partial\,(\varphi+\psi)\,(x)=\partial\varphi\,(x)+\partial\psi\,(x)$ for all $x\in Dom\,(\partial\varphi)$ (this result is a consequence of the Theorem 2.8.7 from [10]).

2. THE RESULTS

THEOREM 1. *Let $\mathbb{X}$ be a Banach space and $\varphi:\mathbb{X}\to]-\infty,+\infty]$ be a proper convex lower semicontinuous function. Then $\partial\varphi:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is proper maximal monotone operator.*

Proof. Using the definition of $\partial\varphi$ it is very easy to prove that $\partial\varphi:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is a monotone operator. Let us prove that $\partial\varphi$ is a proper maximal monotone operator.

Let $(z,z^*)\in\mathbb{X}\times\mathbb{X}^*$ and $\lambda>0$ be arbitrary fixed. Consider the function $\Psi:\mathbb{X}\to]-\infty,+\infty]$ be defined by

$$\Psi(x)=\frac{1}{2}\,\|x-z\|^2+\lambda\varphi(x)-\lambda\,\langle x,z^*\rangle\,.$$

Then Ψ is a proper convex lower semicontinuous function and Ψ is bounded from below. From Ekeland principle [3] (see also [10, Th. 1.4.1], or [2], p. 29, Th. 3.2), for every $\varepsilon > 0$ there exists $x_\varepsilon \in \mathbb{X}$ such that

(1) $$\Psi(x_\varepsilon) \leq \inf\{\Psi(x) : x \in \mathbb{X}\} + \varepsilon^2 \quad \text{and}$$

(2) $$\Psi(x_\varepsilon) \leq \Psi(x) + \varepsilon \; \|x - \; x_\varepsilon\|_{\mathbb{X}} \; , \text{ for all } x \in \mathbb{X}.$$

Remark that the sequence $\{x_\varepsilon : 0 < \varepsilon \leq 1\}$ is bounded, since $\lim\limits_{\|x\|\to\infty} \Psi(x) = +\infty$. From (2) we deduce that

$$0 \in \partial\Psi(x_\varepsilon) + \varepsilon U_{\mathbb{X}^*} = J_{\mathbb{X}}(x_\varepsilon - z) + \lambda\partial\varphi(x_\varepsilon) - \lambda z^* + \varepsilon U_{\mathbb{X}^*} \; ,$$

where $U_{\mathbb{X}^*} = \{u^* \in \mathbb{X}^* : \|u^*\|_{\mathbb{X}^*} \leq 1\}$ and $J_{\mathbb{X}} : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is the duality mapping, that is

$$J_{\mathbb{X}}(x) = \left\{x^* \in \mathbb{X}^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2_{\mathbb{X}^*}\right\} = \partial\left(\frac{1}{2}\|\cdot\|^2\right)(x).$$

So, there exist $u^*_\varepsilon \in U_{X^*}$, $y^*_\varepsilon \in J_{\mathbb{X}}(x_\varepsilon - z)$ and $x^*_\varepsilon \in \partial\varphi(x_\varepsilon)$ (in particular $\partial\varphi$ is a proper point-to-set operator) such that

(3) $$\lambda z^* - \lambda x^*_\varepsilon = y^*_\varepsilon + \varepsilon \; u^*_\varepsilon \; .$$

It follows that $\|\lambda z^* - \lambda x^*_\varepsilon\| \leq \|y^*_\varepsilon\| + \varepsilon = \|x_\varepsilon - z\| + \varepsilon$.

Let now $(z, z^*) \in \mathbb{X} \times \mathbb{X}^*$ such that $\langle z - x, z^* - x^*\rangle \geq 0$ for all $(x, x^*) \in \partial\varphi$. Then

$$\begin{aligned} 0 \leq \langle z - x_\varepsilon, z^* - x^*_\varepsilon\rangle \; &= \langle z - x_\varepsilon, y^*_\varepsilon\rangle + \varepsilon\,\langle z - x_\varepsilon, u^*_\varepsilon\rangle \\ &\leq -\|x_\varepsilon - z\|^2 + \varepsilon\,\|x_\varepsilon - z\|. \end{aligned}$$

Hence $\|x_\varepsilon - z\| \leq \varepsilon$ and $\|x^*_\varepsilon - z^*\| \leq \dfrac{2\varepsilon}{\lambda}$; in particular $x_\varepsilon \to z$ and $x^*_\varepsilon \to z^*$ as $\varepsilon \to 0$. Passing to $\liminf_{\varepsilon\to 0}$ in

$$\langle x^*_\varepsilon, y - x_\varepsilon\rangle + \varphi(x_\varepsilon) \leq \varphi(y), \quad \forall\, y \in \mathbb{X},$$

we obtain $(z, z^*) \in \partial\varphi$. $\square$

From this proof (the equality (3) corresponding to $z = 0$) we deduce a Rockafellar's type surjectivity result in general Banach spaces:

Corollary 2. *If $\mathbb{X}$ is a Banach space and $\varphi : \mathbb{X} \to]-\infty, +\infty]$ is a proper convex lower semicontinuous function then for all $\lambda > 0$,*

$$\overline{R\,(J_{\mathbb{X}} + \lambda\partial\varphi)} = \mathbb{X}^*.$$

If $\mathbb{X}$ is a reflexive Banach space, then

$$R\,(J_{\mathbb{X}} + \lambda\partial\varphi) = \mathbb{X}^*.$$

Proof. From (3) with $z=0$ we deduce that $R\left(\partial\varphi+\lambda J_{\mathbb{X}}\right)+\varepsilon U_{\mathbb{X}^*}=\mathbb{X}^*$. Hence

$$\overline{R\left(J_{\mathbb{X}}+\lambda\partial\varphi\right)}=\bigcap_{\varepsilon>0}\left[R\left(J_{\mathbb{X}}+\lambda\partial\varphi\right)+\varepsilon U_{\mathbb{X}^*}\right]=\mathbb{X}^*.$$

If $\mathbb{X}$ is a reflexive Banach space, then the boundedness of $\{x_\varepsilon : 0<\varepsilon\leq 1\}$ yields that there exists a subsequence $\varepsilon_n\to 0$ such that $x_{\varepsilon_n}\rightharpoonup x_0$ (weakly on $\mathbb{X}$) and

$$\Psi(x_0)\leq\liminf_{\varepsilon_n\to 0}\Psi(x_{\varepsilon_n})=\inf\{\Psi(x):x\in\mathbb{X}\}.$$

Hence $0\in\partial\Psi(x_0)=J_{\mathbb{X}}(x_0)+\lambda\partial\varphi(x_0)-\lambda z^*$ that is $\lambda z^*\in J_{\mathbb{X}}(x_0)+\lambda\partial\varphi(x_0)$. Hence $\mathbb{X}^*=\lambda\mathbb{X}^*\subset R\left(J_{\mathbb{X}}+\lambda\partial\varphi\right)\subset\mathbb{X}^*$. $\square$

Definition 3. Given a monotone operator $A:\mathbb{X}\rightrightarrows\mathbb{X}^*$, the associated Fitzpatrick function is defined as $\mathcal{H}=\mathcal{H}_A:\mathbb{X}\times\mathbb{X}^*\to\,]-\infty,+\infty]$,

$$\begin{aligned}\mathcal{H}\left(x,x^*\right)&\overset{\text{def}}{=}\left\langle x,x^*\right\rangle-\inf\left\{\left\langle x-u,x^*-u^*\right\rangle:(u,u^*)\in A\right\}\\&=\sup\left\{\left\langle u,x^*\right\rangle+\left\langle x,u^*\right\rangle-\left\langle u,u^*\right\rangle:(u,u^*)\in A\right\}\end{aligned}\tag{4}$$

Clearly $\mathcal{H}\left(x,x^*\right)=\left\langle x,x^*\right\rangle$, for all $(x,x^*)\in A$ and

$$\mathcal{H}=\mathcal{H}_A:\mathbb{X}\times\mathbb{X}^*\to\,]-\infty,+\infty]\quad\text{is a proper convex l.s.c. function.}$$

Let $(x^*,x)\in\partial\mathcal{H}\left(u,u^*\right)$. Then, from the definition of a subdifferential operator this means that

$$\left\langle\left(x^*,x\right),\left(z,z^*\right)-\left(u,u^*\right)\right\rangle+\mathcal{H}\left(u,u^*\right)\leq\mathcal{H}\left(z,z^*\right),\ \forall\ (z,z^*)\in\mathbb{X}\times\mathbb{X}^*,$$

or, equivalently,

$$\begin{aligned}&\left\langle u-x,u^*-x^*\right\rangle-\inf_{(y,y^*)\in A}\left\langle u-y,u^*-y^*\right\rangle\\&\leq\left\langle z-x,z^*-x^*\right\rangle-\inf_{(y,y^*)\in A}\left\langle z-y,z^*-y^*\right\rangle,\quad\forall\ (z,z^*)\in\mathbb{X}\times\mathbb{X}^*.\end{aligned}\tag{5}$$

Since the operator A is maximal monotone, then

$$\inf_{(y,y^*)\in A}\left\langle u-y,u^*-y^*\right\rangle\leq 0$$

and

$$\inf_{(y,y^*)\in A}\left\langle z-y,z^*-y^*\right\rangle=0,\ \forall\ (z,z^*)\in A;$$

consequently, we have

$$(x^*,x)\in\partial\mathcal{H}\left(u,u^*\right)\implies\left\langle u-x,u^*-x^*\right\rangle\leq\inf_{(z,z^*)\in A}\left\langle z-x,z^*-x^*\right\rangle.\tag{6}$$

Also, by the monotonicity of A, from (5) follows

$$(x^*,x)\in A\implies(x^*,x)\in\partial\mathcal{H}\left(x,x^*\right).$$

Hence, if $A:\mathbb{X}\rightrightarrows\mathbb{X}^*$ is a maximal monotone operator then $\mathcal{H}_A$ characterizes A as follows.

Theorem 4 (Fitzpatrick). (see Fitzpatrick [4] and Simons-Zălinescu [9]) *Let $\mathbb{X}$ be a Banach space, $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ be a maximal monotone operator and $\mathcal{H}$ its associated Fitzpatrick function. Then, for all $(x, x^*) \in \mathbb{X} \times \mathbb{X}^*$*

$$\mathcal{H}(x, x^*) \geq \langle x, x^* \rangle .$$

Moreover, the following assertions are equivalent

(a) $(x, x^*) \in A$;

(b) $\mathcal{H}(x, x^*) = \langle x, x^* \rangle$;

(c) $\exists\ (u, u^*) \in Dom\,(\partial\mathcal{H})$ *such that*

$$(x^*, x) \in \partial\mathcal{H}\,(u, u^*) \ \text{ and } \ \langle u - x, u^* - x^* \rangle \geq 0;$$

(d) $(x^*, x) \in \partial\mathcal{H}\,(x, x^*)$.

Proof. It's not difficult to show that $(b) \Leftrightarrow (a) \Rightarrow (d) \Rightarrow (c) \Rightarrow (a)$. □

Corollary 5. *Let $\mathbb{X}$ be a Banach space and $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ be a maximal monotone operator. Then*

$$0 \in R\,(J_{\mathbb{X}} + A) \quad \Longleftrightarrow \quad (0, 0) \in R\left(J_{\mathbb{X}} \otimes J_{\mathbb{X}}^{-1} + \partial\mathcal{H}_A\right)$$

Proof. Since $J_{\mathbb{X}}\,(-x) = -J_{\mathbb{X}}\,(x)$, then we clearly have the following equivalences: $0 \in R\,(J_{\mathbb{X}} + A) \Longleftrightarrow \exists\ (x, x^*) \in A$ such that $-x^* \in J_{\mathbb{X}}\,(x) \Longleftrightarrow \exists\ (x, x^*) \in \mathbb{X} \times \mathbb{X}^*$ such that $(0, 0) \in (-x^*, -x) + \partial\mathcal{H}\,(x, x^*)$ and $(-x^*, -x) \in \left(J_{\mathbb{X}}\,(x), J_{\mathbb{X}}^{-1}\,(x^*)\right) \Longleftrightarrow (0, 0) \in R\left(J_{\mathbb{X}} \otimes J_{\mathbb{X}}^{-1} + \partial\mathcal{H}_A\right)$. □

Now from Corollary 2 we have for all $\lambda > 0$,

$$\overline{R\,(J_{\mathbb{X}\times\mathbb{X}^*} + \lambda\partial\mathcal{H}_A)} = \mathbb{X}^* \times \mathbb{X}^{**}$$

When $\mathbb{X}$ is a reflexive Banach space $J_{\mathbb{X}\times\mathbb{X}^*}\,(x, x^*) = J_{\mathbb{X}}\,(x) \otimes J_{\mathbb{X}}^{-1}\,(x^*)$ and therefore, by Corollary 2,

$$R\left(J_{\mathbb{X}} \otimes J_{\mathbb{X}}^{-1} + \lambda\partial\mathcal{H}_A\right) = \mathbb{X}^* \times \mathbb{X} \tag{7}$$

In this sequence of ideas we can rewrite simplifying the approach from [9] of Simons and Zălinescu for the Rockafellar's surjectivity result of maximal monotone operators, as follows

Theorem 6 (Rockafellar). *Let $\mathbb{X}$ be a reflexive Banach space. If $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ is a maximal monotone operator, then $R\,(J_{\mathbb{X}} + \lambda A) = \mathbb{X}^*$, for all $\lambda > 0$.*

Proof. Let $\lambda > 0$ and $z^* \in \mathbb{X}^*$. Since A is maximal monotone if and only $\tilde{A} = \lambda A - z^*$ is maximal monotone, then to prove $z^* \in R\left(J_{\mathbb{X}} + \lambda A\right)$ is equivalent to prove $0 \in R\left(J_{\mathbb{X}} + A\right)$. But by (7) and Corollary 5 the relation $0 \in R\left(J_{\mathbb{X}} + A\right)$ is equivalent to the true assertion $(0,0) \in R\left(J_{\mathbb{X}} \otimes J_{\mathbb{X}}^{-1} + \partial\mathcal{H}_A\right) = \mathbb{X}^* \times \mathbb{X}$. $\square$

Finally we note (see [5]) that in the case of a non-reflexive Banach space $\mathbb{X}$, there exists a maximal monotone operator $A : \mathbb{X} \rightrightarrows \mathbb{X}^*$ and $\lambda > 0$ such that $\overline{R\left(J_{\mathbb{X}} + \lambda A\right)} \subsetneq \mathbb{X}^*$.

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