Dedicated to the memory of Professor Nicu Boboc

HOMOGENIZATION OF SYMMETRIC LÉVY PROCESSES ON $\mathbb{R}^d$

RENÉ L. SCHILLING and TOSHIHIRO UEMURA

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In this short note we study homogenization of symmetric d-dimensional Lévy processes. Homogenization of one-dimensional pure jump Markov processes has been investigated by Tanaka *et al.* in [5]; their motivation was the work by Benssousan *et al.* [1] on the homogenization of diffusion processes in $\mathbb{R}^d$, see also [2] and [11]. We investigate a similar problem for a class of symmetric pure-jump Lévy processes on $\mathbb{R}^d$ and we identify - using Mosco convergence - the limit process.

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A symmetric Lévy process $(X_t)_{t\geq 0}$ is a stochastic process in $\mathbb{R}^d$ with stationary and independent increments, càdlàg paths and symmetric laws $X_t \sim -X_t$. We can characterize the (finite-dimensional distributions of the) process by its characteristic function $\mathbb{E}e^{i\langle \xi, X_t\rangle}$, $\xi \in \mathbb{R}^d$, $t > 0$, which is of the form $\exp(-t\psi(\xi))$; due to the symmetry of X_t, the characteristic exponent ψ is real-valued. It is given by the Lévy–Khintchine formula

$$\psi(\xi) = \frac{1}{2}\langle \xi, \Sigma\xi\rangle + \int_{h\neq 0} \big(1 - \cos\langle \xi, h\rangle\big)\, \nu(\mathrm{d}h), \quad \xi \in \mathbb{R}^d. \tag{1}$$

$\Sigma \in \mathbb{R}^{d\times d}$ is the positive semidefinite *diffusion matrix* and $\nu(\mathrm{d}h)$ is the *Lévy measure*, that is a Radon measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{h\neq 0} \big(1 \wedge |h|^2\big)\, \nu(\mathrm{d}h)$ is finite. It is clear from (1) that we have $\nu(\mathrm{d}h) = \nu(-\mathrm{d}h)$. Throughout this paper we assume that $\Sigma \equiv 0$ and that $\nu(\mathrm{d}h)$ has a (necessarily symmetric) locally bounded density on $\mathbb{R}^d \setminus \{0\}$ w.r.t. Lebesgue measure; in abuse of notation we write $\nu(\mathrm{d}h) = \nu(h)\,\mathrm{d}h$.

Let $Q = (0,1)^d$ be the open unit cube in $\mathbb{R}^d$ and $a : \mathbb{R}^d \to \mathbb{R}$ a function in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for some $1 < p \leq \infty$. We assume that $a(x) = a(-x)$ for $x \in \mathbb{R}^d$ and a is *Q-periodic* in the sense that

$$a(h + ke_i) = a(h) > 0 \;\text{ for all } k \in \mathbb{Z},\; i = 1, 2, \ldots, d \;\text{ and a.a. } h \in Q; \tag{2}$$

as usual, e_i denotes the ith unit vector of $\mathbb{R}^d$. By $\overline{a}$ we denote the *mean value* of a,

$$\overline{a} := \int_Q a(h)\,\mathrm{d}h. \tag{3}$$

We assume that $a_\delta(h) := a\left(\delta^{-1}h\right)$ satisfies

$$\int_{h\neq 0} (1\wedge |h|^2) a_\delta(h)\nu(h)\,\mathrm{d}h < \infty \quad \text{for all} \quad \delta > 0, \tag{4}$$

$$\sup_{\delta>0} \int_{|h|\geq 1} a_\delta(h)\nu(h)\,\mathrm{d}h < \infty. \tag{5}$$

For each $\delta > 0$ we consider the following quadratic form on $L^2(\mathbb{R}^d)$ defined for Lipschitz continuous functions with compact support $u, v \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$

$$\mathcal{E}^\delta(u,v) := \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \big(u(x)-u(y)\big)\big(u(x)-u(y)\big) a_\delta(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x. \tag{6}$$

From the assumptions (2) and (4), we easily see that $(\mathcal{E}^\delta, C_0^{\mathrm{lip}}(\mathbb{R}^d))$ is a closable symmetric form in $L^2(\mathbb{R}^d)$ which is translation invariant, see [4]. Its closure $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ is a translation invariant regular symmetric Dirichlet form in $L^2(\mathbb{R}^d)$, and the associated Markov process is a symmetric Lévy process. If we use (1) and some elementary Fourier analysis, we obtain the following characterization of the Dirichlet form $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ based on the characteristic exponent ψ_δ, cf. [6, Example 4.7.28] and [4, Example 1.4.1],

$$\begin{cases} \mathcal{E}^\delta(u,v) = \displaystyle\int_{\mathbb{R}^d} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\psi_\delta(\xi)\,\mathrm{d}\xi \\ \qquad\mathcal{F}^\delta = \left\{u \in L^2(\mathbb{R}^d) : \displaystyle\int_{\mathbb{R}^d} \left|\widehat{u}(\xi)\right|^2 \psi_\delta(\xi)\,\mathrm{d}\xi < \infty\right\}, \end{cases}$$

$\widehat{u}(\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{-i\langle\xi,x\rangle}u(x)\,\mathrm{d}x$ denotes the Fourier transform and

$$\psi_\delta(\xi) = \int_{h\neq 0} \big(1-\cos\langle\xi,h\rangle\big) a_\delta(h)\nu(h)\,\mathrm{d}h, \quad \xi \in \mathbb{R}^d. \tag{7}$$

Condition (4) ensures that $a_\delta(h)\nu(h)$ is the density of a Lévy measure. If $\nu(h)$ is the density of a Lévy measure and if a is a bounded, nonnegative (and 1-periodic) function, then (4) clearly holds. The following example illustrates that for *unbounded* functions a the situation is different.

Example 1. **a)** Let $0 < \beta < 2$ and pick δ such that $0 < \delta < 1 \wedge (2-\beta)$.

Define functions α_0 on $[0,1/2]$ and α_1 on $[0,1]$ by

$$\alpha_0(x) := \begin{cases} 0, & x=0, \\ x^{-\delta}, & 0<x\le \frac14, \\ 4^\delta, & \frac14 \le x \le \frac12, \end{cases} \quad \text{and} \quad \alpha_1(x) := \begin{cases} \alpha_0(x), & 0 \le x \le \frac12, \\ \alpha_0(1-x), & \frac12 \le x \le 1. \end{cases}$$

Denote by $a : \mathbb{R} \to \mathbb{R}$ the 1-periodic extension of α_1 to the real line. It is obvious that $a \in L^p_{\text{loc}}(\mathbb{R})$ for all $1 < p < 1/\delta$. Define a further function $b = b(x)$ on $\mathbb{R}$ by $b(x) := a(x-1/2)$ for $x \in \mathbb{R}$ and set

$$\nu(h) = \frac{b(h)}{|h|^{1+\beta}}, \quad h \neq 0.$$

Clearly, $\nu(h) = \nu(-h)$; let us show that $a(h)\nu(h)$ is the density of a Lévy measure, *i.e.* $\int_{h\neq 0} (1 \wedge h^2) a(h)\nu(h)\,\mathrm{d}h < \infty$.

Since a and ν are even functions, we see

$$\int_{h\neq 0} (1\wedge h^2) a(h)\nu(h)\,\mathrm{d}h = 2\int_0^1 h^2 a(h)\nu(h)\,\mathrm{d}h + 2\sum_{\ell=1}^{\infty}\int_\ell^{\ell+1} a(h)\nu(h)\,\mathrm{d}h.$$

For the first term we get

$$\begin{aligned} \int_0^1 h^2 a(h)\nu(h)\,\mathrm{d}h &= \int_0^1 h^2 a(h) b(h) h^{-1-\beta}\,\mathrm{d}h \\ &= 4^\delta \int_0^{1/4} h^{1-\delta-\beta}\,\mathrm{d}h + 4^\delta \int_{1/4}^{1/2} h^{1-\beta}(1/2-h)^{-\delta}\,\mathrm{d}h \\ &\quad + 4^\delta \int_{1/2}^{3/4} h^{1-\beta}(h-1/2)^{-\delta}\,\mathrm{d}h + 4^\delta \int_{3/4}^{1} h^{1-\beta}(1-h)^{-\delta}\,\mathrm{d}h \\ &=: c(\delta) < \infty. \end{aligned}$$

The integrals under the sum appearing in the second term can be estimated using the periodicity of a and b; for all $\ell \geq 1$ we have

$$\begin{aligned} \int_\ell^{\ell+1} a(h)\nu(h)\,\mathrm{d}h &= \int_0^1 a(h+\ell) b(h+\ell)(h+\ell)^{-1-\beta}\,\mathrm{d}h \\ &= \int_0^1 a(h)b(h)(h+\ell)^{-1-\beta}\,\mathrm{d}h \\ &\le \ell^{-1-\beta}\int_0^1 a(h)b(h)\,\mathrm{d}h. \end{aligned}$$

As in the previous calculation, and noting that $0<\delta<1$, we again see that

$$\int_0^1 a(h)b(h)\,\mathrm{d}h = 4^\delta \int_0^{1/4} h^{-\delta}\,\mathrm{d}h + 4^\delta \int_{1/4}^{1/2} (1/2-h)^{-\delta}\,\mathrm{d}h$$

$$+ 4^\delta \int_{1/2}^{3/4} (h-1/2)^{-\delta}\,\mathrm{d}h + 4^\delta \int_{3/4}^{1} (1-h)^{-\delta}\,\mathrm{d}h < \infty.$$

Thus, $c := \int_0^1 a(h)b(h)\,\mathrm{d}h < \infty$, and

$$\int_{h\neq 0} (1\wedge h^2) a(h)\nu(h)\,\mathrm{d}h \le 2c(\delta) + c\sum_{\ell=1}^{\infty} \ell^{-1-\beta} < \infty.$$

On the other hand, we also find that

$$\begin{aligned}\int_{h\neq 0} (1\wedge h^2) a_{1/2}(h)\nu(h)\,\mathrm{d}h &= \int_{h\neq 0} (1\wedge h^2) a(2h)b(h)|h|^{-1-\beta}\,\mathrm{d}h \\ &\ge \int_{3/8}^{1/2} h^2 a(2h)b(h)h^{-1-\beta}\,\mathrm{d}h \\ &= \int_{3/8}^{1/2} h^{1-\beta}(1-2h)^{-\delta}(1/2-h)^{-\delta}\,\mathrm{d}h \\ &= 2^\delta \int_{3/8}^{1/2} h^{1-\beta}(1-2h)^{-2\delta}\,\mathrm{d}h,\end{aligned}$$

and this integral blows up if $0<\beta<3/2$ and $1/2 \le \delta < 1\wedge(2-\beta)$. In a similar way we can show that

$$\int_{h\neq 0} (1\wedge h^2) a_\delta(h)\nu(h)\,\mathrm{d}h = \infty$$

for infinitely many $\delta > 0$.

b) Let $a = a(x)$ on $\mathbb{R}$ be as in the previous part. Set $\nu(h) = |h|^{-1-\beta}$ for $h \neq 0$. Then we can show that this pair (a,ν) satisfies the conditions (2)–(4).

We will now discuss the limit of $(\mathcal{E}^\delta, \mathcal{F}^\delta)$ as $\delta \downarrow 0$. To this end, we take a sequence of positive numbers $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n \downarrow 0$ as $n\to\infty$. The following result is a standard result from homogenization theory. Usually it is stated in terms of L^p convergence (rather than L^p_{loc} convergence), see e.g. [3, Theorem 2.6].

LEMMA 2. *Suppose that* (2) *and* (4) *hold. The family* $\{a_{\delta_n}\}_{n\in\mathbb{N}}$ *converges to the constant* $\overline{a} := \int_Q a(h)\,\mathrm{d}h$ *weakly in* $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $1<p<\infty$, *i.e. for any compact set* K *of* $\mathbb{R}^d$,

$$\lim_{n\to\infty} \int_K g(x)a_{\delta_n}(x)\,\mathrm{d}x = \overline{a}\int_K g(x)\,\mathrm{d}x, \quad g\in L^q(K), \tag{8}$$

where p *and* q *are conjugate* $1/p+1/q=1$.

We will need the following corollary of Lemma 2.

Corollary 3. *Assume that (2)–(4) hold and let $\{\delta_n\}_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$. For any compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d$, let $g_n \in L^q(K)$ be a sequence of functions which converges in L^q to some $g \in L^q(K)$. Then the following limit exists*

$$\lim_{n\to\infty} \iint_K g_n(x,y) a_{\delta_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y = \overline{a} \iint_K g(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

Proof. Note that

$$\begin{aligned}
&\left|\iint_K g_n(x,y)a_{\delta_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y - \overline{a}\iint_K g(x,y)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\le \left|\iint_K \big(g_n(x,y)-g(x,y)\big)a_{\delta_n}(x-y)\,\mathrm{d}x\,\mathrm{d}y\right| + \left|\iint_K g(x,y)\big(a_{\delta_n}(x-y)-\overline{a}\big)\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\le \left[\iint_K \left|g_n(x,y)-g(x,y)\right|^q \mathrm{d}x\,\mathrm{d}y\right]^{\frac{1}{q}} \left[\iint_K a_{\delta_n}(x-y)^p\,\mathrm{d}x\,\mathrm{d}y\right]^{\frac{1}{p}} + \left|\int_{\mathbb{R}^d} H(z)\big(a_{\delta_n}(z)-\overline{a}\big)\,\mathrm{d}z\right|
\end{aligned}$$

where we use

$$H(z) := \int_{\mathbb{R}^d} \mathbb{1}_K(y+z,y)g(y+z,y)\,\mathrm{d}y, \quad z \in \mathbb{R}^d.$$

Since K is a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$, H has compact support, hence $H \in L^q_{\mathrm{loc}}(\mathbb{R}^d)$. Because of Lemma 2, the second term tends to 0; the first term also tends to 0 since $g_n \to g$ in L^q, and $\sup_{0<\delta<1} \iint_K a_\delta(x-y)^p\,\mathrm{d}x\,\mathrm{d}y$ is finite. We prove this only for $d = 1$, the arguments for $d > 1$ just have heavier notation. Without loss of generality we may assume that $K = L \times L$ for $L = [-N, N] \subset \mathbb{R}$ and $N \in \mathbb{N}$. Now take $k := \lfloor 2N/\delta \rfloor + 1 \in \mathbb{N}$, the smallest integer which is bigger or equal $2N/\delta$. We have

$$\begin{aligned}
\iint_K a_\delta(x-y)^p\,\mathrm{d}x\,\mathrm{d}y &= \int_{-N}^{N}\int_{-N}^{N} a_\delta(x-y)^p\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{-N}^{N}\left(\int_{-N+y}^{N+y} a_\delta(x)^p\,\mathrm{d}x\right)\mathrm{d}y \\
&\le \int_{-N}^{N}\left(\int_{-2N}^{2N} a_\delta(x)^p\,\mathrm{d}x\right)\mathrm{d}y \\
&= 2N\int_{-2N}^{2N} a_\delta(x)^p\,\mathrm{d}x =: 2N\cdot \mathrm{I}
\end{aligned}$$

where

$$\mathrm{I} = \int_{-2N}^{2N} a_\delta(z)^p\,\mathrm{d}z = \delta\int_{-2N/\delta}^{2N/\delta} a(z)^p\,\mathrm{d}z$$

$$
\begin{aligned}
&\leq \delta \int_{-k}^{k-1} a(z)^p \, \mathrm{d}z \\
&= \delta \sum_{\ell=-k}^{k-1} \int_{\ell}^{\ell+1} a(z)^p \, \mathrm{d}z.
\end{aligned}
$$

Because of the periodicity of a, we find that

$$
\mathrm{I} \leq \delta \sum_{\ell=-k}^{k-1} \int_0^1 a(z+\ell)^p \, \mathrm{d}z = 2k\delta \int_0^1 a(z)^p \, \mathrm{d}z \leq 2(2N+1) \int_0^1 a(z)^p \, \mathrm{d}z. \quad \square
$$

Recall that a sequence of closed forms $\{(\mathcal{E}^n, \mathcal{F}^n)\}_{n\in\mathbb{N}}$ defined on $L^2(\mathbb{R}^d)$ is called *Mosco-convergent* to a form $(\mathcal{E}, \mathcal{F})$, if the following two conditions are satisfied. As usual, we extend $\mathcal{E}^n$ and $\mathcal{E}$ to the whole space $L^2(\mathbb{R}^d)$ by setting $\mathcal{E}^n(u,u) = \infty$, resp. $\mathcal{E}(u,u) = \infty$, if $u \notin \mathcal{F}^n$, resp. $u \notin \mathcal{F}$.

(M1) For all $u \in L^2(\mathbb{R}^d)$ and all sequences $(u_n)_{n\in\mathbb{N}} \subset L^2(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ (weak convergence in L^2) we have $\liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) \geq \mathcal{E}(u,u)$.

(M2) For every $u \in \mathcal{F}$ there exist elements $u_n \in \mathcal{F}^n$, $n \in \mathbb{N}$, such that $u_n \to u$ (strong convergence in L^2) and $\limsup_{n\to\infty} \mathcal{E}^n(u_n,u_n) \leq \mathcal{E}(u,u)$.

Note that **(M1)** entails that we have $\lim_{n\to\infty} \mathcal{E}^n(u_n,u_n) = \mathcal{E}(u,u)$ in **(M2)**.

We can now state the main result of our paper. Together with Remark 5, it can be seen as the Dirichlet form apporach to the problem discussed in [5] and [11, 8]. The paper [7] has, using completely different techniques, similar results for stable-like operators and forms, which include also some non-symmetric and non-translation invariant settings.

THEOREM 4. *Assume that* (2)–(5) *hold for the functions a and ν, and let ν be locally bounded as a function defined on $\mathbb{R}^d \setminus \{0\}$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a monotonically decreasing sequence of positive numbers such that $\delta_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{N}$ we consider the Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n) := (\mathcal{E}^{\delta_n}, \mathcal{F}^{\delta_n})$ defined in* (6)*. The Dirichlet forms $(\mathcal{E}^n, \mathcal{F}^n)$ converge to $(\mathcal{E}, \mathcal{F})$ in the sense of Mosco. The limit $(\mathcal{E}, \mathcal{F})$ is the closure of $(\mathcal{E}, C_0^{\mathrm{lip}}(\mathbb{R}^d))$ which is given by*

$$
\mathcal{E}(u,v) := \overline{a} \iint_{\mathbb{R}\times\mathbb{R}} \big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\nu(y-x) \, \mathrm{d}y \, \mathrm{d}x.
$$

Proof. We will check the conditions **(M1)** and **(M2)** of Mosco convergence. For **(M1)** we take any $u \in L^2(\mathbb{R}^d)$ and any sequence $\{u_n\} \subset L^2(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ as $n \to \infty$. Without loss, we may assume that $\liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) < \infty$.

We will use the Friedrichs mollifier. This is a family of convolution operators

$$J_\epsilon[u](x) = \int_{\mathbb{R}^d} u(x-y)\rho_\epsilon(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^d,\ \epsilon > 0,$$

given by the kernels $\{\rho_\epsilon\}_{\epsilon>0}$ for a C^∞-kernel $\rho : \mathbb{R}^d \to [0,\infty)$ satisfying

$$0 \le \rho(x) = \rho(-x), \quad \int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1, \quad \operatorname{supp}[\rho] = \left\{x \in \mathbb{R}^d :\ |x| \le 1\right\}$$
$$\text{and} \quad \rho_\epsilon(x) := \rho(x/\epsilon), \quad \text{for } \epsilon > 0 \ \text{ and } \ x \in \mathbb{R}^d.$$

We then have

$$\begin{aligned}
&\mathcal{E}^n(u_n,u_n)\\
&\quad = \iint_{x\neq y} \big(u_n(x)-u_n(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad = \int_{\mathbb{R}^d}\left(\iint_{x\neq y} \big(u_n(x)-u_n(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\right)\rho_\epsilon(z)\,\mathrm{d}z\\
&\quad = \int_{\mathbb{R}^d}\left(\iint_{x\neq y} \big(u_n(x-z)-u_n(y-z)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\right)\rho_\epsilon(z)\,\mathrm{d}z,
\end{aligned}$$

and using the Fubini theorem and Jensen's inequality yields, for any compact set K so that $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$,

$$\begin{aligned}
&\mathcal{E}^n(u_n,u_n)\\
&\quad = \iint_{x\neq y}\left(\int_{\mathbb{R}^d} \big(u_n(x-z)-u_n(y-z)\big)^2 \rho_\epsilon(z)\,\mathrm{d}z\right) a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad \ge \iint_{x\neq y}\left(\int_{\mathbb{R}^d} \big(u_n(x-z)-u_n(y-z)\big) \rho_\epsilon(z)\,\mathrm{d}z\right)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad \ge \iint_{K} \big(J_\epsilon[u_n](x)-J_\epsilon[u_n](y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Note that $\sup_{n\in\mathbb{N}} \|u_n\|_{L^2} < \infty$ because of the weak convergence $u_n \rightharpoonup u$. Using again weak convergence $u_n \rightharpoonup u$, we conclude that $u_{n,\epsilon} = J_\epsilon[u_n]$ converges pointwise to $u_\epsilon := J_\epsilon[u]$. Using the local boundedness of ν on $\mathbb{R}^d \setminus \{0\}$ and the fact that K is a compact set satisfying $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$, we see that $\big(u_{n,\epsilon}(x) - u_{n,\epsilon}(y)\big)^2 \nu(y-x)$ converges in $L^q(K)$ to

$$\big(u_\epsilon(x)-u_\epsilon(y)\big)^2\nu(y-x) \quad \text{as } n \to \infty.$$

From (9) we get

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n,u_n) \ge \liminf_{n\to\infty} \mathcal{E}^n(u_{n,\epsilon},u_{n,\epsilon})$$

$$\geq \liminf_{n\to\infty} \iint_K \big(u_{n,\epsilon}(x) - u_{n,\epsilon}(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \overline{a} \iint_K \big(u_\epsilon(x) - u_\epsilon(y)\big)^2 \nu(y-x)\,\mathrm{d}y\,\mathrm{d}x.$$

Since $K \subset \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ is an arbitrary compact set, we can approximate $\mathbb{R}^d \times \mathbb{R}^d \setminus \{(x,x) : x \in \mathbb{R}^d\}$ by such sets. Using monotone convergence and the fact that the left hand side is independent of K, we arrive at

$$\sup_{0<\epsilon<1} \mathcal{E}(u_\epsilon, u_\epsilon) = \sup_{0<\epsilon<1} \sup_{\substack{K \text{ compact} \\ K\subset\mathbb{R}^d\times\mathbb{R}^d\setminus\{(x,x):x\in\mathbb{R}^d\}}} \overline{a} \iint_K \big(u_\epsilon(x) - u_\epsilon(y)\big)^2 \nu(y-x)\,\mathrm{d}y\,\mathrm{d}x$$
$$\leq \liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) < \infty. \tag{10}$$

Theorem 2.4 in [9] now shows that $u_\epsilon \in \mathcal{F} \cap C^\infty(\mathbb{R}^d)$ for each $\epsilon \in (0,1)$. Since J_ϵ is an L^2-contraction operator for each $\epsilon > 0$, we see that the family $\{u_\epsilon\}_{\epsilon>0}$, $u_\epsilon = J_\epsilon[u]$, is bounded w.r.t. $\mathcal{E}_1(\bullet,\bullet) := \mathcal{E}(\bullet,\bullet) + (\bullet,\bullet)_{L^2}$ by (10). The Banach–Alaoglu theorem guarantees that there is an $\mathcal{E}_1$-weakly convergent subsequence $u_{\epsilon(n)}$, $\epsilon(n) \downarrow 0$, and a function v so that $u_{\epsilon(n)}$ converges $\mathcal{E}_1$-weakly to $v \in \mathcal{F}$. Using the Banach–Saks theorem shows that the Cesàro means $\frac{1}{n}\sum_{k=1}^n u_{\epsilon(n_k)}$ of a further subsequence converge $\mathcal{E}_1$-strongly, hence in $L^2(\mathbb{R}^d)$, to v. As u_ϵ converges to u in $L^2(\mathbb{R}^d)$, we can identify the limit as $u = v$. In particular, $u \in \mathcal{F}$ and

$$\liminf_{n\to\infty} \mathcal{E}^n(u_n, u_n) \geq \mathcal{E}(u,u).$$

In order to see **(M2)**, we use the regularity of the Dirichlet form $(\mathcal{E},\mathcal{F})$; therefore, it is enough to consider $u \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$. Set $u_n = u \in C_0^{\mathrm{lip}}(\mathbb{R}^d)$ for each n, and $L := \operatorname{supp} u$ and $G := L + B_1(0)$. Because of the symmetry of the form we have

$$\mathcal{E}^n(u,u) = \mathcal{E}^n_{G\times G}(u,u) + 2\mathcal{E}^n_{G\times G^c}(u,u)$$

where, using the fact that $L = \operatorname{supp} u \subset G$,

$$\mathcal{E}^n_{G\times G}(u,u) = \iint_{G\times G} \big(u(x) - u(y)\big)^2 a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x,$$
$$\mathcal{E}^n_{G\times G^c}(u,u) = \iint_{L\times G^c} u^2(x) a_{\delta_n}(y-x)\nu(y-x)\,\mathrm{d}y\,\mathrm{d}x.$$

Using Corollary 3 we see that

$$\lim_{n\to\infty} \mathcal{E}^n_{G\times G}(u,u) = \overline{a} \iint_{G\times G} \big(u(x) - u(y)\big)^2 \nu(y-x)\,\mathrm{d}y\,\mathrm{d}x.$$

For the other part we get

$$\mathcal{E}^n_{G\times G^c}(u,u) = \int_{\mathbb{R}^d} \left[\int_L u^2(x)\mathbb{1}_{G^c}(x+h)\,\mathrm{d}x\right] \nu(h) a_{\delta_n}(h)\,\mathrm{d}h$$

$$\le \epsilon + \int_{|h|\le R} \left[\int_L u^2(x) \mathbb{1}_{G^c}(x+h)\,\mathrm{d}x \right] \nu(h) a_{\delta_n}(h)\,\mathrm{d}h$$

for any $\epsilon > 0$ and some suitable $R = R_\epsilon$; note that ϵ and R can be chosen independently of n. This is due to our assumption (5) and the fact that the expression in the square brackets is a continuous bounded function in h. Now we can use Lemma 2 for the limit $n \to \infty$; if we then let $R \to \infty$ and $\epsilon \to 0$, we get

$$\begin{aligned} \limsup_{n\to\infty} \mathcal{E}^n_{G\times G^c}(u,u) &\le \overline{a} \iint_{\mathbb{R}^d \times L} u^2(x) \mathbb{1}_{G^c}(x+h) \nu(h)\,\mathrm{d}x\,\mathrm{d}h \\ &= \overline{a} \iint_{L\times G^c} u^2(x)\,\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

Combining all of the above calculations, it follows that

$$\begin{aligned} \limsup_{n\to\infty} \mathcal{E}^n(u_n,u_n) &= \limsup_{n\to\infty} \Big(\mathcal{E}^n_{G\times G}(u,u) + 2\mathcal{E}^n_{G\times G^c}(u,u) \Big) \\ &\le \overline{a} \iint_{G\times G} \big(u(x)-u(y)\big)^2 \nu(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad + 2\overline{a} \iint_{L\times G^c} \big(u(x)-u(y)\big)^2 \nu(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \mathcal{E}(u,u), \end{aligned}$$

finishing the proof. □

Remark 5. Suppose that the function a on $\mathbb{R}$ satisfies (2)–(4), and ν is given by $\nu(x) = |x|^{-1-\alpha}$, $x \in \mathbb{R} \setminus \{0\}$, for some $0 < \alpha < 2$. Then the following quadratic form defines a translation invariant regular symmetric Dirichlet form on $L^2(\mathbb{R})$:

$$\tilde{\mathcal{E}}(u,v) := \iint\limits_{x\ne y} \big(u(x)-u(y)\big)\big(u(x)-u(y)\big) \frac{a(x-y)}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y, \quad u,v \in C_0^{\mathrm{lip}}(\mathbb{R}).$$

Let $\tilde{X} = (\tilde{X}(t))_{t\ge 0}$ be the symmetric Lévy process on $\mathbb{R}$ associated with the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ on $L^2(\mathbb{R})$. For any $n \in \mathbb{N}$, set

$$X^{(n)}(t) := \epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t), \quad t > 0.$$

Then $X^{(n)} = (X^{(n)}(t))_{t\ge 0}$ is also a symmetric Lévy process and we denote for each $n \in \mathbb{N}$ by $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ the corresponding Dirichlet form. The semigroup $\{T_t^{(n)}\}_{t>0}$ generated by $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ is given by

$$\begin{aligned} T_t^{(n)} f(x) &= \mathbb{E}\left[f(X^{(n)}(t)) \,\middle|\, X^{(n)}(0) = x \right] \\ &= \mathbb{E}_{x/\epsilon_n}\left[f(\epsilon_n \tilde{X}(\epsilon_n^{-\alpha} t)) \right] = \Big(\tilde{T}_{\epsilon_n^{-\alpha} t} f(\epsilon_n \cdot) \Big)(\epsilon_n^{-1} x), \quad x \in \mathbb{R}. \end{aligned}$$

Since the Dirichlet form $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ can be obtained by

$$\mathcal{E}^{(n)}(u,v) = \lim_{t\downarrow 0} \frac{1}{t}\big(u - T_t^{(n)}u, v\big)_{L^2},$$

it follows for $t > 0$ that

$$\begin{aligned}
\frac{1}{t}\big(u - T_t^{(n)}u, v\big)_{L^2} &= \frac{1}{t}\int_{\mathbb{R}} \big[u(x) - T_t^{(n)}u(x)\big]v(x)\,\mathrm{d}x \\
&= \frac{1}{t}\int_{\mathbb{R}} \big[u\big(\epsilon_n \cdot \epsilon_n^{-1}x\big) - \big(\tilde{T}_{\epsilon_n^{-\alpha}t}u(\epsilon_n\bullet)\big)\big(\epsilon_n^{-1}x\big)\big]v(x)\,\mathrm{d}x \\
&= \frac{1}{\epsilon_n^{\alpha}}\cdot\frac{1}{s}\int_{\mathbb{R}} \big[u(\epsilon_n\xi) - \big(\tilde{T}_s u(\epsilon_n\bullet)\big)(\xi)\big]v(\epsilon_n\xi)\epsilon_n\,\mathrm{d}\xi
\end{aligned}$$

where we use the notation $\xi = \epsilon_n^{-1}x$ and $s = \epsilon_n^{-\alpha}t$. Letting $s \to 0$, hence $t \to 0$, yields

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\big(u - T_t^{(n)}u, v\big)_{L^2} \\
&= \epsilon_n^{1-\alpha}\cdot\tilde{\mathcal{E}}\big(u(\epsilon_n\bullet), v(\epsilon_n\bullet)\big) \\
&= \epsilon_n^{1-\alpha}\iint_{x\neq y}\big(u(\epsilon_n x) - u(\epsilon_n y)\big)\big(v(\epsilon_n x) - v(\epsilon_n y)\big)\frac{a(x-y)}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \epsilon_n^{1-\alpha}\iint_{x\neq y}\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\frac{a\big(\epsilon_n^{-1}(x-y)\big)}{|x-y|^{1+\alpha}}\epsilon_n^{1+\alpha}\frac{\mathrm{d}x}{\epsilon_n}\frac{\mathrm{d}y}{\epsilon_n} \\
&= \iint_{x\neq y}\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\frac{a\big(\epsilon_n^{-1}(x-y)\big)}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \mathcal{E}^{(n)}(u,v).
\end{aligned}$$

Since Mosco convergence entails the convergence of the semigroups, hence the finite-dimensional distributions of the processes, we may combine the above calculation with Theorem 4 to get the following result: *The processes $X^{(n)}$ associated with $(\mathcal{E}^{(n)}, \mathcal{F}^{(n)})$ - these are obtained by scaling $t \mapsto \epsilon_n^{-\alpha}t$ and $x \mapsto \epsilon_n x$ from the process $\tilde{X}$ given by $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ - converge, in the sense of finite-dimensional distributions, to the process X associated with $(\mathcal{E}, \mathcal{F})$.*

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Technische Universität Dresden
Faculty of Mathematics
Institute of Stochastics,
01062 Dresden, Germany
rene.schilling@tu-dresden.de

Kansai University
Faculty of Engineering Science
Department of Mathematics
Suita-shi, Osaka 564-8680, Japan
t-uemura@kansai-u.ac.jp