Dedicated to the memory of Professor Nicu Boboc

FARADAY CAGE PRINCIPLE FOR CONVOLUTION CAPACITY AND DICHOTOMY OF RIESZ CAPACITY DENSITY

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Let C_k be the convolution capacity with respect to a radial kernel k in $\mathbb{R}^n$. Let $S, T \subset \mathbb{R}^n$. Under a mild condition on k, we show that a weak inequality

$$C_k(S \cap B) \geq \kappa C_k(T \cap B) \quad \text{for all balls } B \text{ with } \kappa \in (0,1) \text{ independent of } B$$

actually implies a stronger inequality

$$C_k(S \cap \Omega) \geq C_k(T \cap \Omega) \quad \text{for all bounded open sets } \Omega.$$

This generalizes the Faraday cage principle for Newtonian capacity proved by Choquet. Our technique also gives the dichotomy of the lower Riesz capacity density of general order.

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1. INTRODUCTION

In analysis we often encounter situations where an a priori uniform weak estimate actually yield a stronger estimate. In 1975, Choquet [6] found such a phenomenon for Newtonian capacity C_2 in $\mathbb{R}^3$.

Theorem A. Let $S, T \subset \mathbb{R}^3$. If there exists a constant $\kappa \in (0,1)$ such that

$$C_2(S \cap Q) \geq \kappa C_2(T \cap Q) \quad \text{for all cubes } Q \text{ in } \mathbb{R}^3,$$

then

$$C_2(S \cap \Omega) \geq C_2(T \cap \Omega) \quad \text{for all bounded open sets } \Omega.$$

For the reminiscence to a physical experiment, he called this observation *"des cages de Faraday grillagées"* or *the grounded Faraday cage*. The main purpose of this note is to show a similar result for a general convolution capacity. Our method has an application to the dichotomy of the lower Riesz capacity density of general order.

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Let μ be a nonnegative Radon measure on $\mathbb{R}^n$, $n \geq 2$, i.e., μ is a Borel measure such that $\mu(K) < \infty$ for every compact set K. The support of μ is denoted by $\operatorname{supp}\mu$. Let $\|\mu\|$ be the total mass of μ. We say that μ is a finite Radon measure if $\|\mu\| < \infty$. Obviously, a Radon measure with compact support is finite. Throughout this paper we let $k(t)$ be a positive nonincreasing continuous function for $t > 0$ such that

$$\lim_{t\downarrow 0} k(t) = \infty, \quad \lim_{t\uparrow\infty} k(t) = 0, \quad \int_0 k(t)t^{n-1}dt < \infty.$$

With a slight abuse of notation, we also write $k(x) = k(|x|)$ for $x \in \mathbb{R}^n$. By assumption $k(x)$ is locally integrable in $\mathbb{R}^n$. We define the convolution capacity C_k with respect to k as

$$C_k(E) = \sup\{\|\mu\| : \operatorname{supp}\mu \subset E,\ k * \mu \leq 1 \text{ in } \mathbb{R}^n\}. \tag{1.1}$$

The capacity has the dual definition

$$\inf\{\|\mu\| : k * \mu \geq 1 \text{ on } E\}, \tag{1.2}$$

which is equal to $C_k(E)$ for analytic sets E (see [7]). By $B(x,r)$ we denote the open ball with center at x and radius r.

Theorem 1.1. *Assume that the kernel k satisfies*

$$\lim_{\eta\uparrow 1}\Big(\sup_{0<t<t_0} \frac{k(\eta t)}{k(t)}\Big) = 1 \tag{1.3}$$

for some $0 < t_0 \leq \infty$. Suppose $S, T \subset \mathbb{R}^n$. If there exist $1 \leq \tau < \infty$, $r_0 > 0$ and $0 < \kappa < 1$ such that

$$C_k(S \cap B(x,\tau r)) \geq \kappa C_k(T \cap B(x,r)) \quad \textit{for all } r \in (0, r_0] \textit{ and } x \in \mathbb{R}^n, \tag{1.4}$$

then, we actually obtain a stronger inequality

$$C_k(S \cap \Omega) \geq C_k(T \cap \Omega) \quad \textit{for all bounded open sets } \Omega. \tag{1.5}$$

By $Q(x,r)$ we denote the open cube with center x and sides of length $2r$ parallel to the coordinate axes in $\mathbb{R}^n$, i.e., $Q(x,r) = \{y = (y^1,\dots,y^n) : |x^i - y^i| < r \text{ for } i = 1,\dots,n\}$. Observe that

$$B(x,r) \subset Q(x,r) \subset B(x, r\sqrt{n}).$$

Hence (1.4) is equivalent to

$$C_k(S \cap Q(x,\tau r)) \geq \kappa C_k(T \cap Q(x,r)) \quad \text{for all cubes } Q(x,r) \text{ with } 0 < r \leq r_0,$$

where τ, r_0 and κ are different constants. Thus Theorem 1.1 extends Theorem A. Following Choquet, let us call the implication (1.4) $\Longrightarrow$ (1.5) the *Faraday cage principle*.

Typical examples of radial kernels k are the Riesz kernel $I_\alpha(x) = \gamma_\alpha |x|^{\alpha-n}$ with normalizing constant $\gamma_\alpha = \Gamma((n-\alpha)/2)/(\pi^{n/2}2^\alpha\Gamma(\alpha/2))$ for $0 < \alpha < n$ and the Bessel kernel G_α for $0 < \alpha \le n$. These kernels enjoy

$$I_\alpha = (-\Delta)^{-\alpha/2} \quad \text{and} \quad G_\alpha = (I-\Delta)^{-\alpha/2} \quad \text{in the distribution sense,}$$

and

$$\lim_{|x|\to 0} \frac{G_\alpha(x)}{I_\alpha(x)} = 1 \quad \text{for } 0 < \alpha < n, \quad G_\alpha(x) = O(e^{-c|x|}) \quad \text{as } |x| \to \infty$$

with some $c > 0$. See [1, pp.8–13] for these accounts. The capacities $C_{I_\alpha}(E)$ and $C_{G_\alpha}(E)$ are referred to as the Riesz capacity and the Bessel capacity, respectively. We write $C_\alpha(E)$ for $C_{I_\alpha}(E)$. If $\alpha = 2 < n$, then $C_2(E)$ is the Newtonian capacity of E up to a multiplicative constant. By definition $I_\alpha(\eta t)/I_\alpha(t) = \eta^{\alpha-n}$ so that $k = I_\alpha$ satisfies (1.3) with $t_0 = \infty$. It is easy to see that the Bessel kernel $G_\alpha(t)$ satisfies (1.3) with finite t_0. We cannot take $t_0 = \infty$ for the Bessel kernel, since the Bessel kernel decays exponentially fast at infinity.

Corollary 1.2. *The Faraday cage principle holds both for the Riesz capacity of order α with $0 < \alpha < n$, and for the Bessel capacity of order α with $0 < \alpha \le n$.*

Remark 1.3. The Faraday cage principle was proved for Bessel (α, p)-capacity with $1 < p < \infty$ and $0 < \alpha p \le n$ with the aid of nonlinear fine topology ([1, Theorem 11.4.2]). For a general kernel k, however, we have no fine topology theory. Theorem 1.1 suggests that the Faraday cage principle may still hold for L^p-capacity of a general kernel.

Theorem 1.1 has a strong relation to the dichotomy of capacity density exploited in [2], [3] and [4]. Let φ be an outer measure such that $0 < \varphi(U) < \infty$ for every bounded open set U. For $E \subset \mathbb{R}^n$ and $r > 0$ we define

$$\mathcal{D}(\varphi, E, r) = \inf_{x\in\mathbb{R}^n} \frac{\varphi(E \cap B(x,r))}{\varphi(B(x,r))}.$$

More generally, for a bounded open set Ω, define

$$\mathcal{D}_\Omega(\varphi, E, r) = \inf_{x\in\mathbb{R}^n} \frac{\varphi(E \cap \Omega(x,r))}{\varphi(\Omega(x,r))},$$

where $\Omega(x,r) = x + r\Omega = \{x + ry : y \in \Omega\}$. This notation is consistent with $B(x,r)$ and $Q(x,r)$. With the aid of the idea of the proof of Theorem 1.1, we show the following dichotomy of the lower Riesz capacity density.

Theorem 1.4. *Let $0 < \alpha < n$ and let Ω be a bounded open set. Then we have the dichotomy for the lower density of the Riesz capacity C_α, i.e., $\lim_{r\to\infty} \mathcal{D}_\Omega(C_\alpha, E, r)$ is either 0 or 1; the first case occurs if and only if $\mathcal{D}_\Omega(C_\alpha, E, r)$ is identically equal to 0 for all $r > 0$. The dichotomy is independent of Ω, i.e., $\lim_{r\to\infty} \mathcal{D}_\Omega(C_\alpha, E, r) = \lim_{r\to\infty} \mathcal{D}(C_\alpha, E, r)$.*

Remark 1.5. The case $\alpha = 2 < n$ is classical. In case $0 < \alpha < 2$, the previous paper [2] gave the same result under the additional assumption that Ω satisfies the interior corkscrew condition. Bogdan's probabilistic estimates for α-harmonic measure ([5]) played an important role. In case $2 < \alpha < n$, however, α-harmonic measure is unavailable. The lack of Frostman's maximum principle (see Remark 2.4) is a serious problem. The proof of Theorem 1.4 is completely different from [2]. This paper employs the capacity definition (1.1), whereas [2] does (1.2). The present method enables us to dispense with the interior corkscrew condition on Ω.

Our proof of Theorem 1.4 depends on the homogeneity of the Riesz kernel, i.e., $I_\alpha(rx) = r^{\alpha-n} I_\alpha(x)$. The Bessel kernel is inhomogeneous and integrable over $\mathbb{R}^n$. From this fact we see that the Bessel capacity density has no dichotomy. More generally we have the following proposition. See [3, Example 7.2] for variational capacity in the metric measure setting.

Proposition 1.6. *If the convolution kernel k is integrable over $\mathbb{R}^n$, then the capacity density with respect to C_k has no dichotomy, i.e., for every bounded open set Ω, there exists a Borel set $E \subset \mathbb{R}^n$ such that*

$$0 < \liminf_{r\to\infty} \mathcal{D}_\Omega(C_k, E, r) \le \limsup_{r\to\infty} \mathcal{D}_\Omega(C_k, E, r) < 1. \tag{1.6}$$

2. PROOF OF THEOREM 1.1

Let μ and ν be Radon measures. We write $\nu \le \mu$ if $\mu = \nu + \lambda$ with another Radon measure λ. The restriction of μ on a Borel set A is defined by

$$\mu|_A(E) = \mu(E \cap A) \quad \text{for Borel sets } E.$$

Obviously, $\mu|_A \le \mu$. We recall some basic properties of convolution potentials.

Lemma 2.1. *Let μ and ν be Radon measures such that $\nu \le \mu$ and $k * \mu < \infty$ in $\mathbb{R}^n$. Let E be a Borel set. If $(k * \mu)|_E$ is continuous, then so is $(k * \nu)|_E$. In particular, if $k * \mu$ is continuous in $\mathbb{R}^n$, then so is $k * \nu$.*

Proof. Write

$$(k * \nu)|_E = (k * \mu)|_E - (k * (\mu - \nu))|_E.$$

Since $k * \nu$ and $k * (\mu - \nu)$ are lower semicontinuous in $\mathbb{R}^n$, it follows that $(k * \nu)|_E$ is lower semicontinuous and that $(k * \mu)|_E - (k * (\mu - \nu))|_E$ is upper semicontinuous, as $(k * \mu)|_E$ is continuous by assumption. Hence $(k * \nu)|_E$ is continuous. □

Our proof of Theorem 1.1 is based on the following

Lemma 2.2 (Continuity Principle [8, Theorem 1.7]). *Let μ be a Radon measure with compact support. If $(k * \mu)|_{\operatorname{supp}\mu}$ is continuous, then $k * \mu$ is continuous in $\mathbb{R}^n$.*

Landkof gives a proof of the continuity principle only for the Riesz kernel; it can be easily generalized with the aid of the boundedness principle or weak maximum principle.

LEMMA 2.3 (Boundedness Principle [1, Theorem 2.6.2]). *There exists an absolute constant M_0 depending only on the dimension n such that if μ is a Radon measure, then*

$$k * \mu \le M_0 \sup_{\operatorname{supp}\mu} k * \mu \quad \text{in } \mathbb{R}^n.$$

Remark 2.4. If k is the Riesz kernel I_α with $0 < \alpha \le 2$ and $\alpha < n$, then M_0 in Lemma 2.3 can be taken as 1. This is known as Frostman's maximum principle.

Let us give more specific lemmas. For a Radon measure μ we denote the restriction of μ on $B(x, r)$ by $\mu_{x,r}$, i.e., $\mu_{x,r}(E) = \mu(E \cap B(x, r))$ for Borel sets E.

LEMMA 2.5. *Let μ be a Radon measure with compact support. If $k * \mu$ is continuous and $k * \mu \le 1$ in $\mathbb{R}^n$, then for each $\eta > 0$, there exists $r > 0$ such that $k * \mu_{x,r} \le \eta$ in $\mathbb{R}^n$ uniformly for $x \in \mathbb{R}^n$.*

Proof. We may assume that $\operatorname{supp}\mu \subset B(0, M/2)$ for some $M > 2$. Let $\eta > 0$. By making M large, we may assume that $k * \mu(y) \le \eta$ for $|y| \ge M$ by the standing assumption $\lim_{t\uparrow\infty} k(t) = 0$. It is sufficient to find $0 < r < 1$ such that $k * \mu_{x,r}(y) \le \eta$ for $|x| \le M$ and $|y| \le M$. Let $k_t(x) = \min\{k(x), t\}$. Then k_t is continuous in $\mathbb{R}^n$, and so is $k_t * \mu$. By the monotone convergence theorem $k_t * \mu(y) \uparrow k * \mu(y)$ for each $y \in \mathbb{R}^n$; and by Dini's theorem the convergence is uniform for $|y| \le M$. Hence we can choose $t > 0$ such that $(k - k_t) * \mu(y) \le \eta/2$ for $|y| \le M$. We have

$$k * \mu_{x,r}(y) = k_t * \mu_{x,r}(y) + (k - k_t) * \mu_{x,r}(y) \le t\mu(\overline{B}(x, r)) + \eta/2 \quad \text{for } |y| \le M. \tag{2.1}$$

Observe that $x \mapsto \mu(\overline{B}(x, r))$ is an upper semicontinuous function for each $r > 0$. Note that μ is non-atomic, i.e., $\mu(\{x\}) = 0$ for every $x \in \mathbb{R}^n$ by $k * \mu \le 1$ and by the standing assumption $\lim_{t\downarrow 0} k(t) = \infty$. Hence, for every x fixed, $\mu(\overline{B}(x, r)) \downarrow 0$ as r decreases to 0. By using a slightly generalized version of Dini's theorem (continuity is replaced by upper semicontinuity), the convergence is uniform for $|x| \le M$, so that there exists $0 < r < 1$ such that $\mu(\overline{B}(x, r)) \le \eta/(2t)$ for $|x| \le M$. Plugging in this inequality to (2.1), we obtain $k * \mu_{x,r}(y) \le \eta$ for $|x| \le M$ and $|y| \le M$, as required. □

LEMMA 2.6. *Let μ be a finite Radon measure such that $k * \mu < \infty$ in $\mathbb{R}^n$. Then, for each $\varepsilon > 0$, there exists a compact subset K of* $\operatorname{supp}\mu$ *such that $\mu(K) > \|\mu\| - \varepsilon$ and $k * (\mu|_K)$ is continuous in $\mathbb{R}^n$.*

Proof. By taking the restriction of μ on a compact set, we may assume that μ has a compact support. By Lusin's theorem we find a compact subset K of $\operatorname{supp}\mu$ such that $(k * \mu)|_K$ is continuous and $\mu(K) > \|\mu\| - \varepsilon$. By Lemma 2.1 $(k * (\mu|_K))|_K$ is continuous, and hence $k * (\mu|_K)$ is continuous in $\mathbb{R}^n$ by Lemma 2.2. □

Lemma 2.7. *Assume that k satisfies* (1.3). *Let $0 < R < \infty$ and $\varepsilon > 0$. Then there exists $M_1 > 1$ depending only on R, ε, k and n with the following property: if $M \ge M_1$, $0 < r < R/M$ and μ and ν are Radon measures supported in $B(x,r)$ with the same total mass $\|\mu\| = \|\nu\| > 0$, then*

$$k * \nu \le (1+\varepsilon) k * \mu \quad \text{in } B(x,R) \setminus B(x,Mr).$$

Proof. First we claim that (1.3) can be replaced by

$$\lim_{\eta \uparrow 1} \Big(\sup_{0<t\le t_1} \frac{k(\eta t)}{k(t)} \Big) = 1$$

with any finite positive number t_1. In fact, let $t_0 < t_1 < \infty$. Observe that the continuous function $k(\eta t)/k(t)$ of $t \in [t_0, t_1]$ decreases to 1 pointwise as $\eta \uparrow 1$. The convergence is uniform by Dini's theorem, so that the claim follows. The claim yields $M_1 > 1$ depending only on R, ε and k such that

$$\frac{k(t - t/M_1)}{k(t + t/M_1)} \le 1 + \varepsilon \quad \text{for } 0 < t \le R. \tag{2.2}$$

Now let $M \ge M_1$, $0 < r < R/M$ and $x \in \mathbb{R}^n$. Suppose μ and ν are Radon measures supported in $B(x,r)$ with the same total mass $\|\mu\| = \|\nu\| > 0$. Take $y \in B(x,R) \setminus B(x,Mr)$. We find $z_1, z_2 \in \overline{B}(x,r)$ such that

$$|y - z_1| = \inf_{z \in B(x,r)} |y - z| \le \sup_{z \in B(x,r)} |y - z| = |y - z_2|.$$

Let $t = |y - x|$. Then $M_1 r \le Mr \le t < R$, so that

$$\frac{k * \nu(y)}{k * \mu(y)} \le \frac{k(|y - z_1|)\,\|\nu\|}{k(|y - z_2|)\,\|\mu\|} = \frac{k(t - r)}{k(t + r)} \le \frac{k(t - t/M_1)}{k(t + t/M_1)}.$$

Hence the required inequality follows from (2.2). □

Proof of Theorem 1.1. Take an arbitrary bounded open set $\Omega \ne \emptyset$. By translation we may assume that $0 \in \Omega$. Choose $R > 4\,\mathrm{diam}(\Omega)$ such that $k(R/4) \le 1/C_k(\Omega)$. Then

$$k * \nu \le 1 \quad \text{in } \mathbb{R}^n \setminus B(0, R/2), \tag{2.3}$$

whenever $\operatorname{supp} \nu \subset \Omega$, and $\|\nu\| \le C_k(\Omega)$.

Now, let us prove (1.5). Without loss of generality we may assume that $C_k(T \cap \Omega) > 0$. Let $0 < \varepsilon < C_k(T \cap \Omega)$. By definition we find a finite Radon measure ξ supported in $T \cap \Omega$ such that $\|\xi\| \ge C_k(T \cap \Omega) - \varepsilon/2$ and $k * \xi \le 1$ in $\mathbb{R}^n$. Lemma 2.6 yields a compact subset K of $T \cap \Omega$ such that $\mu = \xi|_K$ satisfies

$$\begin{aligned} &C_k(T \cap \Omega) - \varepsilon < \|\mu\| \le C_k(T \cap \Omega), \\ &k * \mu \le 1 \text{ in } \mathbb{R}^n, \end{aligned} \tag{2.4}$$

and $k * \mu$ is continuous in $\mathbb{R}^n$.

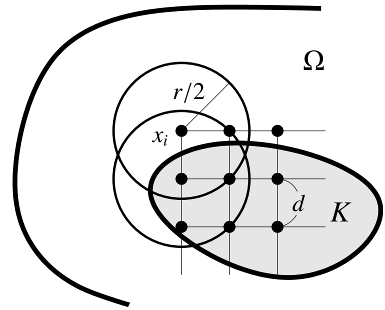

Figure 1: $K \subset \bigcup_{i=1}^N B(x_i, r/2) \subset \bigcup_{i=1}^N \tau B_i \subset \Omega$.

Let $M_1 > 1$ be as in Lemma 2.7. Choose $M \geq M_1$ such that

$$\frac{(4\sqrt{n}(1 + M\tau))^n}{M^{n+1}\kappa} \leq \varepsilon. \tag{2.5}$$

For simplicity let $\eta = 1/M^{n+1}$. By Lemma 2.5 we can choose $r > 0$ so small that

$$\begin{aligned} &0 < r < \min\left\{r_0, \frac{R}{M\tau}, \frac{\operatorname{dist}(\Omega^c, K)}{2\tau}\right\}, \\ &k * \mu_{x,r} \leq \eta \text{ in } \mathbb{R}^n \text{ for every } x \in \mathbb{R}^n, \end{aligned} \tag{2.6}$$

where $r_0 > 0$ is as in Theorem 1.1.

Let $d = r/(2\sqrt{n})$ and cover $\mathbb{R}^n$ by $\{B(z, r/2)\}_{z\in(d\mathbb{Z})^n}$. By compactness we find finitely many points $x_1, \ldots, x_N \in (d\mathbb{Z})^n$ such that $K \subset \bigcup_{i=1}^N B(x_i, r/2)$. See Figure 1. Define Radon measures $\mu_1, \ldots, \mu_N$ by $\mu_1(E) = \mu(E \cap B(x_1, r/2))$ and, for $2 \leq j \leq N$,

$$\mu_j(E) = \mu\Big(E \cap B(x_j, r/2) \setminus \bigcup_{i=1}^{j-1} B(x_i, r/2)\Big) \quad \text{for Borel sets } E.$$

Then $\mu = \sum_{i=1}^N \mu_i$ and $\operatorname{supp}\mu_i \subset K \cap \overline{B}(x_i, r/2) \subset K \cap B(x_i, r)$ for each $1 \leq i \leq N$. Without loss of generality we may assume that $\mu_i \neq 0$, and hence $K \cap B(x_i, r) \neq \emptyset$ for every i. For simplicity write $B_i = B(x_i, r)$ and $\tau B_i = B(x_i, \tau r)$. Observe from (2.6) that $\tau B_i \subset \Omega$, and that $k * \mu_i \leq \eta$ in $\mathbb{R}^n$. Hence $C_k(K \cap B_i) \geq \|\mu_i\|/\eta$ by definition. Therefore (1.4) yields

$$C_k(S \cap \tau B_i) \geq \kappa C_k(T \cap B_i) \geq \frac{\kappa}{\eta}\|\mu_i\| > 0.$$

By definition we find a Radon measure λ_i supported in $S \cap \tau B_i$ such that

$$\begin{aligned} &k * \lambda_i \leq 1 \quad \text{in } \mathbb{R}^n, \\ &\|\lambda_i\| \geq C_k(S \cap \tau B_i) - \frac{\kappa}{2\eta}\|\mu_i\| \geq \frac{\kappa}{2\eta}\|\mu_i\|. \end{aligned} \tag{2.7}$$

Let

$$\nu_i = \frac{\|\mu_i\|}{\|\lambda_i\|}\lambda_i.$$

Then $\|\nu_i\| = \|\mu_i\|$ and $\operatorname{supp}\nu_i \subset S \cap \tau B_i \subset \Omega$. Moreover, by (2.7),

$$k * \nu_i = \frac{\|\mu_i\|}{\|\lambda_i\|} k * \lambda_i \le \frac{2\eta}{\kappa} \quad \text{in } \mathbb{R}^n. \tag{2.8}$$

Let $\nu = \sum_{i=1}^N \nu_i$. It follows from (2.4) that $\|\nu\| = \|\mu\| \le C_k(T \cap \Omega) \le C_k(\Omega)$, and that $\operatorname{supp}\nu \subset S \cap \Omega$.

Let us estimate the upper bound of $k * \nu$ in $\mathbb{R}^n$. Since $k * \nu \le 1$ in $\mathbb{R}^n \setminus B(0, R/2)$ by (2.3), it is sufficient to estimate $k * \nu(x)$ for $x \in B(0, R/2)$. For each $k * \nu_i$ we have

$$k * \nu_i \le (1 + \varepsilon) k * \mu_i \quad \text{in } B(x_i, R) \setminus B(x_i, M\tau r)$$

by (2.6) and Lemma 2.7 with τr in place of r. Since $0 \in \Omega$ and $B(x_i, r) \subset \Omega$, we have $|x_i| \le \operatorname{diam}(\Omega) < R/4$, so that $B(x_i, R) \supset B(0, R - |x_i|) \supset B(0, R/2)$. Hence

$$k * \nu_i \le (1 + \varepsilon) k * \mu_i \quad \text{in } B(0, R/2) \setminus B(x_i, M\tau r). \tag{2.9}$$

We claim that the multiplicity of $\{B(x_i, M\tau r)\}$ is bounded by

$$\frac{((1 + M\tau)r)^n}{(r/(4\sqrt{n}))^n} = (4\sqrt{n}(1 + M\tau))^n. \tag{2.10}$$

In fact, if $y \in B(x_i, M\tau r)$, then $|y - x_i| < M\tau r$, so that $B(y, (1 + M\tau)r) \supset B(x_i, r) \supset B(x_i, d/2)$ with $d = r/(2\sqrt{n})$. Observe that $\{B(z, d/2)\}_{z \in (d\mathbb{Z})^n}$ are mutually disjoint, and so are $\{B(x_i, d/2)\}_{i=1}^N$. Hence we have

$$\sum_{i: y \in B(x_i, M\tau r)} |B(x_i, d/2)| \le |B(y, (1 + M\tau)r)|,$$

so that the multiplicity of $\{B(x_i, M\tau r)\}$ is bounded by (2.10).

Now let $x \in B(0, R/2)$. Combining (2.4), (2.5), (2.8), (2.9), and (2.10) altogether, we obtain

$$\begin{aligned} k * \nu(x) &= \sum_{i: x \in B(x_i, M\tau r)} k * \nu_i(x) + \sum_{i: x \notin B(x_i, M\tau r)} k * \nu_i(x) \\ &\le (4\sqrt{n}(1 + M\tau))^n \frac{2\eta}{\kappa} + (1 + \varepsilon) \sum_{i: x \notin M\tau B_i} k * \mu_i(x) \\ &\le 2\varepsilon + (1 + \varepsilon) k * \mu(x) \le 1 + 3\varepsilon. \end{aligned}$$

This, together with (2.3), implies that $k * \nu \le 1 + 3\varepsilon$ in $\mathbb{R}^n$. Since $\operatorname{supp}\nu \subset S \cap \Omega$, we have

$$C_k(S \cap \Omega) \ge \frac{\|\nu\|}{1 + 3\varepsilon} = \frac{\|\mu\|}{1 + 3\varepsilon} \ge \frac{C_k(T \cap \Omega) - \varepsilon}{1 + 3\varepsilon}$$

by definition and by (2.4). Since $\varepsilon > 0$ is arbitrary, we obtain the required inequality.
□

3. PROOF OF THEOREM 1.4

Throughout this section we let $0 < \alpha < n$. The homogeneity of the Riesz kernel plays a crucial role. We present two easy lemmas. For the completeness we provide their proofs.

LEMMA 3.1. *Let $x \in \mathbb{R}^n$ and $R > 0$. Suppose μ is a Radon measure supported in E. Define the Radon measure $\tilde{\mu}$ by*

$$\tilde{\mu}(A) = R^{n-\alpha}\mu(\tfrac{1}{R}(-x + A)) \quad \text{for } A \subset \mathbb{R}^n. \tag{3.1}$$

Then $\tilde{\mu}$ is supported in $E(x, R)$ and

$$I_\alpha\mu(y) = I_\alpha\tilde{\mu}(\tilde{y}) \quad \text{for every } y \in \mathbb{R}^n \text{ with } \tilde{y} = x + Ry.$$

Proof. We see that $\tilde{\mu}$ is supported in $E(x, R) = x+RE$ since $\frac{1}{R}(-x+(x+RE)) = E$. Let $\tilde{y} = x + Ry$ and $\tilde{z} = x + Rz$. Then (3.1) yields

$$\begin{aligned} I_\alpha\tilde{\mu}(\tilde{y}) &= \int_{E(x,R)} |\tilde{y} - \tilde{z}|^{\alpha-n} d\tilde{\mu}(\tilde{z}) \\ &= \int_E |x + Ry - (x + Rz)|^{\alpha-n} R^{n-\alpha} d\mu(z) \\ &= \int_E |y - z|^{\alpha-n} d\mu(z) = I_\alpha\mu(y), \end{aligned}$$

as required. □

This lemma readily gives the following homogeneity of the Riesz capacity.

LEMMA 3.2. *Let $x \in \mathbb{R}^n$ and $R > 0$. Then $C_\alpha(E(x, R)) = R^{n-\alpha}C_\alpha(E)$ for every Borel set $E \subset \mathbb{R}^n$.*

Proof. Let μ be a Radon measure supported in E such that $I_\alpha * \mu \le 1$ in $\mathbb{R}^n$. Define the Radon measure $\tilde{\mu}$ by (3.1). Lemma 3.1 yields that $\tilde{\mu}$ is supported in $E(x, R) = x + RE$ and $I_\alpha * \tilde{\mu} \le 1$ in $\mathbb{R}^n$. Hence $C_\alpha(E(x, R)) \ge \|\tilde{\mu}\| = R^{n-\alpha}\|\mu\|$ by definition. Taking the supremum with respect to μ, we obtain $C_\alpha(E(x, R)) \ge R^{n-\alpha}C_\alpha(E)$. Exchanging the roles of E and $E(x, R)$, we obtain the opposite inequality. □

We see that $\mathcal{D}_\Omega(C_\alpha, E, r)$ and $\mathcal{D}(C_\alpha, E, r)$ are comparable.

LEMMA 3.3. *Let Ω be a bounded open set. Then there exist constants $a_1, a_2 > 0$ and $A \ge 1$ depending only on Ω, α and n such that*

$$A^{-1}\mathcal{D}(C_\alpha, E, a_1r) \le \mathcal{D}_\Omega(C_\alpha, E, r) \le A\mathcal{D}(C_\alpha, E, a_2r)$$

for every $E \subset \mathbb{R}^n$ and $r > 0$.

Proof. By translation we may assume that $0 \in \Omega$. Let $a_1 = \operatorname{dist}(0, \partial\Omega)$ and $a_2 = \operatorname{diam}(\Omega)$. Then $B(x, a_1 r) \subset \Omega(x, r) \subset B(x, a_2 r)$, and so

$$r^{n-\alpha} C_\alpha(B(0, a_1)) \le C_\alpha(\Omega(x, r)) \le r^{n-\alpha} C_\alpha(B(0, a_2)). \tag{3.2}$$

Hence

$$\frac{C_\alpha(E \cap \Omega(x, r))}{C_\alpha(\Omega(x, r))} \le \frac{C_\alpha(E \cap B(x, a_2 r))}{C_\alpha(B(x, a_1 r))} = \Big(\frac{a_2}{a_1}\Big)^{n-\alpha} \frac{C_\alpha(E \cap B(x, a_2 r))}{C_\alpha(B(x, a_2 r))}.$$

Similarly,

$$\frac{C_\alpha(E \cap \Omega(x, r))}{C_\alpha(\Omega(x, r))} \ge \frac{C_\alpha(E \cap B(x, a_1 r))}{C_\alpha(B(x, a_2 r))} = \Big(\frac{a_1}{a_2}\Big)^{n-\alpha} \frac{C_\alpha(E \cap B(x, a_1 r))}{C_\alpha(B(x, a_1 r))}.$$

Taking the infima of the above inequalities with respect to $x \in \mathbb{R}^n$, we obtain the required estimates with $A = (a_2/a_1)^{n-\alpha}$. □

Proof of Theorem 1.4. For the first assertion of the theorem, it is sufficient to show that if $\mathcal{D}_\Omega(C_\alpha, E, r) > 0$ for some $r > 0$, then $\lim_{r\to\infty} \mathcal{D}_\Omega(C_\alpha, E, r) = 1$. In view of Lemma 3.3, it suffices to show that if there exist $\rho > 0$ and $\kappa > 0$ such that

$$C_\alpha(E \cap B(x, \rho)) \ge \kappa C_\alpha(B(x, \rho)) \quad \text{for every } x \in \mathbb{R}^n, \tag{3.3}$$

then $\lim_{R\to\infty} \mathcal{D}_\Omega(C_\alpha, E, R) = 1$. This also implies that the dichotomy is independent of a bounded open set Ω. Hereafter, we assume that (3.3) holds. Without loss of generality we may assume that $0 \in \Omega$.

Let $\varepsilon > 0$ and choose $M > 1$ such that

$$\Big(\frac{M-1}{M+1}\Big)^{\alpha-n} \le 1 + \varepsilon, \tag{3.4}$$

$$\frac{(4\sqrt{n}(1+M))^n}{M^{n+1}\kappa} \le \varepsilon. \tag{3.5}$$

Let $\eta = 1/M^{n+1}$. We find a compact set $K \subset \Omega$ and a measure μ supported in K such that $\|\mu\| \ge C_\alpha(\Omega) - \varepsilon$ and $I_\alpha * \mu \le 1$ in $\mathbb{R}^n$. By Lemma 2.6 we may assume that $I_\alpha * \mu$ is continuous in $\mathbb{R}^n$. So, Lemma 2.5 yields $r > 0$ such that $I_\alpha * \mu_{w,r} \le \eta$ in $\mathbb{R}^n$ for every $w \in \mathbb{R}^n$. Let

$$R > \max\Big\{\frac{\rho}{r}, \frac{2\rho}{\operatorname{dist}(K, \partial\Omega)}\Big\} \tag{3.6}$$

and define the measure $\tilde{\mu}$ as in (3.1) from μ with $x \in \mathbb{R}^n$ and this R. Lemma 3.1 shows that $\tilde{\mu}$ is supported in $K(x, R)$, and that

$$I_\alpha * \tilde{\mu} \le 1 \quad \text{in } \mathbb{R}^n. \tag{3.7}$$

By (3.6) we have $\rho \le Rr$, so that by Lemma 3.1

$$I_\alpha * \tilde{\mu}_{\tilde{w},\rho}(\tilde{y}) \le I_\alpha * \tilde{\mu}_{\tilde{w},Rr}(\tilde{y}) = I_\alpha * \mu_{w,r}(y) \le \eta \quad \text{for every } y, w \in \mathbb{R}^n, \tag{3.8}$$

where $\tilde{y} = x + Ry$ and $\tilde{w} = x + Rw$.

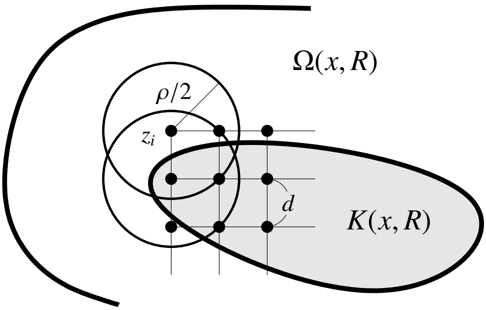

Figure 2: $K(x,R) \subset \bigcup_{i=1}^N B(z_i,\rho/2) \subset \bigcup_{i=1}^N \tilde{B}_i \subset \Omega(x,R)$.

We observe from Lemma 3.2 and (3.2) with R in place of r that

$$\|\tilde{\mu}\| = R^{n-\alpha}\|\mu\| \geq C_\alpha(\Omega(x,R)) - \varepsilon R^{n-\alpha} \geq C_\alpha(\Omega(x,R))\Big(1 - \frac{\varepsilon}{C_\alpha(B(0,a_1))}\Big). \tag{3.9}$$

Let $d = \rho/(2\sqrt{n})$ and cover $\mathbb{R}^n$ by $\{B(z,\rho/2)\}_{z\in(d\mathbb{Z})^n}$. By compactness we find finitely many points $z_1,\dots,z_N \in (d\mathbb{Z})^n$ such that $K(x,R) \subset \bigcup_{i=1}^N B(z_i,\rho/2)$. See Figure 2. Define Radon measures $\tilde{\mu}_1,\dots,\tilde{\mu}_N$ by $\tilde{\mu}_1(A) = \tilde{\mu}(A\cap B(z_1,\rho/2))$ and, for $2 \leq j \leq N$,

$$\tilde{\mu}_j(A) = \tilde{\mu}\Big(A \cap B(z_j,\rho/2) \setminus \bigcup_{i=1}^{j-1} B(z_i,\rho/2)\Big) \quad \text{for Borel sets } A.$$

Then $\tilde{\mu} = \sum_{i=1}^N \tilde{\mu}_i$ and $\operatorname{supp}\tilde{\mu}_i \subset \overline{B}(z_i,\rho/2) \subset B(z_i,\rho)$ for each $1 \leq i \leq N$. Without loss of generality we may assume that $\tilde{\mu}_i \neq 0$ for every i. For simplicity write $\tilde{B}_i = B(z_i,\rho)$.

Observe from (3.8) that $I_\alpha * \tilde{\mu}_i \leq \eta$ in $\mathbb{R}^n$. Hence $C_\alpha(\tilde{B}_i) \geq \|\tilde{\mu}_i\|/\eta$ by definition. Therefore (3.3) yields

$$C_\alpha(E\cap\tilde{B}_i) \geq \kappa C_\alpha(\tilde{B}_i) \geq \frac{\kappa}{\eta}\|\tilde{\mu}_i\|.$$

By definition we find a Radon measure $\tilde{\lambda}_i$ such that $\operatorname{supp}\tilde{\lambda}_i \subset E\cap\tilde{B}_i$ and

$$\begin{aligned} &I_\alpha * \tilde{\lambda}_i \leq 1 \quad \text{in } \mathbb{R}^n, \\ &\|\tilde{\lambda}_i\| \geq C_\alpha(E\cap\tilde{B}_i) - \frac{\kappa}{2\eta}\|\tilde{\mu}_i\| \geq \frac{\kappa}{2\eta}\|\tilde{\mu}_i\|. \end{aligned} \tag{3.10}$$

Let

$$\tilde{\nu}_i = \frac{\|\tilde{\mu}_i\|}{\|\tilde{\lambda}_i\|}\tilde{\lambda}_i.$$

Then $\|\tilde{\nu}_i\| = \|\tilde{\mu}_i\| > 0$ and $\operatorname{supp}\tilde{\nu}_i \subset E\cap\tilde{B}_i$. Observe that $\operatorname{dist}(K(x,R),\partial\Omega(x,R)) = R\operatorname{dist}(K,\partial\Omega)$. Since $\tilde{B}_i\cap K(x,R) \neq \emptyset$, it follows from (3.6) that $\operatorname{supp}\tilde{\nu}_i \subset E\cap\Omega(x,R)$. Moreover, (3.10) gives

$$I_\alpha * \tilde{\nu}_i = \frac{\|\tilde{\mu}_i\|}{\|\tilde{\lambda}_i\|} I_\alpha * \tilde{\lambda}_i \leq \frac{2\eta}{\kappa} \quad \text{in } \mathbb{R}^n. \tag{3.11}$$

Let $\tilde{\nu} = \sum_{i=1}^N \tilde{\nu}_i$. Observe that $\|\tilde{\nu}\| = \|\tilde{\mu}\|$, and that $\operatorname{supp}\tilde{\nu} \subset E \cap \Omega(x,R)$.

Let us estimate the upper bound of $I_\alpha * \tilde{\nu}$ in $\mathbb{R}^n$. If $y \in \mathbb{R}^n \setminus B(z_i, M\rho)$, then

$$\frac{I_\alpha * \tilde{\nu}_i(y)}{I_\alpha * \tilde{\mu}_i(y)} \le \frac{(|y-z_i|-\rho)^{\alpha-n}\,\|\tilde{\nu}_i\|}{(|y-z_i|+\rho)^{\alpha-n}\,\|\tilde{\mu}_i\|} = \Big(\frac{1-\rho/|y-z_i|}{1+\rho/|y-z_i|}\Big)^{\alpha-n} \le \Big(\frac{1-1/M}{1+1/M}\Big)^{\alpha-n} \le 1+\varepsilon$$

by (3.4). Hence

$$I_\alpha * \tilde{\nu}_i \le (1+\varepsilon) I_\alpha * \tilde{\mu}_i \quad \text{in } \mathbb{R}^n \setminus B(z_i, M\rho). \tag{3.12}$$

In the same way as in the proof of Theorem 1.1, we see that the multiplicity of $\{B(z_i, M\rho)\}$ is bounded by

$$\frac{((1+M)\rho)^n}{(\rho/(4\sqrt{n}))^n} = (4\sqrt{n}(1+M))^n. \tag{3.13}$$

Now let $y \in \mathbb{R}^n$. Combining (3.5), (3.7), (3.11), (3.12) and (3.13) altogether, we obtain

$$\begin{aligned} I_\alpha * \tilde{\nu}(y) &= \sum_{i:y\in B(z_i,M\rho)} I_\alpha * \tilde{\nu}_i(y) + \sum_{i:x\notin B(z_i,M\rho)} I_\alpha * \tilde{\nu}_i(y) \\ &\le (4\sqrt{n}(1+M))^n \frac{2\eta}{\kappa} + (1+\varepsilon) \sum_{i:y\notin M\tilde{B}_i} I_\alpha * \tilde{\mu}_i(y) \\ &\le 2\varepsilon + (1+\varepsilon) I_\alpha * \tilde{\mu}(y) \le 1 + 3\varepsilon. \end{aligned}$$

Hence (3.9) yields

$$C_\alpha(E \cap \Omega(x,R)) \ge \frac{\|\tilde{\nu}\|}{1+3\varepsilon} = \frac{\|\tilde{\mu}\|}{1+3\varepsilon} \ge \frac{1-\varepsilon/C_\alpha(B(0,a_1)}{1+3\varepsilon} C_\alpha(\Omega(x,R)).$$

Thus, if R satisfies (3.6), then

$$\mathcal{D}_\Omega(C_\alpha, E, R) \ge \frac{1-\varepsilon/C_\alpha(B(0,a_1))}{1+3\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary, we have $\liminf_{R\to\infty} \mathcal{D}_\Omega(C_\alpha, E, R) \ge 1$, as required. □

4. PROOF OF PROPOSITION 1.6

The construction of E is essentially the same as in [3, Example 7.2].

Proof of Proposition 1.6. Suppose $\|k\|_1 < \infty$. Then $\|k * \chi_E\|_\infty \le \|k\|_1 < \infty$, and by definition

$$C_k(E) \ge \frac{|E|}{\|k\|_1} \quad \text{for every Borel set } E, \tag{4.1}$$

where $|E|$ stands for the Lebesgue measure of E. On the other hand, if $r \ge 1$, then

$$k * \chi_{B(x,2r)} \ge \int_{|y|\le 1} k(y)dy > 0 \quad \text{on } B(x,r),$$

so that the dual definition of capacity yields that

$$C_k(B(x,r)) \le \frac{\|\chi_{B(x,2r)}\|_1}{\int_{|y|\le 1} k(y)dy} = A_1|B(x,r)| \tag{4.2}$$

with A_1 depending only on k and n. Let $M > 10$ and $E = \bigcup_{z\in(M\mathbb{Z})^n} B(z,1)$. If $r > 10M$, then the number of small balls $B(z,1)$ intersecting $B(x,r)$ is bounded by $A_2(r/M)^n$ and the measure density $|E \cap B(x,r)|/|B(x,r)|$ is bounded from below by A_2M^{-n} with A_2 depending only on n.

First consider the case $\Omega = B(0,1)$. By (4.1), (4.2) and the subadditivity of C_k we have

$$A^{-1}M^{-n} \le \frac{|E\cap B(x,r)|/\|k\|_1}{A_1|B(x,r)|} \le \frac{C_k(E\cap B(x,r))}{C_k(B(x,r))} \le \frac{A_2(r/M)^n C_k(B(0,1))}{|B(x,r)|/\|k\|_1} \le AM^{-n}$$

with $A > 1$ depending only on k and n. Hence, if M is large, then

$$0 < \liminf_{r\to\infty} \mathcal{D}(C_k,E,r) \le \limsup_{r\to\infty} \mathcal{D}(C_k,E,r) < 1.$$

Now let Ω be a general bounded open set. In view of Lemma 3.3, we obtain (1.6) by making M large. □

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