# THE HEAT EQUATION FOR THE DIRICHLET FRACTIONAL LAPLACIAN WITH HARDY'S POTENTIALS: PROPERTIES OF MINIMAL SOLUTIONS AND BLOW-UP 

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#### Abstract

Local and global properties of minimal solutions for the heat equation generated by the Dirichlet fractional Laplacian negatively perturbed by Hardy's potentials on open subsets of $\mathbb{R}^{d}$ are analyzed. As a byproduct we obtain instantaneous blow-up of nonnegative solutions in the supercritical case.


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## 1. INTRODUCTION

In this paper, we discuss local and global properties of nonnegative solutions for the heat equation generated by Dirichlet fractional Laplacian negatively perturbed by the potentials $\frac{c}{|x|^{\alpha}}, c>0$. As a consequence we shall prove complete instantaneous blow-up of nonnegative solutions provided $c$ is larger than some critical value $c^{*}$.
Let $0<\alpha<\min (2, d)$ and $\Omega$ be an open subset $\Omega \subset \mathbb{R}^{d}$ containing zero. We denote by $L_{0}^{\Omega}:=\left.(-\Delta)^{\frac{\alpha}{2}}\right|_{\Omega}$ the fractional Laplacian with zero Dirichlet condition on $\Omega^{c}$ (as explained in the next section).
We consider the perturbed heat equation

$$
\left\{\begin{array}{c}
-\frac{\partial u}{\partial t}=L_{0}^{\Omega} u-\frac{c}{|x|^{\alpha}} u \text { in }(0, T) \times \Omega  \tag{1.1}\\
u(t, \cdot)=0 \text { in } \Omega^{c} \text { for all } 0<t<T \leq \infty, \\
u(0, x)=u_{0}(x) \text { a.e. in } \Omega
\end{array}\right.
$$

where $c>0$ and $u_{0}$ is a nonnegative Borel measurable square integrable function on $\Omega$. The meaning of a solution for the equation (1.1) will be explained in the next section.

In [4], the authors establish existence of nonnegative exponentially bounded solutions on bounded Lipschitz domains provided

$$
\begin{equation*}
0<c \leq c^{*}:=\frac{2^{\alpha} \Gamma^{2}\left(\frac{d+\alpha}{4}\right)}{\Gamma^{2}\left(\frac{d-\alpha}{4}\right)} . \tag{1.2}
\end{equation*}
$$

They also prove that for $c>c^{*}$ (the supercritical case) complete instantaneous blow-up takes place, provided $\Omega$ is a bounded Lipschitz domain.
Concerning properties of solutions, only partial information is available in the literature. For example, in [3, Corollary 5.1] the authors prove that under some conditions any nonnegative solution $u(t, x)$ behaves asymptotically like

$$
\begin{equation*}
u(t, x) \sim c_{t}|x|^{-\beta(c)} \delta^{\alpha / 2}(x), \text { a.e. for large } t . \tag{1.3}
\end{equation*}
$$

where $0<\beta(c) \leq \frac{d-\alpha}{2}$ and $\delta$ is the distance function to the complement of the domain.
In the case $\Omega=\mathbb{R}^{d}$, sharp estimates of the heat kernel of the considered equation are recently established in [6]. Relying on these estimates one can derive precise information about nonnegative solutions of (1.1). Moreover, in [6, Corollary 4.11] the authors prove complete instantaneous blow-up of the heat kernel in the supercritical case.
However, as far as we know, the problem whether a blow-up phenomena takes place in the supercritical case is still unsolved for general domains.
In these notes we solve definitively both problems: Sharp local estimates in $\Omega$, of a special nonnegative solution (the minimal solution) of the heat equation will be established in the subcritical case. The estimates are then used to obtain global sharp $L^{p}$ regularity property. We also prove complete instantaneous blow-up in the supercritical case for arbitrary domains, regardless of boundedness and regularity of the boundary.
Our strategy is as follows: At first stage we show that the considered semigroups have heat kernels in the subcritical case. Then sharp estimates for the heat kernels on bounded sets are established. These estimates lead in turns to sharp pointwise estimate of the minimal solution of (1.1) near zero.
The main ingredients at this stage are Doob transform that will transform the forms related to the considered semigroups into Dirichlet forms and the celebrated improved Hardy-Sobolev inequality.
Besides, we exploit the heat kernel estimates to extend the $L^{2}$-semigroups to semigroups on some (weighted) $L^{p}$-spaces.
Finally, we use the heat kernel lower bound on balls to establish blow-up on open sets.
The inspiring points for us are the papers [14, 2, 7] where the problem was addressed and solved for the Dirichlet Laplacian (i.e., $\alpha=2$ ).

Despite the nonlocal nature of the fractional Laplacian, we record many resemblances between our results and those established in $[14,2,7]$ for the Laplacian.

## 2. PREPARATORY RESULTS

From now on we fix an open subset $\Omega \subset \mathbb{R}^{d}$ containing zero and a real number $\alpha$ such that $0<\alpha<\min (2, d)$. The Lebesgue spaces $L^{2}\left(\mathbb{R}^{d}, d x\right)$, resp., $L^{2}(\Omega, d x)$ will be denoted by $L^{2}$, resp., $L^{2}(\Omega)$ and their respective norms, will be denoted by $\|\cdot\|_{L^{2}},\|\cdot\|_{L^{2}(\Omega)}$. We shall write $\int \cdots$ as a shorthand for $\int_{\mathbb{R}^{d}} \cdots$. The letters $C, C^{\prime}, c_{t}, \kappa_{t}$ will denote generic nonnegative finite constants which may vary in value from line to line.
Consider the bilinear symmetric form $\mathcal{E}$ with domain in $L^{2}$, defined by

$$
\begin{align*}
\mathcal{E}(f, g) & =\frac{1}{2} \mathcal{A}(d, \alpha) \iint \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}} d x d y \\
D(\mathcal{E}) & =W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right):=\left\{f \in L^{2}: \mathcal{E}[f]:=\mathcal{E}(f, f)<\infty\right\} \tag{2.1}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{A}(d, \alpha)=\frac{\alpha \Gamma\left(\frac{d+\alpha}{2}\right)}{2^{1-\alpha} \pi^{d / 2} \Gamma\left(1-\frac{\alpha}{2}\right)} \tag{2.2}
\end{equation*}
$$

is a normalizing constant.
Using Fourier transform $\hat{f}(\xi)=(2 \pi)^{-d / 2} \int e^{-i x \cdot \xi} f(x) d x$, a straightforward computation yields the following identity (see [9, Lemma 3.1])

$$
\begin{equation*}
\int|\xi|^{\alpha}|\hat{f}(\xi)|^{2} d \xi=\mathcal{E}[f] \text { for all } f \in W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right) \tag{2.3}
\end{equation*}
$$

It is well known that $\mathcal{E}$ is a Dirichlet form, i.e., it is a densely defined (in $L^{2}$ ) bilinear symmetric and closed form moreover it holds,

$$
f \in W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right) \Rightarrow f_{0,1}:=(0 \vee f) \wedge 1 \in W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right) \text { and } \mathcal{E}\left[f_{0,1}\right] \leq \mathcal{E}[f]
$$

Furthermore $\mathcal{E}$ is regular, i.e., $C_{c}\left(\mathbb{R}^{d}\right) \cap W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right)$ is dense in both spaces $C_{c}\left(\mathbb{R}^{d}\right)$ and $W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right)$ (see [10, Example 1.4.1]). For further information on Dirichlet forms we refer the reader to [10].
The form $\mathcal{E}$ is related (via Kato representation theorem [11, Theorem 2.1, p.322]) to the selfadjoint operator commonly named the fractional Laplacian on $\mathbb{R}^{d}$, which we denote by $L_{0}:=(-\Delta)^{\alpha / 2}$. We note that the domain of $L_{0}$ is the fractional Sobolev space $W^{\alpha, 2}\left(\mathbb{R}^{d}\right)$.
For later purposes we recall the sharp Hardy's inequality (see [15])

$$
\begin{equation*}
\int \frac{f^{2}(x)}{|x|^{\alpha}} d x \leq \frac{1}{c^{*}} \mathcal{E}[f] \text { for all } f \in W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right) \tag{2.4}
\end{equation*}
$$

with $c^{*}$ is the constant given by (1.2).
Henceforth we denote by $L_{0}^{\Omega}$, the operator related to the Dirichlet form in $L^{2}(\bar{\Omega}, d x)$ given by

$$
\begin{aligned}
D\left(\mathcal{E}_{\Omega}\right)= & W_{0}^{\alpha / 2,2}(\Omega):=\left\{f \in W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right): f=0 \quad \text { q.e. on } \Omega^{c}\right\} \\
\mathcal{E}_{\Omega}(f, g)= & \mathcal{E}(f, g) \\
= & \frac{1}{2} \mathcal{A}(d, \alpha) \int_{\Omega} \int_{\Omega} \frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{d+\alpha}} d x d y \\
& +\int_{\Omega} f(x) g(x) \kappa_{\Omega}(x) d x
\end{aligned}
$$

where

$$
\begin{equation*}
\kappa_{\Omega}(x):=\mathcal{A}(d, \alpha) \int_{\Omega^{c}} \frac{1}{|x-y|^{d+\alpha}} d y \tag{2.5}
\end{equation*}
$$

and 'q.e.' means quasi-everywhere (see [10]).
For every $t \geq 0$ we denote by $e^{-t L_{0}^{\Omega}}$ the operator semigroup generated by $L_{0}^{\Omega}$. In the case $\Omega=\mathbb{R}^{d}$ we omit the superscript $\Omega$ in the notations. It is a known fact (see [5]) that $e^{-t L_{0}^{\Omega}}, t>0$ has a kernel (the heat kernel) $p_{t}^{L_{0}^{\Omega}}(x, y)$ which is symmetric jointly continuous and $p_{t}^{L_{0}^{\Omega}}(x, y)>0$, for all $x, y \in \Omega$.
Let us introduce the notion of solution for problem (1.1).
Definition 2.1. Let $V \in L_{l o c}^{1}(\Omega)$ be nonnegative, $u_{0} \in L^{2}(\Omega)$ be nonnegative as well and $0<T \leq \infty$. We say that a Borel measurable function $u:[0, T) \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ is a solution of the heat equation

$$
\left\{\begin{array}{c}
-\frac{\partial u}{\partial t}=L_{0}^{\Omega} u-V u \quad \text { in }(0, T) \times \Omega  \tag{2.6}\\
u(t, \cdot)=0 \text { in } \quad \Omega^{c}, \text { for all } t \in(0, T) \\
\\
u(0, \cdot)=u_{0} \quad \text { on } \Omega
\end{array}\right.
$$

if

1. $u \in \mathcal{L}_{l o c}^{2}\left([0, T), L_{l o c}^{2}(\Omega)\right)$, where $\mathcal{L}^{2}$ is the Lebesgue space of square integrable functions.
2. $u \in L_{l o c}^{1}((0, T) \times \Omega, d t \otimes V(x) d x)$.
3. For every $0 \leq t<T, u(t, \cdot)=0$, a.e. on $\Omega^{c}$.
4. For every $0 \leq t<T$ and every Borel function $\phi:[0, T) \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ such that $\operatorname{supp} \phi \subset[0, T) \times \Omega, \phi, \frac{\partial \phi}{\partial t} \in L^{2}((0, T) \times \Omega)$ and $\phi(t, \cdot) \in D\left(L_{0}\right)$ and

$$
\int_{0}^{t} \int_{\Omega}\left|u(s, x) L_{0} \phi(s, x)\right| d s d x<\infty
$$

the following identity holds true

$$
\begin{align*}
& \int\left((u \phi)(t, x)-u_{0}(x) \phi(0, x)\right) d x+ \\
& \quad \int_{0}^{t} \int u(s, x)\left(-\phi_{s}(s, x)+L_{0}^{\Omega} \phi(s, x)\right) d x d s  \tag{2.7}\\
& =\int_{0}^{t} \int u(s, x) \phi(s, x) V(x) d x d s
\end{align*}
$$

For every $c>0$ we denote by $V_{c}$ the Hardy potential

$$
V_{c}(x):=\frac{c}{|x|^{\alpha}}, x \neq 0
$$

In [4] it is proved that for bounded $\Omega, V=V_{c}$ and $0<c \leq c^{*}$ equation (1.1) has an exponentially bounded nonnegative solution. However, for $c>c^{*}$ and $\Omega$ a bounded Lipschitz domain then any nonnegative solution, $u(t, x)$ blows up completely and instantaneously, i.e.,

$$
u(t, x)=\infty, t>0 \text { a.e. } x
$$

The same statement was recently proved in $[6]$ for $\Omega=\mathbb{R}^{d}$.
In these notes we shall, among others, fill the gap between the cases of bounded $\Omega$ and $\Omega=\mathbb{R}^{d}$.
In the next section we shall be concerned with properties of a special nonnegative solution which is called minimal solution or semigroup solution in the subcritical case, i.e., $0<c<c^{*}$ and in the critical case, i.e., $c=c^{*}$. The term minimal solution comes from the following observation: If $u_{k}$ is the semigroup solution for the heat equation with potential $V_{c} \wedge k, k \in \mathbb{N}$ and if $u$ is any nonnegative solution of (1.1) then $u_{\infty}:=\lim _{k \rightarrow \infty} u_{k}$ is a nonnegative solution of (1.1) and $u_{\infty} \leq u$ a.e.. The observation is proved in [2] for the Dirichlet Laplacian with Hardy potentials, in [4] for the Dirichlet fractional Laplacian on bounded domains and in Lemma 4.1 for general domains. Finally it is proved in [12] in a different context.
We name $u_{\infty}$ the minimal solution and we denote it by $u$ instead of $u_{\infty}$. Let $0<c<c^{*}$. We denote by $\mathcal{E}_{\Omega}^{V_{c}}$ the quadratic form defined by

$$
\begin{equation*}
\operatorname{dom}\left(\mathcal{E}_{\Omega}^{V_{c}}\right)=W_{0}^{\alpha / 2,2}(\Omega), \mathcal{E}_{\Omega}^{V_{c}}[f]=\mathcal{E}_{\Omega}[f]-\int_{\Omega} f^{2}(x) V_{c}(x) d x \tag{2.8}
\end{equation*}
$$

whereas for $c=c^{*}$, we set
(2.9) $\operatorname{dom}\left(\dot{\mathcal{E}_{\Omega}}{ }^{V_{c^{*}}}\right)=W_{0}^{\alpha / 2,2}(\Omega), \dot{\mathcal{E}_{\Omega}}{ }^{V_{c^{*}}}[f]=\mathcal{E}_{\Omega}[f]-\int_{\Omega} f^{2}(x) V_{c^{*}}(x) d x$.

In the case $\Omega=\mathbb{R}^{d}$ we omit the subscript $\Omega$.
As the closability of $\dot{\mathcal{E}}_{\Omega} V_{c^{*}}$ in $L^{2}(\Omega)$ is not obvious we shall apply a method
that enables us to prove in a unified manner the closedness of $\mathcal{E}_{\Omega}^{V_{c}}$ as well as the closability of $\dot{\mathcal{E}_{\Omega}}{ }^{V_{c}}$ in $L^{2}(\Omega)$. To that end we recall some known facts concerning harmonic functions of $L_{0}-\frac{c}{|x|^{\alpha}}$.
We know from [3, Lemma 2.2] that for every $0<c \leq c^{*}$ there is a unique $\beta=\beta(c) \in\left(0, \frac{d-\alpha}{2}\right]$ such that $w_{c}(x):=|x|^{-\beta(c)}, x \neq 0$ solves the equation

$$
\begin{equation*}
(-\Delta)^{\alpha / 2} w-c|x|^{-\alpha} w=0 \text { in the sense of distributions. } \tag{2.10}
\end{equation*}
$$

That is,

$$
\begin{equation*}
<\hat{w},|\xi|^{\alpha} \hat{\varphi}>-c<|x|^{-\alpha} w, \varphi>=0, \varphi \in \mathcal{S} . \tag{2.11}
\end{equation*}
$$

Making use of Riesz potential it is proved in [3, Lemma 2.2] that equation (2.10) is equivalent to

$$
\begin{equation*}
\int \frac{w_{c}(y)}{|x-y|^{d-\alpha}}|y|^{-\alpha} d y=c w_{c}(x), x \neq 0 \tag{2.12}
\end{equation*}
$$

Furthermore for $\beta=\beta_{*}:=\frac{d-\alpha}{2}$, we have $c=c^{*}$. Thereby, $w_{c^{*}}(x)=|x|^{-\frac{d-\alpha}{2}}$, $x \neq 0$.
Next let $c$ be a real number in $\left(0, c^{*}\right]$.
For $0<c<c^{*}$ let $Q_{\Omega}^{c}$ be the $w_{c}$-transform of $\mathcal{E}_{\Omega}^{V_{c}}$, and for $c=c^{*}$ let $\dot{Q}_{\Omega}^{c^{*}}$ be the $w_{c^{*}}$-transform of $\dot{\mathcal{E}}_{\Omega}^{V_{c}}$. Accordingly we have:

$$
\begin{aligned}
\operatorname{dom}\left(Q_{\Omega}^{c}\right) & :=\left\{f \in L^{2}\left(\Omega, w_{c}^{2} d x\right): w_{c} f \in W_{0}^{\alpha / 2,2}(\Omega)\right\}, Q_{\Omega}^{c}[f]=\mathcal{E}_{\Omega}^{V_{c}}\left[w_{c} f\right] \\
& =\mathcal{E}_{\Omega}\left[w_{c} f\right]-c \int_{\Omega} \frac{\left(w_{c} f\right)^{2}}{|x|^{\alpha}} d x, f \in \operatorname{dom}\left(Q_{\Omega}^{c}\right)
\end{aligned}
$$

Whereas

$$
\begin{aligned}
\operatorname{dom}\left(\dot{Q}_{\Omega}^{c^{*}}\right) & :=\left\{f \in L^{2}\left(\Omega, w_{c^{*}}^{2} d x\right): w_{c^{*}} f \in W_{0}^{\alpha / 2,2}(\Omega)\right\}, \dot{Q}_{\Omega}^{c^{*}}[f]=\dot{\mathcal{E}}_{\Omega}^{V_{c}}\left[w_{c^{*}} f\right] \\
& =\mathcal{E}_{\Omega}\left[w_{c^{*}} f\right]-c^{*} \int_{\Omega} \frac{\left(w_{c^{*}} f\right)^{2}}{|x|^{\alpha}} d x, f \in \operatorname{dom}\left(\dot{Q}_{\Omega}^{c^{*}}\right)
\end{aligned}
$$

In the case $\Omega=\mathbb{R}^{d}$ we shall omit the subscript $\mathbb{R}^{d}$ in the above notations.
Lemma 2.2. 1. For every $0<c \leq c^{*}$, the sets $C_{c}^{\infty}(\Omega \backslash\{0\})$ and $\operatorname{dom}\left(Q_{\Omega}^{c}\right) \cap C_{c}(\Omega)$ are cores for $Q_{\Omega}^{c}$.
2. For every $0<c<c^{*}$, the form $Q_{\Omega}^{c}$ is a Dirichlet form in $L^{2}\left(\Omega, w_{c}^{2} d x\right)$ and

$$
\begin{array}{r}
Q_{\Omega}^{c}[f]=\frac{\mathcal{A}(d, \alpha)}{2} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \frac{(f(x)-f(y))^{2}}{|x-y|^{d+\alpha}} w_{c}(x) w_{c}(y) d x d y  \tag{2.13}\\
f \in \operatorname{dom}\left(Q_{\Omega}^{c}\right) .
\end{array}
$$

3. For $c=c^{*}$ the form $\dot{Q}_{\Omega}^{c^{*}}$ is closable in $L^{2}\left(\Omega, w_{c^{*}}^{2} d x\right)$ and

$$
\begin{array}{r}
\dot{Q}_{\Omega}^{c^{*}}[f]=\frac{\mathcal{A}(d, \alpha)}{2} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \frac{(f(x)-f(y))^{2}}{|x-y|^{d+\alpha}} w_{c_{*}}(x) w_{c_{*}}(y) d x d y  \tag{2.14}\\
f \in \operatorname{dom}\left(\dot{Q}_{\Omega}^{c^{*}}\right)
\end{array}
$$

Let $Q_{\Omega}^{c^{*}}$ be the closure of $\dot{Q}_{\Omega}^{c^{*}}$. Then $Q_{\Omega}^{c^{*}}$ is a Dirichlet form.
4. The form $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ is closable.
5. The form $Q_{\Omega}^{c}$ is regular for every $0<c \leq c^{*}$.

Remark 2.3. 1. In the case $\Omega=\mathbb{R}^{d}$ Lemma 2.2 (and Lemma 2.7 below) can be seen as the reciprocal of [6, Theorem 5.4]. In fact the form $\overline{\mathcal{E}}$ considered in [6] is nothing else but the $w_{c}^{-1}$-transform of $Q^{c}$.
2. We show in Remark 3.5 that $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ is in fact, not closed.

Proof. (of Lemma 2.2)
To prove the first claim, we recall some facts about the fractional Sobolev spaces.
By assumptions $0<\alpha<\min (2, d)$. Thus points have zero $\mathcal{E}$-capacity. Accordingly and from the definition we obtain $W_{0}^{\alpha / 2,2}(\Omega \backslash\{0\})=W_{0}^{\alpha / 2,2}(\Omega)$. Moreover, $C_{c}^{\infty}(\Omega \backslash\{0\})$ is dense in $W_{0}^{\alpha / 2,2}(\Omega \backslash\{0\})$ with respect to the norm $\sqrt{\mathcal{E}_{\Omega}+\|\cdot\|_{L^{2}(\Omega)}^{2}}$. Here we refer to [1, Theorem 10.1.1, Corollary 10.1.2, p.281], where the space $W^{\alpha / 2,2}\left(\mathbb{R}^{d}\right)$ coincides with the space $L^{\alpha / 2,2}\left(\mathbb{R}^{d}\right)$ with an equivalent norm. Thus $C_{c}^{\infty}(\Omega \backslash\{0\})$ is a core for $\mathcal{E}_{\Omega}$.
We recall that $W_{0}^{\alpha / 2,2}(\Omega) \cap C_{c}(\Omega)$ is also a core for $\mathcal{E}_{\Omega}$. Consequently all spaces $C_{c}^{\infty}(\Omega \backslash\{0\})$ and $W_{0}^{\alpha / 2,2}(\Omega) \cap C_{c}(\Omega)$ are also cores for $\mathcal{E}_{\Omega}^{V_{c}}$ and $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$, since both forms are dominated by $\mathcal{E}_{\Omega}$. On the other side $f \mapsto w_{c}^{-1} f$ maps $C_{c}^{\infty}(\Omega \backslash\{0\})$ into $C_{c}^{\infty}(\Omega \backslash\{0\})$ and $W_{0}^{\alpha / 2,2}(\Omega) \cap C_{c}(\Omega)$ into $\operatorname{dom}\left(Q_{\Omega}^{c}\right) \cap C_{c}(\Omega), \operatorname{dom}\left(\dot{Q}_{\Omega}^{c^{*}}\right) \cap C_{c}(\Omega)$. All these considerations together with the fact that $\operatorname{dom}\left(\dot{Q}_{\Omega}^{c_{*}}\right)$ is a core for $Q_{\Omega}^{c^{*}}$ lead to assertion 1.
The proof of formulae (2.13)-(2.14) follows the lines of the proof of Lemma 2.7, so we omit it.
We turn our attention now to prove the rest of the lemma.
Let $0<c<c^{*}$. Utilizing Hardy's inequality we obtain

$$
\begin{equation*}
\left(1-\frac{c}{c^{*}}\right) \mathcal{E}_{\Omega}[f] \leq \mathcal{E}_{\Omega}^{V_{c}} \leq \mathcal{E}_{\Omega}[f], f \in W_{0}^{\alpha / 2,2}(\Omega) \tag{2.15}
\end{equation*}
$$

As $\mathcal{E}_{\Omega}$ is closed, inequality (2.15) yields the closedness of $\mathcal{E}_{\Omega}^{V_{c}}$, which in turns implies the closedness of $Q_{\Omega}^{c}$.

By inequality (2.15) once again and the fact that $\mathcal{E}_{\Omega}$ is densely defined in $L^{2}(\Omega, d x)$, we conclude that $\mathcal{E}_{\Omega}^{V_{c}}$ is also densely defined in $L^{2}(\Omega, d x)$. Thus $Q_{\Omega}^{c}$ is densely defined in $L^{2}\left(\Omega, w_{c}^{2} d x\right)$. Furthermore, on the light of formula (2.13) it is obvious that the normal contraction acts on $\operatorname{dom}\left(Q_{\Omega}^{c}\right)$ and hence $Q_{\Omega}^{c}$ is a Dirichlet form.
Let us consider the critical case. As $\mathcal{E}_{\Omega}$ is densely defined in $L^{2}(\Omega, d x)$ the inequality

$$
0 \leq \dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}[f] \leq \mathcal{E}_{\Omega}[f], f \in W_{0}^{\alpha / 2,2}(\Omega)
$$

implies that $\mathcal{E}_{\Omega}^{V_{C^{*}}}$ is densely defined in $L^{2}(\Omega)$. Consequently the form $\dot{Q}_{\Omega}^{c^{*}}$ is densely defined in $L^{2}\left(\Omega, w_{c^{*}}^{2} d x\right)$. Besides formula (2.14) indicates that $\dot{Q}_{\Omega}^{c^{*}}$ is Markovian and closable, by means of Fatou's lemma. Thus $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ is closable as well. Moreover, according to [10, Theorem 3.1.1] $Q_{\Omega}^{c^{*}}$ is Markovian and hence is a Dirichlet form.
Having assertion 1 in hand, it suffices to prove that $\operatorname{dom}\left(Q^{c}\right) \cap C_{c}(\Omega)$ and $\operatorname{dom}\left(\dot{Q}^{c^{*}}\right) \cap C_{c}(\Omega)$ are uniformly dense in $C_{c}(\Omega)$. But this follows from the regularity of $\mathcal{E}_{\Omega}$ together with the fact that $f \mapsto w_{c}^{-1} f$ maps $C_{c}(\Omega)$ into $C_{c}(\Omega)$.

Henceforth, we denote by $\mathcal{E}_{\Omega^{C^{*}}}^{V^{*}}$ the closure of $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$, by $L_{V_{c}}^{\Omega}$ the selfadjoint operator associated to $\mathcal{E}_{\Omega}^{V_{c}}$ for every $0<c \leq c^{*}$ and we let $e^{-t L_{V_{c}}^{\Omega}}, t \geq 0$ the related semigroups.
Similarly, for every $0<c \leq c^{*}$, we let $A_{\Omega}^{w_{c}}$ the operator associated to $Q_{\Omega}^{c}$ in the weighted Lebesgue space $L^{2}\left(\Omega, w_{c}^{2} d x\right)$ and $T_{t, \Omega}^{w_{c}}, t \geq 0$ its semigroup. Then

$$
\begin{equation*}
A_{\Omega}^{w_{c}}=w_{c}^{-1} L_{V_{c}}^{\Omega} w_{c} \text { and } T_{t, \Omega}^{w_{c}}=w_{c}^{-1} e^{-t L_{V_{c}}^{\Omega}} w_{c}, t \geq 0 \tag{2.16}
\end{equation*}
$$

The next proposition explains why are minimal solutions also semigroup solutions.

Proposition 2.4. For every $0<c \leq c^{*}$, the minimal solution is given by $u(t):=e^{-t L_{V_{c}}^{\Omega}} u_{0}, t \geq 0$, and for each $t>0, u(t) \in D\left(L_{V_{c}}^{\Omega}\right)$ and $u \in$ $C\left(\left[0, \infty, L^{2}(\Omega)\right) \cap C^{1}\left(\left(0, \infty, L^{2}(\Omega)\right)\right.\right.$. Furthermore u fulfills Duhamel's formula

$$
\begin{array}{r}
u(t, x)=e^{-t L_{0}^{\Omega}} u_{0}(x)+\int_{0}^{t} \int_{\Omega} p_{t-s}^{L_{0}^{\Omega}}(x, y) u(s, y) V_{c}(y) d y d s, t>0  \tag{2.17}\\
\text { a.e. } x
\end{array} \in \Omega .
$$

Proof. Let $\left(h_{k}\right)_{k}$ be the sequence of closed quadratic forms in $L^{2}(\Omega)$ defined by

$$
h_{k}:=\mathcal{E}_{\Omega}-V_{c} \wedge k
$$

and $\left(H_{k}\right)_{k}$ be the related selfadjoint operators. Then $\left(h_{k}\right)_{k}$ is uniformly lower semibounded and $h_{k} \downarrow \mathcal{E}_{\Omega}^{V_{c}}$ in the subcritical case, whereas $h_{k} \downarrow \dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ in the
critical case. As both forms $\mathcal{E}_{\Omega}^{V_{c}}, \dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ are closable, we conclude by [11, Theorem 3.11] that $\left(H_{k}\right)$ converges in the strong resolvent sense to $L_{V_{c}}^{\Omega}$ for every $0<c \leq$ $c^{*}$. Hence $e^{-t H_{k}}$ converges strongly to $e^{-t L_{V_{c}}^{\Omega}}$ and then the monotone sequence $u_{k}:=e^{-t H_{k}} u_{0}$ converges to $e^{-t L_{V_{c}}^{\Omega}} u_{0}$ which is nothing else but the minimal solution.
The remaining claims of the proposition follow from the standard theory of semigroups.

As minimal solutions are given in term of semigroups we are led to analyze properties of the latter objects to gain information about the first ones. Here is the first result in this direction.

Proposition 2.5. For every $t>0$ the semigroup $e^{-t L_{\Omega c}^{V_{c}}}, t>0$ has a measurable nonnegative symmetric absolutely continuous kernel, $p_{t}^{L_{V_{c}}^{\Omega}}$, in the sense that for every $v \in L^{2}(\Omega)$ it holds

$$
\begin{equation*}
e^{-t L_{\Omega}^{V_{c}}} v=\int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(, y) v(y) d y, t>0, \text { a.e. } x, y \in \Omega \tag{2.18}
\end{equation*}
$$

We call $p_{t}^{L_{V_{c}}^{\Omega}}$ the heat kernel of $e^{-t L_{\Omega}^{V_{c}}}$. Let us emphasize that formula (2.18) implies that the heat kernel $p_{t}^{L_{V_{c}}^{\Omega}}$ is finite a.e..

Proof. Owing to the known facts that $e^{-t L_{0}^{\Omega}}, t>0$ has a nonnegative heat kernel and $V_{c} \wedge k$ is bounded we deduce that $e^{-t H_{k}}$ has a nonnegative heat kernel as well, which we denote by $P_{t, k}$. Moreover, since the sequence $\left(V_{c} \wedge k\right)_{k}$ is monotone increasing, we obtain with the help of Duhamel's formula that the sequence $\left(P_{t, k}\right)_{k}$ is monotone increasing as well. Set

$$
\begin{equation*}
p_{t}^{L_{V_{c}}^{\Omega}}(x, y):=\lim _{k \rightarrow \infty} P_{t, k}(x, y), t>0, \text { a.e. } x, y \in \Omega \tag{2.19}
\end{equation*}
$$

Then $p_{t}^{L_{V_{c}}^{\Omega}}$ enjoys the properties mentioned in the proposition.
Let $v \in L^{2}(\Omega)$ be nonnegative. Then by monotone convergence theorem, together with Proposition (2.4) we get

$$
\begin{align*}
e^{-t L_{\Omega c}^{V_{c}} v} & =\lim _{k \rightarrow \infty} u_{k}(t)=\lim _{k \rightarrow \infty} e^{-t H_{k}} v=\lim _{k \rightarrow \infty} \int_{\Omega} P_{t, k}(, y) v(y) d y \\
& =\int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(, y) v(y) d y, t>0, \text { a.e. } x, y \in \Omega \tag{2.20}
\end{align*}
$$

For an arbitrary $v \in L^{2}(\Omega)$ formula (2.18) follows from the last step by decomposing $v$ into its positive and negative parts.

Remark 2.6. From Proposition 2.5 in conjunction with formula (2.16), we obtain existence of an absolutely kernel for the semigroups $T_{t, \Omega}^{w_{c}}$ for each $t>0$ and each $0<c \leq c^{*}$. We shall denote by $q_{t}^{\Omega}$ the already mentioned kernel and we call it the heat kernel of $Q_{\Omega}^{c}$. For the particular case $\Omega=\mathbb{R}^{d}$ we will omit the subscript $\mathbb{R}^{d}$. Let us stress that the kernels $q_{t}^{\Omega}$ depend on $c$.
Once again, formula (2.16) leads to

$$
\begin{equation*}
q_{t}^{\Omega}(x, y)=\frac{p_{t}^{L_{V_{c}}^{\Omega}}(x, y)}{w_{c}(x) w_{c}(y)}, t>0, \text { a.e. } x, y \in \Omega \tag{2.21}
\end{equation*}
$$

In the particular case $\Omega=\mathbb{R}^{d}$, we proceed to show that forms $Q^{c}$ enjoy conservativeness property. Namely,

$$
\begin{equation*}
T_{t}^{w_{c}} 1=1, t>0, \text { a.e. } x \in \mathbb{R}^{d} \tag{2.22}
\end{equation*}
$$

To achieve our goal we introduce the following forms. For any $0<c \leq c_{*}$ we define the forms $\dot{\mathcal{F}}^{c}$ by:

$$
\begin{aligned}
& \operatorname{dom}\left(\dot{\mathcal{F}}^{c}\right)=C_{c}^{\infty}\left(\mathbb{R}^{d} \backslash\{0\}\right) \\
& \dot{\mathcal{F}}^{c}[f]=\frac{\mathcal{A}(d, \alpha)}{2} \iint \frac{(f(x)-f(y))^{2}}{|x-y|^{d+\alpha}} w_{c}(x) w_{c}(y) d x d y
\end{aligned}
$$

Lemma 2.7. The quadratic form $Q^{c}$ is well defined, moreover it is closable in $L^{2}\left(\mathbb{R}^{d}, w_{c}^{2} d x\right)$.
Let

$$
\mathcal{F}^{c}=\text { the closure of } \dot{\mathcal{F}}^{c} \text { in } L^{2}\left(\mathbb{R}^{d}, w_{c}^{2} d x\right) .
$$

Then for every $0<c \leq c^{*}$, it holds $Q^{c}=\mathcal{F}^{c}$.
Proof. The fact that $\dot{\mathcal{F}}^{c}$ is well defined is indeed equivalent to the following two conditions (see [10, Example 1.2.1]): for every compact set $K \subset \mathbb{R}^{d} \backslash\{0\}$ and every open set $\Omega_{1} \subset \mathbb{R}^{d} \backslash\{0\}$ with $K \subset \Omega_{1}$ one should have

$$
\begin{aligned}
& \int_{K \times K}|x-y|^{2-d-\alpha} w_{c}(x) w_{c}(y) d x d y<\infty \\
& \int_{K} \int_{\Omega_{1}^{c}}|x-y|^{-d-\alpha} w_{c}(x) w_{c}(y) d x d y<\infty
\end{aligned}
$$

The first condition is proved for bounded sets in [3, Lemma 3.1]. Let us prove the finiteness of the second integral.
Since $0 \notin K$, we obtain $\sup _{x \in K} w_{c}(x)<\infty$. Let $x \in K$. Making use of the identity (2.12) we obtain

$$
\begin{equation*}
\int_{\Omega_{1}^{c} \cap B_{1}}|x-y|^{-d-\alpha} w_{c}(y) d y \leq \delta^{-2 \alpha} \int_{\Omega_{1}^{c} \cap B_{1}} \frac{w_{c}(y)}{|x-y|^{d-\alpha}}|y|^{-\alpha} \leq C w_{c}(x)<\infty \tag{2.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega_{1}^{c} \cap B_{1}^{c}}|x-y|^{-d-\alpha} w_{c}(y) d y \leq \int_{\{|x-y|>\delta\}}|x-y|^{-d-\alpha} d y<\infty \tag{2.24}
\end{equation*}
$$

Thereby, the first part of assertion 1. is proved.
The proof of closability is a standard matter so we omit it.
We already know from Lemma 2.2 that $C_{c}^{\infty}\left(\mathbb{R}^{d} \backslash\{0\}\right)$ is a core for $Q^{c}$ and it is by construction a core for $\mathcal{F}^{c}$ as well. Thereby to prove $Q^{c}=\mathcal{F}^{c}$ it suffices to prove $Q^{c}=\mathcal{F}^{c}$ on $C_{c}^{\infty}\left(\mathbb{R}^{d} \backslash\{0\}\right)$. We mention that the rest of the proof fills a gap in the proof of [3, Lemma 3.1].
Let $f \in C_{c}^{\infty}\left(\mathbb{R}^{d} \backslash\{0\}\right)$. Set
(2.25)

$$
F(x, y):=w_{c}(x) f^{2}(x) \frac{\left(w_{c}(y)-w_{c}(x)\right)}{|x-y|^{d+\alpha}} \text { and } G(x, y):=F(x, y)+F(y, x)
$$

An elementary computation leads to

$$
\frac{(f(x)-f(y))^{2}}{|x-y|^{d+\alpha}} w_{c}(x) w_{c}(y)=\frac{\left(w_{c}(x) f(x)-w_{c}(y) f(y)\right)^{2}}{|x-y|^{d+\alpha}}+G(x, y)
$$

According to the former steps, the first and the second functions are in $L^{1}\left(\mathbb{R}^{d} \times\right.$ $\left.\mathbb{R}^{d}, d x d y\right)$. Hence the function $G(x, y)$ is also in $L^{1}\left(\mathbb{R}^{d} \times \mathbb{R}^{d}, d x d y\right)$. Now we follow the lines of the proof of $[3$, Lemma 3.1]. Let us quote

$$
\int_{\{|x-y|>\epsilon\}} G(x, y) d y=2 \int_{\{|x-y|>\epsilon\}} F(x, y) d y
$$

By dominated convergence theorem we obtain

$$
\begin{aligned}
\iint G(x, y) d x d y & =2 \int w_{c}^{2}(x) f^{2}(x)\left(\lim _{\epsilon \rightarrow 0} \int_{\{|x-y|>\epsilon\}} \frac{\left(w_{c}(y)-w_{c}(x)\right)}{|x-y|^{d+\alpha}} d y\right) d x \\
& =-2 \int w_{c}^{2}(x) f^{2}(x)(-\Delta)^{\alpha / 2}\left(w_{c}\right)(x) d x \\
& =-\frac{2}{\mathcal{A}(d, \alpha)} \int f^{2} w_{c}^{2} V_{c} d x
\end{aligned}
$$

Finally multiplying identity $(2.25)$ by $\frac{\mathcal{A}(d, \alpha)}{2}$ and integrating we obtain $Q^{c}[f]=$ $\mathcal{F}^{c}[f]$ for all $f \in C_{c}\left(\mathbb{R}^{d} \backslash\{0\}\right)$ and all $0<c \leq c^{*}$, which completes the proof.

THEOREM 2.8. Assume that $\Omega=\mathbb{R}^{d}$. Then for every $0<c \leq c^{*}$ the form $Q^{c}$ is conservative. It follows, in particular,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} p_{t}^{L_{V_{c}}^{\mathbb{R}^{d}}}(x, y) w_{c}(y) d y=w_{c}(x), \quad x \neq 0 \tag{2.27}
\end{equation*}
$$

Remark 2.9. Theorem 2.8 was proved in [6, Theorem 3.1], however with a different method using integral analysis.

Proof. Identity (2.27) is an immediate consequences of the conservativeness property which we proceed to prove, together with formula (2.21). As a first step we shall prove conservativeness in the subcritical case.
The subcritical case. Let $0<c<c^{*}$. On the light of Lemma 2.7-2, we use Masamune's result [13] which reads in our special case: If

$$
\begin{equation*}
\sup _{x} w_{c}^{-1}(x) \int_{\mathbb{R}^{d}}\left(1 \wedge|x-y|^{2}\right)|x-y|^{-d-\alpha} w_{c}(y) d y<\infty \tag{2.28}
\end{equation*}
$$

and for some $a>0$,

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} e^{-a|x|} w_{c}^{2}(x) d x<\infty \tag{2.29}
\end{equation*}
$$

then the form $Q^{c}$ is conservative.
Clearly condition (2.29) is fulfilled. Let us show that condition (2.28) is satisfied as well. We recall $w_{c}(x)=|y|^{-\beta(c)}$ for some $\beta:=\beta(c) \in\left(0, \frac{d-\alpha}{2}\right)$. Let

$$
\begin{equation*}
I_{1}(x):=\int_{B_{1}(x)} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha-2}} d y, \alpha^{\prime}=2-\alpha \tag{2.30}
\end{equation*}
$$

Let $|x| \leq 2$ and $\gamma:=\frac{d-\alpha}{2}$. Then

$$
\begin{align*}
I_{1}(x) & =\int_{B_{1}(x)} \frac{|y|^{-\beta}|y|^{\alpha^{\prime}}}{|x-y|^{d-\alpha^{\prime}}}|y|^{-\alpha^{\prime}} d y \\
& \leq 3^{\alpha^{\prime}} \int_{B_{1}(x)} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha^{\prime}}}|y|^{-\alpha^{\prime}} d y \tag{2.31}
\end{align*}
$$

In the case $\alpha \geq 1$, we obtain

$$
\alpha^{\prime}>0,0<\beta<d-\alpha^{\prime}
$$

Thus we apply [3, Lemma 2.1] to get

$$
\begin{align*}
\int_{B_{1}(x)} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha^{\prime}}}|y|^{-\alpha^{\prime}} d y & \leq \int_{\mathbb{R}^{d}} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha^{\prime}}}|y|^{-\alpha^{\prime}} d y \\
& =C w_{c}(x) \tag{2.32}
\end{align*}
$$

and

$$
I_{1}(x) \leq C w_{c}(x)
$$

In the case $0<\alpha<1$, change $2-\alpha$ by $\alpha_{1}=\frac{1-\alpha}{2}$ to obtain (by similar arguments)

$$
\begin{equation*}
I_{1}(x) \leq C \int_{B_{1}(x)} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha_{1}}}|y|^{-\alpha_{1}} d y \leq C w_{c}(x) \tag{2.33}
\end{equation*}
$$

Let now $|x| \geq 2$. Then for every $y \in B_{1}(x)$ we have $|y| \geq|x|-1 \geq 1$. Thus

$$
I_{1}(x) \leq C \frac{1}{(|x|-1)^{\beta}}
$$

Hence in both cases we obtain

$$
\sup _{x} w_{c}^{-1}(x) I_{1}(x)<\infty
$$

For the remaining integral, let

$$
\begin{equation*}
I_{2}(x):=\int_{B_{1}^{c}(x)} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y \tag{2.34}
\end{equation*}
$$

We decompose the integral into the sum of three integrals

$$
\begin{align*}
I_{2}(x) & =\int_{B_{1}^{c}(x) \cap\{|y|<1\}} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y+\int_{B_{1}^{c}(x) \cap\{|y|>1 \wedge|x| / 2\}} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y \\
.35) & +\int_{B_{1}^{c}(x) \cap\{1<|y|<|x| / 2\}} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y \tag{2.35}
\end{align*}
$$

On the set $B_{1}^{c}(x) \cap\{|y|<1\}$ we have $|x-y|^{-d-\alpha} \leq|x-y|^{-d+\alpha}|y|^{-\alpha}$. Thus

$$
\begin{align*}
\int_{B_{1}^{c}(x) \cap\{|y|<1\}} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y & \leq \int_{B_{1}^{c}(x) \cap\{|y|<1\}} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha}}|y|^{-\alpha} d y \\
& \leq \int_{B_{1}^{c}(x)} \frac{|y|^{-\beta}}{|x-y|^{d-\alpha}}|y|^{-\alpha} d y \leq C|x|^{-\beta} \tag{2.36}
\end{align*}
$$

Furthermore

$$
\begin{align*}
\int_{B_{1}^{c}(x) \cap\{|y|>1 \wedge|x| / 2\}} & \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y  \tag{2.37}\\
\leq 2^{\beta}|x|^{-\beta} & \int_{B_{1}^{c}(x)}|x-y|^{-d-\alpha} d y \leq C|x|^{-\beta}
\end{align*}
$$

For the last integral we have two situations: if the set

$$
E:=B_{1}^{c}(x) \cap\{1<|y|<|x| / 2\}
$$

is empty, then we are done. If not, then on the set $E$, it holds

$$
\begin{equation*}
|x-y| \geq \frac{|x|}{2} \geq|y|>1 \tag{2.38}
\end{equation*}
$$

Hence

$$
\int_{E} \frac{|y|^{-\beta}}{|x-y|^{d+\alpha}} d y \leq 2^{\beta}|x|^{-\beta} \int_{E} \frac{|y|^{-\beta}}{|x-y|^{\frac{d+3 \alpha}{2}}} d y
$$

$$
\begin{equation*}
\leq \quad 2^{\beta}|x|^{-\beta} \int_{E} \frac{|y|^{-\beta}}{|y|^{\frac{d+3 \alpha}{2}}} d y \leq C w_{c}(x) \tag{2.39}
\end{equation*}
$$

Finally we get $\sup _{x} w_{c}^{-1}(x) I_{2}(x)<\infty$.
Putting all together we get that condition (2.28) is fulfilled and the form $Q^{c}$ is conservative.

The critical case: We recall that conservativeness means

$$
T_{t}^{w_{c^{*}}} 1=\int q_{t}(, y) w_{c^{*}}^{2}(y) d y=1, t>0
$$

From the contraction property of the $L^{\infty}$-semigroup related to $Q^{c^{*}}$ we derive

$$
\int q_{t}(, y) w_{c^{*}}^{2}(y) d y \leq 1
$$

which leads to

$$
\begin{equation*}
\int_{\mathbb{R}^{d}} p_{t}^{L_{V_{c^{*}}}^{\mathbb{R}^{d}}}(x, y) w_{c^{*}}(y) d y \leq w_{c^{*}}(x) \text { for all } x \neq 0, t>0 \tag{2.40}
\end{equation*}
$$

Now the first part of the proof yields, for every $0<c<c^{*}$,

$$
w_{c}(x)=\int_{\mathbb{R}^{d}} p_{t}^{L_{V_{c}^{\mathbb{R}_{c}^{d}}}}(x, y) w_{c}(y) d y \leq \int_{\mathbb{R}^{d}} p_{t}^{L_{V_{c}^{\mathbb{R}^{d}}}^{\mathbb{V}_{c^{*}}}}(x, y) w_{c}(y) d y
$$

$$
\begin{equation*}
=\int_{B_{1}} p_{t}^{L_{V_{c}}^{\mathbb{R}_{\mathbb{R}^{*}}}}(x, y) w_{c}(y) d y+\int_{B_{1}^{c}} p_{t}^{L_{V_{c}}^{\mathbb{R}^{d}}}(x, y) w_{c}(y) d y \tag{2.41}
\end{equation*}
$$

Let us observe that the first integrant is increasing, whereas the second one is decreasing with respect to $c$. Hence, letting $c \rightarrow c^{*}$ and combining monotone convergence theorem with inequality (2.40) we achieve $T_{t}^{w_{c}{ }^{*}} 1=1$ and the proof is completed.

## 3. HEAT KERNEL ESTIMATES, LOCAL AND GLOBAL BEHAVIOR OF THE MINIMAL SOLUTION IN SPACE VARIABLE

In this section we assume that $\Omega$ is bounded.
Since the potentials $V_{c}$ are too singular (they are not in the Kato-class, for example), investigations of properties of solutions of the evolution equations related to $L_{0}^{\Omega}-V_{c}$ becomes a delicate problem. In fact, the theory of elliptic regularity is no longer applicable in this context. To overcome the difficulties we shall make use of the Doob transform for forms $\mathcal{E}_{\Omega}^{V_{c}}$ performed in Lemma 2.2 together with an improved Sobolev inequality. This transformation leads us to get a Dirichlet forms and a Markovian ultracontractive semigroup on some
weighted Lebesgue space. The analysis of the transformed forms will then lead us to get satisfactory results concerning estimating their heat kernels (sharply) and hence to reveal properties of minimal solutions.
As a first step we proceed to prove that Sobolev inequality holds for the $w_{c^{-}}$ transform of the form $\mathcal{E}_{\Omega}^{V_{c}}$.

THEOREM 3.1. 1. Let $0<c<c^{*}$ and $p=\frac{d}{d-\alpha}$. Then the following Sobolev inequality holds true

$$
\begin{equation*}
\left\|f^{2}\right\|_{L^{p}\left(w_{c}^{2} d x\right)} \leq A Q_{\Omega}^{c}[f], f \in D\left(Q_{\Omega}^{c}\right) \tag{3.1}
\end{equation*}
$$

2. For $c=c^{*}$ let $1<p<\frac{d}{d-\alpha}$. Then the following Sobolev inequality holds true

$$
\begin{equation*}
\left\|f^{2}\right\|_{L^{p}\left(w_{c^{*}}^{2} d x\right)} \leq A Q_{\Omega}^{c^{*}}[f], f \in D\left(Q_{\Omega}^{c^{*}}\right) \tag{3.2}
\end{equation*}
$$

3. For every $t>0$ and $0<c \leq c^{*}$ the operator $T_{t, \Omega}^{w_{c}}$ is ultracontractive.
4. For every $0<c<c^{*}$ there is a finite constant $C>0$ such that

$$
\begin{equation*}
0<p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \leq \frac{C}{t^{\frac{d}{\alpha}}} w_{c}(x) w_{c}(y), t>0, \text { a.e. on } \Omega \times \Omega \tag{3.3}
\end{equation*}
$$

5. For $c=c^{*}$, there is a finite constant $C>0$ such that

$$
(3.4) 0<p_{t}^{L_{V_{c^{*}}}^{\Omega}}(x, y) \leq \frac{C}{t^{\frac{p}{p-1}}} w_{c^{*}}(x) w_{c^{*}}(y), t>0, \text { a.e. on } \Omega \times \Omega
$$

Proof. 1) and 2): Let $0<c<c^{*}$. From Hardy's inequality we derive

$$
\begin{equation*}
\left(1-\frac{c}{c^{*}}\right) \mathcal{E}_{\Omega}[f] \leq \mathcal{E}_{\Omega}^{V_{c}}[f], f \in W_{0}^{\alpha / 2,2}(\Omega) \tag{3.5}
\end{equation*}
$$

Now we use the known fact that $W_{0}^{\alpha / 2,2}(\Omega)$ embeds continuously into $L^{\frac{2 d}{d-\alpha}}$ to obtain the following Sobolev's inequality

$$
\begin{equation*}
\left(\int_{\Omega}|f|^{\frac{2 d}{d-\alpha}} d x\right)^{\frac{d-\alpha}{d}} \leq C \mathcal{E}_{\Omega}^{V_{c}}[f], f \in W_{0}^{\alpha / 2,2}(\Omega) \tag{3.6}
\end{equation*}
$$

An application of Hölder's inequality together with Lemma 2.2 and the fact that $\Omega$ is bounded, yield then inequality (3.1).
Towards proving Sobolev's inequality in the critical case we use the improved Hardy-Sobolev inequality, due to Frank-Lieb-Seiringer [9, Theorem 2.3]: For every $1 \leq p<\frac{d}{d-\alpha}$ there is a constant $S_{d, \alpha}(\Omega)$ such that

$$
\begin{equation*}
\left(\int|f|^{2 p} d x\right)^{1 / p} \leq S_{d, \alpha}(\Omega)\left(\mathcal{E}_{\Omega}[f]-c^{*} \int_{\Omega} \frac{f^{2}(x)}{|x|^{\alpha}} d x\right), f \in W_{0}^{\alpha / 2,2}(\Omega) \tag{3.7}
\end{equation*}
$$

and the rest of the proof runs as before.
3) and 4): As $Q_{\Omega}^{c}$ is a Dirichlet form, by the standard theory of Markovian semigroups, it is known (see [8, p.75]) that Sobolev inequality implies ultracontractivity of $T_{t, \Omega}^{w_{c}}$ together with the bound

$$
\begin{equation*}
\left\|T_{t, \Omega}^{w_{c}}\right\|_{L^{2}\left(\Omega, w_{c}^{2} d x\right), L^{\infty}(\Omega)} \leq \frac{C}{t^{d / \alpha}}, t>0 \tag{3.8}
\end{equation*}
$$

By [8, p.59]) the ultracontractivity implies in turns

$$
\begin{equation*}
0 \leq q_{t}^{\Omega}(x, y) \leq \frac{C}{t^{d / \alpha}} \text { for } t>0, \text { a.e. } x, y \tag{3.9}
\end{equation*}
$$

Recalling formula (2.21):

$$
\begin{equation*}
q_{t}^{\Omega}(x, y)=\frac{p_{t}^{L_{V_{c}}^{\Omega}}(x, y)}{w_{c}(x) w_{c}(y)} \text { for } t>0, \text { a.e. } x, y \tag{3.10}
\end{equation*}
$$

yields the upper bound (3.3).
The proof of 5 . is similar so we omit it.
At this stage we turn our attention to establish a lower bound for the heat kernel $p_{t}^{L_{V_{c}}^{\Omega}}$.
Let us first observe that from the definition, the Dirichlet form $Q_{\Omega}^{c}$ is nothing else but the part of the form $Q^{c}$ on $\Omega$, i.e., $Q_{\Omega}^{c}=\left.Q^{c}\right|_{\operatorname{dom}\left(Q_{\Omega}^{c}\right)}$.
Since $Q^{c}$ is a Dirichlet form and $q_{t}$ is continuous there exists a Hunt process on $\mathbb{R}^{d}$ such that

$$
\mathbb{P}^{x}\left(X_{t} \in A\right)=\int_{A} q_{t}(x, y) w_{c}^{2}(y) d y, \quad A \in \mathcal{B}\left(\mathbb{R}^{d}\right)
$$

By positivity of $w_{c}$ and the Dynkin-Hunt formula we get

$$
q_{t}^{\Omega}(x, y)=q_{t}(x, y)-\mathbb{E}^{x}\left[\tau_{\Omega}<t, q_{t-\tau_{\Omega}}\left(X_{\tau_{\Omega}}, y\right)\right], \quad x \text { and } y \in \Omega
$$

where $\tau_{\Omega}=\inf \left\{t>0: X_{t} \notin \Omega\right\}$.
Let $S(t, x):=|x|^{\beta(c)}+t^{\beta(c) / \alpha}$ and $H(t, x):=1+w_{c}\left(x t^{-1 / \alpha}\right)=w_{c}(x) S(t, x)$. We know from [6, Lemma 5.1, Theorem 2.1] together with formula (2.21) that

$$
\begin{equation*}
q_{t}(x, y) \approx S(t, x)\left(t^{-d / \alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) S(t, y), \quad t>0, x \text { and } y \in \mathbb{R}^{d} \tag{3.11}
\end{equation*}
$$

Theorem 3.2. For every $0<c \leq c^{*}$, every compact subset $K \subset \Omega$ and every $t>0$, there is a finite constant $\kappa_{t}=\kappa_{t}(K)>0$ such that

$$
\begin{equation*}
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq \kappa_{t} w_{c}(x) w_{c}(y), \text { a.e. on } K \times K \tag{3.12}
\end{equation*}
$$

Proof. Since $0 \in \Omega$ we may and do assume that $0 \in K$ (we can consider infimum of $p_{t}^{L_{V_{c}}^{\Omega}}$ on the larger set).

First we prove the lower bound for $q_{t}^{\Omega}$ on small balls around zero and small $t>0$. Let $0<r<1$ be such that $\overline{B_{4 r}} \subset \Omega$ and $x, y \in B_{r}$. Then Dynkin-Hunt formula leads to

$$
\begin{align*}
q_{t}^{\Omega}(x, y) & =q_{t}(x, y)-\mathbb{E}^{x}\left[t>\tau_{\Omega}, q_{t-\tau_{\Omega}}\left(y, X_{\tau_{\Omega}}\right)\right] \\
& \geq q_{t}(x, y)-\sup _{s \leq t, z \in \Omega^{c}} q_{s}(y, z) \tag{3.13}
\end{align*}
$$

Since $S(\cdot, y)$ is increasing and $|y-z|>|z| / 2>r$ for $z \in \Omega^{c}$ by (3.11) we obtain

$$
\begin{aligned}
\sup _{s \leq t, z \in \Omega^{c}} q_{s}(y, z) & \leq c_{1} \sup _{z \in \Omega^{c}} S(t, y) \frac{t}{|z|^{d+\alpha}} S(t, z) \\
& \leq c_{1} S(t, y) \frac{t}{r^{d+\alpha}}\left(1+t^{\beta(c) / \alpha}\right)
\end{aligned}
$$

Hence and again (3.11) yields for $t \leq 1$

$$
\frac{q_{t}^{\Omega}(x, y)}{S(t, y)} \geq c_{2} t^{\beta(c) / \alpha}\left(t^{-d / \alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)-c_{1} \frac{t}{r^{d+\alpha}}
$$

For $|x-y| \leq t^{1 / \alpha}$ and $t \leq T(r):=\left(\frac{c_{2} r^{d+\alpha}}{2 c_{1}}\right)^{\alpha /(\alpha+d-\beta(c))}<r$ we get

$$
\frac{q_{t}^{\Omega}(x, y)}{S(t, y)} \geq c_{2} t^{(\beta(c)-d) / \alpha}-c_{1} \frac{t}{r^{d+\alpha}} \geq \frac{c_{2}}{2} t^{(\beta(c)-d) / \alpha}
$$

This implies

$$
q_{t}^{\Omega}(x, y) \geq c S(t, x) S(t, y) t^{-d / \alpha}, \quad|x|,|y| \leq \frac{t^{1 / \alpha}}{2}
$$

In consequence

$$
\begin{array}{r}
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq c H(t, x) H(t, y) t^{-d / \alpha} \geq c H(t, x) H(t, y) p_{t}^{L_{0}^{\Omega}}(x, y) \\
|x| \text { and }|y| \leq \frac{t^{1 / \alpha}}{2}
\end{array}
$$

Let $|x| \leq t^{1 / \alpha} / 2<|y| \leq r$. Set $D:=B_{t^{1 / \alpha} / 4} \backslash B_{t^{1 / \alpha} / 8}$. By Duhamel's formula and estimates of $p_{t}^{L_{0}^{\Omega}}$

$$
p_{t}^{L_{V_{c}}^{\Omega}}(z, y) \geq p_{t}^{L_{0}^{\Omega}}(z, y) \geq c \frac{t}{|y|^{d+\alpha}}, \quad z \in D
$$

By the semigroup property

$$
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq \int_{D} p_{t / 2}^{L_{V_{c}}^{\Omega}}(x, z) p_{t / 2}^{L_{V_{c}}^{\Omega}}(z, y) d z \geq c H(t, x) t^{-d / \alpha} \frac{t}{|y|^{d+\alpha}}|D|
$$

$$
\geq c H(t, x) H(t, y) p_{t}^{L_{0}^{\Omega}}(x, y)
$$

For $t^{1 / \alpha} / 2<|y|,|x|$. We get

$$
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq p_{t}^{L_{0}^{\Omega}}(x, y) \geq \inf _{x, y \in K} p_{t}^{L_{0}^{\Omega}}(x, y)=c_{t}(K)>0
$$

For $|x| \leq t^{1 / \alpha} / 2 \leq r<|y|$ one can obtain by the semigroup property $p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq c H(t, x) c_{t}(K)$. In particular we have

$$
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq H(t, x) H(t, y) c_{t}(K), \quad t \leq T(r), x \text { and } y \in K
$$

If $t>T(r)$ we use the semigroup property to obtain

$$
\begin{aligned}
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) & \geq \iint_{|z|,|w| \leq r} p_{T(r) / 4}^{L_{V_{c}}^{\Omega}}(x, z) p_{t-T(r / 2)}^{L_{V_{c}}^{\Omega}}(z, w) p_{T(r) / 4}^{L_{V_{c}}^{\Omega}}(w, y) d z d w \\
& \geq c(r, K) H(t, x) H(t, y) \inf _{|z|,|w|<r} p_{t-T(r / 2)}^{L_{0}^{\Omega}}(z, w)
\end{aligned}
$$

which ends the proof.
We are now in position to describe the exact behavior, in space variable, of the minimal solution of equation (1.1), especially near 0 .

Theorem 3.3. 1. For every $t>0$ there is a finite constant $c_{t}>0$ such that,

$$
\begin{equation*}
u(t, x) \leq c_{t} w_{c}(x), \text { a.e. on } \Omega \tag{3.14}
\end{equation*}
$$

It follows in particular that $u(t, x)$ is essentially bounded away from zero.
2. For every $t>0$, there are finite constants $c_{t}, c_{t}^{\prime}>0$ such that

$$
\begin{equation*}
c_{t}^{\prime} w_{c}(x) \leq u(t, x) \leq c_{t} w_{c}(x), \text { a.e. near } 0 \tag{3.15}
\end{equation*}
$$

Proof. The upper bound (3.14) follows from Theorem 3.1-4). Let us now prove the lower bound.
Let $K$ be a compact subset of $\Omega$ containing 0 such that the Lebesgue measure of the set $\left\{x \in K: u_{0}(x)>0\right\}$ is nonnegative.
Let $\kappa_{t}$ be as in (3.12), then

$$
\begin{aligned}
u(t, x) & =\int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(x, y) u_{0}(y) d y \geq \int_{K} p_{t}^{L_{V_{c}}^{\Omega}}(x, y) u_{0}(y) d y \\
& \geq \kappa_{t} w_{c}(x) \int_{K} w_{c}(y) u_{0}(y) d y \geq c_{t}^{\prime} w_{c}(x), \text { a.e. on } K,
\end{aligned}
$$

with $c_{t}^{\prime}>0$, which was to be proved.

The local sharp estimate (3.15) leads to a sharp global regularity property of the minimal solution, expressing thereby the smoothing effect of the semigroup $e^{-t L_{V_{c}}^{\Omega}}$.

Proposition 3.4. 1. For every $t>0$, the minimal solution $u$ lies in the space $L^{p}(\Omega)$ if and only if $1 \leq p<\frac{d}{\beta}$.
2. For every $t>0$ and every $2 \leq p<\frac{d}{\beta}$, the operator $e^{-t L_{V_{c}}^{\Omega}}$ maps continuously $L^{2}(\Omega)$ into $L^{p}(\Omega)$.
3. For every $t>0$ and every $\frac{d}{d-\beta}<q<p<\frac{d}{\beta}$, the operator $e^{-t L_{V_{c}}^{\Omega}}$ maps continuously $L^{q}(\Omega)$ into $L^{p}(\Omega)$.
4. The operator $L_{V_{c}}^{\Omega}$ has compact resolvent. Set $\left(\varphi_{k}^{L_{V_{c}}}\right)_{k}$ its eigenfunctions. Then $\left(\varphi_{k}^{L_{V_{c}}}\right)_{k} \subset L^{p}(\Omega)$ for every $p<\frac{d}{\beta}$.

Proof. 1) Assume that $1 \leq p<\frac{d}{\beta}$. Let $R>0$ be large enough so that $\Omega \subset B_{R}$. Using inequality (3.14) we obtain

$$
\int_{\Omega}(u(t, x))^{p} d x \leq c_{t} \int_{\Omega}|x|^{-p \beta} d x \leq c_{t} \int_{B_{R}}|x|^{-p \beta} d x \leq c_{t} \int_{0}^{R} r^{-p \beta+d-1} d r<\infty
$$

Conversely assume that $u \in L^{p}(\Omega)$ for some $p \geq 1$. Let $r>0$ be small enough so that $\overline{B_{r}} \subset \Omega$. Then by inequality (3.15) we get
$C \int_{0}^{r} s^{-p \beta+d-1} d s=C \int_{B_{r}}|x|^{-p \beta} d x \leq \int_{B_{r}}(u(t, x))^{p} d x \leq \int_{\Omega}(u(t, x))^{p} d x<\infty$.
But this is possible only if $p<\frac{d}{\beta}$.
2) Let $u_{0} \in L^{2}(\Omega)$ and $p$ as described in the assertion. We recall that $e^{-t L_{V_{c}}^{\Omega}} u_{0}$ is given by

$$
e^{-t L_{V_{c}}^{\Omega}} u_{0}(x)=\int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(x, y) u_{0}(y) d x d y, t>0, \text { a.e. } x .
$$

Owing to the upper bounds (3.3)-(3.4) we obtain

$$
\left|e^{-t L_{V_{c}}^{\Omega}} u_{0}(x)\right|^{p} \leq c_{t} w_{c}^{p}(x)\left(\int_{\Omega} w_{c}(y)\left|u_{0}(y)\right| d y\right)^{p}
$$

Hence

$$
\begin{equation*}
\int_{\Omega}\left|e^{-t L_{V_{c}}^{\Omega}} u_{0}(x)\right|^{p} d x \leq c_{t}\left(\int_{\Omega} w_{c}\left|u_{0}\right| d x\right)^{p} \int_{\Omega} w_{c}^{p} d x \leq c_{t}\left(\int_{\Omega} u_{0}^{2} d x\right)^{p / 2} \tag{3.16}
\end{equation*}
$$

Consequently the semigroup $e^{-t L_{V_{c}}^{\Omega}} u_{0}, t>0$ is bounded from $L^{2}(\Omega)$ into $L^{p}(\Omega)$. 3) Let $2<r<d / \beta$ and $r^{\prime}$ be the conjugate of $r$. Assertion 2 together with
the symmetry of $e^{-t L_{V_{c}}^{\Omega}} u_{0}$ imply the boundedness of $e^{-t L_{V_{c}}^{\Omega}} u_{0}$ from $L^{r^{\prime}}(\Omega)$ into $L^{2}(\Omega)$. Using assertion 2 once again and the semigroup property, we obtain that $e^{-t L_{V_{c}}^{\Omega}} u_{0}$ operates as a bounded map from $L^{r^{\prime}}(\Omega)$ into $L^{r}(\Omega)$. Let us observe that $r^{\prime}>\frac{d}{d-\beta}$.
Using Riesz-Thorin interpolation theorem we obtain boundedness from $L^{q}(\Omega)$ into $L^{p}(\Omega)$ where

$$
1 / q=\theta / 2+(1-\theta) / r^{\prime} \text { and } 1 / p=\theta / 2+(1-\theta) / r, \theta \in(0,1)
$$

Obviously $p$ and $q$ fulfill all properties mentioned in the assertion.
4) We claim that for each $t>0$ the operator $e^{-t L_{V_{c}}^{\Omega}}$ is a Hilbert-Schmidt. Indeed, the upper bound (3.3) leads to

$$
\int_{\Omega} \int_{\Omega}\left(p_{t}^{L_{V_{c}}^{\Omega}}\right)^{2}(x, y) d x d y \leq c_{t}\left(\int_{\Omega} w_{c}^{2}(x) d x\right)^{2}<\infty
$$

and the claim is proved. Hence $L_{V_{c}}^{\Omega}$ has compact resolvent. The claim about the eigenfunctions follows from assertion 2.

Remark 3.5. We emphasize that $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ is not closed. Indeed, utilizing the inequality $p_{t}^{L_{V_{c}}^{\Omega}} \geq p_{t}^{L_{0}^{\Omega}}$ we conclude that the semigroup $e^{-t L_{V_{c}}^{\Omega}}$ is irreducible for each $t>0$. Consequently, the smallest eigenvalue of $L_{V_{c}}^{\Omega}$ is nondegenerate, i.e., its eigenspace has dimension one and is generated by a nonnegative function, say $\varphi^{L_{V_{c^{*}}}^{\Omega}}$. If $\dot{\mathcal{E}}_{\Omega}^{V_{c^{*}}}$ were closed, then the ground state $\varphi^{L_{V_{c^{*}}}^{\Omega}}$ would be in the space $W_{0}^{\alpha / 2,2}(\Omega)$. Hence by Hardy's inequality we would get $\int_{\Omega}\left(\varphi^{L_{V_{c^{*}}}^{\Omega}}\right)^{2}(x) V_{c^{*}}(x) d x<\infty$. However, from the lower bound (3.12), we obtain for each small ball $B$ around zero

$$
\int_{\Omega}\left(\varphi^{L_{V_{c^{*}}}^{\Omega}}\right)^{2}(x) V_{c^{*}}(x) d x \geq C \int_{B} w_{c^{*}}^{2}(x) V_{c^{*}}(x) d x=\infty
$$

leading to a contradiction.
The already established upper estimate for the heat kernel enables one to extend the semigroup to a larger class of initial data.

Theorem 3.6. 1. The semigroup $e^{-t L_{V_{c}}^{\Omega}}, t>0$ extends to a bounded linear operator from $L^{1}\left(\Omega, w_{c} d x\right)$ into $L^{2}(\Omega)$.
2. The semigroup $e^{-t L_{V_{c}}^{\Omega}}, t>0$ extends to a bounded linear operator from $L^{p}\left(\Omega, w_{c} d x\right)$ into $L^{p}(\Omega)$ for every $1 \leq p<\infty$.
3. The semigroup $e^{-t L_{V_{c}}}, t>0$ extends to a bounded linear semigroup from $L^{p}\left(\Omega, w_{c} d x\right)$ into $L^{p}\left(\Omega, w_{c} d x\right)$ for every $1 \leq p<d / 3$.

Proof. Having estimate (3.3) in hands, a straightforward computation yields

$$
\int_{\Omega}\left(e^{-t L_{V_{c}}^{\Omega}} u_{0}\right)^{2} d x \leq c_{t} \int_{\Omega} w_{c}^{2} d x \cdot\left(\int_{\Omega}\left|u_{0}\right| w_{c} d y\right)^{2}, \text { for all } t>0
$$

and assertion 1. is proved.
Similarly, using Hölder's inequality we achieve

$$
\begin{aligned}
\left|e^{-t L_{V_{c}}^{\Omega}} u_{0}(x)\right|^{p} & \leq \int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(x, y) d y \int_{\Omega} p_{t}^{L_{V_{c}}^{\Omega}}(x, y)\left|u_{0}\right|^{p} d y \\
& \leq c_{t} w_{c}^{2}(x) \int_{\Omega} w_{c}(y) d y \int_{\Omega}\left|u_{0}\right|^{p} w_{c} d y \leq c_{t}\left(\int_{\Omega}\left|u_{0}\right|^{p} w_{c} d y\right) w_{c}^{2}(x)
\end{aligned}
$$

Integrating w.r.t. $x$, we obtain assertion 2.
Assertion 3. can be proved in the same way.

## 4. BLOW-UP OF NONNEGATIVE SOLUTIONS ON OPEN SETS IN THE SUPERCRITICAL CASE

In this section we prove that for $c>c^{*}$ any nonnegative solution of the heat equation (1.1) on arbitrary open sets containing zero blows up completely and instantaneously. The main ingredient for the proof is the lower bound for the heat kernel in the critical case. Our result accomplishes those corresponding to bounded Lipschitz sets [4] and to $\Omega=\mathbb{R}^{d}$ [6].
Besides, our proof deviates from the one developed in [4]. Therein the proof relies on the boundary behavior of the ground state of $L_{0}^{\Omega}$.
Henceforth we fix an open set $\Omega \subset \mathbb{R}^{d}$ containing zero and $c>0$.
Let $V \in L^{1}(\Omega, d x)$ be a nonnegative potential. We set $W_{k}:=V \wedge k$ and $\left(P_{k}\right)$ the heat equation corresponding to the Dirichlet fractional Laplacian perturbed by $-W_{k}$ instead of $-V$ :

$$
\left(P_{k}\right):\left\{\begin{array}{l}
-\frac{\partial u}{\partial t}=L_{0}^{\Omega} u-W_{k} u \quad \text { in }(0, T) \times \Omega  \tag{4.1}\\
\left.u(t, \cdot)=0 \text { in } \quad \Omega^{c} \text { for all } t \in\right] 0, T[ \\
u(0, x)=u_{0}(x) \quad \text { for } \text { a.e. } x \in \mathbb{R}^{d}
\end{array}\right.
$$

Denote by $L_{k}$ the selfadjoint operator associated to the closed quadratic form $\mathcal{E}_{\Omega}^{W_{k}}:$

$$
\operatorname{dom}\left(\mathcal{E}_{\Omega}^{W_{k}}\right)=W_{0}^{\alpha / 2,2}(\Omega), \mathcal{E}_{\Omega}^{W_{k}}[f]=\mathcal{E}_{\Omega}[f]-\int_{\Omega} f^{2}(x) W_{k}(x) d x
$$

Set $u_{k}(t):=e^{-t L_{k}} u_{0}, t \geq 0$ the nonnegative semigroup solution of problem $\left(P_{k}\right)$. Then $u_{k}$ satisfies Duhamel's formula:

$$
u_{k}(t, x):=e^{-t L_{k}} u_{0}(x)=e^{-t L_{0}^{\Omega}} u_{0}(x)
$$

$$
\begin{equation*}
+\int_{0}^{t} \int_{\Omega} p_{t-s}^{L_{0}^{\Omega}}(x, y) u_{k}(s, x) V_{k}(y) d y d s, t>0, \text { a.e. } x \in \Omega \tag{4.2}
\end{equation*}
$$

Let us list the properties of the sequence $\left(u_{k}\right)$ and establish existence of the minimal solution.

Lemma 4.1. i) The sequence $\left(u_{k}\right)$ is increasing.
ii) If $u$ is any nonnegative solution of problem (2.6) then $u_{k} \leq u$, for allk. Moreover $u_{\infty}:=\lim _{k \rightarrow \infty} u_{k}$ is a nonnegative solution of problem (2.6) as well.

The proof runs as the one corresponding to the case of bounded domains (see [4]) so we omit it.
We recall that we use the notation $u(t)$ for the minimal solution $u_{\infty}(t)$.
Remark 4.2. Let $0<c \leq c^{*}$. Owing to the lower bound (3.12) together with the fact that $p_{t}^{L_{V_{c}}^{\Omega}} \geq p_{t}^{L_{V_{c}}^{B}}$ for any open ball $B \subset \Omega$ we automatically get: for every compact subset $K \subset \Omega$ and every $t>0$ there is a finite constant $\kappa_{t}=\kappa_{t}(K)>0$ such that

$$
\begin{equation*}
p_{t}^{L_{V_{c}}^{\Omega}}(x, y) \geq \kappa_{t} w_{c}(x) w_{c}(y), \text { a.e. on } K \times K \tag{4.3}
\end{equation*}
$$

Hence

$$
\begin{equation*}
u(t, x) \geq c_{t} w_{c}(x), \text { a.e. near } 0 \text { for all } t>0 \tag{4.4}
\end{equation*}
$$

Thus $u$ has a singularity at 0 for all $t>0$.
Let us establish a Duhamel formula for the minimal solution.
Lemma 4.3. Let $u$ be the minimal solution of equation (1.1) with $c \geq c^{*}$. Then $u$ satisfies the following Duhamel's formula:

$$
\begin{array}{r}
u(t, x)=e^{-t L_{V_{c^{*}}}^{\Omega}} u_{0}(x)+\left(c-c^{*}\right) \int_{0}^{t} \int_{\Omega} p_{t-s}^{L_{c^{*}}}(x, y) u(s, y)|y|^{-\alpha} d s d y  \tag{4.5}\\
t>0, \text { a.e. } x
\end{array}
$$

Proof. Set $W_{k}^{*}=V_{c^{*}} \wedge k$. Then
$u_{k}(t, x)=e^{-t L_{k}^{\Omega}} u_{0}(x)=e^{-t L_{W_{k}^{*}}^{\Omega}} u_{0}(x)+\int_{0}^{t} \int_{\Omega} p_{t-s}^{L_{W_{k}^{*}}^{\Omega}}(x, y) u_{k}(s)\left(W_{k}-W_{k}^{*}\right) d y d s$.
An easy calculation shows that the sequence $\left(W_{k}-W_{k}^{*}\right)$ is increasing. As the minimal solution is the limit of the $u_{k}$ 's, the result follows by application of monotone convergence theorem.

We have so far collected enough material to announce the main theorem of this section.

THEOREM 4.4. Assume that $c>c^{*}$. Then any nonnegative solution of the heat equation (1.1) blows up completely and instantaneously.

Proof. Assume that a nonnegative solution $u$ exists. Relying on Lemma 4.1, we may and shall suppose that $u=u_{\infty}$. Put $c^{\prime}=c-c^{*}>0$ and let $B$ be an open ball centered at 0 such that $B \subset \Omega$ and $u_{0} \not \equiv 0$ on $B$.
Owing to the fact that $p_{t}^{L_{V_{c^{*}}}^{\Omega}} \geq p_{t}^{L_{V_{c^{*}}}^{B}}$, the identity (4.5) together with the lower bound (3.12) for $p_{t}^{L_{V_{c^{*}}}^{B}}$ lead to

$$
\begin{array}{r}
u(t, x) \geq e^{-t L_{V_{c^{*}}}^{\Omega}} u_{0}(x) \geq e^{-t L_{V_{c^{*}}}^{B}} u_{0}(x) \geq c_{t} w_{c^{*}}(x), \\
\text { a.e. on } B^{\prime}:=\frac{1}{2} B, t>0 . \tag{4.6}
\end{array}
$$

Using (4.3) and (3.12), once again together with the latter lower bound we obtain

$$
\begin{align*}
u(t, x) & \geq c^{\prime} \int_{0}^{t} \int_{B} p_{t-s}^{L_{c^{*}}^{B}}(x, y) u(s, y)|y|^{-\alpha} d s d y \\
& \geq c^{\prime} \int_{0}^{t} c_{s} \int_{B^{\prime}} p_{t-s}^{L_{c^{*}}^{B}}(x, y) w_{c^{*}}(y)|y|^{-\alpha} d s d y  \tag{4.7}\\
& \geq c^{\prime} w_{c^{*}}(x) \int_{0}^{t} c_{s}^{\prime} \int_{B^{\prime}} w_{c^{*}}^{2}(y)|y|^{-\alpha} d s d y
\end{align*}
$$

However,

$$
\begin{equation*}
\int_{B^{\prime}} w_{c^{*}}^{2}(y)|y|^{-\alpha} d y=\infty \tag{4.8}
\end{equation*}
$$

and the solution blows up, which finishes the proof.

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