Dedicated to the memory of Nicu Boboc

ELLIPTIC EQUATIONS DEGENERATING AT INFINITY AND UNIQUENESS OF PROBABILITY SOLUTIONS TO THE KOLMOGOROV EQUATION

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Abstract. We obtain new sufficient conditions for nonuniqueness of probability solutions to stationary Kolmogorov equations. We also study associated second order elliptic equations for the densities of different solutions with respect to a fixed solution and in the case where the fixed solution satisfies the Poincaré inequality we prove a criterion for the existence of a nonconstant solution in the weighted Sobolev space with respect to the fixed solution.

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In this paper we consider the elliptic equation

$$\operatorname{div}(\varrho\nabla v) - \langle a, \nabla v\rangle = 0 \tag{1}$$

with respect to the function $v \in C^\infty(\mathbb{R}^d)$ with $[v^2 + |\nabla v|^2]\varrho \in L^1(\mathbb{R}^d)$, where $\varrho, a^i \in C^\infty(\mathbb{R}^d)$ are given functions such that

$$\varrho(x) > 0, \quad \int_{\mathbb{R}^d} \varrho(x)\, dx = 1, \quad \operatorname{div} a = \sum_{i=1}^d \partial_{x_i} a^i = 0.$$

We do not assume any boundedness or additional integrability of ϱ and a^i on all of $\mathbb{R}^d$, and there are no restrictions on the behaviour of these functions at infinity. In particular, the function ϱ can tend to zero at infinity in arbitrary way. It is well-known that every generalized solution v is a smooth function. In addition, constants obviously satisfy this equation. It is of interest to find necessary and sufficient conditions for the existence of a nonconstant solution integrable with weight ϱ. This problem is addressed in our paper. In particular, we obtain a criterion for the existence of a nonconstant solution belonging to the weighted Sobolev space with respect to the measure $\mu = \varrho\, dx$.

Equation (1) arises naturally in the study of the Kolmogorov equation

$$\Delta\varrho - \operatorname{div}(b\varrho) = 0 \tag{2}$$

with respect to ϱ. We shall see below that if this equation has two different positive solutions, then their ratio satisfies equation (1) with the field a indicated below.

We assume throughout that $b^i \in C^\infty(\mathbb{R}^d)$. An important role in the theory of diffusion processes is played by probability densities ϱ satisfying equation (2). The corresponding probability measure $\mu = \varrho\,dx$ is a stationary measure of some random process for which the generator of the transition semigroup coincides on $C_0^\infty(\mathbb{R}^d)$ with the operator $\Delta + \langle b, \nabla\rangle$ (see [5, Chapter 5], [8]). In general, such a process can explode in finite time. The key questions in the theory of Kolmogorov equations are the existence and uniqueness of probability solutions, that is, solutions which are probability densities. A typical existence and uniqueness condition is the known Hasminskii condition:

$$\langle b(x), x\rangle \to -\infty \qquad \text{as } |x| \to \infty.$$

Generalizations of this condition and other sufficient conditions for existence and uniqueness are presented in detail in [5, Chapters 2, 4, 5]. If a probability solution $\mu = \varrho\,dx$ to equation (2) exists, then in order to guarantee its uniqueness in place of the Hasminskii condition it suffices to have a function $V \in C^2(\mathbb{R}^d)$ such that

$$\lim_{|x|\to\infty} V(x) = +\infty, \quad \Delta V(x) + \langle b(x), \nabla V(x)\rangle \le CV(x)$$

for some constant $C > 0$. Then any nonnegative solution $v \in L^1(\mu)$ to (1) is constant. If $|\nabla V| \le CV$ and the opposite estimate $\Delta V(x) + \langle b(x), \nabla V(x)\rangle \ge -CV(x)$ holds, then in $L^1(\mu)$ there are no nonconstant solutions to (1). To exclude nonconstant nonnegative solutions in $L^1(\mu)$ is also sufficient to require that either $|b(x)|/(1+|x|)$ or $a/\varrho = b - \nabla\varrho/\varrho$ is in $L^1(\mu)$. In particular, this is true if a/ϱ is bounded.

As shown in the papers [6], [7], [13] and in the book [5, Chapter 4], without additional conditions a probability solution to equation (2) can be nonunique. Moreover, the simplex of probability solutions can be infinite-dimensional. The method of obtaining sufficient conditions for uniqueness and constructing examples of nonuniqueness proposed in [13] is based on renorming solutions when one probability solution ϱ to equation (2) is fixed and any other solution is represented as the product $v \cdot \varrho$, so that one has to study densities v of other solutions with respect to this fixed solution ϱ. The function v satisfies equation (1) with the vector field

$$a = b\varrho - \nabla\varrho.$$

Thus, if equation (1) has only constant solutions in the class of nonnegative functions integrable with respect to the measure $\mu = \varrho\,dx$, then ϱ is a unique probability solution to the Kolmogorov equation (2), and if equation (1) has a nonconstant nonnegative solution v integrable with respect to the measure $\mu = \varrho\, dx$, then ϱ and const $v\varrho$ are two different probability solutions to equation (2) with b related to a by the formula

$$b = \frac{a}{\varrho} - \frac{\nabla \varrho}{\varrho}.$$

The key role in the study of equation (1) is played by the bilinear form

$$[f, g] = \int_{\mathbb{R}^d} \langle a, \nabla f \rangle g\, dx. \tag{3}$$

Since $\operatorname{div} a = 0$, by the integration by parts formula for all functions $f, g \in C_0^\infty(\mathbb{R}^d)$ the equality $[f, g] = -[g, f]$ is true, that is, the form (3) is skew-symmetric on the space $C_0^\infty(\mathbb{R}^d)$. The importance of this form in the problems related to uniqueness of solutions to elliptic equations was first noted by V.V. Zhikov in his well-known paper [14], where he studied the equation

$$\operatorname{div}(\nabla v - av) = f, \quad \operatorname{div} a = 0,$$

on a bounded domain. He proved that, in the case of an unbounded vector field a, a solution to the Dirichlet problem with zero boundary condition is in general nonunique, moreover, a sufficient condition for nonuniqueness is the existence of a function φ such that $[\varphi, \varphi] < 0$, where $[f, g]$ is defined as in (3), but the integral is taken over the bounded domain (not over the whole space $\mathbb{R}^d$ as in our case). In the paper [13], with the aid of the form $[f, g]$ sufficient conditions for the existence of a nonconstant bounded solution to equation (1) were given. Moreover, sufficient conditions for the existence of infinitely many linearly independent nonconstant bounded solutions to equation (1) were obtained, which enables one to construct examples of stationary Kolmogorov equations with infinite-dimensional simplices of probability solutions. It should be noted that the theorem on existence of infinitely many linearly independent solutions in the paper [13] and in the book [5] is formulated with an inaccuracy (reproduced also in [4] and [12]), which consists in omitting some assumption used in the proof (although for constructing examples correct considerations are used in the cited works), which we discuss in detail at the end of this paper. The conditions for existence of nonconstant solutions or the conditions under which there are no nonconstant solutions presented in the cited works are only sufficient, but no criterion (i.e., a necessary and sufficient condition) for the existence of nonconstant solutions was obtained there. In the present paper we give the first criterion of this sort for the weighted Sobolev space $W^{2,1}(\mu)$, where $\mu = \varrho\, dx$, under the assumption that the measure μ satisfies the Poincaré inequality. Moreover, we give a criterion for the space of solutions to be infinite-dimensional, obtain some generalizations of conditions from [13] and correct the formulations from [13] and [5] mentioned above.

Throughout $\mu = \varrho\, dx$ and $W^{2,1}(\mu)$ denotes the weighted Sobolev space obtained by completing $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{W^{2,1}(\mu)} = \|u\|_{L^2(\mu)} + \|\nabla u\|_{L^2(\mu)}.$$

Note that $W^{2,1}(\mu)$ is a Hilbert space. Since ϱ is a smooth and strictly positive function, every function $u \in W^{2,1}(\mu)$ locally belongs to the usual Sobolev space, i.e., $u \in W^{2,1}_{loc}(\mathbb{R}^d)$. Under our assumptions about ϱ the class $W^{2,1}(\mu)$ consists of all $u \in W^{2,1}_{loc}(\mathbb{R}^d)$ such that $u, |\nabla u| \in L^2(\mu)$.

We say that a function $\varphi \in W^{1,1}_{loc}(\mathbb{R}^d)$ satisfies condition (H) if there exists a number $C(\varphi) > 0$ such that for every function $\psi \in C_0^\infty(\mathbb{R}^d)$ there holds the estimate

$$[\varphi, \psi] = \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle \psi\, dx \le C\|\nabla\psi\|_{L^2(\mu)}. \tag{H}$$

In this case the mapping $\psi \mapsto [\varphi, \psi]$ extends to a continuous linear functional on the space $W^{2,1}(\mu)$ satisfying the same bound and denoted by the same symbol. However, the extension is not always given by the same integral formula, since the function $\langle a, \nabla\varphi\rangle\psi$ need not be integrable for ψ with noncompact support. We observe that constants belong to the kernel of this functional due to the bound above (but again, the integral above need not exist for $\psi = 1$). Thus, there holds the equality $[\varphi, 1] = 0$. Since a is a smooth vector field, it is clear that, for any function $\psi \in W^{2,1}(\mu)$ vanishing outside some ball, the value $[\varphi, \psi]$, which is understood precisely as the value of the corresponding extended functional, can be calculated by formula (3).

Lemma 1. *Suppose that v is a nonconstant solution of equation* (1) *in the space $W^{2,1}(\mu)$. Then v satisfies condition* (H), $[v, 1] = 0$ *and*

$$[v, v] = -\int_{\mathbb{R}^d} |\nabla v|^2\, d\mu < 0.$$

Proof. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Multiplying the equation by ψ and integrating by parts, we obtain

$$\int_{\mathbb{R}^d} \langle a, \nabla v\rangle \psi\, dx = -\int_{\mathbb{R}^d} \langle \nabla v, \nabla\psi\rangle \varrho\, dx. \tag{4}$$

The right-hand side of this equality is estimated from above by the quantity $\|v\|_{W^{2,1}(\mu)}\|\nabla\psi\|_{L^2(\mu)}$. Therefore, the function v satisfies condition (H). It has already been noted above that $[v, 1] = 0$. We now observe that

$$[v, v] = -\int_{\mathbb{R}^d} |\nabla v|^2 \varrho\, dx. \tag{5}$$

This follows from (4) by approximating v by smooth compactly supported functions ψ and using the continuity of the functional $\psi \mapsto [v, \psi]$. Since the function v is not a constant, the right-hand side is strictly negative. $\square$

We recall that a probability measure σ on $\mathbb{R}^d$ satisfies the Poincaré inequality if there exists a number $C_\sigma > 0$ such that for every function $f \in C_0^\infty(\mathbb{R}^d)$ there holds the inequality

$$\int_{\mathbb{R}^d} \Bigl(f - \int_{\mathbb{R}^d} f \, d\sigma \Bigr)^2 d\sigma \le C_\sigma \int_{\mathbb{R}^d} |\nabla f|^2 \, d\sigma.$$

Note that every probability measure of the form $\sigma = e^{-V} dx$, where V is a convex function, satisfies the Poincaré inequality (see, for example, [2] and [1]).

The next assertion gives a criterion for the existence of a nonconstant solution of equation (1) in the class $W^{2,1}(\mu)$.

THEOREM 2. *Suppose that $\mu = \varrho \, dx$ satisfies the Poincaré inequality. Equation* (1) *has a nonconstant solution in $W^{2,1}(\mu)$ if and only if there exists a function $\varphi \in W^{2,1}(\mu)$ such that φ satisfies condition* (H) *and $[\varphi, \varphi] < 0$.*

Proof. It follows from the lemma that this condition is necessary. We now show that it is sufficient. Suppose that there is a function φ with the stated properties. Let B_n be the ball of radius n centered at zero. Set

$$f = -\mathrm{div}(\varrho \nabla \varphi) + \langle a, \nabla \varphi \rangle = -\mathrm{div}(\varrho \nabla \varphi - \varphi a).$$

According to [9, Theorem 8.3] there exists a function $u_n \in W_0^{2,1}(B_n)$ such that

$$\mathrm{div}(\varrho \nabla u_n) - \langle a, \nabla u_n \rangle = f. \tag{6}$$

We observe that

$$\int_{B_n} |\nabla u_n|^2 \varrho \, dx = -\int_{B_n} \langle a, \nabla u_n \rangle u_n \, dx - \int_{B_n} \langle \nabla \varphi, \nabla u_n \rangle \varrho \, dx - \int_{B_n} \langle a, \nabla \varphi \rangle u_n \, dx.$$

The identity $\mathrm{div}\, a = 0$ and the integration by parts formula yield the equality

$$\int_{B_n} \langle a, \nabla u_n \rangle u_n \, dx = 0.$$

In addition, there holds the estimate

$$-\int_{B_n} \langle \nabla \varphi, \nabla u_n \rangle \varrho \, dx \le \frac{1}{2} \int_{B_n} |\nabla \varphi|^2 \varrho \, dx + \frac{1}{2} \int_{B_n} |\nabla u_n|^2 \varrho \, dx.$$

Therefore, the function u_n satisfies the inequality

$$\int_{B_n} |\nabla u_n|^2 \varrho \, dx \le \int_{B_n} |\nabla \varphi|^2 \varrho \, dx - 2 \int_{B_n} \langle a, \nabla \varphi \rangle u_n \, dx.$$

Extending the function u_n by zero outside of B_n we can assume that $u_n \in W^{2,1}(\mu)$. Then

$$\int_{B_n} \langle a, \nabla\varphi\rangle u_n\, dx = \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle u_n\, dx = [\varphi, u_n],$$

where $[\varphi, u_n]$ denotes the value of the corresponding functional on the function u_n. We have

$$\int_{\mathbb{R}^d} |\nabla u_n|^2 \varrho\, dx \le \int_{\mathbb{R}^d} |\nabla\varphi|^2 \varrho\, dx - 2[\varphi, u_n].$$

Both parts of this inequality do not change if we add constants to u_n. Replacing the function u_n by

$$u_n - \int_{\mathbb{R}^d} u_n \varrho\, dx,$$

we can assume that

$$\int_{\mathbb{R}^d} u_n \varrho\, dx = 0.$$

Note that now u_n belongs to $W^{2,1}(\mu)$, but not to $W_0^{2,1}(B_n)$. However, on the ball B_n the function u_n is still a solution of equation (6). By asumption μ satisfies the Poincaré inequality. We denote the corresponding constant by C_μ. Then

$$\int_{\mathbb{R}^d} u_n^2 \varrho\, dx \le C_\mu \int_{\mathbb{R}^d} |\nabla u_n|^2 \varrho\, dx.$$

Therefore, we have the following estimate on the norm $\|u_n\|_{W^{2,1}(\mu)}$:

$$\|u_n\|^2_{W^{2,1}(\mu)} \le 2(C_\mu + 1)\|\varphi\|_{W^{2,1}(\mu)} - 4(C_\mu + 1)[\varphi, u_n].$$

Since by property (H) there holds the inequality

$$-[\varphi, u_n] \le C\|u_n\|_{W^{2,1}(\mu)},$$

we obtain

$$-4(C_\mu + 1)[\varphi, u_n] \le 8(C_\mu + 1)^2 C^2 + \frac{1}{2}\|u_n\|^2_{W^{2,1}(\mu)}$$

and there holds the estimate

$$\|u_n\|^2_{W^{2,1}(\mu)} \le 4(C_\mu + 1)\|\varphi\|_{W^{2,1}(\mu)} + 16(C_\mu + 1)^2 C^2.$$

Passing to a subsequence, we can assume that the sequence $\{u_n\}$ converges weakly in $W^{2,1}(\mu)$ to some function u. It is clear that u satisfies the equation

$$\operatorname{div}(\varrho\nabla u) - \langle a, \nabla u\rangle = f.$$

Since $\psi \mapsto [\varphi, \psi]$ is a continuous linear functional, the sequence of numbers $[\varphi, u_n]$ converges to $[\varphi, u]$. In addition,

$$\int_{\mathbb{R}^d} |\nabla u|^2 \varrho\, dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^d} |\nabla u_n|^2 \varrho\, dx.$$

Therefore, the function u satisfies the inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 \varrho\, dx \leq \int_{\mathbb{R}^d} |\nabla \varphi|^2 \varrho\, dx - 2[\varphi, u].$$

We observe that if $u = -\varphi + const$, then $0 \leq [\varphi, \varphi]$, which contradicts the hypothesis of the theorem. Thus, the function $v = u + \varphi$ is not a constant and satisfies equation (1). □

Corollary 3. *Suppose in addition to the hypotheses of Theorem* 2 *that* $\varphi \in L^\infty(\mathbb{R}^d)$. *Then there exists a bounded nonconstant solution* $v \in W^{2,1}(\mu)$ *of equation* (1).

Proof. Let $|\varphi| \leq M$, where M is a constant. We observe that the function $v_n = u_n + \varphi + M$ on the ball B_n is a solution to the Dirichlet problem for equation (1), moreover, $\min\{v_n, 0\} \in W_0^{2,1}(B_n)$, that is, $v_n \geq 0$ on ∂B_n. By the maximum principle (see [9, Theorem 8.1]) we have $v_n \geq 0$, hence $u_n \geq -2M$. Similarly we show that $u_n \leq 2M$. Subtracting from u_n the constant

$$\int u_n \varrho\, dx,$$

we obtain a new function not exceeding $3M$ in absolute value. Therefore, the limit function u satisfies the estimate $|u| \leq 3M$. It remains to observe that the solution v constructed in Theorem 2 equals $u + \varphi$ and hence is a bounded function. □

Remark 4. If we have a bounded nonconstant solution v of equation (1), then by adding a sufficiently large constant we can assume that v is a positive function. In this case there exists a number $c > 0$ such that the function $cv\varrho$ is a probability solution to the Kolmogorov equation (2) with $b = a/\varrho - \nabla\varrho/\varrho$ different from ϱ.

Remark 5. Suppose that the measure $\mu = \varrho\, dx$ satisfies the Poincaré inequality. If a function $\varphi \in W^{1,1}_{loc}(\mathbb{R}^d)$ is such that

$$\langle a, \nabla\varphi\rangle \varrho^{-1/2} \in L^2(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle\, dx = 0,$$

then φ satisfies condition (H).

Indeed, for every function $\psi \in C_0^\infty(\mathbb{R}^d)$ there holds the equality

$$\int \langle a, \nabla\varphi\rangle \psi\, dx = \int \langle a, \nabla\varphi\rangle \Bigl(\psi - \int \psi\varrho\, dx\Bigr)\, dx,$$

where the right-hand side is bounded by

$$C\Bigl(\int |\langle a, \nabla\varphi\rangle|^2 \varrho^{-1}\, dx\Bigr)^{1/2} \Bigl(\int |\nabla\psi|^2 \varrho\, dx\Bigr)^{1/2}.$$

Remark 6. The condition $[\varphi,\varphi]<0$ agrees with the classical conditions on low order terms of the elliptic equation with a degenerate principal part. Suppose that we investigate equation (2) on a bounded domain Ω with smooth boundary in place of the whole space $\mathbb{R}^d$ and that ϱ, a^i are smooth functions on the closure $\overline{\Omega}$ of this domain. Suppose that ϱ is strictly positive in the interior of Ω and vanishes along with its first order derivatives on $\partial\Omega$. It is known (see [11, Chapter 1]) that in this case the boundary condition in the Dirichlet problem can be posed only on the part of the boundary where the vector field a has a negative projection on the outer normal ν, i.e., $\langle a,\nu\rangle<0$. Our condition $[\varphi,\varphi]<0$ can be written in the following form:

$$\int_\Omega \langle a,\nabla\varphi\rangle\varphi\,dx=\frac{1}{2}\int_{\partial\Omega}\langle a,\nu\rangle\varphi^2\,dS<0.$$

This inequality implies the existence of a part of the boundary $\partial\Omega$ of positive surface measure on which $\langle a,\nu\rangle<0$. Note that the theory of degenerate elliptic equations can be used directly for constructing examples of nonuniqueness of probability solutions of the Kolmogorov equation (see [10]).

Let us give an example of application of Theorem 2.

Example 7. Let

$$d=2,\quad a(x,y)\equiv(0,-1),\quad \varrho(x,y)=(2\pi)^{-1}e^{-(x^2+y^2)/2}.$$

Let us find a function φ satisfying the condition of Theorem 2. We are looking for φ of the form $\varphi(x,y)=G(y)H(x)$. Then $\langle a,\nabla\varphi\rangle=-G'(y)H(x)$. Let $H,g\in C_0^\infty(\mathbb{R})$, where the function g is a probability density and

$$G(y)=\int_{-\infty}^y g(s)\,ds.$$

The function $\langle a,\nabla\varphi\rangle$ has compact support, hence

$$\langle a,\nabla\varphi\rangle\varrho^{-1/2}\in L^2(\mathbb{R}^d).$$

Suppose now that

$$\int H(x)\,dx=0,\quad \int H(x)^2\,dx>0.$$

Then

$$\int\int\langle a,\nabla\varphi\rangle\,dx\,dy=\int\int G'(y)H(x)\,dx\,dy=0.$$

According to Remark 5 the function φ satisfies condition (H). Let us show that $[\varphi,\varphi]<0$. Indeed,

$$[\varphi,\varphi]=-\int\int G'(y)G(y)H(x)^2\,dx\,dy=-\int H(x)^2\,dx<0.$$

Thus, all conditions of Theorem 2 are fulfilled, so the corresponding equation (1) has a nonconstant solution in $W^{2,1}(\mu)$. Moreover, since the obtained function φ is bounded, by Corollary 3 there exists a positive bounded solution of equation (1) in $W^{2,1}(\mu)$.

Let us consider the subspace $S_{\varrho,a} \subset W^{2,1}(\mu)$ of all solutions of equation (1) in $W^{2,1}(\mu)$. It is readily seen that $S_{\varrho,a}$ is a closed subspace.

LEMMA 8. *For all $u, v \in S_{\varrho,a}$ there holds the equality*

$$[u,v] = -\int \langle \nabla u, \nabla v\rangle \varrho\, dx. \tag{7}$$

Moreover, if the space $S_{\varrho,a}$ is infinite-dimensional, then there exists a sequence of functions $u_k \in S_{\varrho,a}$ such that $[u_k, u_m] = 0$ if $k \neq m$ and $[u_k, u_k] = -1$.

Proof. Equality (7) follows from (4) similarly to (5) (and also follows from (5)). Let us take a sequence of functions $v_k \in S_{\varrho,a}$ such that they are linearly independent along with the function 1. Let us apply to the functions v_k the usual Gram–Schmidt orthogonalization with respect to the skew-symmetric bilinear form $[u,v]$. We observe that there are no constants $c_1, \ldots, c_n$ such that

$$[v_{n+1} - (c_1v_1 + \cdots + c_nv_n), v_{n+1} - (c_1v_1 + \cdots + c_nv_n)] = 0.$$

Indeed, otherwise the function $v_{n+1} - (c_1v_1 + \cdots + c_nv_n)$ is constant, which contradicts the linear idependence of the functions v_k and 1. $\square$

The next assertion gives a criterion for the space of solutions $S_{\varrho,a}$ to be infinite-dimensional.

THEOREM 9. *Suppose that the measure $\mu = \varrho\, dx$ satisfies the Poincaré inequality. The space $S_{\varrho,a}$ is infinite-dimensional if and only if there exists a sequence of functions $\varphi_k \in W^{2,1}(\mu)$ such that each φ_k satisfies condition* (H), *$[\varphi_k, \varphi_m] = 0$ for different k and m and $[\varphi_k, \varphi_k] = -1$.*

Proof. It follows from the previous lemma that the stated conditions are necessary. Let us prove that they are sufficient. For every natural number N we construct N linearly independent solutions. We repeat the reasoning from the proof of Theorem 2. Let $1 \leq k \leq N$ and

$$f_k = -\mathrm{div}(\varrho\nabla\varphi_k) + \langle a, \nabla\varphi_k\rangle.$$

Let B_n be the ball of radius n centered at zero. Let us consider the function $u_{k,n} \in W_0^{2,1}(B_n)$ satisfying on B_n the equation

$$\mathrm{div}(\varrho\nabla u_{k,n}) + \langle a, \nabla u_{k,n}\rangle = f_k.$$

We extend $u_{k,n}$ by zero outside of B_n and subtract a constant in order to make the integral of $u_{k,n}\varrho$ over $\mathbb{R}^d$ zero. So now $u_{k,n}$ is a constant outside of B_n. As in the proof of Theorem 2, for every k the sequence of functions $u_{k,n}$ is bounded in the norm of $W^{2,1}(\mu)$. We pick a sequence $\{n_j\}$ such that for every $k \le N$ the sequence u_{k,n_j} will converge weakly to some function u_k in $W^{2,1}(\mu)$. The functions $v_k = u_k + \varphi_k$ are nonconstant solutions to equation (1). Let us verify that these functions are linearly independent. Suppose that

$$c_1 v_1 + \cdots + c_N v_N = 0.$$

Let us consider the functions $w_n = c_1 u_{1,n} + \cdots + c_N u_{N,n}$. By our construction, for some constant λ_n the function $\widetilde{w}_n = w_n + \lambda_n$ belongs to $W_0^{2,1}(B_n)$ and satisfies the equation

$$\mathrm{div}(\varrho\nabla\widetilde{w}_n) + \langle a, \nabla\widetilde{w}_n\rangle = f, \quad f = -\mathrm{div}(\varrho\nabla\varphi) + \langle a, \nabla\varphi\rangle, \quad \varphi = c_1\varphi_1 + \cdots + c_N\varphi_N.$$

Therefore, there holds the inequality

$$\int |\nabla\widetilde{w}_n|^2 \varrho\, dx \le \int |\nabla\varphi|^2 \varrho\, dx - 2c_1[\varphi_1, \widetilde{w}_n] - \cdots - 2c_N[\varphi_N, \widetilde{w}_n].$$

Since both sides of this inequality do not change when adding a constant to $\widetilde{w}_n$, the last inequality is true for w_n in place of $\widetilde{w}_n$. Therefore, a subsequence of functions w_{n_j} converges weakly to the function $w = c_1 u_1 + \cdots + c_N u_N$, moreover, for w there holds the estimate

$$\int |\nabla w|^2 \varrho\, dx \le \int |\nabla\varphi|^2 \varrho\, dx - 2c_1[\varphi_1, w] - \cdots - 2c_N[\varphi_N, w].$$

Since $c_1 v_1 + \cdots + c_N v_N = 0$, we obtain $w = -\varphi$ and

$$0 \le c_1[\varphi_1, \varphi] + \cdots + c_N[\varphi_N, \varphi] = -c_1^2 - \cdots - c_N^2,$$

which is only possible if $c_1 = c_2 = \cdots = c_N = 0$. $\square$

Example 10. Let

$$d = 2, \quad a(x, y) \equiv (0, -1), \quad \varrho(x, y) = (2\pi)^{-1} e^{-(x^2+y^2)/2}.$$

Let us find functions φ_k satisfying the conditions of Theorem 9. We are looking for functions of the form $\varphi_k(x, y) = G(y)H_k(x)$. Then $\langle a, \nabla\varphi_k\rangle = -G'(y)H_k(x)$. Let $H_k, g \in C_0^\infty(\mathbb{R})$, where the function g is a probability density and

$$G(y) = \int_{-\infty}^{y} g(s)\, ds.$$

Suppose that

$$\int H_k(x)\, dx = 0, \quad \int H_k(x)^2\, dx = 1, \quad \int H_k(x)H_m(x)\, dx = 0,\ k \ne m.$$

Then the functions φ_k satisfy condition (H), $[\varphi_k, \varphi_k] = -1$ and for $k \neq m$ we have

$$[\varphi_k, \varphi_m] = -\int\int G'(y)G(y)H_k(x)H_m(x)\,dx\,dy = -\int H_k(x)H_m(x)\,dx = 0.$$

Thus, all conditions of Theorem 9 are fulfilled, hence the corresponding equation (1) has an infinite-dimensional space of solutions in $W^{2,1}(\mu)$. Since the functions φ_k are bounded, there are infinitely many linearly independent positive bounded solutions (see Corollary 3), which in turn means that the corresponding Kolmogorov equation has an infinite-dimensional simplex of probability solutions.

In Theorems 2 and 9 we assume that the function ϱ satisfies the Poincaré inequality, but this is difficult to verify in the general case. In addition, in spite of Remark 5, condition (H) is also difficult to verify in the general case. For this reason we prove two additional assertions about existence of nonconstant solutions and existence of infinitely many linearly independent solutions to equation (2). Our next assertion generalizes Theorem 2 from [13] (see also [5, Theorem 4.2.2]). Below the form $[f, g]$ is defined by formula (3).

THEOREM 11. *Suppose that there exists a function $\varphi \in C^2(\mathbb{R}^d)$ such that*

$$\varphi \in L^\infty(\mathbb{R}^d), \quad \langle a, \nabla\varphi\rangle \in L^1(\mathbb{R}^d), \quad |\nabla\varphi|^2\varrho \in L^1(\mathbb{R}^d), \quad [\varphi, 1] = 0, \quad [\varphi, \varphi] < 0.$$

Then there is a nonconstant positive bounded solution $v \in W^{2,1}(\mu)$ of equation (1).

Proof. Note that $[\varphi, 1]$ and $[\varphi, \varphi]$ are defined as usual integrals under our assumptions (we have $\varphi \in W^{2,1}(\mu)$, but we do not assume that φ satisfies condition (H) and do not consider now any extensions of our bilinear form). The main steps of the proof are similar to the reasoning from the proof of Theorem 2, but there is some difference in constructing convergent subsequences and justification of limits. Let v_n be the solution to the Dirichlet problem

$$\operatorname{div}(\varrho\nabla v_n) - \langle a, \nabla v_n\rangle = 0, \quad v_n|_{\partial B_n} = \varphi,$$

on the ball B_n of radius n centered at zero. By the maximum principle

$$|v_n| \le \sup_x |\varphi(x)| =: M.$$

Using a priori estimates for solutions to elliptic equations (see [9, Theorem 8.13]) and the diagonal procedure, we can pick a subsequence $\{v_{n_k}\}$ that converges to some function v uniformly on every ball such that its first and second derivatives also converge uniformly on balls to the respective derivatives of the function v. It is clear that v satisfies equation (1). Set

$$u_{n_k} = v_{n_k} - \varphi.$$

The function u_{n_k} satisfies the equation

$$\operatorname{div}(\varrho\nabla u_{n_k}) - \langle a, \nabla u_{n_k}\rangle = -\operatorname{div}(\varrho\nabla\varphi) + \langle a, \nabla\varphi\rangle$$

and vanishes on ∂B_{n_k}. Multiplying the last equality by u_{n_k} and integrating by parts, we obtain

$$\int_{B_{n_k}} |\nabla u_{n_k}|^2 \varrho\, dx = -\int_{B_{n_k}} \langle \nabla u_{n_k}, \nabla\varphi\rangle\, dx - \int_{B_{n_k}} \langle a, \nabla\varphi\rangle u_{n_k}\, dx,$$

where we also use the equality

$$\int_{B_{n_k}} \langle a, \nabla u_{n_k}\rangle u_{n_k}\, dx = 0.$$

Since

$$2|\langle \nabla u_{n_k}, \nabla\varphi\rangle| \le |\nabla u_{n_k}|^2 + |\nabla\varphi|^2,$$

we obtain

$$\int_{B_{n_k}} |\nabla u_{n_k}|^2 \varrho\, dx \le \int_{B_{n_k}} |\nabla\varphi|^2 \varrho\, dx - 2\int_{B_{n_k}} \langle a, \nabla\varphi\rangle u_{n_k}\, dx.$$

Letting $k \to \infty$, we arrive at the estimate

$$\int_{\mathbb{R}^d} |\nabla v - \nabla\varphi|^2 \varrho\, dx \le \int_{\mathbb{R}^d} |\nabla\varphi|^2 \varrho\, dx - 2\int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle (v - \varphi)\, dx.$$

Suppose that the function v is constant. Since

$$\int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle\, dx = 0,$$

we have

$$0 \le \int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle \varphi\, dx.$$

This contradicts the condition $[\varphi, \varphi] < 0$. Thus, v is a nonconstant bounded solution of class $W^{2,1}(\mu)$, which can be made positive by adding a sufficiently large constant. □

Note that in this theorem, unlike [13, Theorem 2] and [5, Theorem 4.2.2], we do not assume that the first and second derivatives of the function φ are bounded.

The next assertion is a corrected version of [13, Theorem 3] and [5, Theorem 4.2.7]).

THEOREM 12. *Suppose that there exist functions $\varphi_1, \ldots, \varphi_N$ satisfying the conditions of Theorem* 11*. If the quadratic form*

$$Q(c) = \sum_{i,j=1}^{N} [\varphi_i, \varphi_j] c_i c_j, \quad c = (c_1, \ldots, c_N) \in \mathbb{R}^N$$

is negative definite, then there exist N linearly independent nonconstant bounded positive solutions of equation (1) *in* $W^{2,1}(\mu)$.

Proof. We use a reasoning similar to the one from the proof of Theorem 9. Repeating the steps of constructing solutions in the proof of Theorem 11, for every $k \le N$ we consider the functions $v_{k,n}$ that are solutions to the Dirichlet problems

$$\operatorname{div}(\varrho \nabla v_{k,n}) - \langle a, \nabla v_{k,n}\rangle = 0, \quad v_{k,n}|_{\partial B_n} = \varphi_k,$$

on the balls B_n of radius n centered at zero. Pick an increasing sequence of numbers n_j such that for every k the functions v_{k,n_j} will converge to some function v_k uniformly on every ball and their first and second derivatives will also converge uniformly on balls. As in Theorem 11, one can show that v_k are nonconstant bounded solutions of equation (1) in $W^{2,1}(\mu)$. Let us show that the functions $1, v_1, \ldots, v_N$ are linearly independent. Suppose that

$$c_0 + c_1 v_1 + \cdots + c_N v_N = 0. \tag{8}$$

The function $w_{n_j} = c_1 v_{1,n_j} + \cdots + c_N v_{N,n_j}$ solves the Dirichlet problem on the ball B_{n_j} for equation (1) with the boundary condition $\varphi = c_1\varphi_1 + \cdots + c_N\varphi_N$. Moreover, the functions w_{n_j} converge to the function $w = c_1 v_1 + \cdots + c_N v_N$ uniformly on every ball and their first and second derivatives also converge uniformly on balls, because the functions v_{k,n_j} converge to v_k for every k. Repeating the reasonong from the proof of Theorem 11, we conclude that the function w satisfies the inequality

$$\int_{\mathbb{R}^d} |\nabla w - \nabla \varphi|^2 \varrho\, dx \le \int_{\mathbb{R}^d} |\nabla \varphi|^2 \varrho\, dx - 2\int_{\mathbb{R}^d} \langle a, \nabla\varphi\rangle (w-\varphi)\, dx.$$

By our assumption (8) the function w equals the constant $-c_0$ and

$$0 \le [\varphi, \varphi] = \Bigl[\sum_{i=1}^N c_i\varphi_i, \sum_{i=1}^N c_i\varphi_i\Bigr] = Q(c).$$

Therefore, $c_1 = \cdots = c_N = 0$ and $c_0 = 0$. $\square$

Remark 13. In [13, Theorem 3] and [5, Theorem 4.2.7] the following is asserted: if functions $\varphi_1, \ldots, \varphi_{N+1} \in C_b^2(\mathbb{R}^d)$ satisfy the conditions

$$\langle a, \nabla \varphi_j\rangle \in L^1(\mathbb{R}^d), \quad [\varphi_j, 1] = 0, \quad [\varphi_j, \varphi_j] < 0,$$

$v_1, \ldots, v_{N+1}$ are solutions of equation (1) constructed by the functions $\varphi_1, \ldots,$ φ_{N+1} according to the algortithm from [13, Theorem 2] and [5, Theorem 4.2.2], respectively (this is precisely the algortithm from Theorem 11 above) and, in addition, the functions $1, v_1, \ldots, v_N$ are linearly independent, then for the linear idependence of the functions $1, v_1, \ldots, v_N, v_{N+1}$ it suffices that

$$[\varphi_{N+1} - (c_1\varphi_1 + \cdots + c_N\varphi_N), \varphi_{N+1} - (c_1\varphi_1 + \cdots + c_N\varphi_N)] < 0$$

for all $c_1, \ldots, c_N \in \mathbb{R}$ (which is equivalent to the negative definitness of the quadration form Q above). This formulation allows v_k to be an arbitrary solution of equation (1) constructed by the functions φ_k, but in principle there might be many such solutions (although we have no such examples), since they are obtained by picking a subsequence in the sequence of solutions to boundary value problems on balls. Nevertheless, in the paper [13] as well as in the book [5], in the proofs of the corresponding assertion we considered not arbitrary solutions v_k, but constructed adaptively, i.e., by picking an increasing sequence of numbers n_j such that for every $k \le N+1$ the solutions v_{k,n_j} to the Dirichlet problems on the balls B_{n_j} with the boundary condition φ_k converge to v_k. These adapted solutions should be also used in the formulation of the theorem. Hence, after making the indicated changes in the formulations, the proofs in the cited works do not need any changes.

Remark 14. It remains an open question whether the space of solutions to equation (1) can have a finite dimension greater than one. It is shown in the recent paper [3] that in the case $d = 2$ the existence of a nonconstant sufficiently regular solution implies the existence of infinitely many linearly independent solutions. However, the condition $d = 2$ is substantially used in the proof, in particular, it is used there that every smooth divergence free vector field on $\mathbb{R}^2$ has the form $(\partial_y H, -\partial_x H)$ for some smooth function H.

Another open question is whether the Kolmogorov equation (2) must have a probability solution if there is a nonzero signed solution in the class of bounded measures.

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