Dedicated to the memory of Nicu Boboc

HOMOGENIZATION OF SYMMETRIC JUMP PROCESSES IN RANDOM MEDIA

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Communicated by Lucian Beznea

This paper surveys some recent progress in [8] for the study of homogenization of symmetric jump processes in a one-parameter stationary ergodic environment. We further present some additional homogenization results under assumptions that are variants of [8], and identify the limiting effective Dirichlet forms explicitly. The jumping kernels of Dirichlet forms are of α-stable-like with $0 < \alpha < 2$, and the associated coefficients as well as the coefficients of symmetrizing measures are allowed to be degenerate and unbounded.

AMS 2010 Subject Classification: 60G51 60G52, 60J25, 60J75.

Key words: homogenization, symmetric non-local Dirichlet form, stationary ergodic environment, α-stable-like operator.

1. INTRODUCTION

1.1. Background

Consider the behavior of particles in inhomogeneous media. Due to the inhomogeneity, their short time behavior may depend on the location of the particles, whereas their long time behavior often tend to be homogeneous due to the averaging effects. Such an averaging process is called homogenization. The aim of homogenization theory is to provide the macroscopic rigorous characterizations of the microscopically heterogeneous media. It has been a very active research area in mathematics for a long time, and a vast literature exists on this topic, see e.g. [1, 4, 21, 23, 34].

The local inhomogeneity of the media can be naturally modelled by random structures of the media, and the problems of stochastic homogenization have been widely studied. The first rigorous result for second order elliptic operators in divergence forms with stochastically homogeneous random coefficients was independently obtained by Kozlov [24] and by Papanicolaou and Varadhan [25]. The crucial points of their approaches are the construction of the so-called corrector field, which is the solution of certain associated elliptic equations, and the proof of sub-linear growth of the corrector. After these two works, a lot of homogenization problems were investigated for various elliptic and parabolic differential equations.

Because there are various communities in mathematics, the goals in the study of homogenization problems are bit different. In probability community, the typical goal is to establish the invariance principle, namely to show $(\varepsilon X_{t/\varepsilon^2})_{t\geqslant 0}$ converges to a constant time change of Brownian motion as $\varepsilon \to 0$, where $(X_t)_{t\geqslant 0}$ is the random process in the random media. Whereas in PDE community, the goal is to prove that the suitably scaled solution of the resolvent equation on the random media converges to the solution of the resolvent equation on the homogeneous media – we will explain it more precisely in Section 1.2, after Lemma 1.2. It is well-known that the convergence of stochastic processes is equivalent to the tightness of the processes and the convergence of finite dimensional distributions. In the symmetric framework, the latter is (more or less) equivalent to the (pointwise) convergence of the resolvent, so the invariance principle is stronger than the convergence of the resolvents. We note that, PDE community treats homogenization problems under much more general framework; indeed there are vast literatures in PDE that consider homogenization for operators where there is no corresponding stochastic processes (for instance homogenization for fully non-linear PDEs).

In order to clarify the problem, let us give one recent result on the quenched invariant principle for random divergence forms by Chiarini and Deuschel [7]. Consider a second order elliptic differential operator $\mathcal{L}^\omega$ of divergence form with random coefficients:

$$\mathcal{L}^\omega u(x) = \mathrm{div}(a^\omega(x)\nabla u(x)), \quad x \in \mathbb{R}^d,$$

where $a^\omega(\cdot)$ is a symmetric d-dimensional matrix with $\omega \in \Omega$ being a realization of the random environment. Assume that $a^\omega(x) = a^{\tau_x\omega}(0)$, where τ_x is the shift of the environment (see Section 1.2 for details). Suppose the following hold:

(i) *There exist* $\lambda, \Lambda : \Omega \to [0,\infty]$ *with* $x \mapsto \lambda(\tau_x\omega)^{-1} + \Lambda(\tau_x\omega) \in L^\infty_{loc}(\mathbb{R}^d; dx)$ *for a.e.* $\omega \in \Omega$ *such that*

$$\lambda(\omega)|\xi|^2 \leqslant (a^\omega(0)\xi, \xi) \leqslant \Lambda(\omega)|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^d \textit{ and a.e. } \omega \in \Omega.$$

(ii) *There exist* $p, q \in [1,\infty]$ *satisfying* $1/p + 1/q < 2/d$ *such that*

$$\mathbb{E}\left[\lambda^{-q} + \Lambda^p\right] < \infty.$$

Theorem 1.1. ([7, Theorem 1.1]) *Assume* (i) *and* (ii) *above, and let* $(X_t^\omega)_{t\geqslant 0}$ *be the diffusion process whose generator is* $\mathcal{L}^\omega$. *Then, for a.e.* $\omega \in \Omega$, *the law of the process* $(\varepsilon X^{\omega}_{t/\varepsilon^2})_{t\geqslant 0}$ *on* $C([0,\infty),\mathbb{R}^d)$ *converges weakly as* $\varepsilon \to 0$ *to Brownian motion with the covariance matrix equal to* $D=(d_{ij})_{1\leqslant i,j\leqslant d}$*, where*

$$d_{ij} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}^{\omega}_0\left[X^{i,\omega}_t X^{j,\omega}_t\right] \quad 1\leqslant i,j\leqslant d,$$

which exist and are deterministic constants. Here $X^{\omega}_t = (X^{i,\omega}_t,\cdots,X^{d,\omega}_t)$*. Moreover,* D *is a positive definite matrix.*

The moment condition (ii) plays an important role in the quenched invariance principle. The quenched invariance principle for nearest neighbor random walk on random conductance model is established in [5] under the moment condition with $p=q=1$ when $d=1,2$. It is conjectured that the optimal moment condition for the quenched invariance principle for symmetric diffusions in stationary ergodic environments to hold is $p=q=1$; see [2] for the recent study subject to the periodic environment. Note that for the homogenization in the PDE literature, based on the two-scale convergence method in [36], the convergence of resolvent under L^2-norm may be established under the moment condition $p>1$ and $q>1$, see [17, 29] for related results in the discrete setting.

The study of homogenization for non-local operators can be traced back to the paper [20], where homogenization for one-dimensional pure jump processes with periodic coefficients was considered by using the probabilistic approach. See [35] for a multi-dimensional generalization with diffusion terms involved. For further developments on homogenization of non-local operators with periodic coefficients, the reader may refer to [15, 16, 30] for probabilistic approaches, and [26, 32, 31] for analytical approaches (even in the setting of nonlinear integro-differential equations). See [11, 17, 18, 22, 27] and the references therein for recent development on homogenization of non-local operators with random coefficients. In a recent preprint [8], we studied homogenization problem for symmetric non-local operators with random coefficients and gave a characterization of the homogenized limiting operators. In this paper, we survey the results obtained in [8], and present some additional homogenization results for non-local operators with random coefficients under conditions that are variants of those in [8]. In a recent paper [22], Kassmann, Piatnitski and Zhizhina investigated homogenization of a class of symmetric stable-like processes in ergodic environment whose jumping kernels are of product form. In that paper, homogenization problem of symmetric stable-like processes in two-parameter ergodic environment was also studied. In [22], random coefficients of the jumping kernel are assumed to be uniformly elliptic and bounded. In fact, all known results concerning stochastic homogenization of jump processes in one-parameter ergodic environment requires that the coefficients are of very special forms (such as the product form). The contribution of [8] is to study homogenization problem for symmetric non-local operators in one-parameter ergodic environment systematically under more general settings. In particular, the corresponding random coefficients can be degenerate and unbounded. We will survey the main results of [8] in Section 2. We will also present homogenization results under some variant settings of [8]. In Subsection 1.2, we will describe the precise setting, and in Subsection 1.3 we will give main theorems of this paper.

1.2. **Setting**

Throughout the paper, we let $d \geqslant 1$, $0 < \alpha < 2$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space that describes the random environment. Let $\{\tau_x\}_{x\in\mathbb{R}^d}$ be a measurable group of transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\tau_0 = \mathrm{id}$ and $\tau_x \circ \tau_x = \tau_{x+y}$ for every $x, y \in \mathbb{R}^d$. $\tau_x\omega := \tau_x(\omega)$ is the environment $\omega \in \Omega$ 'seen from' the point $x \in \mathbb{R}^d$. We assume that $\{\tau_x\}_{x\in\mathbb{R}^d}$ is stationary and ergodic; namely,

(i) $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and $x \in \mathbb{R}^d$;

(ii) if $A \in \mathcal{F}$ and $\tau_x A = A$ for all $x \in \mathbb{R}^d$, then $\mathbb{P}(A) \in \{0, 1\}$;

(iii) the function $(x, \omega) \mapsto \tau_x\omega$ is $\mathscr{B}(\mathbb{R}^d) \times \mathcal{F}$-measurable.

Consider a random variable $\mu : \Omega \to [0, \infty)$ such that for every $\omega \in \Omega$ $\mu(\tau_x\omega) > 0$ for a.e. $x \in \mathbb{R}^d$, and $\mathbb{E}[\mu] = 1$, and a random function $\kappa : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to [0, \infty)$ that satisfies

(1.1)
$$\kappa(x, y; \omega) = \kappa(y, x; \omega),\ \kappa(x+z, y+z; \omega) = \kappa(x, y; \tau_z\omega) \quad \text{for } x, y, z \in \mathbb{R}^d,\ \omega \in \Omega$$

and

$$(1.2) \qquad x \mapsto \int (1 \wedge |z|^2) \frac{\kappa(x, x+z; \omega)}{|z|^{d+\alpha}}\, dz \in L^1_{loc}(\mathbb{R}^d; dx) \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

We write $\mu^\omega(dx) := \mu(\tau_x\omega)\, dx$, which has full support on $\mathbb{R}^d$. Let Γ be an infinite cone in $\mathbb{R}^d$ having non-empty interior that is symmetric with respect to the origin; namely, Γ is a non-empty open subset of $\mathbb{R}^d$ so that $rx \in \Gamma$ for every $x \in \Gamma$ and $r \in \mathbb{R}$.

We now define a regular symmetric Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^w)$ on $L^2(\mathbb{R}^d; \mu^\omega(dx))$ for each $\omega \in \Omega$ as follows. For $\alpha \in (0, 2)$, define

$$\mathcal{E}^\omega(f, g) := \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta} (f(x) - f(y))(g(x) - g(y)) \frac{\kappa(x, y; \omega)}{|x-y|^{d+\alpha}} \mathbb{1}_{\{y-x\in\Gamma\}}\, dx\, dy,$$

where $\Delta := \{(x,x) \in \mathbb{R}^d\}$ is the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$, and $\mathcal{F}^\omega$ the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm $\mathcal{E}_1^\omega(\cdot,\cdot)^{1/2}$, where

$$\mathcal{E}_1^\omega(f,f) := \mathcal{E}^\omega(f,f) + \int_{\mathbb{R}^d} f(x)^2\, \mu^\omega(dx). \tag{1.3}$$

It holds that under (1.2), $\mathcal{E}^\omega(f,f) < \infty$ for all $f \in C_c^\infty(\mathbb{R}^d)$. Clearly, $(\mathcal{E}^\omega, \mathcal{F}^w)$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d; \mu^\omega(dx))$. Hence there are a Borel subset $\mathcal{N}^\omega \subset \mathbb{R}^d$ having zero $\mathcal{E}^\omega$-capacity, and a symmetric Hunt process $X^\omega := \left\{X_t^\omega, t \geqslant 0; \mathbb{P}^x, x \in \mathbb{R}^d \setminus \mathcal{N}^\omega\right\}$ on $\mathbb{R}^d \setminus \mathcal{N}^\omega$; see for instance [19, Chapter 7]. We note that X^ω is a time change of the Hunt process corresponding to the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d; dx)$. When $\Gamma = \mathbb{R}^d$ and $\kappa(x,y;\omega)$ is bounded from above and below by positive constants, this Hunt process is a symmetric α-stable-like process studied in [12].

For any $\varepsilon > 0$, define $X^{\varepsilon,\omega} = \{X_t^{\varepsilon,\omega}; t \geqslant 0\} := \{\varepsilon X^\omega_{t/\varepsilon^\alpha}; t \geqslant 0\}$. We have the following.

Lemma 1.2. ([8, Lemma 1.1]) *For any $\varepsilon > 0$, the scaled process $X^{\varepsilon,\omega}$ has a symmetrizing measure $\mu^{\varepsilon,\omega}(dx) = \mu(\tau_{x/\varepsilon}\omega)\, dx$, and the associated regular Dirichlet form $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))$ is given by*

(1.4)

$$\mathcal{E}^{\varepsilon,\omega}(f,g) = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta} (f(x)-f(y))(g(x)-g(y)) \frac{\kappa(x/\varepsilon, y/\varepsilon;\omega)}{|x-y|^{d+\alpha}} \mathbb{1}_{\{x-y\in\Gamma\}}\, dx\, dy,$$

and $\mathcal{F}^{\varepsilon,\omega}$ is the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm $\mathcal{E}_1^{\varepsilon,\omega}(\cdot,\cdot)^{1/2}$, where the $\mathcal{E}_1$-norm is defined on $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))$ similarly to (1.3).

Denote by $(\mathcal{L}^\omega, \mathrm{Dom}(\mathcal{L}^w))$ (resp. $(\mathcal{L}^{\varepsilon,\omega}, \mathrm{Dom}(\mathcal{L}^{\varepsilon,\omega}))$) the L^2-generator of the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^\omega)$ on $L^2(\mathbb{R}^d; \mu^\omega)$ (resp. $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega})$). That is, for $f \in \mathrm{Dom}(\mathcal{L}^w)$,

$$\mathcal{L}^\omega f(x) = \lim_{\delta\to 0} \frac{1}{\mu(\tau_x\omega)} \int_{\{y\in\mathbb{R}^d: |y-x|>\delta\}} (f(y)-f(x)) \frac{\kappa(x,y;\omega)}{|y-x|^{d+\alpha}} \mathbb{1}_{\{y-x\in\Gamma\}} dy,$$

and for $f \in \mathrm{Dom}(\mathcal{L}^{\varepsilon,\omega})$,

$$\mathcal{L}^{\varepsilon,\omega} f(x) = \lim_{\delta\to 0} \frac{1}{\mu(\tau_{x/\varepsilon}\omega)} \int_{\{y\in\mathbb{R}^d: |y-x|>\delta\}} (f(y)-f(x)) \frac{\kappa(x/\varepsilon, y/\varepsilon;\omega)}{|y-x|^{d+\alpha}} \mathbb{1}_{\{y-x\in\Gamma\}} dy.$$

It is easy to see that for each $\varepsilon > 0$, $g(\cdot) \in \mathrm{Dom}(\mathcal{L}^{\varepsilon,\omega})$ if and only of $g^{(\varepsilon)}(\cdot) := g(\varepsilon\cdot) \in \mathrm{Dom}(\mathcal{L}^\omega)$, and

$$\mathcal{L}^{\varepsilon,\omega} g(x) = \varepsilon^{-\alpha} \mathcal{L}^\omega g^{(\varepsilon)}(x/\varepsilon).$$

For any $\lambda > 0$ and $f \in C_c(\mathbb{R}^d)$, let $u_f^{\varepsilon,\omega}$ be the solution to the following resolvent equation

$$(\lambda - \mathcal{L}^{\varepsilon,\omega}) u_f^{\varepsilon,\omega} = f$$

in $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))$. We like to investigate under what circumstances, there is a subset $\Omega_0 \subset \Omega$ of full probability so that for every $\omega \in \Omega$ and for every $f \in C_c(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} \|u_f^{\varepsilon,\omega} - u_f\|_{L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))} = 0,$$

where u_f is the solution of

$$(\lambda - \mathcal{L})u_f = f.$$

Here $\mathcal{L}$ is the L^2-generator of certain regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$ whose jumping kernel is non-random but can be degenerate. This is a standard framework in homogenization problems in the community of PDE; see for instance [28, 34] for backgrounds and [3, 6, 22] for recent study on homogenization problems related to non-local operators.

Let $K(z)$ be a non-negative bounded and symmetric measurable function on $\mathbb{R}^d$. Define a regular Dirichlet form $(\mathcal{E}^K, \mathcal{F}^K)$ on $L^2(\mathbb{R}^d; dx)$ by

(1.5)
$$\mathcal{E}^K(f,g) = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta} (f(x)-f(y))(g(x)-g(y))\frac{K(x-y)}{|x-y|^{d+\alpha}}\mathbb{1}_{\{x-y\in\Gamma\}}\,dx\,dy,$$

and $\mathcal{F}^K$ is the closure of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm $\mathcal{E}_1^K(\cdot,\cdot)^{1/2}$, where the $\mathcal{E}_1$-norm is defined on $L^2(\mathbb{R}^d; dx)$ similarly to (1.3). The limiting Dirichlet form $(\mathcal{E}, \mathcal{F})$ for the homogenization problems considered in this paper is of this type. We emphasize that the symmetric cone Γ in (1.4) and (1.5) can be a proper subset of $\mathbb{R}^d$ in this paper.

1.3. **Main theorems**

Unlike elliptic differential operators, we have a variable $(y-x)/\varepsilon$ by shifting operators $\tau_{x/\varepsilon}$ and $\tau_{y/\varepsilon}$ in the coefficient

$$\kappa(x/\varepsilon, y/\varepsilon; \omega) = \kappa(0, (y-x)/\varepsilon; \tau_{x/\varepsilon}\omega) = \kappa(0, (x-y)/\varepsilon; \tau_{y/\varepsilon}\omega)$$

of the scaled process X^ε which corresponds to the long range property of the jumping kernel (see (1.4)). This prevents us to directly applying the ergodic theorem to deduce the almost sure convergence as indicated below. We need to impose some reasonable conditions on $\kappa(x, y; \omega)$.

Our main theorems in this paper are variants of [8, Theorem 1.3]. We assume the following conditions on the coefficients $\kappa(x, y; \omega)$.

(C1) *For every $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$,*

(1.6)
$$\kappa(x, y; \omega) = \nu(y-x; \tau_x\omega) + \nu(x-y; \tau_y\omega),$$

where $\nu : \mathbb{R}^d \times \Omega \mapsto [0, \infty)$ *satisfies that for a.s.* $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, z) \left(\nu\left(z/\varepsilon; \tau_{x/\varepsilon}\omega\right) - \bar{\nu}(z; \tau_{x/\varepsilon}\omega)\right) dz\, dx = 0 \qquad \text{for every } h \in C_c^\infty(\mathbb{R}^{2d}). \tag{1.7}$$

Here, $\bar{\nu}(x; \omega)$ *is a non-negative measurable function on* $\mathbb{R}^d \times \Omega$ *so that for any* $z \in \mathbb{R}^d$,

$$\mathbb{E}[\bar{\nu}(z; \cdot)] \geqslant C_1, \quad \mathbb{E}[\bar{\nu}(z; \cdot)^\gamma] \leqslant C_2 \tag{1.8}$$

for some constants $C_1, C_2 > 0$ *and* $\gamma > 1$.

(C2) *There are a constant* $p > 1$ *and non-negative random variables* $\Lambda_1 \leqslant \Lambda_2$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that*

$$\mathbb{E}\left[\Lambda_1^{-1} + \Lambda_2^p\right] < \infty, \tag{1.9}$$

and for a.s. $\omega \in \Omega$,

$$\Lambda_1(\tau_x\omega) + \Lambda_1(\tau_y\omega) \leqslant \kappa(x, y; \omega) \leqslant \Lambda_2(\tau_x\omega) + \Lambda_2(\tau_y\omega) \qquad \text{for every } x, y \in \mathbb{R}^d. \tag{1.10}$$

We have four remarks concerning the above condition.

Remark 1.3. (i) It is easy to see that any $\kappa(x, y; \omega)$ of form (1.6), which satisfies (1.7) with some non-negative $\bar{\nu} : \mathbb{R}^d \times \Omega \to [0, +\infty)$, enjoys the property (1.1) and that for *a.s.* $\omega \in \Omega$,

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \times \mathbb{R}^d} h(x, z) \left(\kappa\left(0, z/\varepsilon; \tau_{x/\varepsilon}\omega\right) - \bar{\kappa}_\varepsilon\left(z; \tau_{x/\varepsilon}\omega\right)\right) dz\, dx = 0 \tag{1.11}$$

for every $h \in C_c^\infty(\mathbb{R}^{2d})$, where $\bar{\kappa}_\varepsilon(z, \omega) := \bar{\nu}(z; \omega) + \bar{\nu}(-z; \tau_{z/\varepsilon}\omega)$.

On the other hand, any $\kappa(x, y; \omega)$, satisfying (1.1) and (1.11) with $\bar{\kappa}_\varepsilon$ being some non-negative $\bar{\nu} : \mathbb{R}^d \times \Omega \to [0, +\infty)$ (independent of ε), admits a representation of the form (1.6) so that (1.7) is satisfied. This is because $\kappa(x, y; \omega) = \kappa(0, y - x; \tau_x\omega)$ and so by the symmetry of $\kappa(x, y; \omega)$ in (x, y) we have

$$\kappa(x, y; \omega) = \tfrac{1}{2}(\kappa(x, y; \omega) + \kappa(y, x; \omega)) = \tfrac{1}{2}(\kappa(0, y - x; \tau_x\omega) + \kappa(0, x - y; \tau_y\omega)).$$

Hence we can write $\kappa(x, y; \omega)$ as

$$\kappa(x, y; \omega) = \nu(y - x; \tau_x\omega) + \nu(x - y; \tau_y\omega),$$

where

$$\nu(x; \omega) := \kappa(0, x; \omega)/2.$$

(ii) Schwab [33] studied the stochastic homogenization for some fully nonlinear integro-differential equations associated with (non-symmetric) α-stable-like operators, where the coefficient $k(x,z;\omega)$ satisfies for any $\omega \in \Omega$, $x,z\in \mathbb{R}^d$ and $\varepsilon>0$,

(a) $k(x,z;\omega)=k(x,-z;\omega)$;

(b) $k(x,z/\varepsilon;\omega)=k(x,z;\omega)$.

See [33, (1.14) and (1.13)]. Clearly, (1.7) is more general than (b) above. (To see this, we take $\nu(z;\tau_x\omega)=k(x,z;\omega)$, and then $\bar\nu(z;\tau_x\omega)=\nu(z;\tau_x\omega)$.) From the viewpoint of assumption (a), **(C1)** can be viewed as a symmetrized version of [33]. See [32] for related works on the periodic homogenization.

(iii) **(C2)** is just **(A2)** in [8]. Under **(C2)**, by using (the continuous version of) the Birkhoff ergodic theorem (see [21, Theorem 7.2] or [8, Proposition 2.1]) and the Hölder inequality, one can verify that (1.7) implies that for a.s. $\omega\in\Omega$, the function

$$(x,z)\mapsto \nu\left(z/\varepsilon;\tau_{x/\varepsilon}\omega\right)-\bar\nu(z;\tau_{x/\varepsilon}\omega)$$

weakly converges to 0 in $L^1_{loc}(\mathbb{R}^{2d};dx\,dz)$ as $\varepsilon\to 0$; that is, for a.s. $\omega\in\Omega$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d\times\mathbb{R}^d} h(x,z)\left(\nu\left(z/\varepsilon;\tau_{x/\varepsilon}\omega\right)-\bar\nu(z;\tau_{x/\varepsilon}\omega)\right)\,dz\,dx=0$$

for every $h\in B_c(\mathbb{R}^{2d})$; see the proof of [8, Lemma 3.1] or that of Proposition 3.2 below.

(iv) In our setting we always assume that (1.2) holds true. In fact, (1.2) is a consequence of **(C2)**. Indeed, suppose **(C2)** holds. Then by the Fubini theorem, for any $R\geqslant 1$,

$$\begin{aligned}
&\mathbb{E}\left[\int_{B(0,R)}\int_{\mathbb{R}^d}(1\wedge|z|^2)\frac{\kappa(x,x+z;\omega)}{|z|^{d+\alpha}}\,dz\,dx\right]\\
&\leqslant \int_{B(0,R)}\int_{\mathbb{R}^d}(1\wedge|z|^2)\frac{\mathbb{E}[\Lambda_2(\tau_x\omega)]+\mathbb{E}[\Lambda_2(\tau_{x+z}\omega)]}{|z|^{d+\alpha}}\,dz\,dx\\
&\leqslant 2\mathbb{E}[\Lambda_2]\int_{B(0,R)}\int_{\mathbb{R}^d}\frac{1\wedge|z|^2}{|z|^{d+\alpha}}\,dz\,dx<\infty.
\end{aligned}$$

In particular, we have $\mathbb{P}$-a.s.,

$$\int_{B(0,R)}\int_{\mathbb{R}^d}(1\wedge|z|^2)\frac{\kappa(x,x+z;\omega)}{|z|^{d+\alpha}}\,dz\,dx<\infty$$

for every $R>0$.

For $\varepsilon > 0$, let $U_\lambda^{\varepsilon,\omega}$ be the λ-order resolvent of the Dirichlet form $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ given by (1.4). Our first main theorem is the following.

THEOREM 1.4. *Suppose that* **(C1)** *and* **(C2)** *hold and that* $\mathbb{E}[\mu^p]$ *for some* $p > 1$. *Then there is* $\Omega_0 \subset \Omega$ *of full probability so that for every* $\omega \in \Omega_0$, $f \in C_c(\mathbb{R}^d)$ *and* $\lambda > 0$,

$$U_\lambda^{\varepsilon,\omega} f \text{ converges to } U_\lambda^K f \text{ locally in } L^1(\mathbb{R}^d; dx) \quad \text{as } \varepsilon \to 0,$$

and

$$\lim_{\varepsilon \to 0} \|U_\lambda^{\varepsilon,\omega} f - U_\lambda^K f\|_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega})} = 0, \tag{1.12}$$

where U_λ^K *is the* λ*-order resolvent of the symmetric Dirichlet form* $(\mathcal{E}^K, \mathcal{F}^K)$ *on* $L^2(\mathbb{R}^d; dx)$ *given by* (1.5) *with*

$$K(z) = \mathbb{E}[\bar\nu(z; \cdot)] + \mathbb{E}[\bar\nu(-z; \cdot)].$$

Clearly, by taking the smaller one, we can assume $p > 1$ in the condition $\mathbb{E}[\mu^p] < \infty$ is the same as the $p > 1$ in **(C2)**. We note that since for any $g \in C_c^1(\mathbb{R}^d)$,

$$\begin{aligned}
&\mathcal{E}^{\varepsilon,\omega}(U_\lambda^{\varepsilon,\omega} f, g) + \lambda \langle U_\lambda^{\varepsilon,\omega} f, g\rangle_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))} = \langle f, g\rangle_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))},\\
&\mathcal{E}^K(U_\lambda^K f, g) + \lambda \langle U_\lambda^K f, g\rangle_{L^2(\mathbb{R}^d; dx)} = \langle f, g\rangle_{L^2(\mathbb{R}^d; dx)},
\end{aligned}$$

using the Birkhoff ergodic theorem, we have

$$\lim_{\varepsilon \to 0} \langle U_\lambda^K f, g\rangle_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))} = \langle U_\lambda^K f, g\rangle_{L^2(\mathbb{R}^d; dx)}$$

and

$$\lim_{\varepsilon \to 0} \langle f, g\rangle_{L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))} = \langle f, g\rangle_{L^2(\mathbb{R}^d; dx)}.$$

We conclude from (1.12) that

$$\lim_{\varepsilon \to 0} \mathcal{E}^{\varepsilon,\omega}(U_\lambda^{\varepsilon,\omega} f, g) = \mathcal{E}^K(U_\lambda^K f, g).$$

The same result as Theorem 1.4 holds for the case where the jump variable z is periodic. To be precise, consider the following assumption:

(C1*) *The coefficient* $\kappa(x, y; \omega)$ *is given by* (1.6) *for some non-negative measurable function* $\nu(z; \omega)$ *on* $\mathbb{R}^d \times \Omega$, *which satisfies that the function* $z \mapsto \nu(z; \omega)$ *is* 1*-periodic in the sense that it can be seen as a function defined on the* d*-dimensional torus* $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$, *and* $\mathbb{E}[\bar\nu] < \infty$ *with*

$$\bar\nu(\omega) := \int_{\mathbb{T}^d} \nu(z; \omega)\, dz.$$

Here is our second main theorem.

THEOREM 1.5. *Suppose that* **(C1*)** *and* **(C2)** *hold and that* $\mathbb{E}[\mu^p]$ *for some* $p > 1$. *Then the conclusion of Theorem* 1.4 *holds with*

$$K(z) := 2\mathbb{E}[\bar{\nu}].$$

The proofs of Theorems 1.4 and 1.5 are similar to that of [8, Theorem 1.3] and we will give a sketch of the proofs in Section 3. We like to mention a recent paper [31] on the study of homogenization of a class of symmetric Lévy processes on $\mathbb{R}^d$ with (deterministic) periodic jumping kernels.

It is natural to consider further the invariance principle of the scaled processes on the path space. In order to obtain it, we need to establish the tightness of the scaled processes, as mentioned in the introduction. In fact, if the initial distribution is absolutely continuous with respect to an invariant measure, then the tightness can be obtained by using the so-called forward-backward martingale decomposition (see [11, Proposition 3.4] for the corresponding statement in the discrete setting). Hence one can obtain the convergence of the processes on the path space under such initial condition (or under some weaker topology), see [11, Theorems 2.2 and 2.3] for more discussions in the discrete case. When $(x, y) \mapsto \kappa(x, y; \omega)$ is bounded between two positive constants, we can use heat kernel estimates from [12] when $\Gamma = \mathbb{R}^d$ or parabolic Harnack inequalities from [13] when $\Gamma \subsetneq \mathbb{R}^d$ to establish the tightness, and therefore the weak convergence of the scaled processes starting from any point. However, it is highly non-trivial to prove such convergence if the process starts at any fixed point (in other word, if the initial distribution is a Dirac measure), when $(x, y) \mapsto \kappa(x, y; \omega)$ is not bounded between two positive constants. We will address this problem in a separate paper.

In this paper, we use := as a way of definition. For all $x \in \mathbb{R}^d$ and $r > 0$, set $B(x, r) = \{z \in \mathbb{R}^d : |z - x| < r\}$. For $p \in [1, \infty]$ and Lebesgue measurable $A \subset \mathbb{R}^d$, we use $|A|$ to denote the d-dimensional Lebesgue measure of A, $C_b(A)$ the space of bounded and continuous functions on A, $L^p(A; dx)$ the space of L^p-integrable functions on A with respect to the Lebesgue measure, and $L^p_{loc}(\mathbb{R}^d; dx)$ the space of locally L^p-integrable functions on $\mathbb{R}^d$ with respect to the Lebesgue measure. Denote $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d; \mu(dx))}$ the inner product in $L^2(\mathbb{R}^d; \mu(dx))$. Denote by $B(\mathbb{R}^d)$ the set of locally bounded measurable functions on $\mathbb{R}^d$, by $B_b(\mathbb{R}^d)$ the set of bounded measurable functions on $\mathbb{R}^d$, and by $B_c(\mathbb{R}^d)$ the set of bounded measurable functions on $\mathbb{R}^d$ with compact support. $C^1_c(\mathbb{R}^d)$ (respectively, $C_c(\mathbb{R}^d)$ or $C^\infty_c(\mathbb{R}^d)$) denotes the space of C^1-smooth (respectively, continuous or C^∞-smooth) functions on $\mathbb{R}^d$ with compact support.

2. SURVEY OF THE RESULTS IN [8]

In this section, we survey the main results from [8]. Consider the following assumption concerning $\kappa(x, y; \omega)$.

(A1) *The coefficient $\kappa(x, y; \omega)$ is given by* (1.6) *for some non-negative measurable function $\nu(z; \omega)$ on $\mathbb{R}^d \times \Omega$, which satisfies that*

(i) *There is a constant $l > 0$ such that for any $n > 0$ and $x, z_1, z_2 \in \mathbb{R}^d$,*

$$
\begin{aligned}
&|\mathrm{Cov}\,(\nu_n(z_1\,; \cdot), \nu_n(z_2; \tau_x(\cdot)))\,| := \\
&\qquad \big|\mathbb{E}\,[\nu_n(z_1; \cdot) \cdot \nu_n(z_2; \tau_x(\cdot))] - \mathbb{E}[\nu_n(z_1; \cdot)]\mathbb{E}[\nu_n(z_2; \cdot)]\big| \\
&\qquad \leqslant C_1(n)\big(1 \wedge |x|^{-l}\big),
\end{aligned}
\tag{2.1}
$$

where $\nu_n = \nu \wedge n$ and $C_1(n)$ is a positive constant depending on n.

(ii) *There is a non-negative measurable function $\bar{\nu}$ on $\mathbb{R}^d$ such that $\mathbb{E}[\nu(z/\varepsilon; \cdot)]$ converges weakly to $\bar{\nu}(z)$ in $L^1_{loc}(\mathbb{R}^d; dx)$ as $\varepsilon \to 0$; that is, for every $h \in L^\infty_{loc}(\mathbb{R}^d; dx)$,*

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} h(z)\mathbb{E}[\nu(z/\varepsilon; \cdot)]\, dz = \int_{\mathbb{R}^d} h(z)\bar{\nu}(z)\, dz.$$

We note that the mixing condition (2.1) in assumption **(A1)** is weaker than the mutually independent stable-like random conductance models investigated in [11, 9, 10]. Indeed, (2.1) only requires the mixing condition on the position variable x, not on the jumping size variable z; while in [11, 9, 10] the mutual independence is imposed on both variables x and z, which was crucial to verify (A4*) (ii) in [11] (see also [9, Section 4]).

Theorem 2.1. ([8, Theorem 1.3]) *Suppose that* **(A1)** *and* **(C2)** *hold, and that $\mathbb{E}[\mu^p] < \infty$ for some $p > 1$. Then the conclusion of Theorem* 1.4 *holds with*

$$K(z) := \bar{\nu}(z) + \bar{\nu}(-z).$$

Another model considered in [8] is $\kappa(x, y; \omega)$ of product form, motivated by [22, (Q1)]. We consider the following assumptions.

(B1) *For every $\omega \in \Omega$ and $x, y \in \mathbb{R}^d$,*

$$\kappa(x, y; \omega) = \nu_1(\tau_x\omega)\nu_2(\tau_y\omega) + \nu_1(\tau_y\omega)\nu_2(\tau_x\omega), \tag{2.2}$$

where ν_1 and ν_2 are non-negative random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

(B2) *There are non-negative random variables $\Lambda_1 \leqslant \Lambda_2$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with*

$$\mathbb{E}\left[\Lambda_1^{-1} + \Lambda_2^2\right] < \infty$$

so that for a.s. $\omega \in \Omega$,

$$\Lambda_1(\tau_x\omega)\Lambda_1(\tau_y\omega) \leqslant \kappa(x,y;\omega) \leqslant \Lambda_2(\tau_x\omega)\Lambda_2(\tau_y\omega) \quad \textit{for every } x,y \in \mathbb{R}^d.$$

The following fact is proved in [8, Proposition 1.4].

PROPOSITION 2.2. *Suppose that* $\kappa(x,y;\omega)$ *is given by* (2.2) *for some non-negative random variables* ν_1 *and* ν_2 *on* $(\Omega,\mathcal{F},\mathbb{P})$. *Then condition* **(B2)** *holds if and only if* ν_1 *and* ν_2 *satisfy that*

$$\mathbb{E}\left[(\nu_1\nu_2)^{-1/2} + (\nu_1+\nu_2)^2\right] < \infty.$$

Any $\kappa(x,y;\omega)$ of form (2.2) enjoys the property (1.1). Similar to Remark 1.3(iv), we can verify that (1.2) is satisfied when **(B1)** and **(B2)** hold. Under (2.2), the corresponding symmetric Dirichlet form $(\mathcal{E}^\omega,\mathcal{F}^\omega)$ has the expression

$$\mathcal{E}^\omega(f,f) := \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\Delta}(f(x)-f(y))^2\frac{\nu_1(\tau_x\omega)\nu_2(\tau_y\omega)}{|x-y|^{d+\alpha}}\mathbb{1}_{\{y-x\in\Gamma\}}\,dx\,dy \qquad \text{for } f \in \mathcal{F}^\omega.$$

In this case, we are able to drop the mixing condition (2.1) from Theorem 2.1.

THEOREM 2.3. ([8, Theorem 1.6]) *Suppose that* **(B1)** *and* **(B2)** *hold, and* $\mathbb{E}[\mu^p] < \infty$ *for some* $p > 1$. *Then the conclusion of Theorem* 1.4 *holds with constant*

$$K(z) := \mathbb{E}[\nu_1]\,\mathbb{E}[\nu_2].$$

As an application of Theorem 2.3, we have the following example that improves [22, Theorem 3, Case (Q1)], where the coefficients $\lambda_i(\tau_x\omega)$ $(i=1,2)$ are assumed to be uniformly bounded between two positive constants and $\Gamma = \mathbb{R}^d$.

Example 2.4. ([8, Example 1.7]) *Let* Γ *be an infinite symmetric cone in* $\mathbb{R}^d$ *that has non-empty interior. For any* $\varepsilon > 0$, *let* $\mathcal{L}^{\varepsilon,\omega}$ *be a Lévy-type operator given by*

$$\mathcal{L}^{\varepsilon,\omega}f(x) = \text{p.v.}\int (f(y)-f(x))\frac{\lambda_1(\tau_{x/\varepsilon}\omega)\lambda_2(\tau_{y/\varepsilon}\omega)}{|y-x|^{d+\alpha}}\mathbb{1}_{\{y-x\in\Gamma\}}\,dy, \tag{2.3}$$

where λ_1 *and* λ_2 *are two non-negative measurable functions on* $(\Omega,\mathcal{F},\mathbb{P})$ *such that*

$$\lambda_2 \in L^2(\Omega;\mathbb{P}), \quad \lambda_2^{-1} \in L^1(\Omega;\mathbb{P}) \quad \textit{and} \quad \lambda_2/\lambda_1 \in L^p(\Omega;\mathbb{P}),$$

for some $p > 1$. *Then as* $\varepsilon \to 0$, $\mathcal{L}^{\varepsilon,\omega}$ *converges in the resolvent topology to*

$$\mathcal{L}f(x) = \text{p.v.}\int (f(y)-f(x))\frac{C_0}{|y-x|^{d+\alpha}}\mathbb{1}_{\{y-x\in\Gamma\}}\,dy,$$

where

$$C_0 = \frac{(\mathbb{E}[\lambda_2])^2}{\mathbb{E}[\lambda_2/\lambda_1]}. \tag{2.4}$$

At the first sight, the constant coefficient C_0 should be $\mathbb{E}[\lambda_1\lambda_2]$, but with the idea of the time change, it turns out the correct one should be (2.4). It is worth emphasizing again that in this site model the mixing condition (2.1) of the media given in Assumption **(A1)** is not needed.

3. PROOFS OF THEOREMS 1.4 AND 1.5

In this section, we give the proofs of Theorems 1.4 and 1.5. In fact, most of the arguments in the proofs except Proposition 3.2 below are the same as those in the proofs of [8, Theorems 1.3 and 1.6], so we will only sketch ideas except the proof of Proposition 3.2.

3.1. Some general results in [8]

In this subsection, we give some general results concerning homogenization of stable-like Dirichlet forms. For any $\varepsilon > 0$, let $\mathcal{L}^{\varepsilon,\omega}$ be the generator of the Dirichlet form $(\mathcal{E}^{\varepsilon,\omega}, \mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))$ given by (1.4). Let $\mathcal{L}^K$ be the generator of the Dirichlet form $(\mathcal{E}^K, \mathcal{F}^K)$ of (1.5) on $L^2(\mathbb{R}^d; dx)$. The goal of homogenization theory is to construct homogenized characteristics and clarify whether the solutions for the operators $\mathcal{L}^{\varepsilon,\omega}$ are close to the solution for the operator $\mathcal{L}^K$. As mentioned in Section 1.2, we are concerned with the following question: when does the solution to the equation

$$(\lambda - \mathcal{L}^{\varepsilon,\omega})u^{\varepsilon,\omega} = f \tag{3.1}$$

on $L^2(\mathbb{R}^d; \mu^{\varepsilon,\omega}(dx))$ for any $\lambda > 0$ and $f \in C_c(\mathbb{R}^d)$ converge in the resolvent topology, as $\varepsilon \to 0$, to the solution to the equation

$$(\lambda - \mathcal{L}^K)u = f \tag{3.2}$$

on $L^2(\mathbb{R}^d; dx)$? We address this question under the following assumption.

Assumption (H): *There is $\Omega_0 \subset \Omega$ of full probability so that*

(i) *For every $\omega \in \Omega_0$ and for any sequence of functions $\{f_\varepsilon : \varepsilon \in (0,1]\}$ such that $f_\varepsilon \in \mathcal{F}^{\varepsilon,\omega}$ for any $\varepsilon \in (0,1]$, and*

$$\limsup_{\varepsilon\to 0} (\|f_\varepsilon\|_\infty + \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon)) < \infty,$$

$\{f_\varepsilon : \varepsilon \in (0,1]\}$ is a pre-compact set as $\varepsilon \to 0$ in $L^1(B(0,r); dx)$ for every $r > 1$ in the sense that for any sequence $\{\varepsilon_n : n \geqslant 1\} \subset (0,1]$ with $\lim_{n\to 0} \varepsilon_n = 0$, there are a subsequence $\{\varepsilon_{n_k} : k \geqslant 1\}$ and a function $f \in L^1_{loc}(\mathbb{R}^d; dx)$ so that f_{n_k} converges to f in $L^1(B(0,r); dx)$ for every $r > 1$.

(ii) *For every* $\omega \in \Omega_0$ *and any* $g \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\iint_{\{0<|x-y|\leqslant\eta\}}(g(x)-g(y))^2\frac{\kappa(x/\varepsilon,y/\varepsilon;\omega)}{|x-y|^{d+\alpha}}\,dx\,dy=0$$

and

$$\lim_{\eta\to 0}\limsup_{\varepsilon\to 0}\iint_{\{|x-y|\geqslant 1/\eta\}}(g(x)-g(y))^2\frac{\kappa(x/\varepsilon,y/\varepsilon;\omega)}{|x-y|^{d+\alpha}}\,dx\,dy=0.$$

(iii) *There is a constant* $p>1$ *such that for every* $\omega\in\Omega_0$ *and* $R>0$,

$$\limsup_{\varepsilon\to 0}\int_{B(0,R)}\left(\int_{B(0,R)}\kappa(x/\varepsilon,y/\varepsilon;\omega)\,dy\right)^p dx<\infty.$$

(iv) *For every* $\omega\in\Omega_0$, *any* $\eta>0$, $f\in B_b(\mathbb{R}^d)$ *and* $g\in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\lim_{\varepsilon\to 0}&\int_{\mathbb{R}^d}\int_{\{\eta<|x-y|<1/\eta\}}(f(x)-f(y))(g(x)-g(y))\frac{\kappa(x/\varepsilon,y/\varepsilon;\omega)}{|x-y|^{d+\alpha}}\mathbb{1}_{\{y-x\in\Gamma\}}\,dx\,dy\\ &=\int_{\mathbb{R}^d}\int_{\{\eta<|x-y|<1/\eta\}}(f(x)-f(y))(g(x)-g(y))\frac{K(x-y)}{|x-y|^{d+\alpha}}\mathbb{1}_{\{y-x\in\Gamma\}}\,dx\,dy,\end{aligned}$$

where $K(z)$ *is a measurable symmetric function on* $\mathbb{R}^d$ *such that* $C_1\leqslant K(z)\leqslant C_2$ *for some constants* $C_1,C_2>0$.

Let $U_\lambda^{\varepsilon,\omega}$ be the λ-order resolvent of the regular Dirichlet form $(\mathcal{E}^{\varepsilon,\omega},\mathcal{F}^{\varepsilon,\omega})$ on $L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))$, and U_λ^K the λ-order resolvent of the regular Dirichlet form $(\mathcal{E}^K,\mathcal{F}^K)$ on $L^2(\mathbb{R}^d;dx)$. It is well known that $U_\lambda^{\varepsilon,\omega}f$ and $U_\lambda^K f$ are the unique solution to (3.1) and (3.2), respectively.

The following theorem concerning the convergence in $L^1_{loc}(\mathbb{R}^d;dx)$ and the resolvent topology is given in [8, Theorems 2.2 and 2.3].

THEOREM 3.1. *Suppose that assumption* **(H)** *holds and* $\mathbb{E}[\mu^p]<\infty$ *for some* $p>1$. *Then, there is a subset* $\Omega_1\subset\Omega$ *of full probability measure so that for every* $\omega\in\Omega_1$ *and* $f\in C_c(\mathbb{R}^d)$,

$$U_\lambda^{\varepsilon,\omega}f \text{ converges to } U_\lambda^K f \text{ in } L^1_{loc}(\mathbb{R}^d;dx)\qquad\text{as } \varepsilon\to 0$$

and

$$\lim_{\varepsilon\to 0}\|U_\lambda^{\varepsilon,\omega}f-U_\lambda^K f\|_{L^2(\mathbb{R}^d;\mu^{\varepsilon,\omega}(dx))}=0.$$

Thanks to this theorem, in order to prove Theorems 1.4 and 1.5 it is enough to prove that assumption **(C)** implies assumption **(H)**. We will prove it in the following subsections.

3.2. **Weak convergence of bilinear forms**

In this subsection, we give a proposition to guarantee the weak convergence of non-local bilinear forms. Recall that $\Gamma \subset \mathbb{R}^d$ is an infinite symmetric cone that has non-empty interior. Note that when $d = 1$, $\Gamma = \mathbb{R}$.

PROPOSITION 3.2. (i) *Suppose that* **(C1)** *holds. Then there is a subset $\Omega_1 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_1$, any $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}}\frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\kappa(x/\varepsilon,(x+z)/\varepsilon;\omega)\,dz\,dx$$
$$= \int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}}\frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\,(\mathbb{E}[\bar\nu(z;\cdot)] + \mathbb{E}[\bar\nu(-z;\cdot)])\,dz\,dx.$$

(ii) *Suppose that* **(C1*)** *holds and that there is a non-negative random variables Λ on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[\Lambda^p] < \infty$ for some $p > 1$ so that for a.s. $\omega \in \Omega$,*

$$\kappa(x,y;\omega) \leqslant \Lambda(\tau_x\omega) + \Lambda(\tau_y\omega) \quad \text{for every } x, y \in \mathbb{R}^d. \tag{3.3}$$

Then there is a subset $\Omega_2 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_2$, any $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}}\frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\kappa\left(x/\varepsilon,(x+z)/\varepsilon;\omega\right)\,dz\,dx$$
$$= 2\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}}\frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}}\mathbb{E}[\bar\nu]\,dz\,dx.$$

Clearly **(C2)** implies (3.3).

The proof of the proposition will use the following lemma from [8], which is an extension of the Birkhoff ergodic theorem, and the ideas of its proof.

LEMMA 3.3. ([8, Lemma 3.1(i)]) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which there is a stationary and ergodic measurable group of transformations $\{\tau_x\}_{x\in\mathbb{R}^d}$ with $\tau_0 = \mathrm{id}$. Suppose that $\nu(z;\omega)$ is a non-negative measurable function on $\mathbb{R}^d \times \Omega$ such that the function $z \mapsto \mathbb{E}\left[\nu(z;\cdot)^p\right]$ is locally integrable for some $p > 1$. Then there is a subset $\Omega_0 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_0$ and every compactly supported $f \in L^q(\mathbb{R}^d \times \mathbb{R}^d; dx\,dy)$ with $q = p/(p-1)$,*

$$\lim_{\varepsilon\to 0}\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x,z)\nu(z;\tau_{x/\varepsilon}\omega)\,dz\,dx = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f(x,z)\mathbb{E}\left[\nu(z;\cdot)\right]\,dz\,dx. \tag{3.4}$$

Proof of Proposition 3.2. (i) Under **(C1)**, there is $\Omega_0 \subset \Omega$ of full probability so that for every $\omega \in \Omega_0$, for any η, $\varepsilon > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\kappa\left(x/\varepsilon,(x+z)/\varepsilon;\omega\right)\,dz\,dx\\
&= \int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\nu\left(z/\varepsilon;\tau_{x/\varepsilon}\omega\right)\,dz\,dx\\
&+ \int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}\nu\left(-z/\varepsilon;\tau_{(x+z)/\varepsilon}\omega\right)\,dz\,dx\\
&=: \sum_{i=1}^2 I_i^\varepsilon.
\end{aligned}$$

By changing variables $x+z \mapsto x$ and $z \mapsto -z$ in the term I_2^ε, it holds that $I_1^\varepsilon = I_2^\varepsilon$, where we used the fact that $\Gamma = -\Gamma$.

Note that for every $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,

$$F(x,z) := \mathbb{1}_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}}\frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}$$

is a bounded and compactly supported function on $\mathbb{R}^d \times \mathbb{R}^d$. Recall that we assume (1.7) with $\mathbb{E}[\bar\nu(z;\cdot)^\gamma] \leqslant C_2$ for all $z \in \mathbb{R}^d$ and some $\gamma > 1$. According to (1.7), Remark 1.3(iii) and Lemma 3.3, there is a subset $\Omega_1 \subset \Omega_0$ of full probability measure so that for any $\omega \in \Omega_1$, $\eta > 0$, $f \in B(\mathbb{R}^d)$ and $g \in B_c(\mathbb{R}^d)$,

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}}\bar\nu(z;\tau_{x/\varepsilon}\omega)\,dz\,dx\\
&= \int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}}\mathbb{E}[\bar\nu(z;\cdot)]\,dz\,dx.
\end{aligned}$$

Putting all these estimates above together immediately yields that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}}\kappa(x/\varepsilon,(x+z)/\varepsilon;\omega)\,dz\,dx\\
&= 2\int_{\mathbb{R}^d}\int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}}\mathbb{E}[\bar\nu(z;\cdot)]\,dz\,dx.
\end{aligned}$$

Again by changing variables $x+z \mapsto x$ and $z \mapsto -z$ in the right hand side of the equality above, we obtain the desired assertion.

(ii) Suppose that **(C1*)** holds. Then, we can still define I_i^ε for $i = 1, 2$ as in (i). As before, we only need to consider the term I_1^ε.

For any bounded set $D = D_1 \times D_2 \subset \mathbb{R}^d \times \mathbb{R}^d$ with D_i $(i = 1, 2)$ being a connected interval in $\mathbb{R}^d$, we have

$$\begin{aligned}
&\iint_{D_1\times D_2} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx \\
&\quad = \sum_{i:Q_i^\varepsilon \subset D_2} \int_{D_1}\int_{Q_i^\varepsilon} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx \\
&\qquad + \sum_{i:Q_i^\varepsilon\cap D_2\neq\emptyset, Q_i^\varepsilon\cap D_2^c\neq\emptyset} \int_{D_1}\int_{Q_i^\varepsilon\cap D_2} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx \\
&\quad =: \sum_{j=1}^2 J_j^\varepsilon,
\end{aligned}$$

where $Q_i^\varepsilon = [z_i^\varepsilon - \varepsilon/2, z_i^\varepsilon + \varepsilon/2]^d$ with $z_i^\varepsilon \in \varepsilon\mathbb{Z}^d$. Note that

$$\begin{aligned}
\int_{Q_i^\varepsilon} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz =& \varepsilon^d \int_{[z_i^\varepsilon/\varepsilon-1/2, z_i^\varepsilon/\varepsilon+1/2]^d} \nu(z; \tau_{x/\varepsilon}\omega)\, dz \\
=& \varepsilon^d \int_{\mathbb{T}^d} \nu(z; \tau_{x/\varepsilon}\omega)\, dz = \varepsilon^d \bar\nu(\tau_{x/\varepsilon}\omega),
\end{aligned} \tag{3.5}$$

where $\bar\nu(\omega) := \int_{\mathbb{T}^d} \nu(z;\omega)\, dz$. We have

$$J_1^\varepsilon = \int_{D_1} \Big(\sum_{i:Q_i^\varepsilon\subset D_2} |Q_i^\varepsilon| \Big) \bar\nu\left(\tau_{x/\varepsilon}\omega\right)\, dx.$$

Observe that for connected interval D_2,

$$\lim_{\varepsilon\to 0} \Big| \Big(\sum_{i:Q_i^\varepsilon\subset D_2} |Q_i^\varepsilon| \Big) - |D_2| \Big| = 0.$$

Thus, thanks to the Birkhoff ergodic theorem, for a.s. $\omega \in \Omega$,

$$\limsup_{\varepsilon\to 0} \Big| J_1^\varepsilon - |D_1 \times D_2| \cdot \mathbb{E}[\bar\nu] \Big| = 0.$$

Furthermore, note that $|\{i : Q_i^\varepsilon \cap D_2 \neq \emptyset, Q_i^\varepsilon \cap D_2^c \neq \emptyset\}| \leqslant c_3\varepsilon^{-(d-1)}$ for some constant $c_3 > 0$ independent of ε. Then, by (3.5) and Birkhoff ergodic theorem again, we can get that for a.s. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} |J_2^\varepsilon| \leqslant \lim_{\varepsilon\to 0} c_3\varepsilon \int_{D_1} \bar\nu(\tau_{x/\varepsilon}\omega) dx = 0.$$

Putting both estimates for J_1^ε and J_1^ε together, and then letting $\varepsilon \to 0$ we conclude that for a.s. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \iint_{D_1\times D_2} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx = |D| \cdot \mathbb{E}[\bar\nu]. \tag{3.6}$$

With the aid of (3.6), we now can use the idea of the proof for [8, Lemma 3.1(i)] to obtain the desired assertion. Indeed, let

$$\mathscr{S} := \left\{ F(x,z) = \sum_{i=1}^{m} a_i \mathbb{1}_{A_i \times B_i}(x,z) : m \in \{1,2,\cdots\},\ a_i \in \mathbb{Q},\ A_i, B_i \in \mathscr{B}_{\mathbb{Q}}(\mathbb{R}^d) \right\}.$$

Here $\mathbb{Q}$ denotes the set of all rational numbers, and $\mathscr{B}_{\mathbb{Q}}(\mathbb{R}^d)$ denotes the collection of all bounded cubes in $\mathbb{R}^d$ whose end points are rational numbers.

According to (3.6), it is easy to see that there exists a subset $\Omega_0 \subset \Omega$ with full probability measure such that for every $\omega \in \Omega_1$ and $F \in \mathscr{S}$,

$$\lim_{\varepsilon \to 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx = \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\mathbb{E}[\bar{\nu}]\, dz\, dx. \tag{3.7}$$

Similarly, by (3.3) with $\mathbb{E}[\Lambda^p] < \infty$ and the argument for (3.7), we can also find a subset $\Omega_2 \subset \Omega_1$ with full probability measure such that for every $\omega \in \Omega_2$ and $A, B \in \mathscr{B}_{\mathbb{Q}}(\mathbb{R}^d)$,

$$\lim_{\varepsilon \to 0} \iint_{A \times B} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)^p\, dz\, dx = \iint_{A \times B} \mathbb{E}[\bar{\nu}^p]\, dz\, dx, \tag{3.8}$$

where $p > 1$ is from the assumption.

For general bounded compactly supported function F on $\mathbb{R}^d \times \mathbb{R}^d$, take $A, B \in \mathscr{B}_{\mathbb{Q}}(\mathbb{R}^d)$ so that $\mathrm{supp}[F] \subset A \times B$. Since $C_b(A \times B)$ is dense in $L^q(A \times B; dx\, dy)$ for $q = p/(p-1)$ and $\mathscr{S}$ is dense in $C_b(A \times B)$ under uniform norm, there is a sequence of functions $\{F_n\}_{n \geqslant 1} \subset \mathscr{S}$ such that

$$\lim_{n \to \infty} \|F_n - F\|_{L^q(A \times B; dx\, dy)} = 0.$$

Hence, by (3.7) and (3.8), we know that for every $\omega \in \Omega_2$,

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx - \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\mathbb{E}\left[\bar{\nu}\right]\, dz\, dx \right| \\
&\leqslant \limsup_{\varepsilon \to 0} \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} (F(x,z) - F_n(x,z))\nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx \right| \\
&\quad + \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} (F(x,z) - F_n(x,z))\mathbb{E}[\bar{\nu}]\, dz\, dx \right| \\
&\quad + \limsup_{\varepsilon \to 0} \left| \iint_{A \times B} F_n(x,z)\nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx - \iint_{A \times B} F_n(x,z)\mathbb{E}\left[\bar{\nu}\right]\, dz\, dx \right| \\
&\leqslant \|F - F_n\|_{L^q(A \times B; dx\, dy)} \\
&\quad \times \left[\limsup_{\varepsilon \to 0} \left(\iint_{A \times B} \nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)^p\, dz\, dx \right)^{1/p} + \left(\iint_{A \times B} \left(\mathbb{E}\left[\bar{\nu}\right]\right)^p\, dz\, dx \right)^{1/p} \right] \\
&\leqslant 2\|F - F_n\|_{L^q(A \times B; dx\, dy)} \left(\iint_{A \times B} \mathbb{E}\left[\bar{\nu}^p\right]\, dz\, dx \right)^{1/p}.
\end{aligned}$$

Letting $n \to \infty$, we get that for any $\omega \in \Omega_2$ and any bounded function F with compact support,

$$\limsup_{\varepsilon \to 0} \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\nu(z/\varepsilon; \tau_{x/\varepsilon}\omega)\, dz\, dx - \iint_{\mathbb{R}^d \times \mathbb{R}^d} F(x,z)\mathbb{E}[\bar{\nu}]\, dz\, dx \right| = 0.$$

Now, taking

$$F(x,z) := \mathbb{1}_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{(f(x+z)-f(x))(g(x+z)-g(x))}{|z|^{d+\alpha}}$$

we arrive at that for every $\omega \in \Omega_2$,

$$\lim_{\varepsilon \to 0} I_1^\varepsilon = \int_{\mathbb{R}^d} \int_{\{\eta<|z|<1/\eta,\, z\in\Gamma\}} \frac{\big(f(x+z)-f(x)\big)\big(g(x+z)-g(x)\big)}{|z|^{d+\alpha}} \mathbb{E}[\bar{\nu}]\, dz\, dx,$$

which in turn yields the required assertion. $\square$

3.3. **Pre-compactness of functions in L^1-space**

In this subsection, we give the compactness for a sequence of uniformly bounded functions whose associated scaled Dirichlet forms are also uniformly bounded. The following is a key compactness result established in [8, Proposition 3.4].

Proposition 3.4. *Suppose that* **(C2)** *holds. Then there is a subset $\Omega_0 \subset \Omega$ of full probability measure so that for every $\omega \in \Omega_0$, any collection of functions $\{f_\varepsilon : \varepsilon \in (0,1]\}$ with $f_\varepsilon \in \mathcal{F}^{\varepsilon,\omega}$ for any $\varepsilon \in (0,1]$ having*

$$\limsup_{\varepsilon \to 0} \left(\|f_\varepsilon\|_\infty + \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon)\right) < \infty$$

Then, $\{f_\varepsilon : \varepsilon \in (0,1]\}$ is pre-compact as $\varepsilon \to 0$ in $L^1(B(0,r); dx)$ for every $r > 1$.

Below is the key lemma for the proof.

Lemma 3.5. ([8, Lemma 3.5]) *Suppose that* **(C2)** *holds. Then there is subset $\Omega_0 \subset \Omega$ of full probability measure so that the following holds for every $\omega \in \Omega_0$. Suppose that $\{f_\varepsilon : \varepsilon \in (0,1]\}$ is a collection of functions with $f_\varepsilon \in \mathcal{F}^{\varepsilon,\omega}$ for $\varepsilon \in (0,1]$ and*

$$\limsup_{\varepsilon \to 0} \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon) < \infty.$$

Then for every $r > 1$, $0 < |h| \leqslant r/3$ and $1 \leqslant i \leqslant d$,

$$\begin{aligned}\limsup_{\varepsilon \to 0} \sup_{x_0 \in B(0,r)} &\int_{B_{2h}(x_0,r)} |f_\varepsilon(x+he_i) - f_\varepsilon(x)|\, dx \\ &\leqslant c_0(r) h^{\alpha/2} \left(\limsup_{\varepsilon \to 0} \mathcal{E}^{\varepsilon,\omega}(f_\varepsilon, f_\varepsilon)^{1/2}\right),\end{aligned} \tag{3.9}$$

where $\{e_i : 1 \leqslant i \leqslant d\}$ *is the orthonormal basis of* $\mathbb{R}^d$, $B_{2h}(x_0, r) := \{y \in B(x_0, r) : |y - \partial B(x_0, r)| > 2h\}$ *and* $c_0(r)$ *is a positive measurable function depending on* r *but independent of* h *and* ω*.*

The proof of this lemma is partly motivated by (and hence resembles) the proof of the compact embeddings in fractional Sobolev spaces; see [14, Theorem 4.54, p. 216].

Given Lemma 3.5, the proof of Proposition 3.4 is relatively easy.

Proof of Proposition 3.4. According to [14, Theorem 1.95, p. 37], the following (a) and (b) imply that $\{f_\varepsilon : \varepsilon \in (0, 1]\}$ is a pre-compact set, as $\varepsilon \to 0$, in $L^1(B(0, r); dx)$ for every $r > 1$.

(a) For every $r > 1$ and $\zeta > 0$, there exists a constant $\delta_1 := \delta_1(r, \zeta; \omega)$ such that for every $h \in \mathbb{R}^d$ with $|h| < \delta_1$ such that

$$\limsup_{\varepsilon \to 0} \int_{B_{\delta_1}(0,r)} |f_\varepsilon(x + h) - f_\varepsilon(x)| \, dx \leqslant \zeta. \tag{3.10}$$

(b) For every $r > 1$ and $\zeta > 0$, there exists a constant $\delta_2 := \delta_2(r, \zeta)$ such that for every $h \in \mathbb{R}^d$ with $|h| < \delta_2$ such that

$$\limsup_{\varepsilon \to 0} \int_{B(0,r) \setminus B_{\delta_2}(0,r)} |f_\varepsilon(x)| \, dx \leqslant \zeta. \tag{3.11}$$

(3.10) can be shown by (3.9) and the triangular inequalities. For any $\delta > 0$, we have

$$\int_{B(0,r) \setminus B_\delta(0,r)} |f_\varepsilon(x)| \, dx \leqslant \|f_\varepsilon\|_\infty |B(0, r) \setminus B_\delta(0, r)| \leqslant c_2(r) \|f_\varepsilon\|_\infty \delta,$$

where $c_2(r)$ is a positive constant independent of δ and ε. This implies (3.11). □

Note that we use the negative moment condition on Λ_1 in (1.9) only in arguments for the statements in this part (in particular, Lemma 3.5).

3.4. **Proofs of Theorems 1.4 and 1.5**

Proofs of Theorems 1.4 *and* 1.5. By Theorem 3.1, it is enough to verify that assumption **(C)** implies assumption **(H)**.

According to Proposition 3.2(i) (resp. Proposition 3.2(ii)) and Proposition 3.4, assumption **(C)** implies (resp. **(C1*)** and **(C2)** imply) conditions (iv) and (i) in assumption **(H)**. On the other hand, it follows from the proof of [8, Theorem 1.3] that conditions (iii) and (ii) in assumption **(H)** hold true under condition **(C2)**. For example, by **(C2)**, for any bounded sets $A, B \subset \mathbb{R}^d$ and $x \in A$, we have

$$\int_B \kappa(x/\varepsilon, y/\varepsilon;\omega)\,dy \leqslant |B|\Lambda_2\big(\tau_{x/\varepsilon}\omega\big) + \int_B \Lambda_2(\tau_{y/\varepsilon}\omega)\,dy.$$

Hence, using the Birkhoff ergodic theorem and (1.9), the condition (iii) of assumption **(H)** holds true. Using the Birkhoff ergodic theorem again and **(C2)** (though requires some more delicate computations), we can show that property (ii) of assumption **(H)** holds as well; see the proof of [8, Theorem 1.3] for related details. This completes the proof. □

Acknowledgments. We are grateful to the referee for helpful comments. The research of Xin Chen is supported by the National Natural Science Foundation of China (No. 11871338). The research of Zhen-Qing Chen is partially supported by Simons Foundation grant 520542 and a Victor Klee Faculty Fellowship at UW. The research of Takashi Kumagai is supported by JSPS KAKENHI Grant Number JP17H01093 and by the Alexander von Humboldt Foundation. The research of Jian Wang is supported by the National Natural Science Foundation of China (No. 11831014 and 12071076), the Program for Probability and Statistics: Theory and Application (No. IRTL1704) and the Program for Innovative Research Team in Science and Technology in Fujian Province University (IRTSTFJ).

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