VECTOR BUNDLES ON NON-KÄHLER ELLIPTIC SURFACES AND INTEGRABLE SYSTEMS

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We survey some parts of the description of the moduli spaces of stable rank-two vector bundles on non-Kähler elliptic surfaces and we give some examples of algebraically completely integrable systems.

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1. NON-KÄHLER ELLIPTIC SURFACES

Let $X \xrightarrow{\pi} B$ be a minimal non-Kähler elliptic surface with B a smooth curve of genus g. It is well-known that $X \xrightarrow{\pi} B$ is a quasi-bundle over the base B, that is, all the smooth fibres are isomorphic to a fixed elliptic curve E and the singular fibres (in a finite number) are multiples of elliptic curves (see, for example, [6], [4]) .

If $m_1T_1, m_2T_2, ..., m_lT_l$ are the singular fibers and m is the least common multiple of $m_1, m_2, ..., m_l$, then there exists an m- cyclic covering $\epsilon : B' \to B$ and a principal elliptic bundle (with fibre E), $\pi' : X' \to B'$, with an m-cyclic covering $\psi' : X' \to X$ over $\epsilon : B' \to B$. Using this construction, the study of vector bundles over a non-Kähler elliptic surface is reduced to the case when $X \xrightarrow{\pi} B$ is a principal elliptic bundle, which we suppose for the rest of the talk.

Remark 1. Let $X \to B$ be principal elliptic bundle over the smooth curve B of genus g. For $g = 0$, X is a Hopf surface, for $g = 1$, X is a primary Kodaira surface and, for $g \geq 2$, X is called a properly elliptic surface.

Let E^* denote the dual of E (we fix a non-canonical identification $E^* = Pic^0(E)$ by fixing an origin on E); in fact we can identify $E^* \cong E$. The Jacobian surface associated to $X \xrightarrow{\pi} B$ is

$$J(X) = B \times E^* \xrightarrow{p_1} B,$$

and X is obtained from the relative Jacobian $J(X)$ by a finite number of logarithmic transformations [18]. We have the following result (see [5], [6], [7]):

THEOREM 1. *For any minimal non-Kähler elliptic surface we have the isomorphism:*

$$NS(X)/Tors(NS(X)) \cong Hom(J_B, Pic^0(E)),$$

where $NS(X)$ is the Neron - Severi group of the surface and J_B denotes the Jacobian variety of the curve B.

This result was extended by Brînzănescu - Ueno for torus quasi-bundles over curves, see [12].

Remark 2. In the case of elliptic surfaces, from the above theorem we get:

For any Chern class $c = c_1(L)$, with $L \in Pic(X)$ a line bundle, the class $\bar{c} \in NS(X)/Tors(NS(X))$, if it is non-zero, defines a covering map $\bar{c} : B \to Pic^0(E)$, which gives us a section of the Jacobian $J(X)$. This is exactly the *spectral curve* associated to the line bundle L, defined by Hitchin (see [17]).

2. VECTOR BUNDLES ON NON-KÄHLER ELLIPTIC SURFACES

Let V be a holomorphic rank-2 vector bundle on X, with fixed $c_1(V) = c_1 \in NS(X)$ and $c_2(V) = c_2 \in \mathbb{Z}$. Now, we fix also the determinant line bundle of V, denoted by $\delta = det(V)$. It defines an involution on the relative Jacobian $J(X) = B \times E^*$ of X:

$$i_\delta : J(X) \to J(X),\ (b, \lambda) \to (b, \delta_b \otimes \lambda^{-1}),$$

where δ_b denotes the restriction of δ to the fibre $E_b = \pi^{-1}(b)$, which has degree zero (see Lemma 2.2 in [9]). Taking the quotient of $J(X)$ by this involution, each fibre of p_1 becomes $E^*/i_\delta \cong \mathbb{P}^1$ and the quotient $J(X)/i_\delta$ is isomorphic to a ruled surface $\mathbb{F}_\delta$ over B. Let $\eta : J(X) \to \mathbb{F}_\delta$ be the canonical map.

We need some notation. The Chern classes and the rank can be defined for any analytic coherent sheaf $\mathcal{F}$ over X. If $\mathcal{F}$ is locally free, then we have $c_1(\mathcal{F}) = c_1(\det(\mathcal{F})) \in NS(X)$. Generally, by the [22], any analytic coherent sheaf $\mathcal{F}$ over a complex surface has a resolution

$$0 \to V_2 \to V_1 \to V_0 \to \mathcal{F} \to 0,$$

with V_i locally free sheaves. Then

$$c_1(\mathcal{F}) = c_1(V_0) - c_1(V_1) + c_1(V_2) \in NS(X).$$

Now, let $\mathcal{F}$ be an analytic coherent sheaf over a surface X of rank $r > 0$, with Chern classes $c_1(\mathcal{F})$ and $c_2(\mathcal{F})$. The *discriminant* $\Delta(\mathcal{F})$ is defined by

$$\Delta(\mathcal{F}) := \frac{1}{r}\left(c_2(\mathcal{F}) - \frac{r-1}{2r}c_1^2(\mathcal{F})\right).$$

For a non-algebraic surface X, $a \in NS(X)$ and r a positive integer we can define the following rational positive number (see [4], [8], [2])

$$m(r,a) := -\frac{1}{2r}max\{\Sigma_1^r(a/r - \mu_i)^2,\ \ \mu_i \in NS(X)\ \text{ with }\ \Sigma_1^r\mu_i = a\}.$$

The main existence result of holomorphic rank-2 vector bundles over non-Kähler elliptic surfaces is the following (see [9]):

THEOREM 2. *Let X be a minimal non-Kähler elliptic surface over a smooth curve B of genus g and fix a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. Set $m_{c_1} := m(2, c_1)$ and denote $\overline{c}_1$ the class of c_1 in $NS(X)$ modulo $2NS(X)$; moreover, let $e_{\overline{c}_1}$ be the invariant of the ruled surface $\mathbb{F}_{\overline{c}_1}$ determined by $\overline{c}_1$. Then, there exists a holomorphic rank-2 vector bundle on X with Chern classes c_1 and c_2 if and only if*

$$\Delta(2, c_1, c_2) \geq (m_{c_1} - d_{\overline{c}_1}/2),$$

where $d_{\overline{c}_1} := (e_{\overline{c}_1} + 4m_{c_1})/2$. Note that both $d_{\overline{c}_1}$ and $(m_{c_1} - d_{\overline{c}_1}/2)$ are non-negative numbers. Furthermore, if

$$(m_{c_1} - d_{\overline{c}_1}/2) \leq \Delta(2, c_1, c_2) < m_{c_1},$$

then the corresponding vector bundles are non-filtrable.

3. MODULI SPACES OF STABLE VECTOR BUNDLES

The main tool to study vector bundles on any elliptic surface X is by taking restrictions to the smooth fibres. Note that if X is non-Kähler, then the restriction of any line bundle on X to a smooth fibre of π always has degree zero; see [9]. For a rank two vector bundle V over X, its restriction to a generic fibre of π is semistable; more precisely, its restriction to a fibre $\pi^{-1}(b)$ is unstable on at most an isolated set of points $b \in B$ and, these isolated points are called the *jumps* of the bundle. Furthermore, there exists a divisor S_V in the relative Jacobian $J(X) = B \times E^*$ of X, called the *spectral curve* or *cover* of the bundle, that encodes the isomorphism class of the bundle over each fibre of π. The spectral curve is the following divisor

$$S_V := \Sigma_1^k(\{x_i\} \times E^*) + \overline{C},$$

where $\overline{C}$ is a bisection of $J(X)$ (i.e. $\overline{C}.E^* = 2$) and $x_1, x_2, ..., x_k$ are points in B that correspond to the jumps of V.

The spectral curve is constructed by using a twisted Fourier - Mukai transform. For more details, see [10], Section 3, Theorem 3.1, [15] and [13].

By construction, the spectral curve S_V of the bundle V is invariant by the involution i_δ of $J(X)$, and descends to the quotient $\mathbb{F}_\delta$; in fact, it is a pullback via η of a divisor on $\mathbb{F}_\delta$ of the form

$$\mathcal{G}_V := \Sigma_1^k f_i + A,$$

where f_i is the fibre of the ruled surface over the point x_i and A is a section of the ruling such that $\eta^* A = \overline{C}$. The divisor $\mathcal{G}_V$ is called *the graph of* V.

The degree of a vector bundle can be defined on any compact complex manifold M of dimension d. A theorem of Gauduchon's [16] states that any hermitian metric on M is conformally equivalent to a metric (called now a *Gauduchon metric*), whose associated $(1,1)$-form ω satisfies $\partial\overline{\partial}\omega^{d-1} = 0$. Suppose that M is endowed with such a metric and let L be a holomorphic line bundle on M. The *degree of L with respect to* ω is defined (see [14]), up to a constant factor, by

$$\deg(L) := \int_M F \wedge \omega^{d-1},$$

where F is the curvature of a hermitian connection on L, compatible with $\overline{\partial}_L$. Any two such forms differ by an exact $\partial\overline{\partial}$- exact form. Since $\partial\overline{\partial}\omega^{d-1} = 0$, the degree is independent of the choice of connection and is therefore well-defined. This degree is an extension of that in the Kähler case, where we get the usual topological degree. In general, this degree is not a topological invariant, for it can take values in a continuum.

Having defined the degree of holomorphic line bundles, we define the *degree* of a torsion-free coherent sheaf $\mathcal{V}$ by $\deg(\mathcal{V}) := \deg(\det \mathcal{V})$, where $\det \mathcal{V}$ is the determinant line bundle of $\mathcal{V}$, and the *slope of* $\mathcal{V}$ by

$$\mu(\mathcal{V}) := \deg(\mathcal{V})/rank(\mathcal{V}).$$

Now, we define the notion of stability: A torsion-free coherent sheaf $\mathcal{V}$ on M is *stable* if and only if for every coherent subsheaf $\mathcal{S} \subset \mathcal{V}$ with $0 < rk(\mathcal{S}) < rk(\mathcal{V})$, we have $\mu(\mathcal{S}) < \mu(\mathcal{V})$.

Fix a rank-2 vector bundle V on a minimal non-Kähler elliptic surface X and let δ be its determinant line bundle; there exists a sufficient condition on the spectral cover of V that ensures its stability (see [11]):

Proposition 1. *Suppose that the spectral cover of V includes an irreducible bisection $\overline{C}$ of $J(X)$. Then V is irreducible, and hence it is also stable with respect to any Gauduchon metric.*

Let X be a minimal non-Kähler elliptic surface and consider a pair (c_1, c_2) in $NS(X) \times \mathbb{Z}$. We fix a Gauduchon metric on X. For a fixed line bundle δ on X with $c_1(\delta) = c_1$, let $\mathcal{M}_{\delta,c_2}$ be the moduli space of stable (with respect to the fixed Gauduchon metric) holomorphic rank-2 vector bundles with invariants $\det(V) = \delta$ and $c_2(V) = c_2$. Note that, for any $c_1 \in NS(X)$, one can choose a line bundle δ on X such that

$$c_1(\delta) \in c_1 + 2NS(X) \quad \text{and} \quad m(2, c_1) = -\frac{1}{2}(c_1(\delta)/2)^2;$$

moreover, if there exist line bundles a and δ' such that $\delta = a^2\delta'$, then there is a natural isomorphism between the moduli spaces $\mathcal{M}_{\delta,c_2}$ and $\mathcal{M}_{\delta',c_2}$, defined by $V \to a \otimes V$.

This moduli space can be identified, via the Kobayashi - Hitchin correspondence, with the moduli space of gauge-equivalence classes of Hermitian - Einstein connections in the fixed differentiable rank-2 vector bundle determined by δ and c_2 (see, for example, [14], [19]). In particular, if the determinant δ is the trivial line bundle $\mathcal{O}_X$, then there is a one-to-one correspondence between $\mathcal{M}_{\mathcal{O}_X,c_2}$ and the moduli space of $SU(2)$-instantons, that is, anti-selfdual connections.

We can define the map

$$G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$$

that associates to each stable vector bundle its graph in $Div(\mathbb{F}_\delta)$, called the *graph map*. In [3], [20], the stability properties of vector bundles on Hopf surfaces were studied by analysing the image and the fibres of this map. In particular, it was shown [20] that the moduli space admits a natural Poisson structure with respect to which the graph map is a Lagrangian fibration whose generic fibre is an abelian variety, i.e. the map G admits an algebraically completely integrable system structure. For the general case, the moduli spaces $\mathcal{M}_{\delta,c_2}$ are studied by Brînzănescu - Moraru in [11].

We have the following results (see [11]):

Theorem 3. *Let $X \xrightarrow{\pi} B$ be a non-Kähler elliptic surface and let $\mathcal{M}_{\delta,c_2}$ be defined as above. Then:*

(i) *There are necessary and sufficient conditions such that $\mathcal{M}_{\delta,c_2}$ is nonempty (see Theorem 2).*

(ii) *If* $c_2 - c_1^2/2 > g - 1$ *(*g *is the genus of* B*), the moduli space* $\mathcal{M}_{\delta,c_2}$ *is smooth on the open dense subset of regular bundles (a regular bundle is a vector bundle for which its restriction to any fibre has its automorphism group of the smallest dimension).*

(iii) *The generic fibre of the graph map* $G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$ *is a Prym variety (for Prym varieties, see [21]).*

(iv) *Let* $\mathbb{P}_{\delta,c_2}$ *be the set of divisors in* $\mathbb{F}_\delta$ *of the form* $\Sigma_1^k f_i + A$*, where* A *is a section of the ruling and the* f_i*'s are fibres of the ruled surface, that are numerically equivalent to* $\eta_*(B_0) + c_2 f$*. For* $c_2 \geq 2$*, the graph map is surjective on* $\mathbb{P}_{\delta,c_2}$*. For* $c_2 < 2$*, see [11].*

(v) *Explicit descriptions of the the singular fibres of* G *are given, see [11].*

Special results on the moduli space $\mathcal{M}_{\delta,c_2}$ in the case of primary Kodaira surfaces are given in [1].

4. **INTEGRABLE SYSTEMS**

Let $X \xrightarrow{\pi} B$ be a minimal non-Kähler elliptic surface with B a smooth curve of genus g. For $g \leq 1$ the surface X has a Poisson structure. By using this Poisson structure, in the case of a principal elliptic bundle, one defined a Poisson structure on the moduli space $\mathcal{M}_{\delta,c_2}$; see, for details [11]. Then:

Theorem 4. *Let* $X \xrightarrow{\pi} B$ *a principal elliptic bundle. If* $g \leq 1$*, the moduli space* $\mathcal{M}_{\delta,c_2}$ *is smooth of dimension* $8\Delta(2, c_1, c_2)$ *and* $G : \mathcal{M}_{\delta,c_2} \to Div(\mathbb{F}_\delta)$ *is an algebraically completely integrable Hamiltonian system.*

For details, see [20] and [11].

We finish with the following problem: Is it possible to define a Poisson structure on the moduli space $\mathcal{M}_{\delta,c_2}$ in the case $g \geq 2$?

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