FOUR-DIMENSIONAL GEOMETRIC SUPERGRAVITY AND ELECTROMAGNETIC DUALITY: A BRIEF GUIDE FOR MATHEMATICIANS

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We give a gentle introduction to the global geometric formulation of the bosonic sector of four-dimensional supergravity on an oriented four-manifold M of arbitrary topology, providing a geometric characterization of its U-duality group. The geometric formulation of four-dimensional supergravity is based on a choice of a vertically Riemannian submersion π over M equipped with a flat Ehresmann connection, which determines the non-linear *section* sigma model of the theory, and a choice of flat symplectic vector bundle $\mathcal{S}$ equipped with a positive complex polarization over the total space of π, which encodes the inverse gauge couplings and theta angles of the theory and determines its gauge sector. The classical fields of the theory consist of Lorentzian metrics on M, global sections of π and two-forms valued in $\mathcal{S}$ satisfying an algebraic relation which defines the notion of *twisted* self-duality in four Lorentzian dimensions. We use this geometric formulation to investigate the group of *electromagnetic duality transformations* of supergravity, also known as the continuous classical U-duality group, which we characterize using a certain short exact sequence of automorphism groups of vector bundles. Moreover, we discuss the general structure of the Killing spinor equations of four-dimensional supergravity, providing several explicit examples and remarking on a few open mathematical problems. This presentation is aimed at mathematicians working in differential geometry.

1. INTRODUCTION

Supergravity theories are supersymmetric theories of gravity which, aside from their intrinsic phenomenological interest, are of fundamental importance in high energy physics, since they describe the low-energy limit of string and M-theory and their associated supersymmetric compactifications [10, 13, 32, 40, 58]. In addition to their role in theoretical physics, supergravity theories have been the source of important developments and activity in mathematics, especially in geometry and topology, see for instance [31, 33, 37, 42, 56, 61] as well as

their references and citations. Indeed, the local formulation of supergravity is well-known to involve important mathematical structures and objects interacting in a delicate equilibrium dictated by supersymmetry, such as Kähler-Hodge manifolds, Riemannian manifolds of special holonomy, harmonic maps, exceptional Lie groups, gerbes and Courant algebroids or gauge-theoretic moduli spaces, to name only a few. This makes the mathematical study of supergravity into a rich and quite formidable endeavour. Supergravity theories can be defined in various dimensions and signatures and can be deformed through various mechanisms while preserving their supersymmetric structure (see [60, 35] and references therein for details on the deformation of such theories through *gauging*). In this short review, we will consider exclusively four-dimensional *ungauged* supergravity theories in Lorentzian signature, where the term ungauged indicates that we will not consider any gauging of the theory. Such theories are particularly relevant for several reasons, both from the physical and mathematical point of view, among which we can mention the following:

• Four-dimensional supergravity theories describe the effective dynamics of the massless modes of string and M-theory compactifications to four-dimensions, which is the observed physical dimension of spacetime and therefore yields the adequate set-up for phenomenological applications [10, 41].

• Supergravity theories in four-dimensions enjoy a type of duality called *electromagnetic U-duality*, which is inherited from ordinary electromagnetic duality in four Lorentzian dimensions and has deep connections with string theory U-duality groups [44]. Furthermore, electromagnetic duality gives rise to interesting mathematical structures of gauge-theoretic type [51].

• Four-dimensional supergravity theories involve rich non-linear sigma models with Riemannian target spaces of special type [32, 58, 43], whose moduli spaces of solutions can be expected to enjoy interesting applications in the differential topology of Riemannian three and four manifolds.

• The dimensional reduction of $\mathcal{N} = 2$ four-dimensional supergravity to three-dimensions is the origin of the celebrated c-map in quaternionic-Kähler and projective special Kähler geometry [28, 26, 42], see also [24] for a related construction called the *the r-map* in the literature. In particular, $\mathcal{N} = 2$ supergravity has a deep connection with Quaternionic-Kähler manifolds whose mathematical investigation has been already initiated in several pioneering works, see [1, 2, 18, 42, 55] and their citations.

In contrast to higher-dimensional supergravities, which receive increasing mathematical attention and whose geometric formulation and structure is be-

ing actively investigated [7, 9, 12, 29, 37, 38, 59], the mathematics community has paid little attention to low-dimensional supergravity, in particular four-dimensional supergravity. This might be due in part to the inaccessibility of the relevant physics literature to mathematicians. Nonetheless, four-dimensional supergravity is an extremely rich subject from a mathematical standpoint, and there exists indeed a *plethora* of such theories involving interesting modern mathematical structures and leading to novel mathematical problems, most of which have not emerged into the mathematical community. The distinction between higher and low dimensional supergravity is akin to that between higher and low-dimensional differential topology, the latter yielding a remarkably rich and subtle theory [27]. Given their importance, the local structure and properties of four dimensional supergravities have been extensively studied in the physics literature, in a long-term effort that evolved myriad of ramifications. We refer the reader to [3, 4, 6, 14, 15, 19, 20, 23, 25, 30, 54] for more details and references. Despite all this work, the global geometric formulation and proper mathematical theory of four-dimensional supergravity are poorly understood and remain open for investigation and exploration.

It would be desirable to develop the complete mathematical foundations of all four-dimensional supergravities, including their bosonic and fermionic sectors. In our opinion, this may be currently out of reach. Fortunately, most applications of supergravity to differential geometry and topology only require the mathematical theory of the bosonic sector together and the Killing spinor equations, which fully capture the geometry and topology of supersymmetric solutions and associated moduli spaces. Solutions of the equations of motion of a supergravity theory which satisfy the supergravity Killing spinor equations are called *supersymmetric solutions* and have been intensively studied in the physics [39] and mathematics literature [33], the latter focusing almost entirely on higher-dimensional Riemannian signature. The global geometrization of bosonic supergravity together with its associated Killing spinor equations on oriented manifolds of arbitrary topology was named *geometric supergravity* in [49, 50, 16], which initiated a long-term program devoted to systematically developing the mathematical foundations of four-dimensional (ungauged) geometric supergravity. The first step in this program concerns the bosonic sector, paying special attention to its Dirac quantization, electromagnetic U-duality group and various reductions to three-dimensional Riemannian manifolds and Riemann surfaces. We note that the mathematical theory of geometric supergravity is far from finished, and [49, 50, 16] constitute only a first few steps towards its completion.

In this short review we will discuss some of the results of [49, 50, 16] concerning the global mathematical formulation and symplectic duality struc-

ture of geometric supergravity. Roughly speaking, the generic bosonic sector of four-dimensional supergravity consists of three sub-sectors, namely:

- The gravitational sector, which corresponds to the Einstein-Hilbert term of the local Lagrangian.

- The scalar sector, which corresponds locally to a non-linear sigma model coupled to gravity.

- The gauge sector, which corresponds locally to a theory of an arbitrary number of *abelian gauge fields* coupled to the scalars fields of the scalar sector.

Therefore, four-dimensional supergravity can be though of as the unification, using supersymmetry as a guiding principle, of three cornerstones of differential geometry, namely the theory of Einstein metrics, the theory of harmonic maps and Yang-Mills theory.

The Killing spinor equations of four-dimensional supergravity are first order differential equations involving the bosonic fields of the theory and a *supersymmetry parameter*, which is mathematically described as a section of an appropriate bundle of Clifford modules over the underlying Lorentzian manifold. Supergravity Killing spinor equations generalize, through the principle of supersymmetry, well-known spinorial equations studied intensively in the literature, such as Hermite-Yang-Mills equations, instanton equations, the Seiberg-Witten equations, generalized Killing spinor equations or the pseudo-holomorphicity equations. The study of supergravity Killing spinor equations makes contact with modern areas of mathematics under current development and brings supergravity into *mathematical gauge theory*, a field of mathematics whose tools and methods are specially well-adapted to the study of supersymmetric solutions and their moduli spaces. We hope that the development of the mathematical theory of four-dimensional supergravity can clarify this relation and bring new problems and perspectives into mathematical gauge theory.

An important remark is in order: we do not discuss the Dirac quantization of geometric supergravity in this report, since it is yet to be fully developed and it is work in progress [51]. As shown in Op. Cit., implementing Dirac quantization is a fundamental step in order to properly understand the geometric structure of four-dimensional supergravity as well as the global structure of its solutions and associated moduli spaces.

The outline of this manuscript is as follows. In Section 2 we review the well-known local formulation of four-dimensional bosonic supergravity, giving a rigorous seemingly novel description of its electromagnetic U-duality group. In Section 3 we explain the global geometric formulation of bosonic four-dimensional supergravity together with the necessary geometric background.

In Section 4 we describe the global electromagnetic U-duality group of geometric supergravity, characterizing it in terms of a certain short exact sequence and discussing some examples. Finally, in Section 5 we briefly discuss the Killing spinor equations of four-dimensional supergravity and present some explicit examples, mentioning along the way some open mathematical problems.

2. **LOCAL BOSONIC SUPERGRAVITY**

In this section we review the local formulation of the generic bosonic sector of four-dimensional supergravity, paying special attention to the electromagnetic *U-duality* group of the theory, which consists of electromagnetic duality transformations of the abelian gauge fields coupled to scalars and gravity. The local formulation of the bosonic sector of four-dimensional supergravity was considered in detail in references [3, 4, 5, 34], where the duality transformations of the local theory were investigated. The reader is referred to [6, 14, 30, 54] for comprehensive reviews and exhaustive lists of references.

Let U be a contractible non-empty oriented and relatively compact open subset of $\mathbb{R}^4$ with coordinates $\{x^a\}$, where $a = 1, \ldots, 4$. Fixing non-negative integers n_s, n_v, the configuration space of the local bosonic sector of extended four-dimensional supergravity with n_s scalar fields and n_v abelian gauge fields is defined as the set of triples (g, ϕ, A) consisting of:

- A Lorentzian metric g defined on U.

- An $\mathbb{R}^{n_s}$-valued function $\phi \colon U \to \mathbb{R}^{n_s}$ defined on U. We denote the components of ϕ by $\phi^i \colon U \to \mathbb{R}$, with $i = 1, \ldots, n_s$ and fix an oriented open subset $V \subset \mathbb{R}^{n_s}$ containing $\phi(U)$. The real functions $\{\phi^i\}$ are the (locally-defined) *scalar fields* of the theory. We will refer to such functions $\phi \colon U \to \mathbb{R}^{n_s}$ as *scalar maps*.

- An $\mathbb{R}^{n_v}$-valued one-form $A \in \Omega^1(U, \mathbb{R}^{n_v})$. When necessary, we will denote the components of A by $A^\Lambda \in \Omega^1(U)$, with $\Lambda = 1, \ldots, n_v$, which correspond to the local U(1) *gauge fields* of the theory. We denote by:

$$F \stackrel{\text{def.}}{=} \mathrm{d}A \in \Omega^2(U, \mathbb{R}^{n_v}),$$

the *field strength* associated to A, whose components will be denoted by $F^\Lambda = \mathrm{d}A^\Lambda \in \Omega^2(U)$.

The *local* bosonic sector of extended four-dimensional supergravity is defined

through the following action functional:

$$S_l[g_U, \phi, A] \stackrel{\text{def.}}{=} \int_U \left\{ -R_{g_U} + \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j + \mathcal{R}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda * F^{\Sigma\, ab} \right.$$

(1)

$$\left. + \mathcal{I}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda F^{\Sigma\, ab} \right\} \nu_{g_U},$$

where:

- ν_{g_U} is the Lorentzian volume form associated to g and the given orientation on U.

- $\mathcal{G} \in \Gamma(T^*V \odot T^*V)$ is a Riemannian metric on V. We denote by:

$$\mathcal{G}(\phi) \stackrel{\text{def.}}{=} \mathcal{G} \circ \phi \colon U \to \text{Sym}(n_s, \mathbb{R})$$

the composition of $\mathcal{G}$ with ϕ and by $\mathcal{G}_{ij}(\phi)$ the components of $\mathcal{G}(\phi)$ in the Cartesian coordinates of $V \subset \mathbb{R}^{n_s}$.

- $\mathcal{R}, \mathcal{I} \colon V \to \text{Sym}(n_v, \mathbb{R})$ are smooth functions on V valued in the vector space of $n_v \times n_v$ square symmetric matrices with real entries. We denote by:

$$\mathcal{R}(\phi) \stackrel{\text{def.}}{=} \mathcal{R} \circ \phi \colon U \to \text{Sym}(n_v, \mathbb{R}), \qquad \mathcal{I}(\phi) \stackrel{\text{def.}}{=} \mathcal{I} \circ \phi \colon U \to \text{Sym}(n_v, \mathbb{R})$$

the compositions of $\mathcal{R}$ and $\mathcal{I}$ with ϕ and by $\mathcal{I}_{\Lambda\Sigma}(\phi)$, $\mathcal{R}_{\Lambda\Sigma}(\phi)$ the entries of the corresponding symmetric matrices. Furthermore, $\mathcal{I}$ is required to be positive definite, a condition which is imposed in order to have a consistent kinetic term for the gauge fields A^Λ.

Therefore, the local bosonic sector of supergravity on the oriented open sets (U, V) is uniquely determined by a choice of Riemannian metric $\mathcal{G}$ on V and matrix-valued functions $\mathcal{I}$ and $\mathcal{R}$ as described above. In some cases a scalar potential can occur in (1), but we have set it to zero for simplicity. The functional S_l can be naturally written as a sum of three pieces:

$$S_l = S_l^e + S_l^s + S_l^v,$$

where:

$$S_l^e[g] \stackrel{\text{def.}}{=} -\int_U R_g \nu_g$$

is the Einstein-Hilbert action on U,

$$S_l^s[g, \phi] \stackrel{\text{def.}}{=} \int_U \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j\, \nu_g$$

is a local non-linear sigma model with target space metric $\mathcal{G}$, and

$$S_l^v[g, \phi, A] \stackrel{\text{def.}}{=} \int_U \left\{ \mathcal{R}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda * F^{\Sigma\, ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda F^{\Sigma\, ab} \right\} \nu_g,$$

is a local Abelian Yang-Mills theory coupled to the scalars $\{\phi^i\}_{i=1,\dots,n_s}$.

Remark 2.1. In standard supergravity terminology, S_l^e defines the *gravity sector* of the theory, S_l^s defines the *scalar sector* of the theory and S_l^v defines the *gauge sector* of the theory.

The matrix $\mathcal{I}$ generalizes the inverse of the squared coupling constant appearing in ordinary four-dimensional gauge theories, whereas $\mathcal{R}$ generalizes the *theta angle* of quantum chromodynamics. All together, the generic bosonic sector of extended supergravity couples Einstein-Hilbert's action to a non-linear sigma model with Riemannian target space $(V, \mathcal{G})$ and to a given number of *abelian gauge fields*. In supergravity terminology, the Riemannian manifold $(V, \mathcal{G})$ is called the *scalar manifold* of the theory and $\mathcal{G}$ its *scalar metric*.

Definition 2.2. We define a **local electromagnetic structure** on V to be a pair $(\mathcal{R}, \mathcal{I})$, where both $\mathcal{R}$ and $\mathcal{I}$ are symmetric $n_v \times n_v$ matrix-valued functions on V with $\mathcal{I}$ positive-definite. We will denote by $\mathfrak{E}_V$ the set of all electromagnetic structures on V. We define a **local scalar-electromagnetic structure** on V to be a triple $(\mathcal{G}, \mathcal{R}, \mathcal{I})$, where $\mathcal{G}$ is a Riemannian metric on V and $(\mathcal{R}, \mathcal{I})$ is an electromagnetic structure. We will refer to the local supergravity with scalar metric $\mathcal{G}$ and gauge couplings $(\mathcal{R}, \mathcal{I})$ simply as the local supergravity associated to $(\mathcal{G}, \mathcal{R}, \mathcal{I})$.

Supersymmetry constrains the local isometry type of the Riemannian manifold $(V, \mathcal{G})$ that can be considered as the target space of the non-linear sigma model of a given supergravity theory. Depending on the amount $\mathcal{N}$ of supersymmetry preserved, the local isometry type of $(V, \mathcal{G})$ is given as follows [4]:

Table 1 – Isometry type of the scalar manifolds of four-dimensional supergravity, depending on the amount $\mathcal{N}$ of supersymmetry of the theory. The symbol n_c denotes the number of chiral multiplets, n_v denotes the number of vector multiplets and n_H denotes the number of hypermultiplets

Number of supersymmetries	Isometry type of $(V, \mathcal{G})$	Dimension
$\mathcal{N} = 1$	$\mathcal{M}_{\text{KH}}$	$2n_c$
$\mathcal{N} = 2$	$\mathcal{M}_{\text{PSK}} \times \mathcal{M}_{\text{QK}}$	$2n_v + 4n_H$
$\mathcal{N} = 3$	$\text{SU}(3, n)/\text{S}(\text{U}(3) \times \text{U}(n))$	$6n_v$
$\mathcal{N} = 4$	$\text{SU}(1, 1)/\text{U}(1) \times \text{SO}(6, n)/\text{S}(\text{O}(6) \times \text{O}(n))$	$6n_v + 2$
$\mathcal{N} = 5$	$\text{SU}(1, 5)/\text{S}(\text{U}(1) \times \text{U}(5))$	10
$\mathcal{N} = 6$	$\text{SO}^*(12)/(\text{U}(1) \times \text{SU}(6))$	30
$\mathcal{N} = 8$	$\text{E}_{7(7)}/(\text{SU}(8)/\mathbb{Z}_2)$	70

Remark 2.3. The case $\mathcal{N} = 7$ does not appear in the previous list because

$\mathcal{N} = 7$ supergravity can be shown to always admit an additional supersymmetry which automatically makes it into $\mathcal{N} = 8$ supergravity [13].

The symbol $\mathcal{M}_{KH}$ denotes a Kähler-Hodge manifold, or more precisely a complex manifold equipped with a chiral triple [16], whereas $\mathcal{M}_{PSK}$ and $\mathcal{M}_{QK}$ respectively denote a projective special Kähler manifold and a Quaternionic-Kähler manifold. For $\mathcal{N} > 2$, the scalar manifolds appearing in the previous table are all simply connected and non-compact symmetric manifolds equipped with a certain Riemannian metric. All of them are diffeomorphic to $\mathbb{R}^k$ for an appropriate k. In the $\mathcal{N} = 8$ case, $E_{7(7)}$ denotes the maximally non-compact real form of the complex exceptional Lie group E_7 and $SU(8)/\mathbb{Z}_2 \subset E_{7(7)}$ is its maximal compact subgroup.

2.1. Equations of motion

Let $G_{ab}^g \overset{\text{def.}}{=} R_{ab}^g - \frac{1}{2}g_{ab}R^g$ denote the Einstein tensor associated to g. The equations of motion that follow from the action functional (1) for a given local scalar-electromagnetic structure $(\mathcal{G}, \mathcal{R}, \mathcal{I})$ are the following:

- The *Einstein equations*:

(2)
$$\begin{aligned}
G_{ab}^g =\,& \mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial_b\phi^j - \frac{1}{2}g_{ab}\mathcal{G}_{ij}(\phi)\partial_c\phi^i\partial^c\phi^j \\
& + 2\mathcal{I}_{\Lambda\Sigma}(\phi)F_{ac}^\Lambda F_b^{\Sigma c} - \frac{1}{2}g_{ab}\mathcal{I}_{\Lambda\Sigma}(\phi)F_{cd}^\Lambda F^{\Sigma cd}\,.
\end{aligned}$$

- The *scalar equations*:

(3)
$$\begin{aligned}
\nabla_a^g(\mathcal{G}_{ik}(\phi)\partial^a\phi^i) =\,& \frac{1}{2}\partial_k\mathcal{G}_{ij}(\phi)\partial_a\phi^i\partial^a\phi^j + \frac{1}{2}\partial_k\mathcal{R}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda * F^{\Sigma ab} \\
& + \frac{1}{2}\partial_k\mathcal{I}_{\Lambda\Sigma}(\phi)F_{ab}^\Lambda F^{\Sigma ab}\,.
\end{aligned}$$

- The *Maxwell equations*:

(4)
$$\nabla_a^g(\mathcal{R}_{\Lambda\Sigma}(\phi) * F^{\Sigma ab} + \mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Sigma ab}) = 0\,.$$

The variables of the supergravity equations consist on Lorentzian metrics g on U, n_s scalars $\{\phi^i\}$ and n_v closed two-forms $\{F^\Lambda\}$. Conditions $dF^\Lambda = 0$, $\Lambda = 1, \ldots, n_v$, are known as the *Bianchi identities*, and ensure that $F = dA$ for a vector valued one-form A on U. It can be easily seen that the Maxwell equations are equivalent to:

$$d(\mathcal{R}_{\Lambda\Sigma}(\phi)F^\Sigma) = d(\mathcal{I}_{\Lambda\Sigma}(\phi) * F^\Sigma)\,.$$

Define now the two-forms:

$$G_\Lambda(\phi) \stackrel{\text{def.}}{=} \mathcal{R}_{\Lambda\Sigma}(\phi)F^\Sigma - \mathcal{I}_{\Lambda\Sigma}(\phi) * F^\Sigma \in \Omega^2(U), \qquad \Lambda = 1, \ldots, n.$$

For ease of notation, we will sometimes drop the explicit dependence of $G_\Lambda(\phi)$ on ϕ. Then, the Bianchi identities and Maxwell equations (4) are given by:

$$\mathrm{d}F^\Sigma = 0, \qquad \mathrm{d}G_\Lambda = 0, \qquad \Lambda = 1, \ldots, n$$

which in turn can be equivalently written simply as:

$$\mathrm{d}\mathcal{V}(\phi) = 0,$$

where $\mathcal{V}(\phi) \in \Omega^2(U, \mathbb{R}^{2n})$ denotes the following vector of two-forms:

$$\mathcal{V}(\phi) = \begin{pmatrix} F \\ G(\phi) \end{pmatrix} \in \Omega^2(U, \mathbb{R}^{2n_v}).$$

The following important lemma follows by direct computation.

LEMMA 2.4. *Let $(\mathcal{R}, \mathcal{I})$ be a local electromagnetic structure. A vector-valued two-form $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})$ can be written as:*

$$\mathcal{V} = \begin{pmatrix} F \\ G(\phi) \end{pmatrix},$$

for $F \in \Omega^2(U, \mathbb{R}^{n_v})$, where $\phi \colon U \to \mathbb{R}^{n_s}$ and $G(\phi) = \mathcal{R}(\phi)F - \mathcal{I}(\phi) * F$, if and only if:

$$(5) \qquad\qquad *\mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V}),$$

where $\mathcal{J} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R})$ is the matrix-valued map defined as follows

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R}),$$

and $\mathcal{J}(\phi) \stackrel{\text{def.}}{=} \mathcal{J} \circ \phi \colon U \to \mathrm{Gl}(2n_v, \mathbb{R})$. In particular, we have $\mathcal{J}^2 = -1$.

Remark 2.5. The matrix-valued map $\mathcal{J} \colon V \to \mathrm{Gl}(2n_v, \mathbb{R})$ can be understood as a fiber-wise complex structure on the trivial vector bundle of rank $2n_v$ over V.

Equation (5) is known in the literature as the *twisted self-duality condition* for the field strength $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})$. The following proposition gives the geometric interpretation of condition (5), which in turn unveils the global geometric interpretation of the twisted self-duality condition, as we will see in Section 3. For future reference, we define the *standard symplectic form* of $\mathbb{R}^{2n}$ to have the matrix representation:

$$(6) \qquad\qquad \omega = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}$$

in the canonical basis of $\mathbb{R}^{2n_v}$. More precisely, if we denote the canonical basis of $\mathbb{R}^{2n_v}$ by $\mathcal{E} = (e_1, \ldots, e_{n_v}, f_1, \ldots, f_{n_v})$, we have:

$$(7) \qquad \omega \stackrel{\text{def.}}{=} \sum_a f_a^* \wedge e_a^*, \qquad a = 1, \ldots, n_v,$$

where $\mathcal{E}^* = (e_1^*, \ldots, e_{n_v}^*, f_1^*, \ldots, f_{n_v}^*)$ is the basis dual to $\mathcal{E} = (e_1, \ldots, e_{n_v}, f_1, \ldots, f_{n_v})$.

PROPOSITION 2.6. *Let ω be the standard symplectic form on $\mathbb{R}^{2n}$. A matrix-valued map $\mathcal{J} \colon V \to \mathrm{Aut}(\mathbb{R}^{2n})$ can be written as:*

$$(8) \qquad \mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} \colon V \to \mathrm{Aut}(\mathbb{R}^{2n})$$

for a local electromagnetic structure $(\mathcal{R}, \mathcal{I})$ if and only if $\mathcal{J}|_p$ is a compatible taming of ω for every $p \in V$.

Remark 2.7. We recall that a complex structure J on $\mathbb{R}^{2n_v}$ is said to be a *compatible taming* of ω if:

$$\omega(J\xi_1, J\xi_2) = \omega(\xi_1, \xi_2), \qquad \forall\, \xi_1, \xi_2 \in \mathbb{R}^{2n_v},$$

and:

$$\omega(\xi, J\xi) > 0, \qquad \forall\, \xi \in \mathbb{R}^{2n_v} \setminus \{0\}.$$

In the following we shall always consider $\mathbb{R}^{2n_v}$ to be endowed with the symplectic form ω as introduced above.

Proof. If $\mathcal{J}$ is taken as in equation (8) for a certain local electromagnetic structure $(\mathcal{R}, \mathcal{I})$ then a direct computation shows that it is a compatible taming of ω. Conversely, assume that $\mathcal{J}$ is a complex structure on $\mathbb{R}^{2n_v}$ taming ω at every point in V (we omit the evaluation at a point for ease of notation). Let $\mathcal{E} = (e_1, \ldots, e_{n_v}, f_1, \ldots, f_{n_v})$ the canonical basis of $\mathbb{R}^{2n}$, which is symplectic with respect to ω. The vectors $\mathcal{E}_f = (f_1, \ldots, f_{n_v})$ form a basis of the complex vector space $(\mathbb{R}^{2n}, \mathcal{J})$, and there exists a unique map $\mathcal{N} \colon V \to \mathrm{Mat}(n, \mathbb{C})$ valued in the complex $n \times n$ square matrices and satisfying:

$$(9) \qquad e_a = \sum_b \mathcal{N}_{ab}\, f_b, \qquad a, b = 1, \ldots, n_v.$$

Then:

$$\omega(e_a, f_b) = \sum_c \omega(\mathcal{N}_{ac}\, f_c, f_b) = \sum_c \mathrm{Im}\,(\mathcal{N})_{ac}\omega(Jf_c, f_a) = -\delta_{ab},$$

which implies that $\mathrm{Im}\,(\mathcal{N})$ is a symmetric and positive definite $n \times n$ real matrix. Moreover, using the previous equation and the compatibility of $\mathcal{J}$ with ω, we compute:

$$0 = \omega(e_a, e_b) = \mathrm{Re}(\mathcal{N})_{ab} + \mathrm{Im}(\mathcal{N})_{ac}\omega(J(e_b), f_c) = \mathrm{Re}(\mathcal{N})_{ab} - \mathrm{Re}(\mathcal{N})_{ba},$$

whence $\mathcal{N}\colon V \to \mathrm{Mat}(n_v, \mathbb{C})$ is valued in the symmetric complex square matrices of positive-definite imaginary part. Hence, setting:

$$\mathcal{R} \overset{\mathrm{def.}}{=} -\mathrm{Re}(\mathcal{N})\,, \qquad \mathcal{I} \overset{\mathrm{def.}}{=} \mathrm{Im}(\mathcal{N})\,,$$

we obtain a well-defined local electromagnetic structure $(\mathcal{R}, \mathcal{I})$. Using again equation (9) to compute the action of $\mathcal{J}$ on the basis $\mathcal{E}$ we obtain:

$$\mathcal{J}(e_a) = -\mathcal{R}_{ab}\mathcal{I}_{bc}^{-1}\, e_c - \mathcal{R}_{ab}\mathcal{I}_{bc}^{-1}\mathcal{R}_{cd}\, f_d - \mathcal{I}_{ad}\, f_d\,, \qquad \mathcal{J}(f_a) = \mathcal{I}_{ab}^{-1}\, e_b + \mathcal{I}_{ab}^{-1}\mathcal{R}_{bc}\, f_c\,,$$

which is equivalent to (8).   □

Remark 2.8. The map $\mathcal{N}\colon V \to \mathrm{Mat}(n_v, \mathbb{C})$ constructed in the proof of the previous proposition is called the *period matrix* in the literature and can be used to obtain a convenient local formulation of bosonic supergravity, as we will explain in Section 2.3.

For future convenience we introduce the following definition.

Definition 2.9. A **taming map** $\mathcal{J}$ on V is a smooth map $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$ such that $\mathcal{J}|_p$ is a complex structure on $\mathbb{R}^{2n_v}$ which compatibly tames ω, where the latter is the standard symplectic structure on $\mathbb{R}^{2n_v}$ as defined in (7) in terms of the canonical basis of $\mathbb{R}^{2n_v}$. We will denote the space of all taming maps by $\mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega)$.

PROPOSITION 2.10. *There is a one-to-one correspondence between taming maps $\mathcal{J}\colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$ and local electromagnetic structures $(\mathcal{R}, \mathcal{I})$, i.e. there exists a canonical bijection:*

$$\gamma\colon \mathfrak{E}_V \to \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega)\,, \qquad (\mathcal{R}, \mathcal{I}) \mapsto \mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix}\,.$$

Proof. The statement follows directly from the proof of proposition 2.6. The inverse:

$$\gamma^{-1}\colon \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega) \to \mathfrak{E}_V\,,$$

maps a taming map $\mathcal{J}\colon \in \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega)$ to the electromagnetic structure $\gamma^{-1}(\mathcal{J}) = (\mathcal{R}, \mathcal{I})$ given by:

$$\mathcal{R} \overset{\mathrm{def.}}{=} -\mathrm{Re}(\mathcal{N})\,, \qquad \mathcal{I} \overset{\mathrm{def.}}{=} \mathrm{Im}(\mathcal{N})\,,$$

where $\mathcal{N}$ is the complex matrix uniquely defined by $e_a = \sum_b \mathcal{N}_{ab} f_b$ in terms of the canonical symplectic basis $\mathcal{E} = (e_1, \ldots, e_{n_v}, f_1, \ldots, f_{n_v})$ of $(\mathbb{R}^{2n_v}, \omega)$.   □

By the previous proposition, an electromagnetic structure can be equivalently described in terms of a taming map $\mathcal{J}$ and a local scalar-electromagnetic structure can be denoted simply by pair $(\mathcal{G}, \mathcal{J})$. This description of local electromagnetic structures is particularly convenient for the global geometric formulation of bosonic supergravity, as we discuss in section 3.

The gauge fields $\{A_\Lambda\}$ integrating $\{G_\Lambda\}$ are usually referred to as the *electric gauge fields*, whereas the one-forms $\{A^\Lambda\}$ integrating $\{F^\Lambda\}$ are usually referred as their dual *magnetic gauge fields*.

Definition 2.11. A vector-valued one-form $A \in \Omega^1(U, \mathbb{R}^{2n_v})$ is said to be **twisted selfdual** with respect to the Lorentzian metric g, the scalar map $\phi \colon U \to \mathbb{R}^{n_s}$ and a taming map $\mathcal{J}$ if:

$$*_g \mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V}),$$

where $\mathcal{V} = \mathrm{d}A$ and $*_g$ is the Hodge dual associated to g and the given orientation on U.

The global geometric interpretation of the local gauge fields (A^Λ, A_Λ) and their role in the formulation of supergravity is investigated in [51] through the implementation of Dirac quantization. Since we will not consider Dirac quantization in this review, we will consider instead the *classical* configuration space of local bosonic supergravity.

Definition 2.12. The **classical configuration space** $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ of the local bosonic supergravity on U associated to $(\mathcal{G}, \mathcal{J})$, where $\mathcal{G}$ is a Riemannian metric on V and $\mathcal{J}$ is a taming map on V, is defined as the following set:

$$\mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) \overset{\text{def.}}{=} \{(g, \phi, \mathcal{V}) \mid *\mathcal{V} = -\mathcal{J}(\phi)(\mathcal{V}),$$
$$g \in \mathrm{Lor}(U), \ \phi \in C^\infty(U, V), \ \mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})\}.$$

The **solution space** $\mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$ of the local bosonic supergravity associated to $(\mathcal{G}, \mathcal{J})$ is the subset of $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ whose elements satisfy the equations of motion of bosonic supergravity.

Definition 2.13. Let $(\mathcal{G}, \mathcal{R}, \mathcal{I})$ be a scalar-electromagnetic structure. The **scalar energy momentum tensor** associated to $(\mathcal{G}, \mathcal{R}, \mathcal{I})$ is the following map:

$$\mathcal{T}(\mathcal{G}) \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{R}, \mathcal{I}) \to \Gamma(T^*U \odot T^*U),$$
$$(g, \phi, \mathcal{V}) \mapsto \mathcal{G}_{ij}(\phi)\mathrm{d}\phi^i \odot \mathrm{d}\phi^j - \frac{1}{2}g\mathcal{G}_{ij}(\phi)\partial_c\phi^i\partial^c\phi^j.$$

The **gauge energy momentum tensor** associated to $(\mathcal{G}, \mathcal{R}, \mathcal{I})$ is the following map:

$$\mathcal{T}(\mathcal{R}, \mathcal{I}) = \mathcal{T}(\mathcal{J}) \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{R}, \mathcal{I}) \to \Gamma(T^*U \odot T^*U),$$
$$(g, \phi, \mathcal{V}) \mapsto 2\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{ac}F^{\Sigma c}_b\mathrm{d}x^a \odot \mathrm{d}x^b - \frac{1}{2}g\mathcal{I}_{\Lambda\Sigma}(\phi)F^\Lambda_{cd}F^{\Sigma cd}.$$

The sum:

$$\mathcal{T}(\mathcal{G}, \mathcal{R}, \mathcal{I}) \overset{\text{def.}}{=} \mathcal{T}(\mathcal{G}) + \mathcal{T}(\mathcal{R}, \mathcal{I}) \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{R}, \mathcal{I}) \to \Gamma(T^*U \odot T^*U),$$

is the **energy momentum tensor** of the local supergravity associated to $(\mathcal{G}, \mathcal{R}, \mathcal{I})$.

Remark 2.14. The Einstein equations of the local bosonic supergravity associated to $(\mathcal{G}, \mathcal{J})$ can be written in terms of the energy momentum tensor simply as:

$$\mathrm{G}^g = \mathcal{T}(\mathcal{G}, \mathcal{J})(g, \phi, \mathcal{V}),$$

for $(g, \phi, \mathcal{V}) \in \mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$.

The gauge energy momentum tensor admits a convenient formulation in terms of the taming associated to $(\mathcal{R}, \mathcal{I})$.

LEMMA 2.15. *The following formula holds:*

$$\mathcal{T}(\mathcal{J})(g, \phi, \mathcal{V}) = \omega(\mathcal{V}_{ac}, \mathcal{J}\mathcal{V}_b{}^c)\, \mathrm{d}x^a \odot \mathrm{d}x^b,$$

for every $(g, \phi, \mathcal{V}) \in \mathrm{Conf}_U(\mathcal{G}, \mathcal{R}, \mathcal{I})$, *where* $\mathcal{J} = \gamma(\mathcal{R}, \mathcal{I})$.

Proof. Write $\mathcal{V} = (F, G(\phi))^t$. We compute:

$$\omega(\mathcal{V}_{ac}, \mathcal{J}\mathcal{V}_b{}^c) = F_{ac}^t(\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R})F_b{}^c + G_{ac}^t\mathcal{I}^{-1}G_b{}^c - 2F_{ac}^t\mathcal{R}\mathcal{I}^{-1}G_b{}^c$$
$$= F_{ac}^t\mathcal{I}F_b{}^c + *F_{ac}^t\mathcal{I} * F_b{}^c.$$

Using the relation:

$$*F_{ac}^t\mathcal{I} * F_b{}^c = F_{ac}^t\mathcal{I}F_b{}^c - \frac{1}{2}F_{cd}^t\mathcal{I}F^{cd}g_{ab},$$

we conclude. $\quad\square$

Since the Maxwell equations reduce simply to the condition $\mathrm{d}\mathcal{V} = 0$, every solution $\mathcal{V}$ is locally integrable and thus we can write:

$$(10) \qquad \mathcal{V} = \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} = \begin{pmatrix} \mathrm{d}A^\Lambda \\ \mathrm{d}A_\Lambda \end{pmatrix}, \qquad \Lambda = 1, \ldots, n_v.$$

2.2. Duality transformations of the local equations

A precise understanding of the group of duality transformations of the local equations is crucial to construct the global geometric formulation of bosonic extended supergravity. By *duality transformations* we refer here to symmetries of the local supergravity equations which do not involve diffeomorphisms of U, that is, symmetries that *cover* the identity on U. We are especially interested in global symmetries of the equations of motion that may not preserve the action functional (1). These symmetries extend to supergravity the well-known electromagnetic duality transformations occurring in standard electromagnetism [21, 22] and are a key ingredient of bosonic supergravity in

four Lorentzian dimensions and its connection to string theory. Denote by $\mathrm{Sp}(2n_v, \mathbb{R}) \subset \mathrm{Aut}(\mathbb{R}^{2n_v})$ the group of automorphisms preserving the standard symplectic form ω and by $\mathrm{Diff}(V)$ the group of diffeomorphisms of V preserving its fixed orientation. In order to characterize the duality transformations of local bosonic supergravity we recall first that the group $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ has a natural action $\mathbb{A}$ on $\mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v})$ given by:

$$\mathbb{A} \colon \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R}) \times \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v})$$
$$\to \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v}),$$
$$(f, \mathfrak{A}, g, \phi, \mathcal{V}) \mapsto (g, f \circ \phi, \mathfrak{A}\mathcal{V}).$$

For every $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ we define:

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v}) \to \mathrm{Lor}(U) \times C^\infty(U, V) \times \Omega^2(U, \mathbb{R}^{2n_v}),$$
$$(g, \phi, \mathcal{V}) \mapsto (g, f \circ \phi, \mathfrak{A}\mathcal{V}).$$

This action does not preserve the configuration space $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ of a given local supergravity associated to the local electromagnetic structure $(\mathcal{G}, \mathcal{J})$. There is however an important subgroup of $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$, the so-called *U-duality group*, which does preserve both the configuration and solution spaces of the given supergravity. In order to characterize it we recall first the natural left-action of the group $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ on the set of taming maps, given by:

$$\mathcal{J} \mapsto \mathfrak{A} \left(\mathcal{J} \circ f^{-1} \right) \mathfrak{A}^{-1},$$

for every $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and every taming map

$$\mathcal{J} \colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v}).$$

Given $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and a taming map $\mathcal{J} \colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$, in the following we define:

$$\mathcal{J}_{\mathfrak{A}}^f \overset{\mathrm{def.}}{=} \mathfrak{A} \left(\mathcal{J} \circ f^{-1} \right) \mathfrak{A}^{-1}.$$

LEMMA 2.16. *The total energy momentum tensor of local bosonic supergravity satisfies:*

$$\mathcal{T}(f_* \mathcal{G}, \mathcal{J}_{\mathfrak{A}}^f)(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = \mathcal{T}(\mathcal{G}, \mathcal{J})(g, \phi, \mathcal{V}),$$

where $(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = (g, f \circ \phi, \mathfrak{A}\mathcal{V})$ *for* $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$.

Proof. We compute:

$$\hat{\mathcal{G}}_{ij}(\hat{\phi}) \partial_a \hat{\phi}^i \partial_b \hat{\phi}^j = \hat{\mathcal{G}}_{ij}(f \circ \phi) \partial_a (f \circ \phi)^i \partial_b (f \circ \phi)^j$$
$$= \hat{\mathcal{G}}_{ij}(f \circ \phi) \partial_k f^i \partial_l f^j \, \partial_a \phi^k \partial_b \phi^l = \mathcal{G}_{ij}(\phi) \partial_a \phi^k \partial_b \phi^l,$$

where we have used that:

$$\hat{\mathcal{G}}_{ij} = (f_* \mathcal{G})_{ij} = \mathcal{G}_{kl} \circ f^{-1} \, \partial_i (f^{-1})^k \partial_j (f^{-1})^l.$$

This proves the statement for the scalar energy momentum tensor, that is:

$$\mathcal{T}(f_*\mathcal{G})(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = \mathcal{T}(\mathcal{G})(g, \phi, \mathcal{V}).$$

The statement for the gauge energy momentum tensor follows directly from Lemma 2.15, which implies:

$$\mathcal{T}(\mathcal{J}_{\mathfrak{A}}^{f})(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = \mathcal{T}(\mathcal{G})(g, \phi, \mathcal{V}),$$

and hence we conclude. □

THEOREM 2.17. *For every* $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$, *the map* $\mathbb{A}_{f,\mathfrak{A}}$ *induces by restriction a bijection:*

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}_U(f_*\mathcal{G}, \mathcal{J}_A^f),$$

such that it further restricts to a bijection of the corresponding spaces of solutions:

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Sol}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Sol}_U(f_*\mathcal{G}, \mathcal{J}_{\mathfrak{A}}^f),$$

where $f_*\mathcal{G}$ *is the push-forward of* $\mathcal{G}$ *by* $f \colon V \to V$ *and* $\mathcal{J}_{\mathfrak{A}}^f \overset{\mathrm{def.}}{=} \mathfrak{A}(\mathcal{J} \circ f^{-1})\mathfrak{A}^{-1}$

Remark 2.18. If we consider a pair $(f, \mathfrak{U}) \in \mathrm{Diff}(V) \times \mathrm{Aut}(\mathbb{R}^{2n_v})$, with $\mathfrak{A}$ not necessarily preserving ω, then $\mathcal{J}_A^f$ is not guaranteed to be a taming map for the fixed standard symplectic structure ω, a condition which is necessary for $\mathcal{J}_{\mathfrak{A}}^f$ to define a local electromagnetic structure. The group $\mathrm{Diff}(V) \times \mathrm{Aut}(\mathbb{R}^{2n_v})$ was discussed in [45] as the group of *pseudo-dualities* of four-dimensional supergravity.

Proof. We compute:

$$\mathcal{J}(\phi)\mathcal{V} = -*_g \mathcal{V} \iff (\mathcal{J} \circ f^{-1})(f \circ \phi)\mathfrak{A}^{-1}\mathfrak{A}\mathcal{V} = -\mathfrak{A}^{-1}\mathfrak{A} *_g \mathcal{V}$$
$$\iff \mathfrak{A}(\mathcal{J} \circ f^{-1})(f \circ \phi)\mathfrak{A}^{-1}\mathfrak{A}\mathcal{V} = -*_g(\mathfrak{A}\mathcal{V}),$$

which is equivalent to $\mathcal{J}_{\mathfrak{A}}^f(f \circ \phi)\mathfrak{A}\mathcal{V} = -*_g(\mathfrak{A}\mathcal{V})$, whence:

$$\mathbb{A}_{f,\mathfrak{A}}(g, \phi, \mathcal{V}) \in \mathrm{Conf}_U(f_*\mathcal{G}, \mathcal{J}_A^f), \qquad \forall (g, \phi, \mathcal{V}) \in \mathrm{Conf}_U(\mathcal{G}, \mathcal{J}).$$

The fact that $\mathbb{A}_{f,\mathfrak{A}}$ is a bijection is now clear. In order to prove that $\mathbb{A}_{f,\mathfrak{A}}$ preserves the corresponding spaces of solutions consider $(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) \in \mathrm{Conf}_U(f_*\mathcal{G}, \mathcal{J}_{\mathfrak{A}}^f)$ such that:

$$(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = (g, f \circ \phi, \mathfrak{A}\mathcal{V}),$$

for $(f, \mathfrak{A}) \in \mathrm{Diff}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and $(g, \phi, \mathcal{V}) \in \mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$. Write now:

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

in terms of $n_v \times n_v$ blocks and:

$$\hat{\mathcal{V}} = \begin{pmatrix} \hat{F} \\ \hat{G}(\hat{\phi}) \end{pmatrix}, \qquad \mathcal{V} = \begin{pmatrix} F \\ G(\phi) \end{pmatrix}.$$

In particular $\hat{F} = a\,F + b\,G(\phi)$. The fact that the solution spaces of the Einstein equations are preserved by $(f, \mathfrak{A})$ follows directly from Lemma 2.16, since it implies:

$$\mathrm{G}^g = \mathcal{T}(f_*\mathcal{G}, \mathcal{J}_{\mathfrak{A}}^f)(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = \mathcal{T}(\mathcal{G}, \mathcal{J})(g, \phi, \mathcal{V}),$$

We consider now the scalar equations (3), which we evaluate on $(\hat{g}, \hat{\phi}, \hat{\mathcal{V}})$ and rewrite for convenience as follows:

$$\nabla_a^{g, \hat{\mathcal{G}}(\hat{\phi})} \partial^a \hat{\phi}^i + \frac{\hat{\mathcal{G}}^{ik}(\hat{\phi})}{2} (\partial_k \hat{\mathcal{R}}_{\Lambda\Sigma}(\hat{\phi}) \hat{F}_{ab}^\Lambda * \hat{F}^{\Sigma ab} + \partial_k \hat{\mathcal{I}}_{\Lambda\Sigma}(\hat{\phi}) \hat{F}_{ab}^\Lambda \hat{F}^{\Sigma ab}) = 0,$$

where $\nabla^{g, \hat{\mathcal{G}}(\hat{\phi})}$ is the product connection of the Levi-Civita connection of g and the Levi-Civita connection of $\hat{\mathcal{G}}(\hat{\phi})$. Using the equivariance properties of the Levi-Civita connection under metric pull-back we obtain:

$$\nabla_a^{g, \hat{\mathcal{G}}(\hat{\phi})} \partial^a \hat{\phi}^i = \partial_k f^i \, \nabla_a^{g, \mathcal{G}(\phi)} \partial^a \phi^k.$$

On the other hand, a tedious calculation shows that:

$$\frac{\hat{\mathcal{G}}^{ik}(\hat{\phi})}{2} (\partial_k \hat{\mathcal{R}}_{\Lambda\Sigma}(\hat{\phi}) \hat{F}_{ab}^\Lambda * \hat{F}^{\Sigma ab} + \partial_k \hat{\mathcal{I}}_{\Lambda\Sigma}(\hat{\phi}) \hat{F}_{ab}^\Lambda \hat{F}^{\Sigma ab})$$
$$= \frac{\hat{\mathcal{G}}^{ik}(\hat{\phi})}{2} \omega(\hat{\mathcal{V}}, \partial_k \hat{\mathcal{J}}(\hat{\phi}) \hat{\mathcal{V}}) = \partial_l f^i \frac{\mathcal{G}^{lk}(\phi)}{4} \omega(\mathcal{V}, \partial_k \mathcal{J}(\phi) \mathcal{V}),$$

whence $(f, \mathcal{G})$ maps solutions of the scalar equations to solutions of the scalar equations. Finally, the solution spaces of the Maxwell equations (4) are also clearly preserved since:

$$\mathrm{d}\hat{\mathcal{V}} = \mathrm{d}(\mathfrak{A}\mathcal{V}) = \mathfrak{A}\mathrm{d}\mathcal{V},$$

Therefore, $(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) \in \mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$ if and only if $(g, \phi, \mathcal{V}) \in \mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$. $\square$

The previous theorem can be used to characterize which elements $(f, \mathfrak{U}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ define through the action $\mathbb{A}$ symmetries of a the local supergravity associated to a given local scalar-electromagnetic structure $(\mathcal{G}, \mathcal{J})$. Denote in the following by $\mathrm{Iso}(V, \mathcal{G})$ the isometry group of $\mathcal{G}$.

COROLLARY 2.19. *Let* $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ *such that* $f \in \mathrm{Iso}(V, \mathcal{G})$ *and:*

$$(11) \qquad \mathcal{J}_{\mathfrak{A}}^f = \mathfrak{A}\,(\mathcal{J} \circ f^{-1})\,\mathfrak{A}^{-1} = \mathcal{J}.$$

Then $\mathbb{A}_{f, \mathfrak{A}} \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ *is a bijection of the configuration space of the supergravity defined by the scalar-electromagnetic structure* $(\mathcal{G}, \mathcal{J})$. *In particular,* $\mathbb{A}_{f, \mathfrak{A}}$ *preserves the solution space* $\mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$, *that is, it maps solutions to solutions.*

A pair $(f, \mathfrak{A}) \in \mathrm{Iso}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R})$ satisfying equation (11) will be called a *duality transformation*. It is easy to see that two duality transformations $(f_1, \mathfrak{A}_1)$ and $(f_2, \mathfrak{A}_2)$ can be composed in the natural way:

$$(f_1, \mathfrak{A}_1) \circ (f_2, \mathfrak{A}_2) = (f_1 \circ f_2, \mathfrak{A}_1 \circ \mathfrak{A}_2),$$

whence the set of all duality transformation of the local bosonic supergravity associated to $(\mathcal{G}, \mathcal{J})$, which we denote by $\mathrm{U}(\mathcal{G}, \mathcal{J})$, becomes naturally a group.

Definition 2.20. The **electromagnetic U-duality group**, or **U-duality group** for short, of the local bosonic supergravity associated to $(\mathcal{G}, \mathcal{J})$ is given by:

$$(12) \qquad \mathrm{U}(\mathcal{G}, \mathcal{J}) \overset{\mathrm{def.}}{=} \left\{ (f, \mathfrak{A}) \in \mathrm{Iso}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A} \, \mathcal{J} \, \mathfrak{A}^{-1} = \mathcal{J} \circ f \right\}.$$

Remark 2.21. By corollary 2.19, for every element $(f, \mathfrak{A}) \in \mathrm{U}(\mathcal{G}, \mathcal{J})$, the map $\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Conf}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ restricts to a bijection

$$\mathbb{A}_{f,\mathfrak{A}} \colon \mathrm{Sol}_U(\mathcal{G}, \mathcal{J}) \to \mathrm{Sol}_U(\mathcal{G}, \mathcal{J}).$$

Denote by $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) \subset \mathrm{Sp}(2n_v, \mathbb{R})$ the subgroup of $\mathrm{Sp}(2n_v, \mathbb{R})$ preserving the given taming map $\mathcal{J}$, that is:

$$\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) \overset{\mathrm{def.}}{=} \left\{ \mathfrak{A} \in \mathrm{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A} \, \mathcal{J} \, \mathfrak{A}^{-1} = \mathcal{J} \right\}.$$

Then, for every electromagnetic structure $(\mathcal{G}, \mathcal{J})$ we have the following short exact sequence:

$$(13) \qquad 1 \to \mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) \to \mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Iso}_{pr}(V, \mathcal{G}) \to 1,$$

where $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J})$ embeds in $\mathrm{U}(\mathcal{G}, \mathcal{J})$ through the map $\mathfrak{A} \mapsto (\mathrm{Id}, \mathfrak{A})$ and $\mathrm{Iso}_{pr}(V, \mathcal{G}) \subset \mathrm{Iso}(V, \mathcal{G})$ is the subgroup of the isometry group of $(V, \mathcal{G})$ that is obtained by projecting $\mathrm{U}(\mathcal{G}, \mathcal{J})$ onto its first component in the presentation (12). On the other hand, by definition we have a canonical surjective map:

$$\mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \subset \mathrm{Sp}(2n_v, \mathbb{R}), \qquad (f, \mathfrak{A}) \mapsto \mathfrak{A},$$

fitting in the following short exact sequence:

$$(14) \qquad 1 \to \mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) \to \mathrm{U}(\mathcal{G}, \mathcal{J}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \to 1,$$

where:

$$\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) \overset{\mathrm{def.}}{=} \{ f \in \mathrm{Iso}(V, \mathcal{G}) \mid \mathcal{J} \circ f = \mathcal{J} \},$$

is the stabilizer of $\mathcal{J}$ in $\mathrm{Iso}(V, \mathcal{G})$ and $\mathrm{Sp}_{pr}(2n_v, \mathbb{R})$ the subgroup of $\mathrm{Sp}(2n_v, \mathbb{R})$ that is obtained by projecting $\mathrm{U}(\mathcal{G}, \mathcal{J})$ onto its second component in the presentation (12). All together, we obtain the following proposition.

PROPOSITION 2.22. *The electromagnetic U-duality group* $\mathrm{U}(\mathcal{G}, \mathcal{J})$ *of the local supergravity theory associated to the electromagnetic structure* $(\mathcal{G}, \mathcal{J})$ *canonically fits into the short exact sequences* (13) *and* (14).

The short exact sequences (13) and (14) are very useful to compute the electromagnetic U-duality group of a given local bosonic supergravity. In particular, we obtain the following corollary.

COROLLARY 2.23. *If* $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) = \mathrm{Id}$ *then* $\mathrm{U}(\mathcal{G}, \mathcal{J}) = \mathrm{Iso}_{pr}(V, \mathcal{G}) \subset \mathrm{Iso}(V, \mathcal{G})$ *canonically becomes a subgroup of the isometry group of the scalar manifold* $(V, \mathcal{G})$. *If* $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) = \mathrm{Id}$ *is trivial then* (14) *yields a canonical embedding* $\mathrm{U}(\mathcal{G}, \mathcal{J}) \hookrightarrow \mathrm{Sp}(2n_v, \mathbb{R})$ *in the symplectic group* $\mathrm{Sp}(2n_v, \mathbb{R})$. *If both* $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{J}) = \mathrm{Id}$ *and* $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{J}) = \mathrm{Id}$ *then the U-duality group* $\mathrm{U}(\mathcal{G}, \mathcal{J})$ *is canonically isomorphic to a subgroup of* $\mathrm{Iso}(V, \mathcal{G})$ *embedded in* $\mathrm{Sp}(2n_v, \mathbb{R})$.

The previous corollary puts on firm grounds the validity of a folklore statement made in the literature which states that the *U-duality group consists of a copy of the isometry group of the scalar manifold into the symplectic group.* Before presenting some examples it is useful to formulate local bosonic supergravity in terms of the *period matrix map*.

2.3. **The period matrix map**

For computational as well as conceptual purposes it is convenient to develop a local formulation of the theory in terms of complexified field strength and couplings, a formulation which gives rise to the concept of *period matrix* (whose name will be justified in a moment). We define the *complexified field strengths*:

$$F^+ \stackrel{\mathrm{def.}}{=} F - i * F, \qquad F^- \stackrel{\mathrm{def.}}{=} F + i * F,$$

in terms of which the gauge sector of the theory (associated to a given electromagnetic structure $(\mathcal{R}, \mathcal{I})$) is conveniently written, using matrix notation, as follows:

$$\mathrm{S}_l^v[g_U, \phi, A] \stackrel{\mathrm{def.}}{=} \frac{i}{4} \int_U \left\{ F^{+T} \mathcal{N} F^+ - F^{-T} \mathcal{N}^* F^- \right\} \nu_{g_U},$$

where we have defined the *period matrix map*:

$$\mathcal{N} \stackrel{\mathrm{def.}}{=} -\mathcal{R} + i\mathcal{I} \colon V \to \mathrm{Sym}(n_v, \mathbb{C}).$$

Since $(\mathcal{R}, \mathcal{I})$ is a local electromagnetic structure, the map $\mathcal{N}$ is in fact a function on V valued in Siegel upper space $\mathbb{SH}(n_v) \subset \mathrm{Mat}(n_v, \mathbb{C})$ of square $n_v \times n_v$ complex matrices with positive definite imaginary part. For ease of notation, we define:

$$\mathcal{N}(\phi) = -\mathcal{R}(\phi) + i\mathcal{I}(\phi) \stackrel{\mathrm{def.}}{=} \mathcal{N} \circ \phi \colon U \to \mathbb{SH}(n_v),$$

to which we will refer as the *scalar* period matrix map.

Remark 2.24. The term *period matrix* is motivated by the role played by $\mathcal{N}$ when the bosonic supergravity theory under consideration corresponds to

the effective theory of a Type-IIB compactification on a Calabi-Yau three-fold X. In such situation, $\mathcal{N}$ establishes linear relations between the periods on the moduli space of complex structures of X with respect to a symplectic basis of the third homology group of X [30].

The twisted self-duality condition can be described in a very natural manner by introducing complexified field strengths. Recall that the complexification of the vector space of real vector-valued two-forms $\Omega^2(U, \mathbb{R}^{2n_v})$ is given by $\Omega^2(U, \mathbb{C}^{2n_v})$, the vector space of two-forms taking values in the complex vector space $\mathbb{C}^{2n_v}$. We define:

$$G^\pm \overset{\text{def.}}{=} \frac{1}{2}(G \mp i * G) \in \Omega^2(U, \mathbb{C}^{n_v}), \qquad \mathcal{V}^\pm \overset{\text{def.}}{=} \frac{1}{2}(\mathcal{V} \mp i * \mathcal{V}) \in \Omega^2(U, \mathbb{C}^{2n_v}).$$

Note that we have:

$$*F^\pm = \pm i F^\pm.$$

PROPOSITION 2.25. *A vector-valued two-form $\mathcal{V} \in \Omega^2(U, \mathbb{R}^{2n_v})$ is twisted self-dual with respect to a scalar map ϕ and a taming map:*

$$\mathcal{J} = \begin{pmatrix} -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \\ -\mathcal{I} - \mathcal{R}\mathcal{I}^{-1}\mathcal{R} & \mathcal{R}\mathcal{I}^{-1} \end{pmatrix} : V \to \mathrm{Aut}(\mathbb{R}^{2n_v}),$$

if and only if:

(15)
$$\mathcal{V}^+ = \begin{pmatrix} F^+ \\ \mathcal{N}^*(\phi)F^+ \end{pmatrix}$$

for a complex self-dual two-form $F^+ = \frac{1}{2}(F - i * F)$, where $\mathcal{N} = \mathcal{R} + i\mathcal{I} : V \to \mathbb{SH}(n_v)$.

Proof. By Lemma 2.4, $\mathcal{V}$ is twisted self-dual with respect to $\mathcal{J}$ if and only if:

$$\mathcal{V} = \begin{pmatrix} F \\ \mathcal{R}F - \mathcal{I} * F \end{pmatrix},$$

for $F \in \Omega^2(U, \mathbb{R}^{n_v})$. This equation can be easily shown to be equivalent to:

(16)
$$\mathcal{V}^+ = \begin{pmatrix} F^+ \\ \mathcal{N}^*(\phi)F^+ \end{pmatrix}$$

by computing $\mathcal{V}^+ = \frac{1}{2}(\mathcal{V} - i * \mathcal{V})$. $\square$

With these provisos in mind, we obtain:

$$G^+ = \mathcal{N}^*(\phi)F^+, \qquad G^- = \mathcal{N}(\phi)F^-, \qquad \mathcal{V}^+ = (F^+, \mathcal{N}^*F^+)^t,$$

where the superscript $*$ denotes complex conjugation. These conditions are equivalent with:

$$\mathcal{V} = \frac{1}{2}(\mathcal{V}^+ + \mathcal{V}^-),$$

being twisted self-dual with respect to the corresponding $\mathcal{J}$.

Remark 2.26. Since a period matrix $\mathcal{N}$ is equivalent to the data $(\mathcal{R}, \mathcal{I})$, which in turn is equivalent to its associated taming map $\mathcal{J}$, we will sometimes denote the electromagnetic structure $(\mathcal{R}, \mathcal{I})$ simply by $\mathcal{N}$.

Due to the fact that the period matrix $\mathcal{N}$ takes values in the Siegel upper space, the real symplectic group $\mathrm{Sp}(2n_v, \mathbb{R})$ acts on $\mathcal{N}$ through the natural left action of $\mathrm{Sp}(2n_v, \mathbb{R})$ on $\mathbb{SH}(n_v)$ via *fractional transformations*. Recall that the fractional transformation of $\tau \in \mathbb{SH}(n_v)$ by a matrix $\mathfrak{A} \in \mathrm{Sp}(2n_v, \mathbb{R})$ is, by definition, given by:

$$\mathfrak{A} \cdot \tau = \frac{c + d\tau}{a + b\tau} \overset{\text{def.}}{=} (c + d\tau)(a + b\tau)^{-1}, \qquad \tau \in \mathbb{SH}(n_v),$$

where we wrote A in $n_v \times n_v$ blocks as follows:

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is the natural generalization of the action of $\mathrm{Sl}(2, \mathbb{R})$ on the upper-half plane. Hence, a symplectic matrix $\mathfrak{A} \in \mathrm{Sp}(2n, \mathbb{R})$ acts point-wise on the period matrix $\mathcal{N}(\phi) \colon U \to \mathbb{SH}(n_v)$:

$$\mathfrak{A} \cdot \mathcal{N}(\phi) = \frac{c + d\mathcal{N}(\phi)}{a + b\mathcal{N}(\phi)}.$$

Remark 2.27. More generally, $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ has a natural left action on the set of period matrix maps as follows:

$$\mathcal{N} \mapsto \mathfrak{A} \cdot \mathcal{N} \circ f^{-1},$$

for every $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and every period matrix map $\mathcal{N} \colon V \to \mathbb{SH}(n_v)$.

Since a period matrix map $\mathcal{N} \colon V \to \mathbb{SH})(n_v)$ is equivalent to a taming map $\mathcal{J} \colon V \to \mathrm{Aut}(\mathbb{R}^{2n_v})$ we can describe the electromagnetic U-duality group defined in 2.20 in terms of a period matrix map. For this, we recall first that, as an immediate consequence of Proposition 2.10, there exists a bijection:

$$\mu \colon \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega) \to C^\infty(V, \mathbb{SH}(n_v)),$$

which, to every compatible taming $\mathcal{J} \in \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega)$ assigns the following period matrix:

$$\mu(\mathcal{J}) = \mathcal{R} + i\mathcal{I} \colon V \to \mathbb{SH}(n_v),$$

where $\gamma^{-1}(\mathcal{R}, \mathcal{I}) = \mathcal{J}$ is the electromagnetic structure associated to $\mathcal{J}$ by means of the bijection γ. The inverse of μ maps every period matrix $\tau = \mathrm{Re}(\tau) + i\mathrm{Im}(\tau) \colon V \to \mathbb{SH}(n_v)$ to the taming defined explicitly by equation (8) by identifying $\mathcal{R} = \mathrm{Re}(\tau)$ and $\mathcal{I} = \mathrm{Im}(\tau)$.

LEMMA 2.28. *The map* $\mu\colon \mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega) \to C^\infty(V, \mathbb{SH}(n_v))$ *is equivariant with respect to the natural action of* $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R}$ *on* $\mathfrak{J}_V(\mathbb{R}^{2n_v}, \omega)$ *and* $C^\infty(V, \mathbb{SH}(n_v))$, *respectively. That is, the following relation holds:*

$$\mu(\mathcal{J}_{\mathfrak{A}}^f) = \mathfrak{A} \cdot \mu(\mathcal{J}) \circ f^{-1}\,,$$

for every $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$.

PROPOSITION 2.29. *Let* $\mathcal{J}$ *be a taming map and set* $\mathcal{N} \overset{\text{def.}}{=} \mu(\mathcal{J})$. *The electromagnetic U-duality group* $\mathrm{U}(\mathcal{G}, \mathcal{J})$ *of the local bosonic supergravity associated to* $(\mathcal{G}, \mathcal{J})$ *is canonically isomorphic to:*
(17)
$$\mathrm{U}(\mathcal{G}, \mathcal{N}) \overset{\text{def.}}{=} \{(f, \mathfrak{A}) \in \mathrm{Iso}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A} \cdot \mathcal{N} = \mathcal{N} \circ f\}\,, \quad \mathcal{N} = \mu(\mathcal{J})\,,$$

through the identity map $\mathrm{U}(\mathcal{G}, \mathcal{N}) \ni (f, \mathfrak{A}) \mapsto (f, \mathfrak{A}) \in \mathrm{U}(\mathcal{G}, \mathcal{J})$.

Proof. It is enough to note that $(f, \mathfrak{A}) \in \mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ satisfies $\mathcal{J}_{\mathfrak{A}}^f = \mathcal{J}$ if and only if $\mu(\mathcal{J}_{\mathfrak{A}}^f) = \mu(\mathcal{J})$, which in turn is equivalent to:

$$\mathfrak{A} \cdot \mu(\mathcal{J}) = \mu(\mathcal{J}) \circ f\,,$$

by Lemma 2.28. $\square$

Remark 2.30. Equation (17), which defines the duality group in terms of the period matrix $\mathcal{N}$, can be alternatively obtained as follows, which is the way in which this group is usually described in the literature. Consider:

$$(\hat{g}, \hat{\phi}, \hat{\mathcal{V}}) = (g, f \circ \phi, \mathfrak{A}\mathcal{V})\,,$$

for $(f, \mathfrak{A}) \in \mathrm{Diff}(V, \mathcal{G}) \times \mathrm{Sp}(2n_v, \mathbb{R})$ and $(g, \phi, \mathcal{V}) \in \mathrm{Sol}_U(\mathcal{G}, \mathcal{J})$. Write:

$$\mathfrak{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

By Proposition 2.25, we have $\mathcal{V}^+ = (F^+, \mathcal{N}^*(\phi)F^+)$ but in general for an arbitrary $(f, \mathfrak{A})$ there will exist no period matrix $\hat{\mathcal{N}}\colon V \to \mathbb{SH}(n_v)$ such that $\hat{\mathcal{V}}^+ = (\hat{F}^+, \hat{\mathcal{N}}^*(\hat{\phi})\hat{F}^+)$. Imposing that such period matrix $\hat{\mathcal{N}}$ exists we obtain:

$$\hat{\mathcal{N}} = \mathfrak{A} \cdot \mathcal{N} \circ f^{-1}\,,$$

and the condition appearing in (17) follows now by imposing $\hat{\mathcal{N}} = \mathcal{N}$.

Using the previous canonical identification between $\mathrm{U}(\mathcal{G}, \mathcal{N})$ and $\mathrm{U}(\mathcal{G}, \mathcal{J})$ we obtain short exact sequences for $\mathrm{U}(\mathcal{G}, \mathcal{N})$ analogous to (13) and (14), namely:

$$1 \to \mathrm{Stab}_{\mathrm{Sp}}(\mathcal{N}) \to \mathrm{U}(\mathcal{G}, \mathcal{N}) \to \mathrm{Iso}_{pr}(V, \mathcal{G}) \to 1\,,$$
(18)
$$1 \to \mathrm{Stab}_{\mathrm{Iso}}(\mathcal{N}) \to \mathrm{U}(\mathcal{G}, \mathcal{N}) \to \mathrm{Sp}_{pr}(2n_v, \mathbb{R}) \to 1\,,$$

where:

$$\text{Stab}_{\text{Sp}}(\mathcal{N}) \stackrel{\text{def.}}{=} \{\mathfrak{A} \in \text{Sp}(2n_v, \mathbb{R}) \mid \mathfrak{A} \cdot \mathcal{N} = \mathcal{N}\},$$

$$\text{Stab}_{\text{Iso}}(\mathcal{N}) \stackrel{\text{def.}}{=} \{f \in \text{Iso}(V, \mathcal{G}) \mid \mathcal{N} \circ f = \mathcal{N}\}$$

We consider now several examples of importance in supergravity, which are more conveniently studied using the period matrix map rather than the taming map.

Example 2.31. Assume that $\mathcal{N} \in \mathbb{SH}(n_v)$ is a constant matrix. Then:

$$\mathcal{N} \circ f = \mathfrak{A} \cdot \mathcal{N},$$

for every $f \in \text{Iso}(V, \mathcal{G})$ and $\mathfrak{A} \in \text{Stab}_{\text{Sp}}(\mathcal{N}) \subset \text{Sp}(2n_v, \mathbb{R})$. Since the action of $\text{Sp}(2n_v, \mathbb{R})$ on $\mathbb{SH}(n_v)$ is transitive with stabilizer isomorphic to the unitary group $\text{U}(n_v) \subset \text{Sp}(2n_v, \mathbb{R})$ and $\mathcal{N}$ is assumed to be constant, the conjugacy class of the stabilizer of $\mathcal{N}$ in $\text{Sp}(2n_v, \mathbb{R})$ is independent of $\mathcal{N}$. Therefore:

$$\text{U}(\mathcal{G}, \mathcal{N}) \simeq \text{Iso}(V, \mathcal{G}) \times \text{U}(n_v),$$

is by Proposition 2.22 the corresponding U-duality group.

Example 2.32. Set $n_v = 1$. We have $\mathbb{SH}(1) = \mathbb{H}$. Furthermore, assume $V = \mathbb{H}$ is equipped with its Poincaré metric $\mathcal{G}$. Take $\mathcal{N} \colon \mathbb{H} \to \mathbb{H}$ to be the identity map, that is, $\mathcal{N}(\tau) = \tau$ where τ is the global coordinate on $\mathbb{H}$. Notice that this particular period matrix occurs in pure $N = 4$ four-dimensional supergravity [11]. We have $\text{Iso}(\mathbb{H}, \mathcal{G}) = \text{PSl}(2, \mathbb{R})$ acting on $\mathbb{H}$ through fractional transformations. Hence:

$$\mathcal{N} \circ f(\tau) = f \cdot \tau = f \cdot \mathcal{N}(\tau) = \hat{f} \cdot \mathcal{N}(\tau), \qquad \forall\, f \in \text{Iso}(\mathbb{H}, \mathcal{G}),$$

where $\hat{f} \in \text{Sl}(2, \mathbb{R})$ denotes any lift of $f \in \text{PSl}(2, \mathbb{R})$ to $\text{Sl}(2, \mathbb{R})$. This implies that $\text{Iso}_{pr}(\mathcal{M}, \mathcal{G}) = \text{Iso}(\mathcal{M}, \mathcal{G})$. On the other hand, a direct computation shows that:

$$\text{Stab}_{\text{Sp}}(\mathcal{N}) = \mathbb{Z}_2 = \{\text{Id}, -\text{Id}\} \subset \text{Sl}(2, \mathbb{R}).$$

Therefore, by Proposition 2.22 we have the following short exact sequence:

$$(19) \qquad 1 \to \mathbb{Z}_2 \to \text{U}(\mathcal{G}, \mathcal{N}) \to \text{PSl}(2, \mathbb{R}) \to 1,$$

which yields a central extension of $\text{U}(\mathcal{G}, \mathcal{N})$. Using now the fact that $\text{Stab}_{\text{Iso}}(\mathcal{N}) = \text{Id}$ we conclude that the electromagnetic U-duality group is $\text{U}(\mathcal{G}, \mathcal{N}) = \text{Sl}(2, \mathbb{R})$ and (19) is indeed a non-trivial central extension of the isometry group of the scalar manifold.

Example 2.33. Take $n_v = 2$ and set $V = \mathbb{H}$ equipped with its Poincaré metric $\mathcal{G}$. Consider the period matrix $\mathcal{N} \colon \mathbb{H} \to \mathbb{SH}(2)$ defined as follows:

$$\mathcal{N}(\tau) = \begin{pmatrix} \tau & 0 \\ 0 & -\frac{1}{\tau} \end{pmatrix},$$

where τ is the global coordinate on $\mathbb{H}$. Clearly $\mathcal{N}$ is symmetric. Furthermore, its imaginary part is positive definite:

$$\mathrm{Im}\,\mathcal{N}(\tau) = \begin{pmatrix} \mathrm{Im}\,(\tau) & 0 \\ 0 & \frac{\mathrm{Im}\,(\tau)}{|\tau|^2} \end{pmatrix},$$

whence $\mathcal{N}$ is a well-defined period matrix. In fact, $\mathcal{N}$ is the period matrix occurring in the *axio-dilaton model* of $\mathcal{N} = 2$ supergravity, see for example [36, Section 2] for more details. It is unambiguously fixed by supersymmetry and in particular by the projective special Kähler structure of the scalar manifold of the theory. As in the previous example, we have:

$$\mathrm{Iso}(V, \mathcal{G}) = \mathrm{PSl}(2, \mathbb{R}),$$

acting through fractional transformations. Let $\mathfrak{A} \in \mathrm{Sp}(4n_v, \mathbb{R})$. A quick computation shows that:

$$\mathfrak{A} \cdot \mathcal{N} = \mathcal{N},$$

if and only if:

$$\mathfrak{A} = (u, u) \in \mathrm{SO}(2) \times \mathrm{SO}(2) \hookrightarrow \mathrm{Sp}(4, \mathbb{R}),$$

where the embedding is diagonal. Hence: $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{N}) = \mathrm{U}(1)$ diagonally embedded in $\mathrm{Sp}(4, \mathbb{R})$. On other hand, it can be seen that $\mathrm{Iso}_{pr}(\mathbb{H}, \mathcal{G}) = \mathrm{PSl}(2, \mathbb{R})$ and hence, the electromagnetic U-duality group fits into the following short exact sequence:

$$1 \to \mathrm{U}(1) \to \mathrm{U}(\mathcal{G}, \mathcal{N}) \to \mathrm{PSl}(2, \mathbb{R}) \to 1,$$

by Proposition 2.22. Moreover, it can be easily verified that $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{N}) = \mathrm{Id}$. Hence, the U-duality group is canonically embedded as $\mathrm{U}(\mathcal{G}, \mathcal{N}) = \mathrm{Sp}_{pr}(4, \mathbb{R}) \hookrightarrow \mathrm{Sp}(4, \mathbb{R})$, which provides an explicit realization of the electromagnetic U-duality group in $\mathrm{Sp}(4, \mathbb{R})$.

Example 2.34. Take $n_v = 2$ and set $V = \mathbb{H}$ equipped with its Poincaré metric $\mathcal{G}$. Consider the period matrix $\mathcal{N} \colon \mathbb{H} \to \mathbb{SH}(2)$ defined as follows:

$$\mathcal{N}(\tau) = \begin{pmatrix} \frac{\tau^2}{2}(\tau + 3\tau^*) & -\frac{3}{2}\tau(\tau + \tau^*) \\ -\frac{3}{2}\tau(\tau + \tau^*) & 3(\tau + \tau^*) + \frac{3}{2}(\tau - \tau^*) \end{pmatrix},$$

where τ is the global coordinate on $\mathbb{H}$. Clearly $\mathcal{N}$ is symmetric. Its imaginary part can be computed to be:

$$\mathrm{Im}\,\mathcal{N}(\tau) = \begin{pmatrix} \mathrm{Im}\,(\tau)^3 + 3\mathrm{Re}\,(\tau)^2\mathrm{Im}\,(\tau) & -3\mathrm{Re}\,(\tau)\mathrm{Im}\,(\tau) \\ -3\mathrm{Re}\,(\tau)\mathrm{Im}\,(\tau) & 3\mathrm{Im}\,(\tau) \end{pmatrix},$$

It is easy to see that $\mathrm{Tr}\,(\mathrm{Im}\,\mathcal{N}(\tau)) > 0$ and $\det(\mathrm{Im}\,\mathcal{N}(\tau)) > 0$ whence $\mathrm{Im}\,(\mathcal{N})$ is positive definite and $\mathcal{N}$ is well-defined as a period matrix. In fact, $\mathcal{N}$ is the period matrix occurring in the t^3 *model* of $\mathcal{N} = 2$ supergravity, see [58,

Section 7] for more details, which is a particularly important supergravity model in the context of Type-II compactifications on Calabi-Yau three-folds. It is unambiguously fixed by supersymmetry and in particular by the projective special Kähler structure of the scalar manifold of the theory. As in the previous example, we have:

$$\mathrm{Iso}(V, \mathcal{G}) = \mathrm{PSl}(2, \mathbb{R}),$$

acting through fractional transformations on τ. We have $\mathcal{N} \circ f = \mathcal{N}$ for $f \in \mathrm{Iso}(V, \mathcal{G})$ if and only if f is the identity, whence $\mathrm{Stab}_{\mathrm{Iso}}(\mathcal{N}) = \mathrm{Id}$ and $\mathrm{U}(\mathcal{G}, \mathcal{N})$ embeds in $\mathrm{Sp}(4, \mathbb{R})$. Let $\mathfrak{A} \in \mathrm{Sp}(4n_v, \mathbb{R})$. A tedious computation shows now that:

$$\mathfrak{A} \cdot \mathcal{N} = \mathcal{N},$$

if and only if $\mathfrak{A} = \mathrm{Id}$ as well as $\mathrm{Iso}_{pr}(V, \mathcal{G}) = \mathrm{Iso}(V, \mathcal{G})$. Therefore, the U-duality group is isomorphic to $\mathrm{PSl}(2, \mathbb{R})$ and is canonically embedded in $\mathrm{Sp}(4, \mathbb{R})$.

Remark 2.35. Definition 2.20 and Proposition 2.22 should be compared with the characterization of U-duality groups already existing in the supergravity literature, see for instance [6, 30]. The general approach considers a *fixed embedding* of the isometry group in the symplectic group of the appropriate dimension in such a way that for each isometry of $(V, \mathcal{G})$ there exists a unique symplectic transformation satisfying Equation (11) and no isometry leaves $\mathcal{J}$ (or the period matrix) invariant. This immediately implies by assumption that $\mathrm{Stab}_{\mathrm{Sp}}(\mathcal{N}) = \mathrm{Stab}_{\mathrm{Iso}} = \mathrm{Id}$ and hence such U-duality group is simply a copy of the isometry group of $(V, \mathcal{G})$ inside $\mathrm{Sp}(2n_v, \mathbb{R})$. This is in general not the case for the U-duality group introduced in Definition 2.20, which therefore differs from the one considered in the literature. This difference becomes more dramatic when considering the global formulation of the theory, see Section 4.

3. GEOMETRIC BOSONIC SUPERGRAVITY

In this section we describe the global geometric formulation of the generic bosonic sector of supergravity on an oriented four-manifold M, to which we will refer simply as *geometric bosonic supergravity*, or geometric supergravity for short, following the terminology introduced in [16]. The key points we have considered when constructing geometric bosonic supergravity are the following:

• We have required geometric bosonic supergravity to be defined in terms of global differential operators acting on the spaces of sections of the appropriate fiber bundles. This is specially important to study the global structure of supergravity solutions and the associated moduli spaces.

- We have required geometric supergravity to implement the electromagnetic U-duality groups described in Section 2.2, in the sense that it must be possible to understand the theory as being the result of *gluing* the local theories introduced in Section 2 by means of a Čech one cocycle valued in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. This point is specially important for geometric supergravity to describe *supergravity U-folds* in a differential-geometric context, as explained in [49], which is particularly relevant for string theory applications.

Since we are mainly interested in the mathematical structure of the gauge sector of the theory (which is responsible for the existence of a *symplectic duality structure*), we will assume for simplicity that the theory is coupled to a standard non-linear sigma model, instead of the more general notion of *section* sigma model considered in [50].

Instead of going through the process of constructing geometric bosonic supergravity we present it in its final form and we discuss its most interesting features. A geometric bosonic supergravity with metrically trivial section sigma model is determined by the following data [49]:

- An oriented and complete Riemannian manifold $(\mathcal{M}, \mathcal{G})$, the so-called *scalar manifold* of the theory.

- A triple $\Delta \overset{\text{def.}}{=} (\mathcal{S}, \omega, \mathcal{D})$ consisting of a vector bundle $\mathcal{S}$ over $\mathcal{M}$ endowed with the symplectic pairing ω and the flat symplectic connection $\mathcal{D}$. We denote the complexification of $\Delta = (\mathcal{S}, \omega, \mathcal{D})$ by $\Delta_{\mathbb{C}} = (\mathcal{S}_{\mathbb{C}}, \omega_{\mathbb{C}}, \mathcal{D}_{\mathbb{C}})$.

- A compatible taming $\mathcal{J}$ on $(\mathcal{S}, \omega, \mathcal{D})$, that is, a complex structure on $\mathcal{S}$ satisfying:

$$\omega(\mathcal{J} s_1, \mathcal{J} s_2) = \omega(s_1, s_2), \qquad Q(s, s) \overset{\text{def.}}{=} \omega(s, \mathcal{J} s) \geq 0,$$

for all $s_1, s_2, s \in \mathcal{S}$, with $Q(s, s) = 0$ if and only if $s = 0$. We will denote by $\Theta \overset{\text{def.}}{=} (\Delta, \mathcal{J})$ a pair consisting of a flat symplectic vector bundle Δ equipped with a compatible taming $\mathcal{J}$.

Following the terminology of [49], and given a scalar manifold $(\mathcal{M}, \mathcal{G})$, we will refer to Δ as a *duality structure* and to $\Theta = (\Delta, \mathcal{J})$ as an *electromagnetic structure*. The notion of morphism of duality structures and electromagnetic structures is the natural one given by a morphism of vector bundle preserving the relevant data data, see [49] for more details. Finally, we will refer to a scalar manifold $(\mathcal{M}, \mathcal{G})$ together with a choice of electromagnetic structure Θ as a scalar-electromagnetic structure Φ:

$$\Phi \overset{\text{def.}}{=} (\mathcal{M}, \mathcal{G}, \Theta).$$

A choice of duality structure Δ together with a scalar manifold $(\mathcal{M}, \mathcal{G})$ will be referred to as a *scalar-duality* structure, being denoted simply by $(\mathcal{M}, \mathcal{G}, \Delta)$. Isomorphism classes of duality structures on a fixed scalar manifold $\mathcal{M}$ are in general not unique and depend on the fundamental group of $\mathcal{M}$. By the standard theory of flat vector bundles, isomorphism classes of duality structures are in one to one correspondence with the character variety:

$$\mathfrak{M}_d \overset{\text{def.}}{=} \mathrm{Hom}(\pi_1(\mathcal{M}), \mathrm{Sp}(2n_v, \mathbb{R})) / \mathrm{Sp}(2n_v, \mathbb{R}).$$

Remark 3.1. The fact that character varieties yield in general *continuous* moduli spaces implies that one can construct an uncountable infinity of inequivalent geometric bosonic supergravites, all of which are however locally equivalent.

A duality structure Δ can be *trivial* in two generally inequivalent senses. We say that Δ is *symplectically trivial* if $(\mathcal{S}, \omega) \in \Delta$ is symplectically trivial, that is, if it admits a global symplectic frame. On the other hand, we will say that Δ is *holonomy trivial* if Δ is symplectically trivial and the holonomy of $\mathcal{D}$ is in addition trivial. Note that if $\mathcal{M}$ is simply connected every duality structure is symplectically trivial and holonomy trivial.

3.1. Geometric background

Let $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$ be a scalar-electromagnetic structure. Smooth maps from M to $\mathcal{M}$ will be called *scalar maps*. For every scalar map $\varphi \colon M \to \mathcal{M}$ we use the superscript $^\varphi$ to denote bundle pull-back by φ. For instance, Δ^φ will denote the pull-back of Δ by φ, which defines a flat symplectic vector bundle over M, and Θ^φ will denote the pull-back of Θ by φ, respectively. For every Lorentzian metric g on M and scalar map φ we define an isomorphism of vector bundles:

$$\star_{g, \mathcal{J}^\varphi} \colon \Lambda T^* M \otimes \mathcal{S}^\varphi \to \Lambda T^* M \otimes \mathcal{S}^\varphi \,,$$

through the following equation:

$$\star_{g, \mathcal{J}^\varphi}(\alpha \otimes s) = *_g \alpha \otimes \mathcal{J}^\varphi(s) \,, \qquad \alpha \in \Lambda T^* M \,, \quad s \in \mathcal{S}^\varphi$$

on homogeneous elements. Since the square of the Hodge operator on two-forms is minus the identity, we obtain by restriction an involutive isomorphism of vector bundles:

$$\star_{g, \mathcal{J}^\varphi} \colon \Lambda^2 T^* M \otimes \mathcal{S}^\varphi \to \Lambda^2 T^* M \otimes \mathcal{S}^\varphi \,,$$

that is, $\star_{g, \mathcal{J}^\varphi}^2 = 1$. Hence we can split the bundle of two-forms taking values in $\mathcal{S}^\varphi$ in eigenbundles of $\star_{g, \mathcal{J}^\varphi}$:

$$\Lambda^2 T^* M \otimes \mathcal{S}^\varphi = (\Lambda^2 T^* M \otimes \mathcal{S}^\varphi)_+ \oplus (\Lambda^2 T^* M \otimes \mathcal{S}^\varphi)_- \,,$$

where the subscript denotes the corresponding eigenvalue. The associated spaces of sections will be denoted accordingly by:

$$\Omega^2(M, \mathcal{S}^\varphi) = \Omega^2_+(M, \mathcal{S}^\varphi) \oplus \Omega^2_-(M, \mathcal{S}^\varphi) \,.$$

Definition 3.2. Elements of $\Omega^2_+(M, \mathcal{S}^\varphi)$ will be called **twisted selfdual two-forms** and elements of $\Omega^2_-(M, \mathcal{S}^\varphi)$ will be called **twisted anti-selfdual two-forms**.

The flat symplectic connection $\mathcal{D}^\varphi \in \Delta^\varphi$ defines a canonical exterior covariant derivative for forms on M taking values in $\mathcal{S}^\varphi$, which we denote by:

$$\mathrm{d}_{\mathcal{D}^\varphi} : \Omega^k(M, \mathcal{S}^\varphi) \to \Omega^{k+1}(M, \mathcal{S}^\varphi) \,,$$

where $k = 0, \ldots 4$. Since $\mathcal{D}^\varphi$ is flat, the operator $\mathrm{d}_{\mathcal{D}^\varphi}$ is a coboundary operator on the complex of forms taking values in $\mathcal{S}^\varphi$. We denote the associated cohomology groups by $H^k(M, \Delta^\varphi)$ and the corresponding total cohomology by $H(M, \Delta^\varphi)$. Denote by:

$$\mathfrak{G}_\Delta(U) \stackrel{\mathrm{def.}}{=} \{s \in \Gamma(U, \mathcal{S}^\varphi) \mid \mathcal{D}^\varphi s = 0\} \,, \qquad U \subset M \,,$$

the sheaf of smooth flat sections of Δ. This is a locally constant sheaf of symplectic vector spaces of rank $2n_v$, whose stalk is isomorphic to the typical fiber of Δ. There exists a natural isomorphism of graded vector spaces:

$$H(M, \Delta^\varphi) \simeq H(M, \mathfrak{G}_\Delta) \,,$$

where $H(M, \mathfrak{G}_\Delta)$ denotes the sheaf cohomology of $\mathfrak{G}_\Delta$.

Note that the definition of electromagnetic structure $\Theta = (\Delta, \mathcal{J})$ does not require $\mathcal{D} \in \Delta$ to be compatible with $\mathcal{J}$; the case when they are non-compatible is in fact crucial for the correct description of geometric bosonic supergravity. The failure of $\mathcal{D}$ to be compatible with $\mathcal{J}$ is measured by the *fundamental form* of an electromagnetic structure.

Definition 3.3. Let $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$ be a scalar-electromagnetic structure. The *fundamental form* Ψ of Θ is the following one-form on $\mathcal{M}$ taking values in $\mathrm{End}(\mathcal{S})$:

$$\Psi \stackrel{\mathrm{def.}}{=} \mathcal{D}\mathcal{J} \in \Omega^1(\mathcal{M}, \mathrm{End}(\mathcal{S})) \,.$$

Remark 3.4. It is not hard to see that $\Psi(X) \in \Gamma(\mathrm{End}(\mathcal{S}))$, $X \in T\mathcal{M}$, is an anti-linear self-adjoint endomorphism of the Hermitian vector bundle $(\mathcal{S}, Q, \mathcal{J})$.

Definition 3.5. An electromagnetic structure Θ is called *unitary* if $\Psi = 0$.

To define geometric bosonic supergravity we need to introduce three natural operations on tensors taking values in a vector bundle. These operations depend on the choice of electromagnetic structure Θ.

Definition 3.6. The *twisted exterior pairing* $(\cdot, \cdot)_{g,Q^\varphi}$ is the unique pseudo-Euclidean scalar product on $\Lambda T^*M \otimes \mathcal{S}^\varphi$ satisfying:

$$(\rho_1 \otimes s_1, \rho_2 \otimes s_2)_{g,Q^\varphi} = (\rho_1, \rho_2)_g Q^\varphi(s_1, s_2),$$

for any $\rho_1, \rho_2 \in \Omega(M)$ and any $s_1, s_2 \in \Gamma(\mathcal{S}^\varphi)$, where $(-,-)_g$ denotes the scalar product induced by g on tensors over M. Recall that $Q(s_1, s_2) = \omega(s_1, \mathcal{J}s_2)$ and the superscript denotes pull-back by φ.

For any vector bundle W over M, we trivially extend the twisted exterior pairing to a W-valued pairing (which for simplicity we denote by the same symbol) between the bundles $W \otimes \Lambda T^*M \otimes \mathcal{S}^\varphi$ and $\Lambda T^*M \otimes \mathcal{S}^\varphi$:

$$(w \otimes \eta_1, \eta_2)_{g,Q^\varphi} \overset{\text{def.}}{=} w \otimes (\eta_1, \eta_2)_{g,Q^s}, \quad \forall\, w \in \Gamma(W), \quad \forall\, \eta_1, \eta_2 \in \Lambda T^*M \otimes \mathcal{S}^\varphi.$$

Definition 3.7. The *inner g-contraction of (2,0) tensors* is the bundle morphism $\oslash_g : (\otimes^2 T^*M)^{\otimes 2} \to \otimes^2 T^*M$ uniquely determined by the condition:

$$(\alpha_1 \otimes \alpha_2) \oslash_g (\alpha_3 \otimes \alpha_4) = (\alpha_2, \alpha_4)_g \alpha_1 \otimes \alpha_3, \quad \forall\, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in T^*M.$$

We define the *inner g-contraction of two-forms* to be the restriction of $\oslash_g$ to $\wedge^2 T^*M \otimes \wedge^2 T^*M \subset (\otimes^2 T^*M)^{\otimes 2}$.

Definition 3.8. We define the *twisted inner contraction* of $\mathcal{S}^\varphi$-valued two-forms to be the unique morphism of vector bundles:

$$\oslash_Q \colon \Lambda^2 T^*M \otimes \mathcal{S}^\varphi \times_M \Lambda^2 T^*M \otimes \mathcal{S}^\varphi \to \otimes^2(T^*M)$$

satisfying:

$$(\rho_1 \otimes s_1) \oslash_Q (\rho_2 \otimes s_2) = Q^\varphi(s_1, s_2)\rho_1 \oslash_g \rho_2,$$

for all $\rho_1, \rho_2 \in \Omega^2(M)$ and all $s_1, s_2 \in \Gamma(\mathcal{S}^\varphi)$.

3.2. Configuration space and equations of motion

In this section, we define geometric bosonic supergravity through a system of partial differential equations which yields a non-trivial extension of local supergravity as described in Section 2. We remark that geometric bosonic supergravity is not expected to admit in general an action functional, which is consistent with the fact that it implements U-duality non-trivially and therefore its globally-defined solutions can be viewed as locally geometric supergravity U-folds. We begin by introducing the configuration space of geometric bosonic supergravity, which yields the space of variables of its system of partial differential equations.

Definition 3.9. Let Φ be a scalar-electromagnetic structure on an oriented four-manifold M. The **configuration space** of geometric bosonic supergravity on (M, Φ) is the set:

$$\mathrm{Conf}_M(\Phi) \overset{\mathrm{def.}}{=} \left\{ (g, \varphi, \mathcal{V}) \mid g \in \mathrm{Lor}(M), \; \varphi \in C^\infty(M, \mathcal{M}), \; \mathcal{V} \in \Omega^2_+(M, \mathcal{S}^\varphi) \right\},$$

where $\mathrm{Lor}(M)$ denotes the space of Lorentzian metrics on M.

Definition 3.10. Let Φ be a scalar-electromagnetic structure on M. The geometric bosonic supergravity on M associated to Φ is defined by the following system of partial differential equations:

- The Einstein equations:

$$(20) \qquad\qquad \mathrm{Ric}^g - \frac{g}{2} R^g = \frac{g}{2} \mathrm{Tr}_g(\mathcal{G}^\varphi) - \mathcal{G}^\varphi + 2\mathcal{V} \oslash_Q \mathcal{V}.$$

- The scalar equations:

$$(21) \qquad\qquad \mathrm{Tr}_g(\nabla \mathrm{d}\varphi) = \frac{1}{2} (*\mathcal{V}, \Psi^\varphi \mathcal{V})_{g, Q^\varphi},$$

where ∇ denotes the connection on $T^*M \otimes T\mathcal{M}^\varphi$ defined as the tensor product of the Levi-Civita connection on (M, g) and the pull-back by φ of the Levi-Civita connection on $(\mathcal{M}, \mathcal{G})$.

- The Maxwell equations:

$$(22) \qquad\qquad \mathrm{d}_{\mathcal{D}^\varphi} \mathcal{V} = 0,$$

for triples $\Phi = (g, \varphi, \mathcal{V}) \in \mathrm{Conf}_M(\Phi)$.

Remark 3.11. The configuration space $\mathrm{Conf}_M(\Phi) = \mathrm{Conf}_M(\mathcal{G}, \Delta, \mathcal{J})$ of geometric bosonic supergravity contains as variables the *field strength* two-form instead of the appropriate notion of gauge potential, which should be described globally by an adequate notion of connection. To identify the geometrically correct notion of gauge potential we have to first *Dirac quantize* the theory, similarly to what is done with standard Maxwell theory. In the latter theory, assuming that the field strength has integral periods allows one to identify the gauge potential as a connection on a certain principal S^1 bundle. The complete Dirac quantization of four-dimensional supergravity and its geometric interpretation has not been developed in the literature and is currently work in progress [51].

Remark 3.12. The fact that geometric bosonic supergravity reduces locally to the standard formulation of local bosonic supergravity was proved in [49], to which we refer the reader for further details.

In the following we will denote by $\mathrm{Sol}_M(\Phi) = \mathrm{Sol}_M(\mathcal{G}, \Delta, \mathcal{J}) \subset \mathrm{Conf}_M(\Phi)$ the solution set of the geometric bosonic supergravity on M associated to the scalar-electromagnetic structure Φ.

4. **THE GLOBAL DUALITY GROUP**

In this section we characterize the *global* duality group of geometric bosonic supergravity for a fixed scalar electromagnetic structure $\Phi = (\mathcal{M}, \mathcal{G}, \Theta)$, which corresponds to the global counterpart of the electromagnetic U-duality group of the local theory, as discussed in Section 2.2. Given a duality structure $\Delta = (\mathcal{S}, \omega, \mathcal{D})$, we denote by $\mathrm{Aut}(\mathcal{S})$ the group of *unbased* automorphisms of the vector bundle $\mathcal{S} \in \Delta$. Given $u \in \mathrm{Aut}(\mathcal{S})$ we will denote by $f_u \colon \mathcal{M} \to \mathcal{M}$ the unique diffeomorphism covered by u. Moreover, we denote by $\mathrm{Aut}(\Delta)$ the group of unbased automorphisms of $\mathcal{S}$ preserving both ω and D, that is:

$$\mathrm{Aut}(\Delta) \overset{\mathrm{def.}}{=} \{ u \in \mathrm{Aut}(\mathcal{S}) \mid \omega^u = \omega \,, \ \mathcal{D}^u = \mathcal{D} \} \,.$$

Given a duality structure Δ over $\mathcal{M}$, the group $\mathrm{Aut}(\Delta)$ has a natural left-action on $\mathrm{Lor}(M) \times C^\infty(M, \mathcal{M}) \times \Omega^2(\mathcal{S})$, given by:

$$\mathbb{A} \colon \mathrm{Aut}(\Delta) \times \mathrm{Lor}(M) \times C^\infty(M, \mathcal{M}) \times \Omega^2(M, \mathcal{S})$$
$$\to \mathrm{Lor}(M) \times C^\infty(M, \mathcal{M}) \times \Omega^2(M, \mathcal{S}) \,,$$
$$(u, g, \varphi, \mathcal{V}) \mapsto (g, f_u \circ \varphi, u \cdot \mathcal{V}) \,,$$

which gives the global counterpart of (11). For every $u \in \mathrm{Aut}(\Delta)$, we define:

$$\mathbb{A}_u \colon \mathrm{Lor}(M) \times C^\infty(M, \mathcal{M}) \times \Omega^2(M, \mathcal{S}) \to \mathrm{Lor}(M) \times C^\infty(M, \mathcal{M}) \times \Omega^2(M, \mathcal{S}) \,,$$
$$(g, \varphi, \mathcal{V}) \mapsto (g, f_u \circ \varphi, u \cdot \mathcal{V}) \,.$$

This action does not preserve the configuration space $\mathrm{Conf}_U(\mathcal{G}, \mathcal{J})$ of a given scalar-electromagnetic structure $\Phi = (\mathcal{G}, \mathcal{J})$. Instead, we have the following result, which gives the global counterpart of Theorem 2.17.

THEOREM 4.1 ([49, Theorem 3.15]). *For every $u \in \mathrm{Aut}(\Delta)$, the map $\mathbb{A}_u$ defines by restriction a bijection:*

$$\mathbb{A}_u \colon \mathrm{Conf}_M(\mathcal{G}, \Delta, \mathcal{J}) \to \mathrm{Conf}_M(f_{u*}\mathcal{G}, \Delta, \mathcal{J}_u) \,,$$

which induces a bijection between the corresponding spaces of solutions:

$$\mathbb{A}_u \colon \mathrm{Sol}_M(\mathcal{G}, \Delta, \mathcal{J}) \to \mathrm{Sol}_M(f_{u*}\mathcal{G}, \Delta, \mathcal{J}_u) \,,$$

where $f_{u*}\mathcal{G}$ is the push-forward of $\mathcal{G}$ by $f_u \colon \mathcal{M} \to \mathcal{M}$ and $\mathcal{J}_u$ is the bundle push-forward of $\mathcal{J}$ by u.

Remark 4.2. Since elements in $\mathrm{Aut}(\mathcal{S})$ may cover non-trivial diffeomorphisms of $\mathcal{M}$, the pull-back/push-forward conditions appearing above must be dealt with care. More explicitly, define the following action of $\mathrm{Aut}(\mathcal{S})$ on sections of $\mathcal{S}$:

$$u \cdot s = u \circ s \circ f_u^{-1} \colon M \to \mathcal{S}, \qquad u \in \mathrm{Aut}(\mathcal{S}), \qquad s \in \Gamma(\mathcal{S}).$$

This action defines an isomorphism of real vector spaces $u \colon \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ for every element $u \in \mathrm{Aut}(\mathcal{S})$. We have $\omega^u = \omega$ if and only if:

$$(\omega^u)(s_1, s_2) \overset{\text{def.}}{=} \omega(u \cdot s_1, u \cdot s_2) \circ f_u = \omega(s_1, s_2), \quad \forall\, s_1, s_2 \in \Gamma(\mathcal{S}).$$

Likewise, $\mathcal{D}^u = \mathcal{D}$ if and only if:

$$\mathcal{D}_X^u(s) \overset{\text{def.}}{=} u^{-1} \cdot (\mathcal{D}_{f_u^* X}(u \cdot s)) = \mathcal{D}_X(s), \qquad \forall\, s \in \Gamma(\mathcal{S}), \qquad \forall\, X \in \mathfrak{X}(\mathcal{M}),$$

where $f_u^* X \in \mathfrak{X}(\mathcal{M})$ is the pull-back of $X \in \mathfrak{X}(\mathcal{M})$ by $f_u \colon \mathcal{M} \to \mathcal{M}$. Moreover, the explicit push-forward of $\mathcal{J}$ by $u \in \mathrm{Aut}(\mathcal{S})$ is given as follows:

$$\mathcal{J}_u(s) \overset{\text{def.}}{=} u \cdot (\mathcal{J}(u^{-1} \cdot s)) = u \circ \mathcal{J}(u^{-1} \circ s),$$

for every $s \in \Gamma(S)$.

Therefore, $\mathrm{Aut}(\Delta)$ yields the global counterpart of the *pseudo-duality group* considered in [45], which is given by $\mathrm{Diff}(\mathcal{M}) \times \mathrm{Sp}(2n_v, \mathbb{R})$, and therefore differs remarkably from the latter if Δ is non-trivial. Every $u \in \mathrm{Aut}(\Delta)$ maps the configuration and solutions spaces of the supergravities associated to $(\mathcal{G}, \Delta, \mathcal{J})$ to those associated to $(f_{u*}\mathcal{G}, \Delta, \mathcal{J}_u)$. Denote by $\mathrm{Aut}_b(\Delta) \subset \mathrm{Aut}(\Delta)$ the subgroup consisting of automorphisms of $\mathrm{Aut}(\Delta)$ covering the identity. We have the short exact sequence:

$$1 \to \mathrm{Aut}_b(\Delta) \to \mathrm{Aut}(\Delta) \to \mathrm{Diff}_\Delta(\mathcal{M}) \to 1,$$

where $\mathrm{Diff}_\Delta(\mathcal{M})$ is the subgroup of the orientation-preserving diffeomorphism group of $\mathcal{M}$ that can be covered by elements in $\mathrm{Aut}(\Delta)$, which necessarily contains the identity component of $\mathrm{Diff}(\mathcal{M})$. The proof of the following important lemma can be found in [27].

LEMMA 4.3. *Let Δ be a duality structure and $m \in \mathcal{M}$. We have a canonical isomorphism:*

$$\mathrm{Aut}_b(\Delta) = \mathrm{C}(\mathrm{Hol}_m(\mathcal{D}), \mathrm{Aut}(S_m, \omega_m)),$$

where $\mathrm{Hol}_m(\mathcal{D})$ denotes the holonomy group of $\mathcal{D}$ at $m \in \mathcal{M}$, $\mathrm{Aut}(S_m, \omega_m) \simeq \mathrm{Sp}(2n_v, \mathbb{R})$ is the automorphism group of the fiber $(S_m, \omega_m) = (S, \omega)|_m$ and $\mathrm{C}(\mathrm{Hol}_m(\mathcal{D}), \mathrm{Aut}(S_m, \omega_m)))$ denotes the centralizer of $\mathrm{Hol}_m(\mathcal{D})$ in $\mathrm{Aut}(S_m, \omega_m)$. In particular, $\mathrm{Aut}_b(\Delta)$ is finite-dimensional.

We now introduce a global counterpart of the local electromagnetic U-duality group which is traditionally studied in the supergravity literature and was discussed in Section 2.

Definition 4.4. Let $\Phi = (\mathcal{G}, \Delta, \mathcal{J})$ be a scalar-electromagnetic structure on $\mathcal{M}$. We define the **electromagnetic U-duality group** $\mathfrak{S}(\Phi)$ of Φ, or **U-duality group** for short, as the subgroup of $\mathrm{Aut}(\Delta)$ which preserves both the metric $\mathcal{G}$ and the taming $\mathcal{J}$. That is:

$$\mathfrak{S}(\Phi) \overset{\mathrm{def.}}{=} \{u \in \mathrm{Aut}(\Delta) \mid f_{u*}\mathcal{G} = \mathcal{G}, \ \mathcal{J}^u = \mathcal{J}\},$$

where $\mathcal{J} \in \Phi$.

Remark 4.5. We have $\mathcal{J}^u = \mathcal{J}$ if and only if:

$$\mathcal{J}(u \circ s) = u \circ \mathcal{J}(s), \qquad \forall\, s \in \Gamma(s),$$

where $\circ$ composition of maps.

We denote by $\mathrm{Aut}_b(\Theta) \subset \mathrm{Aut}_b(\Delta)$ the based automorphisms of Δ, which are vector bundle isomorphisms covering the identity and preserving both Δ and $\mathcal{J}$. The U-duality group $\mathfrak{G}$ fits into the following short exact sequence:

$$1 \to \mathrm{Aut}_b(\Theta) \to \mathfrak{S}(\Phi) \to \mathrm{Iso}_\Phi(\mathcal{M}, \mathcal{G}) \to 1,$$

where $\mathrm{Iso}_\Phi(\mathcal{M}, \mathcal{G}) \subset \mathrm{Iso}(\mathcal{M}, \mathcal{G})$ is the subgroup of the isometry group of $(\mathcal{M}, \mathcal{G})$ that can be covered by elements in $\mathfrak{S}(\Phi)$. Since $\mathrm{Aut}_b(\Theta) \subset \mathrm{Aut}_b(\Delta)$, Lemma 4.3 implies that $\mathrm{Aut}_b(\Theta)$ is finite-dimensional. Moreover, $\mathrm{Iso}_\Phi(\mathcal{M}, \mathcal{G})$ is well-known to be a finite-dimensional Lie group, which in turn implies that $\mathfrak{S}(\Phi)$ is a finite-dimensional Lie group which yields the global counterpart of the local electromagnetic U-duality group defined in (12). The U-duality group of a supergravity theory maps solutions of that theory to solutions and thus it can be used as a *solution generating* mechanism, as the following corollary of Theorem 4.1 states.

COROLLARY 4.6. *The U-duality group* $\mathfrak{S}(\Phi)$ *of the supergravity theory associated to* Φ *preserves* $\mathrm{Sol}_M(\Phi)$, *that is, it maps solutions to solutions. In particular, every* $u \in \mathfrak{S}_M(\Phi)$ *defines a bijection from* $\mathrm{Sol}_M(\Phi)$ *to itself.*

4.1. **Holonomy trivial duality structure**

In this section we consider the U-duality group in the special case when the duality structure Δ is holonomy trivial, that is, when it admits a global flat symplectic frame. Fixing such a frame $\mathcal{E} = (e_1, \ldots, e_{n_v}, f_1, \ldots, f_{n_v})$, whose

dual coframe we denote by $\mathcal{E}^* = (e_1^*, \ldots, e_{n_v}^*, f_1^*, \ldots, f_{n_v}^*)$, we can canonically identify Δ as follows:

$$\mathcal{S} = \mathcal{M} \times \mathbb{R}^{2n_v}\,, \qquad \omega = \sum_j f_j^* \wedge e_j^*\,, \qquad \mathcal{D} = \mathrm{d}\colon \Omega(\mathcal{M}, \mathbb{R}^{2n_v}) \to \Omega(\mathcal{M}, \mathbb{R}^{2n_v})\,,$$

where d denotes the standard exterior derivative acting on forms taking values on $\mathbb{R}^{2n_v}$. A taming $\mathcal{J} \in \mathrm{Aut}(\mathcal{S})$ of Δ is equivalent through this identification to a unique smooth taming map:

$$\mathcal{J}\colon \mathcal{M} \to \mathrm{Aut}(\mathbb{R}^{2n_v})\,.$$

Moreover, $\mathcal{E}$ yields a canonical identification of the unbased automorphism group of $\mathrm{Aut}(\mathcal{S})$:

$$\mathrm{Aut}(S) = \mathrm{Diff}(\mathcal{M}) \times C^\infty(\mathcal{M}, \mathrm{Aut}(\mathbb{R}^{2n_v}))\,,$$

whose action is given by:

$$(f, \mathfrak{U})(p, v) = (f(p), \mathfrak{A}(p)(v))\,,$$

for every $(p, v) \in \mathcal{M} \times \mathbb{R}^{2n_v}$ and

$$(f, \mathfrak{A}) \in \mathrm{Aut}(S) = \mathrm{Diff}(\mathcal{M}) \times C^\infty(\mathcal{M}, \mathrm{Aut}(\mathbb{R}^{2n_v}))\,.$$

An element $(f, \mathfrak{A}) \in \mathrm{Aut}(S)$ preserves ω and $\mathcal{D}$ if and only if $\mathfrak{A}$ is constant and belongs to the symplectic group $\mathrm{Sp}(2n_v, \mathbb{R}) \subset \mathrm{Aut}(\mathbb{R}^{2n_v})$ defined as the stabilizer of ω in $\mathrm{Aut}(\mathbb{R}^{2n_v})$. Therefore:

$$\mathrm{Aut}(\Delta) = \mathrm{Diff}(\mathcal{M}) \times \mathrm{Sp}(2n_v, \mathbb{R})\,,$$

which corresponds to the group $\mathrm{Diff}(V) \times \mathrm{Sp}(2n_v, \mathbb{R})$ considered in section 2. The pullback $\mathcal{J}^u$ of $\mathcal{J}$ by $u = (f, \mathfrak{A}) \in \mathrm{Aut}(\Delta)$ reads:

$$\mathcal{J}^{u^{-1}}|_p(p, v) = (u \cdot \mathcal{J})|_{f^{-1}(p)}(u^{-1}|_p \cdot (p, v)) = u \cdot \mathcal{J}|_{f^{-1}(p)}(f^{-1}(p), \mathfrak{A}^{-1}(v))$$
$$= (p, \mathfrak{A}\mathcal{J}_{f^{-1}(p)}\mathfrak{A}^{-1}(v))\,.$$

Thus an element $(f, \mathfrak{A}) \in \mathrm{Aut}(\Delta)$ preserves $\mathcal{J}\colon \mathcal{M} \to \mathrm{Aut}(\mathbb{R}^{2n_v})$ if and only if:

$$\mathfrak{A}(\mathcal{J} \circ f^{-1})\mathfrak{A}^{-1} = \mathcal{J}\,,$$

which in turn implies that the electromagnetic U-duality group associated to a scalar electromagnetic structure $\Phi = (\mathcal{G}, \Delta, \mathcal{J})$ is given by:

$$(23) \qquad \mathfrak{S}(\Phi) \overset{\mathrm{def.}}{=} \left\{ (f, \mathfrak{A}) \in \mathrm{Aut}(\Delta) \mid f_*\mathcal{G} = \mathcal{G}\,, \ \mathfrak{A}\mathcal{J}\mathfrak{A}^{-1} = \mathcal{J} \circ f \right\}\,,$$

which recovers equation (12).

5. **SUPERGRAVITY KILLING SPINOR EQUATIONS**

In the previous sections we discussed the generic bosonic sector of four-dimensional supergravity, which a priori does not involve supersymmetry and relies only on a consistent coupling of gravity, scalars and abelian gauge fields in a manner compatible with electromagnetic duality. In order to have the complete picture of geometric supergravity and to showcase the power of supersymmetry, we need to discuss the *supergravity* Killing spinor equations, which arise from imposing invariance under supersymmetry transformations on a purely bosonic solution. For the moment, we denote by $\mathscr{B}$ the bosonic fields of a four-dimensional supergravity theory, which we know from Section 3 to consist of Lorentzian metrics, smooth maps into the scalar manifold of the theory and twisted self-dual two-forms taking values in the duality bundle, and let us denote by $\mathcal{F}$ the corresponding *fermionic fields*. The latter depend heavily on the specific supergravity theory under consideration. Given a *supersymmetry parameter* ε, which we can think of as being a spinor on M^*, the *infinitesimal* supersymmetry transformations of $\mathscr{B}$ and $\mathcal{F}$ in the direction ε correspond schematically to an infinitesimal transformation of the form:

$$\delta_\varepsilon \mathscr{B} = \mathcal{F}(\varepsilon) \,, \qquad \delta_\varepsilon \mathcal{F} = \mathscr{B}(\varepsilon)$$

where the right hand side depends linearly on ε. A solution $(\mathscr{B}, \mathcal{F})$ of four-dimensional supergravity is said to be *supersymmetric* if it is invariant under such an infinitesimal transformation, i.e.:

$$(24) \qquad \delta_\varepsilon \mathscr{B} = \mathcal{F}(\varepsilon) = 0 \,, \qquad \delta_\varepsilon \mathcal{F} = \mathscr{B}(\varepsilon) = 0 \,.$$

To the best of our knowledge, there is no fully general and mathematically rigorous formulation of these transformations which could serve to give the basis of a mathematical theory of supergravity including its complete fermionic sector and supersymmetry transformations. As disappointing as this may seem, what is important to us is that if we restrict the previous transformations to a purely bosonic background, that is, if we set $\mathcal{F} = 0$, (24) reduces to an expression of the form:

$$(25) \qquad \delta_\varepsilon \mathcal{F} = \mathscr{B}(\varepsilon) = 0 \,,$$

which is expected to admit a rigorous mathematical formulation using the tools of mathematical gauge theory and global differential geometry and analysis. In the case of four-dimensional ungauged supergravity, equation (25) yields a system of partial differential equations for a metric g, a scalar map φ and a twisted

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[*]More precisely, it is a section of a bundle of real or complex Clifford modules of certain type, which is in general not associated to a spin structure but the more general notion of *Lipschitz structure* instead, see [47, 48] for more details.

self-dual two-form $\mathcal{V}$ coupled to a spinor ε. In agreement with the terminology introduced earlier, a bosonic solution $\mathscr{B}$ is then said to be supersymmetric if Equation (25) holds. The spinorial equations arising from $\delta_\varepsilon \mathcal{F} = 0$ are always of the type:

$$\mathfrak{D}_{\mathscr{B}}\varepsilon = 0\,, \qquad \mathfrak{Q}_{\mathscr{B}}(\varepsilon) = 0\,,$$

where $\varepsilon \in \Gamma(S)$ is a section of an appropriate bundle of real or complex Clifford modules over the underlying manifold M, $\mathfrak{D}_{\mathscr{B}}$ is a connection on S depending on $\mathscr{B}$ and $\mathfrak{Q}_{\mathscr{B}} \in \Gamma(End(S))$ is an endomorphism of S depending also on $\mathscr{B}$. We note that the mathematical theory of supergravity Killing spinor equations is far from being established, so in the following we will content ourselves with presenting some particular examples where such mathematical formulation does exist, see [16] for more details. The main difficulty in developing the mathematical theory of supergravity Killing spinor equations resides in giving global mathematical sense to the local formulas available in the supergravity physics literature for $\mathfrak{D}_{\mathscr{B}}$ and $\mathfrak{Q}_{\mathscr{B}}(\varepsilon)$, which involve state of the art geometric structures subtly coupled through supersymmetry.

5.1. **Pure (AdS) $\mathcal{N} = 1$ supergravity**

We fix an oriented Lorentzian spin four-manifold, which for simplicity in the exposition we will assume to satisfy $H^1(M, \mathbb{Z}_2) = 0$ (so the spin structure is unique up to isomorphism). For every Lorentzian metric g on M, we denote by S_g the unique (modulo isomorphism) bundle of irreducible real Clifford modules over the bundle of Clifford algebras $Cl(M, g)$ of (M, g). Pure (AdS) $\mathcal{N} = 1$ supergravity is the simplest four-dimensional supergravity theory. The scalar manifold consists of a point and the duality structure is trivial of zero rank. The scalar potential is constant. The bosonic *matter content* of the theory, that is, its configuration space, consists therefore simply of a Lorentzian metric and the theory admits the following action functional [58, Chapter 5]:

$$\mathfrak{S}[g] = \int_U \left[\mathrm{R}_g + 6\lambda^2 \right] \mathrm{vol}_g\,.$$

The partial differential equations associated to the variational problem of the previous functional are:

$$\mathrm{Ric}(g) = -3\lambda^2\, g\,.$$

Therefore, bosonic pure AdS $\mathcal{N} = 1$ supergravity is given by Einstein's theory of gravity coupled to a non positive cosmological constant. The Killing spinor equations of the theory read [58, Chapter 5]:

$$(26) \qquad \qquad \nabla_v^g \varepsilon = \frac{\lambda}{2} v \cdot \varepsilon\,, \qquad \forall\, v \in \mathfrak{X}(M)\,,$$

for a real spinor $\varepsilon \in \Gamma(S_g)$. Consequently, an Einstein metric g with Einstein constant $-3\lambda^2$ is a supersymmetric solution of $\mathcal{N} = 1$ pure AdS supergravity if and only if (M, g) admits a spinor ε satisfying (26). Hence, the set of supersymmetric solutions of pure (AdS) $\mathcal{N} = 1$ supergravity is given by:

$$\mathrm{Sol}_S(M, \lambda) = \left\{ (g, \epsilon) \mid \mathrm{Ric}(g) = -3\lambda^2 g, \quad \nabla_v^g \varepsilon = \frac{\lambda}{2} v \cdot \varepsilon, \ \forall v \in \mathfrak{X}(M) \right\}.$$

Equation (26) is a particular case of a *real* Killing spinor equation on a Lorentzian four-manifold, and has been studied in [53, 17]. Reference [53] proves that an oriented and spin Lorentzian four-manifold carrying a solution of (26) such that $\lambda \neq 0$ is locally conformally a Brinkmann space-time. On the other hand, Reference [17] proves the following global result.

THEOREM 5.1 ([17, Theorem 5.3]). (M, g) *admits a nontrivial real Killing spinor with Killing constant* $\frac{\lambda}{2}$ *if and only if it admits a pair of orthogonal one-forms* $u, l \in \Omega^1(M)$ *with* u *lightlike and* l *of positive unit norm satisfying:*

$$\nabla^g u = \lambda \, u \wedge l, \qquad \nabla^g l = \kappa \otimes u + \lambda (l \otimes l - g),$$

for some $\kappa \in \Omega^1(M)$. *In this case,* $u^\sharp \in \mathfrak{X}(M)$ *is a Killing vector field with geodesic integral curves.*

Remark 5.2. Theorem 5.1 immediately implies that κ is closed if and only if Leitner's result holds with respect to u, that is, if and only if every such (M, g) is locally conformally Brinkmann with respect to u. We have not been able to prove that κ is necessarily closed.

Of course, when $\lambda = 0$ equation (26) reduces to the condition of ε being a parallel spinor, which has been extensively studied both in the mathematics and physics literature, see for example [52] and references therein. To the best knowledge of the authors, the differential topology of globally hyperbolic Lorentzian manifolds carrying a solution of (26) has not been investigated in the literature. We believe that the global characterization provided by Theorem 5.1 is a convenient starting point for such a study.

5.2. Chiral $\mathcal{N} = 1$ supergravity with constant scalar map and superpotential

We fix an oriented and spin Lorentzian four-manifold, which we assume to satisfy $H^1(M, \mathbb{Z}_2) = 0$ for the same reasons as in the previous section. For every Lorentzian metric g on M, we denote by $\$_g$ the unique (modulo isomorphism) bundle of irreducible complex Clifford modules over $\mathrm{Cl}(M, g)$. The Lorentzian volume form of (M, g) is denoted by ν, while the complex volume form is denote by $\nu_{\mathbb{C}} = i\nu$.

We have:

$$(27) \qquad \nu_{\mathbb{C}}^2 = 1 \,,$$

Therefore, the complex spinor bundle splits as a sum of chiral bundles:

$$\$_g = \$_g^- \oplus \$_g^+ \,,$$

where the superscript denotes the chirality. To describe chiral $\mathcal{N} = 1$ supergravity with constant scalar map and superpotential (see [32, 58] for more details) we take the scalar manifold to be a point and the duality structure to be trivial of rank zero. Under these assumptions, it can be shown that the theory admits an action functional given by:

$$\mathfrak{S}[g] = \int_U \left[\mathrm{R}_g + 24|w|^2 \right] \mathrm{vol}_g \,,$$

where $w \in \mathbb{C}$ is a complex constant corresponding to the superpotential and $U \subset M$ is any relatively compact open set. The equations of motion associated to the previous functional read:

$$\mathrm{Ric}(g) = -12|w|^2 g \,,$$

which are the standard Einstein equations coupled to a non positive cosmological constant. Using the chiral splitting $\$_g = \$_g^- \oplus \$_g^+$, we define for every $w \in \mathbb{C}$ the following morphism of complex vector bundles:

$$T_w \colon \Omega^0(\$_g) \to \Omega^1(\$_g) \,, \qquad T_w(\epsilon_1 \oplus \epsilon_2)(v) = \gamma(v)(w\epsilon_1 \oplus \bar{w}\epsilon_2) \,,$$

where $\gamma \colon \mathrm{Cl}(M, g) \to \mathrm{End}(\$_g)$ denotes Clifford multiplication. In terms of $T_w \colon \Omega^0(\$_g) \to \Omega^1(\$_g)$ the Killing spinor equations of the theory are given by:

$$(28) \qquad \nabla^g \varepsilon = T_w(\varepsilon) \,, \qquad \mathfrak{c}(\varepsilon) = \varepsilon \,,$$

where ∇^g denotes the lift of the Levi-Civita connection to the spinor bundle and:

$$\mathfrak{c} \colon \$_g \to \$_g$$

denotes the canonical complex-conjugate and spin-equivariant automorphism of the complex spinor bundle $\$_g$ (see [16] for more details). Therefore, the set $\mathrm{Sol}_S(M)$ of supersymmetric solutions on M consists on pairs (g, ϵ), with g a Lorentzian metric and ϵ chiral spinor, such that:

$$\mathrm{Sol}_S(M) = \left\{ (g, \epsilon) \mid \mathrm{Ric}(g) = -12|w|^2 \, g \,, \nabla^g \epsilon = T_w(\epsilon) \,, \mathfrak{c}(\varepsilon) = \varepsilon, \, \forall v \in \mathfrak{X}(M) \right\} \,.$$

It is important to point out that Equation (28) does not correspond to a *standard* Killing spinor equation (for neither real nor imaginary Killing spinors) unless w is real, due to the fact that the endomorphism T_w involves the complex conjugate of w. This in turn implies that the number that occurs as the Einstein constant of the corresponding integrability condition is actually $|w|^2$. This

allows w to be any complex number instead of only real or purely imaginary. To the best of our knowledge, the globally hyperbolic Lorentzian four-manifolds that admit supersymmetric solutions to this supergravity theory have not been investigated in the literature.

5.3. Chiral $\mathcal{N} = 1$ supergravity with vanishing superpotential

We fix an oriented Lorentzian $\mathrm{Spin}^c(3,1)$ four-manifold, which for simplicity of exposition we will assume to satisfy $H^2(M,\mathbb{Z}) = 0$ (so that isomorphism classes of spin^c structures on M are unique). For every Lorentzian metric g on M, we denote by $\$_g$ the unique (modulo isomorphism) bundle of irreducible complex Clifford modules over $\mathrm{Cl}(M,g)$. As before, the complex spinor bundle splits as a sum of chiral bundles:

$$\$_g = \$_g^- \oplus \$_g^+\,,$$

where the superscript denotes chirality. The scalar manifold of $\mathcal{N} = 1$ supergravity with vanishing superpotential and with trivial duality structure of rank zero is a complex manifold $\mathcal{M}$ equipped with a negative Hermitian holomorphic line bundle $(\mathcal{L}, \mathcal{H})$ with Hermitian structure $\mathcal{H}$. The Riemannian metric $\mathcal{G}$ occurring in the non-linear sigma model of four-dimensional supergravity is given by the metric induced by the curvature of the Chern connection of $(\mathcal{L}, \mathcal{H})$, see [8, 16] for more details. Such $\mathcal{N} = 1$ supergravity admits a Lagrangian formulation with Lagrangian given by:

$$\mathfrak{Lag}[g, \varphi] = \mathrm{R}_g - |\mathrm{d}\varphi|^2_{\mathcal{G},g}\,,$$

for pairs (g, φ) consisting of Lorentzian metrics g and scalar maps $\varphi\colon M \to \mathcal{M}$. As it is standard in the theory of harmonic maps (or wave maps), we consider:

$$\mathrm{d}\varphi \in \Omega^1(M, T\mathcal{M}^\varphi)\,,$$

as a one-form on M taking values in the pullback of $T\mathcal{M}^\varphi$ by φ. Therefore, the theory reduces to Einstein gravity coupled to a non-linear sigma model with target space given by the complex manifold $\mathcal{M}$ equipped with the Kähler metric defined by the curvature of the Chern connection of $(\mathcal{L}, \mathcal{H})$. The Killing spinor equations are given by:

$$\nabla^\varphi \epsilon = 0\,, \qquad \mathrm{d}\varphi^{0,1} \cdot \epsilon = 0\,,$$

where $\nabla^\varphi \colon \Gamma(\$_g) \to \Gamma(\$_g)$ is the canonical lift of the Levi-Civita connection on (M,g) together with the pull-back of the Chern connection on $(\mathcal{L}, \mathcal{H})$ by φ. Therefore, the set $\mathrm{Conf}_S(M, \mathcal{M}, \mathcal{L}, \mathcal{H})$ of supersymmetric configurations on M consists on triples (g, φ, ϵ), with g a Lorentzian metric, $\varphi\colon M \to \mathcal{M}$ a scalar map and ϵ a chiral spinor, such that:

$$\mathrm{Conf}_S(M, \mathcal{M}, \mathcal{L}, \mathcal{H}) = \left\{(g, \varphi, \epsilon) \mid \nabla^\varphi \epsilon = 0,\ \mathrm{d}\varphi^{0,1} \cdot \epsilon = 0\right\}\,.$$

Lorentzian manifolds (M, g) admitting a solution (g, φ, ϵ) to the Killing spinor equations stated above are particular instances of Lorentzian $\mathrm{Spin}^c(3, 1)$ manifolds admitting parallel spinors. Simply connected and geodesically complete Lorentzian manifolds admitting spin-c parallel spinors have been studied and classified in the literature, see reference [57] for the Riemannian case and Reference [46] for the pseudo-Riemannian case. It cannot be expected a priori that every $\mathrm{Spin}^c(3, 1)$ Lorentzian four-manifold which admits a parallel spinor also admits a solution to the above Killing spinor equations. Adapting the main Theorem of [46] to our situation we obtain the following result.

PROPOSITION 5.3. *Let M be a simply-connected and geodesically complete Lorentzian four-manifold admitting a supersymmetric solution (g, φ, ϵ) of $\mathcal{N} = 1$ chiral supergravity with vanishing superpotential. Then, one of the following holds:*

1. *(M, g) is isometric to four-dimensional flat Minkowski space.*

2. *(M, g) is isometric to $(M, g) \simeq (\mathbb{R}^2 \times X, \eta_{1,1} \times h)$, where $\eta_{1,1}$ is the flat two-dimensional Minkowski metric and X is a Riemann surface equipped with a Kähler metric h.*

3. *The holonomy group H of (M, g) is a subgroup of $\mathrm{SO}(2) \ltimes \mathbb{R}^2 \subset \mathrm{SO}_0(3, 1)$, where $\mathrm{SO}(2) \ltimes \mathbb{R}^2$ is the stabilizer of a null vector in $\mathbb{R}^4$.*

Therefore, every geodesically complete and simply connected supersymmetric solution must be of the form described by the previous proposition. However, the converse need not be true, since a supersymmetric solution requires (M, g) to admit a parallel spinor with respect to the specific connection ∇^φ, which is coupled to the scalar map φ, which is in turn required to satisfy its corresponding Killing spinor equation. To the best of our knowledge, the problem of classifying globally hyperbolic Lorentzian four-manifolds carrying supersymmetric solutions of this supergravity theory is currently open.

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