HESSE MANIFOLDS AND HESSIAN SYMMETRIES OF MULTIFIELD COSMOLOGICAL MODELS

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I give a brief overview of the mathematical theory of Noether symmetries of multifield cosmological models, which decompose naturally into *visible* and *Hessian* (a.k.a. 'hidden') symmetries. While visible symmetries correspond to those infinitesimal isometries of the Riemannian target space of the scalar field map which preserve the scalar potential, Hessian symmetries have a much deeper theory. The latter correspond to *Hesse functions*, defined as solutions of the so-called *Hesse equation* of the target space. By definition, a *Hesse manifold* is a Riemannian manifold which admits nontrivial Hesse functions – not to be confused with a Hess*ian* manifold (the latter being a Riemannian manifold whose metric is locally the Hessian of a function). All Hesse n-manifolds $\mathcal{M}$ are non-compact and characterized by their *index*, defined as the dimension of the space of Hesse functions, which carries a natural symmetric bilinear pairing. The Hesse index is bounded from above by $n+1$ and, when the metric is complete, this bound is attained iff $\mathcal{M}$ is a Poincaré ball, in which case the space of Hesse functions identifies with $\mathbb{R}^{1,n}$ through an isomorphism constructed from the Weierstrass map. More generally, any elementary hyperbolic space form is a complete Hesse manifold and any Hesse manifold whose local Hesse index is maximal is hyperbolic. In particular, the class of complete Hesse surfaces coincides with that of elementary hyperbolic surfaces and hence any such surface is isometric with the Poincaré disk, the hyperbolic punctured disk or a hyperbolic annulus. Thus Hesse manifolds generalize hyperbolic manifolds. On a complete Hesse manifold $(\mathcal{M}, G)$, the value of any Hesse function Λ can be expressed though the distance from a *characteristic subset* of $\mathcal{M}$ determined by Λ. Moreover, the gradient flow of Λ can be described using the distance function.

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1. INTRODUCTION AND PHYSICS MOTIVATION

Cosmological models with at least two real scalar fields are of increasing interest in theoretical physics. In our previous work [1, 2, 11, 3, 4], we initiated a geometric study of the classical dynamics of multifield cosmological models with arbitrary scalar manifold (which we approached from a mathematically rigorous perspective), summarizing some of our results in [5, 6, 7]. Cosmological models with n real scalar fields and standard kinetic term are parameterized by

a so-called *scalar triple* $(\mathcal{M}, \mathcal{G}, V)$, where $\mathcal{M}$ is a connected smooth manifold (generally non-compact and of non-trivial topology) which is the target space of the scalar field map, $\mathcal{G}$ is a Riemannian metric on $\mathcal{M}$ which specifies the kinetic term of the scalar fields and V is a smooth real-valued function defined on $\mathcal{M}$, which specifies the scalar potential. Such models arise naturally in string theory, where $(\mathcal{M}, \mathcal{G})$ appears as a moduli space of string compactifications and V is induced by a flux on the compactification manifold or by quantum effects. The classical cosmological model parameterized by $(\mathcal{M}, \mathcal{G}, V)$ involves the *scale factor* $a \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_{>0})$ of a simply-connected Friedmann-Lemaitre-Robertson-Walker spacetime and a smooth curve $\varphi : \mathbb{R} \to \mathcal{M}$ (whose parameter $t \in \mathbb{R}$ is called *cosmological time*) subject to a system of ODEs known as the *cosmological equations*:

$$
\begin{aligned}
&3H^2 + 2\dot{H} + \frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V \circ \varphi = 0 \\
&(\nabla_t + 3H)\dot{\varphi} + (\mathrm{grad}_{\mathcal{G}} V) \circ \varphi = 0 \ , \\
&\frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} + V \circ \varphi = 3H^2 \ ,
\end{aligned} \tag{1}
$$

where the dot indicates derivation with respect to t and $H \stackrel{\text{def.}}{=} \frac{\dot{a}}{a} \in \mathcal{C}^\infty(\mathbb{R})$ is the *Hubble parameter*. The last equation in this system is called the *Friedmann equation*. Notice that a enters this system only through its logarithmic derivative H. When H is positive, eliminating it through the Friedmann equation allows one to reduce (1) to the single second order autonomous ODE:

$$
\nabla_t \dot{\varphi}(t) + \sqrt{\frac{3}{2}} \left[||\dot{\varphi}(t)||^2_{\mathcal{G}} + 2V(\varphi(t)) \right]^{1/2} \dot{\varphi}(t) + (\mathrm{grad}_{\mathcal{G}} V)(\varphi(t)) = 0 \ , \tag{2}
$$

which defines a dissipative geometric dynamical system (in the sense of [13]) on the total space of the tangent bundle of $\mathcal{M}$. In general, little is known about the deeper behavior of this dynamical system, some aspects of which were explored in [11, 3, 4] and summarized in [5, 6, 7].

Let $\mathcal{N} \stackrel{\text{def.}}{=} \mathbb{R}_{>0} \times \mathcal{M}$ be the *configuration space* of the variables a and φ. The cosmological equations (1) can be derived from the variational principle of the so-called *minisuperspace Lagrangian*[1] $L_{\mathcal{M},\mathcal{G},V} : T\mathcal{N} \to \mathbb{R}$, which is given by:

$$
L_{\mathcal{M},\mathcal{G},V}(a, \dot{a}, \varphi, \dot{\varphi}) \stackrel{\text{def.}}{=} -3a\dot{a}^2 + a^3 \left[\frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} - V(\varphi) \right] \ , \tag{3}
$$

supplemented by the *Friedmann constraint*:

$$
\frac{1}{2}||\dot{\varphi}||^2_{\mathcal{G}} + V(\varphi) = 3H^2 \ . \tag{4}
$$

[1]The term "minisuperspace" is historically motivated and has nothing to do with supersymmetry.

Here we identity $T\mathcal{N}$ with the first jet bundle of $\mathcal{N}$ and we abuse notation as common in jet bundle theory. Notice that the Friedmann constraint is non-holonomic.

The constrained Lagrangian description given by (3) and (4) allows for systematic study of the Lie symmetries (see [12]) of (2) using the Noether method. In [1], we exploited this point of view to classify those cosmological models with $\dim \mathcal{M} = 2$ which admit Noether symmetries, making the technical assumption that the metric $\mathcal{G}$ is rotationally invariant. As already pointed out in that reference, the latter assumption is purely technical and not needed for the results of loc. cit. Moreover, the approach of [1] generalizes to cosmological models parameterized by arbitrary scalar triples $(\mathcal{M}, \mathcal{G}, V)$, leading to a deep mathematical theory. This generalization is discussed in detail in the preprints [10] and [8]. We summarize some of its results below, focusing on those aspects which are most relevant to Riemannian geometers. For notational simplicity, we rescale the physics-motivated scalar manifold metric $\mathcal{G}$ to:

$$G \stackrel{\text{def.}}{=} \frac{3}{8}\mathcal{G} \ ,$$

thus replacing $(\mathcal{M}, \mathcal{G})$ by the *rescaled scalar manifold* $(\mathcal{M}, G)$ and $(\mathcal{M}, \mathcal{G}, V)$ by the *rescaled scalar triple* $(\mathcal{M}, G, V)$.

Notations and conventions. Throughout this paper, $\mathcal{M}$ will denote a smooth, paracompact, Hausdorff and connected n-manifold (which need not be compact). The differential of a function $f \in \mathcal{C}^\infty(\mathcal{M})$ is denoted by $\mathrm{d}f \in \Omega^1(\mathcal{M})$, while its value at a point $m \in \mathcal{M}$ is denoted by:

$$\mathrm{d}_m f = (\mathrm{d}f)(m) \in T^*_m\mathcal{M} = \mathrm{Hom}_{\mathbb{R}}(T_m\mathcal{M}, \mathbb{R}) \ .$$

We use the notations:

$$Z(f) \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \mid f(m) = 0\} \ , \quad \mathrm{Crit}(f) \stackrel{\text{def.}}{=} \{m \in \mathcal{M} \ \mid \mathrm{d}_m f = 0\}$$

for the zero and critical locus of f and:

$$\mathcal{M}_f(a) \stackrel{\text{def.}}{=} f^{-1}(\{a\}) = \{m \in \mathcal{M} \mid f(m) = a\}$$

for the level set of f at $a \in \mathbb{R}$. We will often use the following two operators determined by a Riemannian metric G on $\mathcal{M}$:

- The *Killing operator* of $(\mathcal{M}, G)$, defined as the $\mathbb{R}$-linear first-order differential operator $\mathcal{K}_G : \mathcal{X}(\mathcal{M}) \to \Gamma(\mathcal{M}, \mathrm{Sym}^2(T^*\mathcal{M}))$ which associates to any vector field $X \in \mathcal{X}(\mathcal{M}) \stackrel{\text{def.}}{=} \Gamma(\mathcal{M}, T\mathcal{M})$ the symmetrization of the covariant derivative of the 1-form $X^\flat \in \Omega^1(\mathcal{M})$:

$$\mathcal{K}(X) \stackrel{\text{def.}}{=} \mathrm{Sym}^2[\nabla(X^\flat)] \ .$$

In local coordinates on $\mathcal{M}$, we have:

$$\mathcal{K}_G(X)_{ij} \stackrel{\text{def.}}{=} \frac{1}{2}\left[\nabla_i X_j + \nabla_j X_i\right] = \frac{1}{2}\left(\partial_i X_j + \partial_j X_i - 2\Gamma^k_{ij} X_k\right) \ ,$$

where Γ^k_{ij} are the Christoffel symbols:

$$\Gamma^k_{ij} = G^{lk}\Gamma_{ijl} = \frac{1}{2} G^{lk}(\partial_j G_{il} + \partial_i G_{jl} - \partial_l G_{ij})$$

and we use implicit summation over repeated indices.

• The *Hessian operator* of $(\mathcal{M}, G)$, defined as the $\mathbb{R}$-linear second order differential operator $\mathrm{Hess}_G : \mathcal{C}^\infty(\mathcal{M}) \to \Gamma(\mathcal{M}, \mathrm{Sym}^2(T^*\mathcal{M}))$ which associates to a smooth real-valued function f defined on $\mathcal{M}$ its Hessian tensor:

$$\mathrm{Hess}_G(f) \stackrel{\text{def.}}{=} \nabla \mathrm{d} f.$$

In local coordinates on $\mathcal{M}$, we have:

$$\mathrm{Hess}_G(f)_{ij} \stackrel{\text{def.}}{=} \mathrm{Hess}_G(f)(\partial_i, \partial_j) = \partial_i \partial_j f - \Gamma^k_{ij}\partial_k f \ .$$

Notice the relation:

$$\mathcal{K}_G(\mathrm{grad}_G f) = \mathrm{Hess}_G(f) \qquad \forall f \in \mathcal{C}^\infty(\mathcal{M}) \ .$$

2. NOETHER SYMMETRIES OF MULTIFIELD COSMOLOGICAL MODELS

Let $(\mathcal{M}, G)$ be a Riemannian manifold and $V \in \mathcal{C}^\infty(\mathcal{M})$ be a smooth real-valued function defined on $\mathcal{M}$. Let $\mathcal{N} \stackrel{\text{def.}}{=} \mathbb{R}_{>0} \times \mathcal{M}$. We have a natural decomposition $T\mathcal{N} = T_{(1)}\mathcal{N} \oplus T_{(2)}\mathcal{N}$, where $T_{(1)}$ and $T_{(2)}$ are the pullbacks of the tangent bundles $T\mathbb{R}_{>0}$ and $T\mathcal{M}$ through the canonical projections $p_1 : \mathcal{N} \to \mathbb{R}_{>0}$ and $p_2 : \mathcal{N} \to \mathcal{M}$:

$$T_{(1)}\mathcal{N} \stackrel{\text{def.}}{=} p_1^*(T\mathbb{R}_{>0}) \ , \quad T_{(2)}\mathcal{N} \stackrel{\text{def.}}{=} p_2^*(T\mathcal{M}) \ .$$

Hence any vector field $X \in \mathcal{X}(\mathcal{N})$ decomposes as: $X = X_{(1)} + X_{(2)}$, with $X_{(i)} \in \Gamma(\mathcal{N}, T_{(i)}\mathcal{N})$. In local coordinates (U, a, φ^i) on $\mathcal{N}$, we have:

$$X_{(1)}(a, \varphi) = X^a(a, \varphi)\frac{\partial}{\partial a} \ , \qquad X_{(2)}(a, \varphi) = X^i(a, \varphi)\frac{\partial}{\partial \varphi^i} \ ,$$

where $X^a, X^i \in \mathcal{C}^\infty(U)$ and $i = 1, ..., n$.

2.1. **The characteristic system of variational symmetries**

The following result reduces the study of Noether symmetries of the minisuperspace Lagrangian to that of certain real-valued functions and vector fields defined on $\mathcal{M}$.

THEOREM 2.1 ([8]). *A vector field $X \in \mathcal{X}(\mathcal{N})$ is a time-independent Noether symmetry of the minisuperspace Lagrangian of the classical cosmological model parameterized by the rescaled scalar manifold $(\mathcal{M}, G)$ and by the scalar potential V iff it has the form:*

$$X := X_{\Lambda,Y} = \frac{\Lambda}{\sqrt{a}}\partial_a + Y - \frac{4}{a^{3/2}}(\mathrm{grad}_G\Lambda) \ ,$$

where the function $\Lambda \in \mathcal{C}^\infty(\mathcal{M})$ and the vector field $Y \in \mathcal{X}(\mathcal{M})$ satisfy the characteristic system *of $(\mathcal{M}, G, V)$:*

$$\mathrm{Hess}_G(\Lambda) = G\Lambda \quad , \qquad \langle \mathrm{d}V, \mathrm{d}\Lambda\rangle_G = 2V\Lambda \tag{5}$$

$$\mathcal{K}_G(Y) = 0 \quad , \qquad Y(V) = 0 \ . \tag{6}$$

Notice that the two equations above containing Λ decouple from those containing Y, so the characteristic system consists of two independent systems of linear PDEs: the Λ-*system* (5) and the Y-*system* (6) of $(\mathcal{M}, G, V)$. In local coordinates, the characteristic system reads:

$$\left(\partial_i\partial_j - \Gamma^k_{ij}\partial_k\right)\Lambda = G_{ij}\Lambda \quad , \qquad G^{ij}\partial_i V\partial_j\Lambda = 2V\Lambda$$

$$\nabla_i Y_j + \nabla_j Y_i = 0 \quad , \qquad Y^i\partial_i V = 0 \ ,$$

where we use Einstein summation over repeated indices $i, j, k = 1, ..., n$.

The solutions of the Y-system coincide with those Killing vector fields Y of $(\mathcal{M}, G)$ which satisfy $L_Y V = 0$, i.e. with infinitesimal isometries of $(\mathcal{M}, G)$ which preserve the scalar potential V. Such solutions form the Lie algebra of the *group of symmetries* of the rescaled scalar triple $(\mathcal{M}, G, V)$, defined as the stabilizer of V inside the group $\mathrm{Iso}(\mathcal{M}, G)$ of isometries of $(\mathcal{M}, G)$:

$$\mathrm{Aut}(\mathcal{M}, G, V) \stackrel{\text{def.}}{=} \{\psi \in \mathrm{Iso}(\mathcal{M}, G) \mid V \circ \psi = V\} \ .$$

Notice that $\mathrm{Aut}(\mathcal{M}, G, V)$ is a Lie group since it is a closed subgroup of $\mathrm{Iso}(\mathcal{M}, G)$. For a generic triple $(\mathcal{M}, G, V)$, we have $\mathrm{Aut}(\mathcal{M}, G, V) = 1$, hence the Y-system of a generic rescaled scalar triple admits only the trivial solution $Y = 0$.

The first equation of the Λ-system will be called the *Hesse equation* of $(\mathcal{M}, G)$. The second equation of that system (which we call the Λ-V *equation*) can be solved explicitly *for* V once we pick a solution of the first.

THEOREM 2.2. *Let Λ be a nontrivial solution of the Hesse equation. Then any smooth solution of the Λ-V-equation of $(\mathcal{M}, G)$ takes the form:*

$$V = \Omega ||\mathrm{d}\Lambda||_G^2 = \Omega \left[\Lambda^2 - (\Lambda, \Lambda)_G\right] \ , \tag{7}$$

where $\Omega \in \mathcal{C}^\infty(\mathcal{M} \setminus \mathrm{Crit}(\Lambda))$ is an arbitrary smooth function which is constant along the gradient flow of Λ:

$$\langle \mathrm{d}\Omega, \mathrm{d}\Lambda \rangle_G = 0 \ . \tag{8}$$

For generic $(\mathcal{M}, G)$, the Hesse equation admits only the trivial solution $\Lambda = 0$, which satisfies the Λ-V equation with any V.

The observations above imply, as expected, that a generic multifield cosmological model has no Noether symmetries. Those special models which *do* admit such symmetries are of particular interest in theoretical physics.

Definition 2.3. A time-independent Noether symmetry $X = X_{\Lambda,Y}$ is called:

- *visible* if $\Lambda = 0$, in which case $X = X_{0,Y} = Y$.
- *Hessian* if $Y = 0$, in which case $X = X_{\Lambda,0} = \frac{\Lambda}{\sqrt{a}}\partial_a - \frac{4}{a^{3/2}}(\mathrm{grad}_G \Lambda)$.

The rescaled scalar triple $(\mathcal{M}, G, V)$ and corresponding cosmological model are called *visibly-symmetric* or *Hessian* if they admit visible or Hessian symmetries, respectively.

Let $\mathfrak{N}_h(\mathcal{M}, G, V)$, $\mathfrak{N}_v(\mathcal{M}, G, V)$ and $\mathfrak{N}(\mathcal{M}, G, V)$ be the linear spaces of Hessian, visible and time-independent Noether symmetries. Then there exists an obvious linear isomorphism:

$$\mathfrak{N}(\mathcal{M}, G, V) \simeq_{\mathbb{R}} \mathfrak{N}_h(\mathcal{M}, G, V) \oplus \mathfrak{N}_v(\mathcal{M}, G, V) \ .$$

Definition 2.4. The cosmological model defined by the rescaled scalar triple $(\mathcal{M}, G, V)$ is called *weakly Hessian* if the Hesse equation of $(\mathcal{M}, G)$ admits nontrivial solutions. It is called *Hessian* if $\mathfrak{N}_h(\mathcal{M}, G, V) \neq 0$.

Theorem 2.2 implies:

COROLLARY 2.5. *The cosmological model defined by the rescaled scalar triple $(\mathcal{M}, G, V)$ is Hessian iff it is weakly Hessian and the scalar potential V has the form* (7)*, with Ω a solution of* (8)*.*

Since the study of visible symmetries reduces to a classical problem in Riemannian geometry, the mathematically interesting problem is to classify all Hessian scalar triples and hence all Hessian cosmological models. By the results above this reduces in turn to the problem of characterizing those Riemannian manifolds whose Hesse equation admits nontrivial solutions. Below, we describe a few results in this direction, whose proof can be found in [10].

3. HESSE FUNCTIONS AND HESSE MANIFOLDS

Let us start by formulating the mathematical problem without reference to its origin in physics.

Definition 3.1. Let $(\mathcal{M}, G)$ be a Riemannian manifold of positive dimension. A *Hesse function* of $(\mathcal{M}, G)$ is a smooth solution $\Lambda \in \mathcal{C}^\infty(\mathcal{M})$ of the following linear second order PDE, which is called the *Hesse equation* of $(\mathcal{M}, G)$:

$$\mathrm{Hess}_G(\Lambda) = G\Lambda \tag{9}$$

and whose space of solutions we denote by $\mathcal{H}_G(\mathcal{M})$. The *Hesse index* of $(\mathcal{M}, G)$ is defined through:

$$\mathfrak{h}_G(\mathcal{M}) \stackrel{\text{def.}}{=} \dim_{\mathbb{R}} \mathcal{H}_G(\mathcal{M}) \ .$$

The Riemannian manifold $(\mathcal{M}, G)$ is called a *Hesse manifold* if $\mathfrak{h}_G(\mathcal{M}) > 0$, i.e. if $\mathcal{H}_G(\mathcal{M}) \neq 0$.

Remark 3.2. The notion of Hesse manifold should not be confused with that of Hess*ian* manifold, which means a Riemannian manifold whose metric is given locally by the Hessian of a function.

We start by studying the Hesse equation.

3.1. Relation to Hessian equations. Non-compactness of Hesse manifolds.

The Hesse equation (9) of $(\mathcal{M}, G)$ is equivalent with a system of so-called *Hessian equations* (see [16, 9], namely a Hessian system which includes both the Helmholtz and Monge-Ampère equations of $(\mathcal{M}, G)$.

For any $f \in \mathcal{C}^\infty(\mathcal{M})$ and $m \in \mathcal{M}$, let:

$$Q^G_m(f)(z) \stackrel{\text{def.}}{=} \det\left[z\,\mathrm{id}_{T_m\mathcal{M}} - \widehat{\mathrm{Hess}}_G(f)(m)\right] = \sum_{k=0}^{n} (-1)^k c^G_k(f)(m) z^{n-k} \in \mathbb{R}[z]$$

be the characteristic polynomial of the G_m-symmetric linear operator

$$\widehat{\mathrm{Hess}}_G(f)(m) \in \mathrm{End}_{\mathbb{R}}(T_m\mathcal{M})$$

obtained by raising an index of the symmetric tensor $\mathrm{Hess}_G(f)(m)$, where z is a formal variable. The characteristic coefficients $c^G_k(f)(m)$ define smooth functions $c^G_k(f) \in \mathcal{C}^\infty(\mathcal{M})$ as m varies in $\mathcal{M}$.

Definition 3.3. The functions $c^G_k(f) \in \mathcal{C}^\infty(\mathcal{M})$ are called the *Hessian functions* of f with respect to G.

Let:

$$\sigma_k(z_1,\ldots,z_n) \stackrel{\text{def.}}{=} \sum_{1\leq i_1<\ldots<i_k\leq n} z_{i_1}\ldots z_{i_k} \in \mathbb{R}[z_1,\ldots,z_n]$$

be the elementary symmetric polynomials in n variables, where k runs from 0 to n. We have:

$$c_k^G(f)(m) = \sigma_k(\lambda_1(f)(m),\ldots,\lambda_n(f)(m)) \ , \ \forall m \in \mathcal{M} \ ,$$

where $\lambda_j(f)$ are functions given by the real eigenvalues of the G-symmetric endomorphism $\widehat{\mathrm{Hess}}_G(f)$ of $T\mathcal{M}$. Let $\wedge^k\widehat{\mathrm{Hess}}_G(f) \in \mathrm{End}_{\mathbb{R}}(\wedge^k T\mathcal{M})$ be the k-exterior power of this endomorphism. The relations:

$$c_k(f) = \mathrm{tr}\left[\wedge^k\widehat{\mathrm{Hess}}_G(f)\right] \ , \ \forall k = 0,\ldots,n$$

show that the correspondence $f \to c_k^G(f)$ gives a differential operator:

$$c_k^G : \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$$

of order $2k$ (which is non-linear for $k > 1$).

Definition 3.4. The differential operator c_k^G is called the *k-th invariant Hessian operator* of $(\mathcal{M}, G)$.

In particular, we have:

$$c_0 = 1 \ , \quad c_1 = \mathrm{tr}\left[\widehat{\mathrm{Hess}}_G(\Lambda)\right] = -\Delta_G\Lambda \ , \quad c_n = \det\left[\widehat{\mathrm{Hess}}_G(\Lambda)\right] = \mathrm{M}_G(\Lambda) \ ,$$

where $\Delta_G = -\mathrm{div}_G\mathrm{grad}_G$ and M_G are respectively the positive Laplacian and the Monge-Ampère operators of $(\mathcal{M}, G)$.

Definition 3.5. A *Hessian equation* on $(\mathcal{M}, G)$ is a PDE of the form:

$$F \circ (f \times c_1^G(f) \times \ldots \times c_n^G(f)) = 0 \ ,$$

where $F \in \mathcal{C}^\infty(\mathbb{R} \times \mathcal{M})$ is given and the unknown f is a smooth real-valued function defined on $\mathcal{M}$.

We refer the reader to [16, 9] for background on Hessian equations.

Proposition 3.6 ([10]). *The Hesse equation* (9) *is equivalent with the following system of Hessian equations:*

$$c_k^G(\Lambda) = \frac{n!}{k!(n-k)!}\Lambda^k \ , \quad \forall k = 1,\ldots,n \ .$$

In particular, any Hesse function Λ satisfies the Helmholtz equation $\Delta_G\Lambda = -n\Lambda$ and the Monge-Ampère equation $\mathrm{M}_G(\Lambda) = \Lambda^n$.

Since the right hand side of the Helmholtz equation has the "wrong sign" for the positive Laplacian Δ_G, this implies:

Corollary 3.7. *Let $(\mathcal{M}, G)$ be a Hesse manifold. Then $(\mathcal{M}, G)$ is non-compact.*

3.2. **The space of Hesse functions**

The space of Hesse functions of any Riemannian manifold is finite-dimensional. More precisely:

PROPOSITION 3.8 ([10]). *For any Riemannian n-manifold $(\mathcal{M}, G)$, we have $\mathfrak{h}_G(\mathcal{M}) \leq n+1$.*

The space of Hesse functions carries a natural symmetric bilinear pairing which is invariant under the action of the isometry group. We start by defining a certain extension of this pairing.

Definition 3.9. The *extended Hesse pairing* of $(\mathcal{M}, G)$ is the symmetric $\mathbb{R}$-bilinear map $(\ ,\ )^e_G : \mathcal{C}^\infty(\mathcal{M}) \times \mathcal{C}^\infty(\mathcal{M}) \to \mathcal{C}^\infty(\mathcal{M})$ defined through:

$$(f_1, f_2)^e_G \stackrel{\text{def.}}{=} f_1 f_2 - \langle \mathrm{d}f_1, \mathrm{d}f_2 \rangle_G = f_1 f_2 - \langle \mathrm{grad}_G f_1, \mathrm{grad}_G f_2 \rangle_G \ .$$

Recall that we assume $\mathcal{M}$ to be connected. An easy computation using the Hesse equation gives:

PROPOSITION 3.10 ([10]). *The function $(\Lambda_1, \Lambda_2)^e_G$ is constant on $\mathcal{M}$ for any Hesse functions $\Lambda_1, \Lambda_2 \in \mathcal{H}_G(\mathcal{M})$. Hence the restriction of the extended Hesse pairing to the subspace $\mathcal{H}_G(\mathcal{M}) \subset \mathcal{C}^\infty(\mathcal{M})$ gives an $\mathbb{R}$-valued bilinear pairing:*

$$(\ ,\ )_G : \mathcal{H}_G(\mathcal{M}) \times \mathcal{H}_G(\mathcal{M}) \to \mathbb{R} \ ,$$

which we shall call the Hesse pairing *of $(\mathcal{M}, G)$.*

Remark 3.11. By Proposition 3.10, any Hesse function $\Lambda \in \mathcal{H}_G(\mathcal{M})$ satisfies the nonlinear first order ODE:

$$||\mathrm{grad}_G \Lambda||^2_G = \Lambda^2 - (\Lambda, \Lambda)_G \ , \tag{10}$$

where $(\Lambda, \Lambda)_G$ is a constant. Notice that $||\mathrm{grad}_G \Lambda||^2_G = ||\mathrm{d}\Lambda||^2_G$. When $(\Lambda, \Lambda)_G = 0$, equation (10) reduces on the complement of the zero locus of Λ to the eikonal equation of $(\mathcal{M}, G)$ for the function $f \stackrel{\text{def.}}{=} \log|\Lambda|$:

$$||\mathrm{grad}_G f||^2_G = 1 \ . \tag{11}$$

Hence (10) can be viewed as a generalization of the eikonal equation.

Definition 3.12. The *Hesse norm* of a Hesse function Λ is the non-negative number $\kappa_\Lambda \stackrel{\text{def.}}{=} \sqrt{|(\Lambda, \Lambda)_G|}$, while its *type indicator* is the sign factor $\epsilon_\Lambda \stackrel{\text{def.}}{=} \mathrm{sign}(\Lambda, \Lambda)_G$. A non-trivial Hesse function Λ is called *timelike*, *spacelike* or *lightlike* when ϵ_Λ equals $+1$, -1 or 0 respectively.

Notice that lightlike Hesse functions form a cone in $\mathcal{H}_G(\mathcal{M})$.

3.3. The Morse property of Hesse functions

PROPOSITION 3.13 ([10]). *Let $\Lambda \in \mathcal{H}_G(\mathcal{M})$ be a nontrivial Hesse function. Then Λ has isolated critical points, i.e. it is a Morse function on $\mathcal{M}$. Moreover, the following statements hold:*

- *If Λ is timelike, then Λ does not have any zeroes on $\mathcal{M}$.*
- *If Λ is spacelike, then Λ does not have any critical points on $\mathcal{M}$.*
- *If Λ is lightlike, then Λ has neither zeroes nor critical points on $\mathcal{M}$.*

Hence Λ can have zeroes iff $(\Lambda, \Lambda)_G < 0$ and it can have critical points iff $(\Lambda, \Lambda)_G > 0$.

4. THE GRADIENT FLOW OF HESSE FUNCTIONS

Let $\Lambda \in \mathcal{H}_G(\mathcal{M})$ be a non-trivial Hesse function and consider the gradient flow equation:

$$\gamma'(q) = -(\mathrm{grad}_{\mathcal{G}}\Lambda)(\gamma(q)) \tag{12}$$

for smooth curves $\gamma : I \to \mathcal{M}$, where I is an interval and $\gamma'(q) \stackrel{\text{def.}}{=} \frac{\mathrm{d}\gamma}{\mathrm{d}q}$. This equation fixes the parameter q of a solution γ (which we shall call the *gradient flow parameter*) up to translation by a constant. The *level set parameter* λ of γ is defined through:

$$\lambda(q) \stackrel{\text{def.}}{=} \Lambda(\gamma(q))$$

and decreases as the gradient flow parameter increases.

PROPOSITION 4.1 ([10]). *The level set and gradient flow parameters of any gradient flow curve γ of Λ satisfy:*

$$\mathrm{d}q = -\frac{\mathrm{d}\lambda}{||\mathrm{d}_{\gamma(\lambda)}\Lambda||_G^2} = \frac{\mathrm{d}\lambda}{(\Lambda, \Lambda)_G - \lambda^2} \tag{13}$$

and are related through:

$$q = \begin{cases} \frac{1}{\kappa_\Lambda} \operatorname{arctanh}\left(\frac{\lambda - \lambda_0}{\kappa_\Lambda}\right) & , \quad \textit{if } \epsilon_\Lambda = +1 \\ -\frac{1}{\kappa_\Lambda} \arctan\left(\frac{\lambda - \lambda_0}{\kappa_\Lambda}\right) & , \quad \textit{if } \epsilon_\Lambda = -1 \\ \frac{\lambda_0}{\lambda} & , \quad \textit{if } \epsilon_\Lambda = 0 \end{cases} . \tag{14}$$

where λ_0 is an integration constant and:

$$\lambda = \begin{cases} \kappa_\Lambda \tanh(\kappa_\Lambda q) & , \quad \textit{if } \epsilon_\Lambda = +1 \\ -\kappa \tan(\kappa_\Lambda q) & , \quad \textit{if } \epsilon_\Lambda = -1 \\ \frac{1}{q} & , \quad \textit{if } \epsilon_\Lambda = 0 \end{cases} , \tag{15}$$

where κ_Λ and ϵ_Λ are the Hesse norm and type indicator of λ. In the formulas above, we chose the integration constant λ_0 such that $q|_{\lambda=\lambda_0} = 0$ when $\epsilon_\Lambda = \{-1, +1\}$ and $q|_{\lambda=\lambda_0} = 1$ when $\epsilon_\Lambda = 0$.

4.1. **The general form of Hesse functions**

The relation to the eikonal equation allows us to express Hesse functions using the distance function of the Riemannian manifold $(\mathcal{M}, G)$, for whose properties we refer the reader to [14]. We need a few preparations before stating this result.

PROPOSITION 4.2 ([10]). *Suppose that the Riemannian manifold $(\mathcal{M}, G)$ is complete and let $\Lambda \in \mathcal{H}_G(\mathcal{M}) \setminus \{0\}$ be a non-trivial Hesse function. Then the following statements hold:*

1. *If Λ is timelike, then the vanishing locus $Z(\Lambda)$ of Λ is empty and hence Λ has constant sign (denoted η_Λ) on $\mathcal{M}$. Moreover, Λ has exactly one critical point, with critical value $\eta_\Lambda \kappa_\Lambda$, which is a global minimum or maximum according to whether $\eta_\Lambda = +1$ or -1.*

2. *If Λ is spacelike, then the set $\mathrm{Crit}(\Lambda)$ of critical points of Λ is empty. Moreover, the vanishing locus of Λ coincides with the κ_Λ-level set of the function $||\mathrm{d}\Lambda||_G$:*
$$Z(\Lambda) = \{m \in \mathcal{M} \,|\, ||\mathrm{d}_m\Lambda||_G = \kappa_\Lambda\} \ ,$$
which is a non-singular hypersurface in $\mathcal{M}$.

3. *If Λ is lightlike, then $Z(\Lambda) = \mathrm{Crit}(\Lambda) = \emptyset$ and hence Λ has constant sign on $\mathcal{M}$, which we denote by η_Λ.*

Definition 4.3. Suppose that $(\mathcal{M}, G)$ is complete. Then a timelike or lightlike non-trivial Hesse function $\Lambda \in \mathcal{H}_G(\mathcal{M}) \setminus \{0\}$ is called *future (resp. past) pointing* when $\eta_\Lambda = +1$ (resp. -1).

Definition 4.4. Let $\Lambda \in \mathcal{H}_G(\mathcal{M}) \setminus \{0\}$ be a non-trivial Hesse function of $(\mathcal{M}, G)$. The *characteristic set* of Λ is the following closed subset of $\mathcal{M}$:
$$Q_\Lambda \overset{\text{def.}}{=} \begin{cases} \mathrm{Crit}(\Lambda) \ , & \text{if } \Lambda \text{ is timelike} \\ Z(\Lambda) \ , & \text{if } \Lambda \text{ is spacelike} \\ \mathcal{M}_{|\Lambda|}(1) \ , & \text{if } \Lambda \text{ is lightlike} \end{cases} .$$
The *characteristic constant* of Λ is defined through:
$$C_\Lambda \overset{\text{def.}}{=} \begin{cases} \kappa_\Lambda \ , & \text{if } \epsilon_\Lambda = +1 \\ 0 \ , & \text{if } \epsilon_\Lambda = -1 \\ 1 \ , & \text{if } \epsilon_\Lambda = 0 \end{cases} .$$

Set $\mathcal{U}_\Lambda \stackrel{\text{def.}}{=} \mathcal{M} \setminus \mathrm{Crit}(\Lambda)$. We have:

$$Q_\Lambda = \{m \in \mathcal{U}_\Lambda \ \ |\,|\Lambda(m)| = C_\Lambda\} \ .$$

Definition 4.5. Let $\Lambda \in \mathcal{H}_G(\mathcal{M}) \setminus \{0\}$ be a non-trivial Hesse function of $\mathcal{M}$. The *characteristic sign function* of Λ is the function $\Theta_\Lambda : \mathcal{M} \to \mathbb{R}$ defined through:

$$\Theta_\Lambda(m) \stackrel{\text{def.}}{=} \begin{cases} 1 \ , & \text{if } \epsilon_\Lambda = +1 \\ \mathrm{sign}(\Lambda(m)) \ , & \text{if } \epsilon_\Lambda = -1 \\ \mathrm{sign}(|\Lambda(m)| - 1) \ , & \text{if } \epsilon_\Lambda = 0 \end{cases} \ . \tag{16}$$

The Λ-*distance function* of $(\mathcal{M}, G)$ is the function $d_\Lambda : \mathcal{M} \to \mathbb{R}$ defined through:

$$d_\Lambda(m) \stackrel{\text{def.}}{=} \Theta_\Lambda(m) \mathrm{dist}_G(m, Q_\Lambda) \ . \tag{17}$$

THEOREM 4.6 ([10]). *Let $\Lambda \in \mathcal{H}_G(\mathcal{M})$ be a non-trivial Hesse function. Then the following relation holds for all $m \in \mathcal{M}$:*

$$\Lambda(m) = \begin{cases} \mathrm{sign}(\Lambda)\kappa_\Lambda \cosh d_\Lambda(m) \ , & \textit{if } \epsilon_\Lambda = +1 \\ \kappa_\Lambda \sinh d_\Lambda(m) \ , & \textit{if } \epsilon_\Lambda = -1 \\ \mathrm{sign}(\Lambda) e^{d_\Lambda(m)} \ , & \textit{if } \epsilon_\Lambda = \ 0 \end{cases} \ .$$

4.2. **Maximally Hesse manifolds are Poincaré balls**

Complete Hesse manifolds of maximal Hesse index turn out to be particularly simple, namely any such manifold is isometric with a Poincaré ball.

Definition 4.7. A Hesse manifold $(\mathcal{M}, G)$ is called *maximally Hesse* if $\mathfrak{h}_G(\mathcal{M}) = n + 1$.

Recall that a Riemannian manifold $(\mathcal{M}, G)$ is *hyperbolic* if its metric G has unit negative sectional curvature. Up to isometry, there exists a unique simply connected and complete hyperbolic n-manifold, namely the Poincaré n-ball, whose description we recall below. Let:

$$\mathrm{D}^n \stackrel{\text{def.}}{=} \{u \in \mathbb{R}^n | 0 \leq ||u||_E < 1\}$$

be the open unit n-ball, where $||\ ||_E$ is the Euclidean norm on $\mathbb{R}^n$. The *Poincaré ball metric* is the complete Riemannian metric G_n on D^n whose squared line element is given by:

$$\mathrm{d}s^2_{G_n} = \frac{4}{(1 - ||u||_E^2)^2} \sum_{i=1}^n (\mathrm{d}u^i)^2 \ .$$

The n-dimensional *Poincaré ball* is the complete hyperbolic manifold $\mathbb{D}^n \stackrel{\text{def.}}{=} (\mathrm{D}^n, G_n)$.

PROPOSITION 4.8 ([10]). *A complete Riemannian n-manifold $(\mathcal{M}, G)$ is maximally Hesse iff it is isometric to the Poincaré ball $\mathbb{D}^n$.*

The space of Hesse functions of $\mathbb{D}^n$ identifies naturally with a Minkowski space, as we explain next. Consider the $(n+1)$-dimensional Minkowski space $\mathbb{R}^{1,n} \stackrel{\text{def.}}{=} (\mathbb{R}^{n+1}, (\ ,\ ))$ where:

$$(X, Y) \stackrel{\text{def.}}{=} X^0 Y^0 - \sum_{i=1}^{n} X^i Y^i = \eta^{\mu\nu} X^\mu Y^\nu \tag{18}$$

is the Minkowski pairing. We denote the canonical basis of $\mathbb{R}^{n+1}$ by:

$$E_0 = (1, 0, 0, \ldots, 0)\ ,\quad E_1 = (0, 1, 0, \ldots, 0)\ ,\quad \ldots\quad ,\quad E_n = (0, 0, 0, \ldots, 1)\ .$$

Let $\vec{X} \stackrel{\text{def.}}{=} (X^1, \ldots, X^n)$, so that $X = (X^0, \vec{X})$ and:

$$(X, X) = X^0 Y^0 - \vec{X} \cdot \vec{Y}\ ,$$

where $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^n$. Let S_n^+ be the future sheet of the hyperboloid defined by the equation $(X, X) = 1$:

$$\mathrm{S}_n^+ \stackrel{\text{def.}}{=} \{X \in \mathbb{R}^{n+1} | (X, X) = 1\ \&\ X^0 > 0\} = \left\{X \in \mathbb{R}^{n+1} | X^0 = \sqrt{1 + ||\vec{X}||_E^2}\right\}.$$

Then S_n^+ is diffeomorphic with D^n through the *Weierstrass map* $\Xi : \mathrm{D}^n \to \mathrm{S}_n^+$, which is defined through:

$$\Xi(u) \stackrel{\text{def.}}{=} \left(\frac{1 + ||u||_E^2}{1 - ||u||_E^2}, \frac{2u}{1 - ||u||_E^2}\right)\ ,\ \forall u \in \mathrm{D}^n \tag{19}$$

and whose inverse $\Xi^{-1} : \mathrm{S}_n^+ \to \mathbb{D}^n$ is given by:

$$\Xi^{-1}(X) = \frac{\vec{X}}{X^0 + 1} = \frac{\vec{X}}{1 + \sqrt{1 + ||\vec{X}||_E^2}}\ ,\ \forall X \in \mathrm{S}_n^+\ .$$

Notice the relations

$$||u||_E^2 = \frac{\Xi^0(u) - 1}{X^0(u) + 1} \iff \Xi^0(u) = \frac{1 + ||u||_E^2}{1 - ||u||_E^2}\ .$$

The components $\Xi^\mu(u)$ (which satisfy the relation $\eta_{\mu\nu}\Xi^\mu(u)\Xi^\nu(u) = -1$) are the classical *Weierstrass coordinates* of the point $u \in \mathrm{D}^n$. The Weierstrass map can be viewed as the projection of D^n onto S_n^+ from the point $-E_0 = (-1, 0, \ldots, 0)$ of $\mathbb{R}^{1,n}$. It is well-known that Ξ is an isometry from $\mathbb{D}^n$ to S_n^+ when S_n^+ is endowed with the Riemannian metric induced by the *opposite* of the Minkowski metric (18). We can now state the result announced above:

THEOREM 4.9 ([10]). *For any $n > 1$, there exists a bijective isometry $\mathbf{\Lambda} : \mathbb{R}^{1,n} \stackrel{\sim}{\to} (\mathcal{H}_{G_n}(\mathrm{D}^n), (\ ,\ )_{G_n})$ such that:*

$$\mathbf{\Lambda}(E_\mu) = \Xi_\mu \stackrel{\text{def.}}{=} \eta_{\mu\nu}\Xi^\nu\quad ,\quad \forall \mu \in \{0, \ldots, n\}\ .$$

4.3. **The Hesse sheaf and local Hesse index**

Let $(\mathcal{M}, G)$ be a Riemannian n-manifold. The Hesse equation naturally defines a sheaf of vector spaces on $\mathcal{M}$.

Definition 4.10. A *local Hesse function* of $\mathcal{M}$ relative to G is a locally defined solution of the Hesse equation of $(\mathcal{M}, G)$. The *Hesse sheaf* of $(\mathcal{M}, G)$ is the sheaf of local Hesse functions of $(\mathcal{M}, G)$.

PROPOSITION 4.11 ([10]). *We have* $\operatorname{rk} \mathcal{H}_G \leq n+1$.

Definition 4.12. We say $(\mathcal{M}, G)$ is *locally Hesse* if its Hesse sheaf does not vanish, i.e. if $\operatorname{rk} \mathcal{H}_G > 0$.

Notice that $\mathcal{H}_G(\mathcal{M}) = H^0(\mathcal{H}_G)$ and hence $\mathfrak{h}_G(\mathcal{M}) = h^0(\mathcal{H}_G) = \dim_{\mathbb{R}} H^0(\mathcal{H}_G)$. Thus $(\mathcal{M}, G)$ is globally Hesse iff its Hesse sheaf admits nontrivial global sections.

4.4. **Locally maximally Hesse manifolds are elementary hyperbolic space forms**

Definition 4.13. A Riemannian manifold $(\mathcal{M}, G)$ is called *locally maximally Hesse* if $\operatorname{rk} \mathcal{H}_G = n+1$.

THEOREM 4.14 ([10]). *A Riemannian manifold is locally maximally Hesse iff it is hyperbolic.*

Note that a general hyperbolic manifold need not be Hesse. The situation is clarified by the following result.

PROPOSITION 4.15 ([10]). *Let* $(\mathcal{M}, G)$ *be a complete Riemannian manifold. The following are equivalent:*

- $(\mathcal{M}, G)$ *is hyperbolic and globally Hesse.*
- $(\mathcal{M}, G)$ *is an elementary hyperbolic space form.*

In this case, $(\mathcal{M}, G)$ *is maximally Hesse iff it is isometric with a Poincaré ball.*

Hyperbolic uniformization and the notion of elementary hyperbolic space form are recalled in Appendix A.

A. HYPERBOLIC UNIFORMIZATION AND ELEMENTARY HYPERBOLIC SPACE FORMS

Recall that the group of orientation-preserving isometries of the Poincaré n-ball is naturally isomorphic with the connected component $\mathrm{SO}_0(1,n)$ of the identity in the Lorentz group $\mathrm{SO}(1,n)$. Indeed, $\mathrm{SO}_0(1,n)$ acts linearly on $\mathbb{R}^{n+1}$ (and hence on the hyperboloid model S_n^+ of $\mathbb{D}^n$) through the fundamental representation $R : \mathrm{SO}_0(1,n) \to \mathrm{Aut}_{\mathbb{R}}(\mathbb{R}^{n+1})$:

$$R_A(x) = AX \quad , \quad \forall A \in \mathrm{SO}_0(1,n) \quad , \quad \forall X \in \mathbb{R}^{n+1} \quad ,$$

where $R_A \stackrel{\text{def.}}{=} R(A)$. Since this action preserves orientation as well as the Minkowski pairing (and hence the Riemannian metric induced on S_n^+), it induces a morphism of groups $\psi : \mathrm{SO}_0(1,n) \to \mathrm{Iso}_+(\mathbb{D}^n)$, which turns out to be an isomorphism. For any $A \in \mathrm{SO}_0(1,n)$, the corresponding isometry $\psi_A \stackrel{\text{def.}}{=} \psi(A)$ of the Poincaré ball is determined uniquely by the following condition, which encodes $\mathrm{SO}_0(1,n)$-equivariance of the Weierstrass map:

$$\Xi \circ \psi = R \circ \Xi \quad , \quad \text{i.e. } \Xi(\psi_A(\vec{u})) = A\Xi(\vec{u}) \ , \quad \forall A \in \mathrm{SO}_o(1,n) \quad , \quad \forall \vec{u} \in \mathrm{D}^n \quad . \tag{20}$$

A general element $A \in \mathrm{SO}_0(1,n)$ has the form:

$$A(\vec{v}) = \begin{bmatrix} \gamma & -\gamma(\vec{v})\vec{v} \\ -\gamma(\vec{v})\vec{v}^T & I_n + (\gamma(\vec{v}) - 1)\hat{v} \otimes \hat{v} \end{bmatrix}$$

where $\vec{v} \in \mathbb{R}^n$ and we defined:

$$v \stackrel{\text{def.}}{=} ||\vec{v}||_E \ , \quad \hat{v} \stackrel{\text{def.}}{=} \frac{\vec{v}}{v} \ , \quad \gamma(\vec{v}) \stackrel{\text{def.}}{=} \frac{1}{\sqrt{1-v^2}} \ ,$$

$$\hat{v} \otimes \hat{v} = (\hat{v}_i \hat{v}_j)_{i,j=1,\dots n} = \left(\frac{v_i v_j}{v^2}\right)_{i,j=1,\dots n} .$$

The following result is classical:

Proposition A.1. *For any $\vec{v} \in \mathbb{R}^n$, $\vec{u} \in \mathrm{D}^n$ and $A \in \mathrm{SO}_0(1,n)$, we have:*

$$\psi_{A(\vec{v})}(\vec{u}) = \frac{2u + 2(\gamma(\vec{v}) - 1)(\hat{v} \cdot \vec{u})\hat{v} - \gamma(\vec{v})(1 + ||\vec{u}||_E^2)\vec{v}}{1 - ||\vec{u}||_E^2 + \gamma(\vec{v})(1 + ||\vec{u}||_E^2 - 2\vec{v} \cdot \vec{u})} \quad . \tag{21}$$

By the uniformization theorem of hyperbolic geometry (see [15]), any oriented and complete hyperbolic n-manifold $(\mathcal{M}, G)$ can be written as the Riemannian quotient of the unit hyperbolic ball $\mathbb{D}^n$ through a discrete subgroup $\Gamma \in \mathrm{Iso}(\mathbb{D}^n) \simeq \mathrm{SO}_0(1,n)$ called the *uniformizing group* of $(\mathcal{M}, G)$. Notice that Γ is isomorphic with the fundamental group of $\mathcal{M}$. We remind the reader of the following classical notions, for which we refer him or her to [15].

Definition A.2. A discrete subgroup Γ of $\mathrm{SO}_0(1,n)$ is called *elementary* if its action on the closure of the Poincaré ball fixes at least one point.

Definition A.3. An n-dimensional *elementary hyperbolic space form* is a complete hyperbolic n-manifold uniformized by a torsion-free elementary discrete subgroup $\Gamma \subset \mathrm{SO}_0(1,n)$.

A torsion-free elementary discrete subgroup $\Gamma \subset \mathrm{SO}_0(1,n)$ is called:

- *elliptic*, if it conjugates to a subgroup of the *canonical rotation group* $\mathcal{R}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_0(1,n)}(E_0) \simeq \mathrm{SO}(n)$. In this case, Γ is finite.
- *hyperbolic*, if it conjugates to a subgroup of the *canonical squeeze group* $\mathcal{T}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_0(1,n)}(E_n) \simeq \mathrm{SO}(1,n-1)$. In this case, Γ is a hyperbolic cyclic group.
- *parabolic*, if it conjugates to a subgroup of the *canonical shear group* $\mathcal{P}_n \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{SO}_0(1,n)}(E_0+E_n) \simeq \mathrm{ISO}(n)$. In this case, Γ is a free Abelian group of rank at most $n-1$.

Any nontrivial torsion-free elementary discrete subgroup of $\mathrm{SO}_0(1,n)$ is either elliptic, parabolic or hyperbolic, while the trivial subgroup of $\mathrm{SO}_0(1,n)$ belongs to each of these classes. An elementary hyperbolic space form different from $\mathbb{D}^n$ is called elliptic, parabolic or hyperbolic if its uniformizing group Γ is of that type.

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