HIGHER SUPERGEOMETRY FOR THE SUPER-σ-MODEL

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The article presents a status report and reviews a programme of a systematic (super)algebraisation and geometrisation of the canonical data (vacua, symmetries, dualities, defects *etc.*) of super-σ-models for super-p-branes of superstring theory and M-theory on homogeneous spaces of supersymmetry Lie supergroups and of a κ-symmetry-equivariant geometrisation of the classes in the corresponding Cartan–Eilenberg cohomology. The programme draws upon and generalises essentially the much successful higher-geometric and -algebraic approach to the two-dimensional non-linear bosonic σ-model of string theory.

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1. **INTRODUCTION**

Higher geometry and algebra pervade modern physics, both in its conceptual and methodological layer, reconquering a domain ruled undividedly by functional analysis and linear algebra for a long time. Ample substantiation of this claim is provided by the so-called geometric approach to lagrangean field theory, by investigation and constructive application of dualities, symmetries and conservation laws – neatly organised and quantified by group and algebra theory as well as cohomology theory, together with the theory of principal (and associated) bundles with compatible connection – in the analysis of the structure of a given dynamical model, the form of its classical configurations and relations to other models as well as in structurisation of its quantum description (*e.g.*, through application of symmetry bootstrap and integrability techniques) and in theoretical model building (*e.g.*, through the use of symmetry invariants in the construction of lagrangean densities), and – finally – by the symmetry-equivariant geometric and functorial schemes of quantisation. Some physical models are 'more equal than others' in their manifestation of the underlying geometric and algebraic structure – emblematic for this way of thinking about the mathematical rendering of physical phenomena is the two-dimensional conformal field theory in the Belavin–Polyakov–Zamolodchikov picture of Ref. [19], with its conformal-bootstrap approach and hidden categorial structure (*cp*, *e.g.*, Refs. [145, 128]), and in particular the rational version thereof, *cp* Refs. [181, 113, 91] and [127, 129], with its current-algebra and quantum-group symmetries (*cp* Refs. [127, 5, 84, 126, 67]), the structure of a holomorphic vector bundle with compatible (Knizhnik–Zamolodchikov–Bernard) connection on the space of conformal blocks of correlators over the moduli space of complex structures on a given two-dimensional spacetime with marked points (operator insertions), *cp* Refs. [113, 13, 14], and a deep relation with the topological quantum field theory of Refs. [183, 9, 170], the three-dimensional Chern–Simons theory of Ref. [46] occupying a prominent position in this context, *cp* Refs. [184, 53, 49, 50]. Quite naturally, and not unrelatedly, the algebro-geometric aspect becomes especially pronounced in a class of field theories modelling propagation of (localised) distributions of mass and cohomological charge of various dimensions in background gravitational and differential-form fields on an ambient target geometry, known as non-linear σ-models. Having originated as effective field theories of bound states of elementary particles, *cp* Ref. [78], such models abound in string theory and related theories of charged branes, and it was realised early on, in Refs. [7, 66], that their rigorous treatment calls for incorporation of basic concepts and tools of higher cohomology theory into the standard field-theoretic arsenal. Subsequent geometrisation of the relevant Beilinson–Deligne cohomology, worked out by Murray *et al.* in Refs. [133, 130, 154, 155, 33, 104, 34] and generalised by Gajer in Ref. [63], has led to a multi-faceted and fruitful entanglement of higher geometry and algebra in the formulation and – after Ref. [66] – symmetry analysis and (pre)quantisation of the field theory, preparing the ground for the development of an independent framework of their description, classification and study over the years that followed (*cp* Sec. 2 for details).

In keeping with the original expectations tied to string theory, the latter having been conceived as a scheme of ultimate unification of all fundamental interactions, the associated σ-models were inscribed in another grand unification programme, to wit, the supersymmetry of Refs. [124, 81, 77, 173, 174, 10, 187, 188], essentially from its inception. This required, at least in principle, introduction of the fundamentally novel formalism of superfield theory embedded in the somewhat heavy sheaf- and category-theoretic framework of supergeometry, as in Refs. [17, 65, 11, 144, 175], endowed with an action of a supersymmetry 'group'. The latter notion was put on a firm footing by Kostant and Koszul in Refs. [108, 109], in which the concept of a Lie supergroup and that of its homogeneous space was shaped. With many of the high hopes for a direct application of the ensuing superstring theory in the modelling of natural phenomena at large undermined severely by high-energy experiments, there remains one very good physical reason – independent of the purely mathematical interest in this field of intensive research activity – to pursue the study of models of superstring and superbrane theory on certain distinguished such supermanifolds with the Graßmann-even body of the general type $\mathrm{AdS}_m \times \mathbb{S}^n$, namely, the application of the perturbative superstring theory in these supergeometries in a predictive description of the behaviour of QCD-type systems at a strong coupling on the basis of the so-called AdS/CFT 'correspondence', *cp* Refs. [120, 74, 185, 119].

The supersymmetric σ-model describing 'propagation' of a massive super-charged $(p+1)$-dimensional brane in a homogeneous space of a supersymmetry Lie supergroup, with a prototype worked out by de Azcárraga, Lukierski, Green and Schwarz in Refs. [37, 82, 83], has long been known to combine the various higher-geometric and -algebraic features alluded to before. In particular, it carries a nontrivial cohomological information, in analogy with its bosonic predecessor. The relevant cohomology is the supersymmetric refinement of the standard de Rham cohomology of the supergeometric target, and that fact has far-reaching consequences due to the inherent non-compactness of the supersymmetry group. Drawing strong motivation and inspiration from tremendous successes of the higher-geometric approach to the two-dimensional σ-model with a non-$\mathbb{Z}/2\mathbb{Z}$-graded target space, developed by Gawędzki, Reis, Waldorf, Schreiber, Schweigert and the Author, one is thus confronted with the challenge of establishing a *physically meaningful* geometrisation of the distinguished classes in the aforementioned cohomology that enter the definition of the supersymmetric σ-model. The present paper is a status report on and an extensive review of a programme, initiated by the Author, of a (super)algebraisation and geometrisation of the canonical data (critical field configurations, symmetries, dualities, defects *etc.*) of the σ-model for the super-charged $(p+1)$-dimensional brane and of a geometrisation of the physically relevant cohomology classes consistent with the various realisations of supersymmetry in that σ-model. Thus, upon recalling the reference bosonic field theory – the two-dimensional non-linear σ-model – and giving a systematic overview of the higher-geometric approach to its study in Sec. 2, with emphasis on the description of global and local symmetries of the theory, we open the supergeometric narrative in Sec. 3 by reassembling the requisite elements of supergeometry with supersymmetry, illustrated on physically relevant examples. This sets the stage for the introduction of the main (super)field-theoretic object of interest – the supersymmetric σ-model – in Sec. 4. The construct is presented in two formulations – a standard 'dynamical' one imitating its bosonic counterpart and a purely topological one – and a correspondence between them is stated that paves the way for subsequent fully fledged algebraisation and geometrisation of the canonical analysis and (higher-)geometrisation of the cohomological data of the model. The former is delineated in Sec. 5 which ends with an in-depth treatment of the fundamental *gauge* supersymmetry of the σ-model that restores equibalance between bosonic and fermionic degrees of freedom in the vacuum of the theory – the κ-symmetry. In the next Sec. 6, a scheme of supersymmetry-equivariant geometrisation of the class of cohomological data of the field theory under consideration is laid out and backed up by concrete examples, and its compatibility with the global supersymmetry of the supergeometric target as well as with asymptotic correspondences between different targets, induced by İnönü–Wigner contractions on the underlying tangent Lie superalgebras, are discussed. Various universal and model-sensitive consistency checks of the proposal are summarised in Sec. 7. The paper concludes with a recapitulation and a brief outline of possible directions of future development of the supergeometric programme reviewed.

2. LESSONS FROM LIFE WITHOUT SPIN

The logic and methodology of the lagrangean approach to the modelling of physical phenomena has played a prime rôle in our theoretical endeavour for over 200 years, ever since its inception marked by Lagrange's seminal treatise *Méchanique Analitique* [114] of 1788/89. The fundamental nature of the classical action functional was revealed in Dirac's insightful study [45] of 1933, later elaborated and ingeniously applied by Feynman [48]. With the indispensability of a rigorous definition of the lagrangean density encoding the dynamics of a system of interest thus firmly established, there came the realisation – hinted at already in Ref. [44], and formalised neatly in the classical régime by Wu and Yang in Ref. [186] and by Lubkin [118] and Trautman [169] – of inadequacy of the commonly employed tools of the global tensorial calculus on configuration 'spaces' of systems with charge whose evolution takes place in the presence of topological singularities, such as, *e.g.*, those engineered in the Aharonov–Bohm experiment of Ref. [2]. Higher cohomology, in various guises, was finally identified as the appropriate tool of description for a large number of such systems by Alvarez in Ref. [7] and Gawędzki in Ref. [66], and the latter work solidified the status of the cohomological analysis as a natural and convenient bridge between the classical and the quantum by canonically associating a prequantisation of the classical model with the cohomological data of the differential-form field coupling to the charge current. Subsequent geometrisation of classes in the relevant real Beilinson–Deligne hypercohomology, originally devised by Murray *et al.* in Refs. [133, 130] and then embedded in a more general scheme by Gajer in Ref. [63], opened up an avenue for a systematic higher-cohomological, -geometric and -categorial treatment of a large class of field theories with a topological charge. A prominent position among these has been occupied by the two-dimensional nonlinear bosonic σ-model with the topological Wess–Zumino term in the action functional, *cp* Ref. [59], with numerous applications in, *i.a.*, the theory of condensed matter, two-dimensional statistical physics near the critical point, the effective field theory of collective excitations of spin chains and critical (bosonic) string theory, and by the related topological (gauge) field theories, such as, *e.g.*, the three-dimensional Chern–Simons field theory of Ref. [46] in the presence of Wilson loops, *cp* Refs. [184, 53]. The present section is dedicated to a concise recapitulation of the main aspects and an enumeration of the most important successes of the higher-cohomological approach to the study of the σ-model initiated in Refs. [7, 66]. It is intended to be a rich source of motivation, useful intuitions and concrete formal constructions and techniques that shall be exploited in the supergeometric setting of later sections.

Let Σ be an oriented compact connected two-dimensional C^∞-manifold without boundary, to be termed the **worldsheet** and regarded as the 'spacetime'[1] of a field theory with the trivial covariant configuration bundle (or 'field bundle') $\mathcal{F}_\sigma \equiv \Sigma \times M \longrightarrow \Sigma$ whose fibre M, to be called the **target space**, is a C^∞-manifold of dimension $\dim M = D+1$, $D \in \mathbb{N}$ endowed with a metric tensor field $\mathrm{g} \in \Gamma(\mathsf{T}^*M \otimes_M^{\mathrm{sym}} \mathsf{T}^*M)$ and a closed 3-form field $\mathrm{H}_3 \in \Gamma(\bigwedge_M^3 \mathsf{T}^*M) \equiv \Omega^3(M)$. The triple

$$\mathfrak{B}^{(\mathrm{NG})} \equiv \big(M, \mathrm{g}, \mathrm{H}_3\big) \tag{2.1}$$

shall be referred to as the **background**. The theory of interest is a lagrangean field theory of mappings $x \in C^\infty(\Sigma, M) \equiv [\Sigma, M]$ determined by an action functional whose critical points are minimal embeddings (for g) deformed by Lorentz-type forces. The source of the latter forces is a purely topological coupling of H_3 to the charged-loop current defined by the embedded worldsheet, locally – *i.e.*, for x such that $x(\Sigma) \subset \mathcal{O}(\subset M)$ with the property $\mathrm{H}_3 \restriction_{\mathcal{O}} = \mathsf{d}\mathrm{B}$ for some $\mathrm{B} \in \Omega^2(\mathcal{O})$ – given by the topological term $S_{\mathrm{top}}[x] \underset{\mathrm{loc.}}{=} \int_\Sigma x^*\mathrm{B}$ of the action functional. The field theory admits a 'dual' Polyakov formulation in which the standard Nambu–Goto term ($\mu_1 \in \mathbb{R}^\times$ is a dimensionless parameter)

$$S^{(\mathrm{NG})}_{\sigma,\mathrm{metr}}[x] = \mu_1 \int_\Sigma \sqrt{\left|\det\big(x^*\mathrm{g}\big)\right|} \tag{2.2}$$

[1] Physically, such 'spacetimes' arise naturally when we consider interference of contributions to the quantum-mechanical transition amplitudes between given intitial and final states of the elementary loop (or string) coming from its cobordant trajectories in M.

in the action functional, with minimal embeddings as extremals, is replaced by an energy functional that depends algebraically on an intrinsic metric γ on Σ. The requirement of existence of a non-anomalous quantum lift of the classical symmetries of the 'dual' Polyakov model, quantified by the semidirect product $\mathrm{Diff}^+(\Sigma)\ltimes\mathrm{Weyl}(\gamma)$ of the group of orientation-preserving diffeomorphisms of Σ and the group of Weyl scalings of the worldsheet metric, translates into differential constraints imposed upon admissible backgrounds which force the Weitzenböck connection on $\mathsf{T}M$ obtained from the standard Levi-Civita one for g by adding to it a torsion term naturally determined by H_3 to be Ricci-flat (up to corrections of higher order in a small parameter of the theory known as the string tension that we drop from our discussion for the sake of simplicity). This implies that we may no longer be at liberty to take H_3 with a trivial class in the de Rham cohomology upon choosing a metric on M. These physical considerations emphasise the relevance of a rigorous definition of the topological term. A prerequisite of such a definition is a geometrisation of the de Rham 3-cocycle H_3 in what is to be understood as a generalisation of the standard assignment of a principal $\mathbb{C}^\times$-bundle with compatible connection to the Maxwell 2-cocycle coupling topologically to the charged-particle current in the lagrangean model of propagation of a point-like particle in external gravitational and electromagnetic fields, *cp* Ref. [66]. Thus, we arrive at

Definition 2.1. [133, 130, 154] Let M be a C^∞-manifold and H_3 a de Rham 3-cocycle on it, with periods $\mathrm{Per}(\mathrm{H}_3)\subseteq 2\pi\mathbb{Z}$. An **abelian 1-gerbe** with a connective structure of **curvature** H_3 over **base** M is a septuple

$$\mathcal{G}^{(1)}=\big(\mathsf{Y}M,\pi_{\mathsf{Y}M},\mathrm{B},L,\pi_L,\mathrm{A}_L,\mu_L\big),\qquad \mathrm{curv}\big(\mathcal{G}^{(1)}\big)\equiv\mathrm{H}_3$$

composed of

- a surjective submersion $\pi_{\mathsf{Y}M}\ :\ \mathsf{Y}M\longrightarrow M$;
- a 2-form $\mathrm{B}\in\Omega^2(\mathsf{Y}M)$, termed the **curving** of $\mathcal{G}^{(1)}$;
- a principal $\mathbb{C}^\times$-bundle $\pi_L\ :\ L\longrightarrow\mathsf{Y}^{[2]}M$ over the fibred square $\mathsf{Y}^{[2]}M\equiv\mathsf{Y}M\times_M\mathsf{Y}M$ (with the canonical projections $\mathrm{pr}_i\ :\ \mathsf{Y}^{[2]}M\longrightarrow\mathsf{Y}M,\ i\in\{1,2\}$);
- a principal $\mathbb{C}^\times$-connection 1-form $\mathrm{A}_L\in\Omega^1(L)$;
- a connection-preserving principal $\mathbb{C}^\times$-bundle isomorphism

$$\mu_L\ :\ \mathrm{pr}_{1,2}^*L\otimes\mathrm{pr}_{2,3}^*L\xrightarrow{\cong}\mathrm{pr}_{1,3}^*L,$$

termed the **groupoid structure**, over $\mathsf{Y}^{[3]}M\equiv\mathsf{Y}M\times_M\mathsf{Y}M\times_M\mathsf{Y}M$ (with $\mathrm{pr}_{i,j}\equiv(\mathrm{pr}_i,\mathrm{pr}_j)\ :\ \mathsf{Y}^{[3]}M\longrightarrow\mathsf{Y}^{[2]}M$), with the following properties:

– the connective structure obeys $\pi_{\mathsf{Y}M}^*{\rm H}_3 = \mathsf{d}{\rm B}$ and $\mathsf{d}{\rm A}_L = \pi_L^*({\rm pr}_2^* - {\rm pr}_1^*){\rm B}$;

– the groupoid structure satisfies the associativity constraint [161, Eq. (2.3)].

Given 1-gerbes $\mathcal{G}_A^{(1)} = (\mathsf{Y}_A M, \pi_{\mathsf{Y}_A M}, {\rm B}_A, L_A, \pi_{L_A}, {\rm A}_{L_A}, \mu_{L_A})$, $A \in \{1,2\}$ over the common base M, a **1-isomorphism** between them is a sextuple

$$\Phi = \big(\mathsf{YY}_{1,2}M, \pi_{\mathsf{YY}_{1,2}M}, E, \pi_E, {\rm A}_E, \alpha_E\big) \;:\; \mathcal{G}_1^{(1)} \xrightarrow{\cong} \mathcal{G}_2^{(1)}$$

composed of

- a surjective submersion $\pi_{\mathsf{YY}_{1,2}M} \;:\; \mathsf{YY}_{1,2}M \longrightarrow \mathsf{Y}_1 M \times_M \mathsf{Y}_2 M \equiv \mathsf{Y}_{1,2}M$;
- a principal $\mathbb{C}^\times$-bundle $\pi_E \;:\; E \longrightarrow \mathsf{YY}_{1,2}M$;
- a principal $\mathbb{C}^\times$-connection 1-form ${\rm A}_E \in \Omega^1(E)$;
- a connection-preserving principal $\mathbb{C}^\times$-bundle isomorphism

$$\alpha_E \;:\; (\pi_{\mathsf{YY}_{1,2}M} \times \pi_{\mathsf{YY}_{1,2}M})^*{\rm pr}_{1,3}^* L_1 \otimes {\rm pr}_2^* E \xrightarrow{\cong} {\rm pr}_1^* E \otimes (\pi_{\mathsf{YY}_{1,2}M} \times \pi_{\mathsf{YY}_{1,2}M})^*{\rm pr}_{2,4}^* L_2$$

over $\mathsf{Y}^{[2]}\mathsf{Y}_{1,2}M \equiv \mathsf{YY}_{1,2}M \times_M \mathsf{YY}_{1,2}M$,

with properties described, *e.g.*, in Ref. [161, Sec. 2.1].

Given 1-isomorphisms $\Phi_A = \big(\mathsf{Y}^A\mathsf{Y}_{1,2}M, \pi_{\mathsf{Y}^A\mathsf{Y}_{1,2}M}, E_A, \pi_{E_A}, {\rm A}_{E_A}, \alpha_{E_A}\big) \;:\; \mathcal{G}_1^{(1)} \xrightarrow{\cong} \mathcal{G}_2^{(1)}$, $A \in \{1,2\}$, a **2-isomorphism** between them is a triple

$$\varphi = \big(\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M, \pi_{\mathsf{YY}_{1,2}\mathsf{Y}_{1,2}M}, \beta\big) \;:\; \Phi_1 \stackrel{\cong}{\Longrightarrow} \Phi_2$$

composed of

- a surjective submersion $\pi_{\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M} \;:\; \mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M \longrightarrow \mathsf{Y}^1\mathsf{Y}_{1,2}M \times_{\mathsf{Y}_{1,2}M} \mathsf{Y}^2\mathsf{Y}_{1,2}M \equiv \mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M$;
- a connection-preserving principal $\mathbb{C}^\times$-bundle isomorphism

$$\beta \;:\; \pi_{\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M}^*{\rm pr}_1^* E_1 \xrightarrow{\cong} \pi_{\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M}^*{\rm pr}_2^* E_2\,,$$

satisfying the coherence condition [161, Eq. (2.5)] over

$$\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M \times_M \mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M,$$

expressing its compatibility with the α_{E_A}. $\diamond$

Abelian 1-gerbes over a given base together with the associated 1-isomorphisms and 2-isomorphisms form the **bicategory of abelian 1-gerbes with a connective structure over** M, denoted as $\mathfrak{BGrb}_\nabla(M)$, *cp* Refs. [133, 130, 154] and Ref. [176]. Among its 0-cells, we encounter the distinguished **trivial 1-gerbes**

$$\mathcal{I}^{(1)}_{\mathrm{B}} = \big(M, \mathrm{id}_M, \mathrm{B}, M \times \mathbb{C}^\times, \mathrm{pr}_1, 0, \mu_0\big)\,,$$

with the groupoid structure $\mu_0 \,:\, (M \times \mathbb{C}^\times) \otimes (M \times \mathbb{C}^\times) \longrightarrow M \times \mathbb{C}^\times \,:\, (x, z_1) \otimes (x, z_2) \longmapsto (x, z_1 \cdot z_2)$. The bicategory admits a monoidal structure (the Deligne tensor product) and pullback along smooth maps, the latter extending to an operation between bicategories associated with different bases.

The reason why we have written the definition of the various geometric constructs *in extenso* is to emphasise the key feature of the geometrisation, to wit, the use of surjective submersions as structure maps and smooth forms over them as tensorial data. The geometrisation scheme outlined above generalises to de Rham cocycles of higher degrees, leading to the emergence of a hierarchy of higher-geometric objects known as **abelian** p**-gerbes** with a connective structure, *cp* Ref. [63] (and Ref. [104]). The hierarchy, starting with principal $\mathbb{C}^\times$-bundles with principal $\mathbb{C}^\times$-connections, now recognised as **abelian 0-gerbes**, is based on a simple principle of recursion by which a $(p+1)$-gerbe of curvature $\mathrm{H}_{p+3} \in Z^{p+3}_{\mathrm{dR}}(M)$ over M (with periods $\mathrm{Per}(\mathrm{H}_{p+3}) \subseteq 2\pi\mathbb{Z}$) is defined in terms of a surjective submersion $\pi_{\mathsf{Y}M} \,:\, \mathsf{Y}M \longrightarrow M$ equipped with a primitive $\mathrm{B}_{p+2} \in \Omega^{p+2}(\mathsf{Y}M)$ of $\pi^*_{\mathsf{Y}M}\mathrm{H}_{p+3}$, or a trivial $(p+1)$-gerbe, and of a p-gerbe over $\mathsf{Y}^{[2]}M$ together with a family of distinguished p-gerbe k-isomorphisms with $k \in \overline{1, p+1}$ over the respective components $\mathsf{Y}^{[k+2]}M$ of the nerve of the (pair) groupoid $\mathsf{Y}^{[2]}M \underset{t\equiv\mathrm{pr}_2}{\overset{s\equiv\mathrm{pr}_1}{\rightrightarrows}} \mathsf{Y}M$ that 'resolve' the $(p+1)$-gerbe, the last one of these, a $(p+1)$-isomorphism, satisfying a coherence condition over $\mathsf{Y}^{[p+4]}M$, *cp* Refs. [154, 155, 178] for an exemplification with $p+1 = 2$. The hierarchical construction has a natural hypercohomological counterpart in which we associate with H_{p+3} the sheaf-theoretic data of its resolution over an open cover $\mathscr{O}_M$ of M (*e.g.*, a good one, whose existence is controlled by the Weil–de Rham Theorem) that compose a $(p+1)$-cocycle in the total cohomology of the Čech–Deligne bicomplex formed by extending the bounded Deligne complex $\mathcal{D}(p+1)^\bullet$:

$$\begin{array}{l} \underline{\mathrm{U}(1)}_M \equiv \mathcal{D}(p+1)^0 \xrightarrow{\ \mathsf{i}\,\mathsf{d}\log\, \equiv\, \mathsf{d}^{(0)}\ } \underline{\Omega}^1(M) \equiv \mathcal{D}(p+1)^1 \\ \xrightarrow{\ \mathsf{d}\, \equiv\, \mathsf{d}^{(1)}\ } \underline{\Omega}^2(M) \equiv \mathcal{D}(p+1)^2 \xrightarrow[\mathsf{d}\, \equiv\, \mathsf{d}^{(2)}]{} \cdots\cdots \xrightarrow[\mathsf{d}\, \equiv\, \mathsf{d}^{(p)}]{} \underline{\Omega^{p+1}}(M) \equiv \mathcal{D}(p+1)^{p+1} \end{array},$$

of sheaves of locally smooth $\mathrm{U}(1)$-valued maps and k-forms (for $k \in \overline{1, p+1}$) on

M in the direction of the standard Čech cohomology associated with $\mathscr{O}_M$, *cp* Ref. [104]. The hypercohomological structure provides us with a natural model of the higher-geometric structure over the nerve of the open cover.

We are now ready to give:

Definition 2.2. In the hitherto notation, a (**monophase**) **two-dimensional nonlinear bosonic σ-model for the Nambu-Goto background** $\mathfrak{B}^{(\mathrm{NG})}$ of Eq. (2.1) is a theory of maps from $[\Sigma, M]$ determined by the principle of least action applied to the **Dirac-Feynman amplitude**

$$\mathcal{A}_{\mathrm{DF}} \ :\ [\Sigma, M] \longrightarrow \mathrm{U}(1) \ :\ x \longmapsto \mathcal{A}_{\mathrm{DF}}^{\mathrm{metr}}[x] \cdot \mathcal{A}_{\mathrm{DF}}^{\mathrm{top}}[x] \,,$$

given by the product of the metric factor

$$\mathcal{A}_{\mathrm{DF}}^{\mathrm{metr}}[x] = \mathrm{e}^{\mathrm{i}\, S^{(\mathrm{NG})}_{\sigma,\mathrm{metr}}[x]} \,, \tag{2.3}$$

with $S^{(\mathrm{NG})}_{\sigma,\mathrm{metr}}$ as in (2.2), and of the topological (or **Wess-Zumino**) factor

$$\mathcal{A}_{\mathrm{DF}}^{\mathrm{top}}[x] = \imath_1\big([x^*\mathcal{G}^{(1)}]\big)$$

given by the image of the isoclass of the pullback $x^*\mathcal{G}^{(1)}$ of an abelian 1-gerbe $\mathcal{G}^{(1)}$ of curvature H_3 over M under the canonical isomorphism $\imath_1 : \mathcal{W}^3(\Sigma; 0) \xrightarrow{\cong} \mathrm{U}(1)$ in which $\mathcal{W}^3(\Sigma; 0)$ is the group of isoclasses of flat gerbes over Σ.

◇

The topological factor $\mathcal{A}_{\mathrm{DF}}^{\mathrm{top}}$ can be expressed explicitly in terms of components of the Čech-Deligne 2-cocycle resolving H_3 over $\mathscr{O}_M$, *cp* Refs. [66, 104]. It is to be viewed as a natural generalisation of the standard line holonomy of a principal $\mathbb{C}^\times$-bundle. In particular, it is readily seen to be a Cheeger–Simons differential character of degree 2 modulo $2\pi\mathbb{Z}$, in the sense of Ref. [32],

$$\mathcal{A}_{\mathrm{DF}}^{\mathrm{top}}[x] =: h_{\mathcal{G}^{(1)}}\big(x(\Sigma)\big) \,, \qquad h_{\mathcal{G}^{(1)}} \in \mathrm{Hom}_{\mathbf{AbGrp}}\big(Z_2(M), \mathrm{U}(1)\big)$$

with the property $h_{\mathcal{G}^{(1)}}(\partial c) = \varepsilon_{\mathcal{G}^{(1)}}(c)$, expressed – for any $c \in C_3(M)$ – in terms of the 3-cochain $\varepsilon_{\mathcal{G}^{(1)}} \equiv \mathrm{e}^{\mathrm{i}\int_{\cdot} \mathrm{curv}(\mathcal{G}^{(1)})} \ :\ C_3(M) \longrightarrow \mathrm{U}(1) \ :\ c \longmapsto \mathrm{e}^{\mathrm{i}\int_c \mathrm{curv}(\mathcal{G}^{(1)})}$. This property justifies the proposed completion of the unproblematic metric term from the purely physical point of view. The ability to write down a *rigorous* definition of the field theory does not exhaust the list of advantages of working with the higher-geometric constructs.

The geometrisation of the 3-cocycle component of the σ-model background is the first step towards geometric quantisation of that theory. Indeed, there exists a canonical **transgression map** $\tau_1 : \mathbb{H}^2(M, \mathcal{D}(2)^\bullet) \longrightarrow \mathbb{H}^1(\mathsf{L}M, \mathcal{D}(1)^\bullet)$ that associates an isoclass of a 0-gerbe over the (single-loop) configuration space $\mathsf{L}M \equiv [\mathbb{S}^1, M]$ of the σ-model to the isoclass of the 1-gerbe $\mathcal{G}^{(1)}$, the former being identified with the class of its sheaf-theoretic local data (a 1-cocycle) in the 1st Beilinson-Deligne hypercohomology group[2] $\mathbb{H}^1(\mathsf{L}M, \mathcal{D}(1)^\bullet)$ of $\mathsf{L}M$ and the latter – with a similar class (a 2-cocycle) in the 2nd Beilinson-Deligne hypercohomology group $\mathbb{H}^2(M, \mathcal{D}(2)^\bullet)$ of the target space M. The transgression mechanism, admitting straightforward higher-dimensional generalisations, was first explicitly described and put to use for $p = 1$ in Ref. [66]. The curvature of the principal $\mathbb{C}^\times$-connection 1-form $\mathcal{A}$ on a **transgression bundle** $\mathscr{L}_{\mathcal{G}^{(1)}} \xrightarrow{\pi_{\mathscr{L}_{\mathcal{G}^{(1)}}}} \mathsf{L}M$ from the image of $[\mathcal{G}^{(1)}]$ under τ_1 reads $\delta\mathcal{A} = \pi^*_{\mathscr{L}_{\mathcal{G}^{(1)}}} \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{curv}\big(\mathcal{G}^{(1)}\big)$, where $\mathrm{ev} : \mathsf{L}M \times \mathbb{S}^1 \longrightarrow M : (\underline{x}, \phi) \longmapsto \underline{x}(\phi)$ is the standard evaluation map. It is to be compared with the (pre)symplectic form[3]

$$\Omega_\sigma = \delta\vartheta_{\mathsf{T}^*\mathsf{L}M} + \pi^*_{\mathsf{T}^*\mathsf{L}M} \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{curv}\big(\mathcal{G}^{(1)}\big) \tag{2.4}$$

of the σ-model over its (single-loop) space of states $\mathsf{T}^*\mathsf{L}M$ that canonically projects to the configuration space, $\pi_{\mathsf{T}^*\mathsf{L}M} : \mathsf{T}^*\mathsf{L}M \longrightarrow \mathsf{L}M$, the 2-form being expressed in terms of the so-called (kinetic-)action 1-form $\vartheta_{\mathsf{T}^*\mathsf{L}M} \in \Omega^1(\mathsf{T}^*\mathsf{L}M)$ with the familiar local presentation $\vartheta_{\mathsf{T}^*\mathsf{L}M}[\underline{x}, \mathrm{p}] = \int_{\mathbb{S}^1} \mathrm{Vol}(\mathbb{S}^1) \wedge \mathrm{p}_\mu(\cdot)\delta\underline{x}^\mu(\cdot)$ in the coordinates $(\underline{x}^\mu, \mathrm{p}_\nu)$ on $\mathsf{T}^*\mathsf{L}M$ (here, p is the *kinetic*-momentum field over the Cauchy contour $\mathbb{S}^1$). Thus, upon tensoring with the trivial principal $\mathbb{C}^\times$-bundle over $\mathsf{T}^*\mathsf{L}M$ with the global connection 1-form $\vartheta_{\mathsf{T}^*\mathsf{L}M}$, the pullback of the transgression bundle along the bundle projection $\pi_{\mathsf{T}^*\mathsf{L}M}$ becomes a natural model for the frame bundle of the **prequantum bundle** of the σ-model. Suitably polarised sections of the latter define the Hilbert space of the two-dimensional field theory. The programme of gerbe-induced geometric quantisation was realised in full for the σ-models with compact Lie groups as targets (the so-called Wess-Zumino-Witten (WZW) σ-models of Ref. [181], *cp* also Ref. [113]) by Gawędzki *et al.* in Refs. [66, 68, 79, 80, 69].

The structural rôle played by the (1-)gerbe in the quantisation of the σ-model legitimates employing gerbe theory in its canonical analysis and in the discussion of dualities between σ-models, rigid symmetries of a given σ-model being a special case thereof induced by a class of isometries of the target space. The correspondence between dualities and distinguished 1-isomorphisms, dubbed 1-gerbe bimodules in Ref. [61], was studied at great length in Refs. [157, 158] in the setting of the polyphase σ-model of Ref. [139] in which monophase

[2] The hypercohomology is the direct limit, over refinements of good open covers, of the total cohomology of the Čech-Deligne bicomplex mentioned earlier.

[3] The (pre)symplectic form can be derived in the first-order formalism of Refs. [64, 105, 106, 111, 168, 112], *cp* also Ref. [141] for a modern treatment.

regions of the spacetime Σ are separated by conformal defects. The study showed that a bimodule canonically determines an isotropic 'correspondence' subspace within the product of monophase (single-loop) spaces of states, over which the bimodule trangresses – through a mechanism fully analogous to that originally discovered by Gawędzki – to an isomorphism between the respective monophase prequantum bundles. It also revealed the field-theoretic function of the distinguished 2-isomorphisms associated with defect junctions in the polyphase σ-model, which is that of 'duality intertwiners' defining isotropic subspaces in the multiple products of monophase spaces of states attached to the junctions and transgressing to isomorphisms between tensor products of the corresponding prequantum bundles for monophase spaces of the in-coming and out-going states, restricted to the isotropics – *cp* also Ref. [140] for an in-depth treatment in the highly symmetric setting of the WZW σ-model.

It is the symmetries of a given σ-model that are of prime relevance to the analysis of the supersymmetric field theories of immediate interest to us, therefore we summarise below the main ideas, constructions and results regarding their gerbe-theoretic aspect. The symmetries are customarily grouped into two categories:

• **global symmetries**, quantifying a correspondence between physical field configurations defined by the constancy of the DF amplitude on orbits of the global-symmetry group G_σ – these are induced by the isometries of (M, g) that compose a group G_σ, assumed to act smoothly on M,

$$\lambda_\cdot \,:\, \mathrm{G}_\sigma \times M \longrightarrow M \,:\, (g, m) \longmapsto \lambda_g(m)\,, \tag{2.5}$$

and so also on field configurations, $\lambda_{\cdot *} \,:\, \mathrm{G}_\sigma \times [\Sigma, M] \longrightarrow [\Sigma, M] \,:\, (g, x) \longmapsto \lambda_g \circ x$, and to lift to 1-isomorphisms

$$\Phi_g \,:\, \lambda_g^* \mathcal{G}^{(1)} \xrightarrow{\cong} \mathcal{G}^{(1)}\,, \qquad g \in \mathrm{G}_\sigma\,; \tag{2.6}$$

• **local** (or **gauge**) **symmetries**, signalling a redundancy of the field degrees of freedom that becomes reflected in the degeneracy of the presymplectic form of the σ-model whose characteristic distribution is spanned by generators of such symmetries, *cp* Ref. [64], and, consequently, calling for a reduction through descent to the orbispace of the action of the gauge-symmetry group in the space of states – generically, these Σ-dependent transformations do *not* preserve the two factors in $\mathcal{A}_{\mathrm{DF}}$ *independently*, and so do not admit a higher-geometric realisation (but *cp* the supersymmetric scenario in Sec. 5).

In the paradigm of local field theory, the ontological gap between the two denotations of 'symmetry' is bridged by the so-called gauging which morally boils down to rendering the global symmetry local. Geometrically, the gauging of the global-symmetry group G_σ consists in replacing the original field bundle $\mathcal{F}_\sigma \longrightarrow \Sigma$ of the theory with the bundle $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathcal{F}_\sigma \equiv (\mathsf{P}_{\mathrm{G}_\sigma} \times_\Sigma \mathcal{F}_\sigma)/\mathrm{G}_\sigma \longrightarrow \Sigma$ associated by $\lambda.$ with a principal G_σ-bundle $\mathsf{P}_{\mathrm{G}_\sigma} \longrightarrow \Sigma$. The associated bundle has the same typical fibre as $\mathcal{F}_\sigma$, and so the very same count of field-theoretic degrees of freedom, but admits an action, locally modelled on $\lambda.$, of the Fréchet–Lie gauge group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}_\sigma})$ of global sections of the adjoint bundle $\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}_\sigma} \equiv \mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathrm{G}_\sigma \longrightarrow \Sigma$ associated with $\mathsf{P}_{\mathrm{G}_\sigma}$ by the adjoint action of G_σ on itself. Whenever $\mathcal{F}_\sigma/\mathrm{G}_\sigma$ is a smooth manifold (which happens, *e.g.*, when $\lambda.$ is free and proper), the procedure descends the original field theory to the quotient bundle, and in the remaining cases we may think of the gauging as a method of modelling the theory with the field bundle given by the orbispace of $\lambda.$ in the space of global sections of the original one subjected to suitable gauge reductions (the so-called untwisted sector) *and* to enhancement by patchwise continuous sections that become continuous only upon descent to the orbispace (the twisted sector(s)). Quantum field-theoretic arguments concerning existence and non-degeneracy of the vacuum of the field theory with the global symmetry gauged, invoked in Refs. [87, 89] where gauging was investigated at great length in the context of the gerbe theory of the two-dimensional σ-model also in the presence of defects, lead to the conclusion that representatives of *all* isoclasses of principal G_σ-bundles ought to be considered in the geometric procedure outlined above. This necessity was elucidated in elementary terms, using a correspondence between – on one hand – the physically favoured principal G_σ-bundles $\mathsf{P}_{\mathrm{G}_\sigma}$ whose associated bundles $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathcal{F}_\sigma$ admit a global section and – on the other hand – principal bundles over Σ with the **action groupoid** $\mathrm{G}_\sigma \ltimes M$ as the structure groupoid (*cp* Ref. [125], but also Refs. [138, 137]), in Refs. [159, 160] in which the isoclass of the trivial bundle was demonstrated to correspond to the aforementioned untwisted sector, and the other isoclasses – to the twisted sector(s) in what may be regarded as a generalisation of the worldsheet-orbifold construction of Refs. [42, 43]. Passing from pure geometry to field-theory dynamics in the case of a continuous symmetry, *i.e.*, for G_σ a Lie group with the tangent Lie algebra $\mathfrak{g}_\sigma$, requires endowing $\mathsf{P}_{\mathrm{G}_\sigma}$ with a principal G_σ-connection 1-form $\mathcal{A}_\sigma \in \Omega^1(\mathsf{P}_{\mathrm{G}_\sigma}) \otimes \mathfrak{g}_\sigma$ and using the latter to consistently transmit gauge transformations – *via* its own variability under the defining action of the structure group – to the dynamical sector of the field theory ('past the derivatives'), an effect that survives the quotienting $\mathsf{P}_{\mathrm{G}_\sigma} \times_\Sigma \mathcal{F}_\sigma \xrightarrow{\pi_\sim} (\mathsf{P}_{\mathrm{G}_\sigma} \times_\Sigma \mathcal{F}_\sigma)/\mathrm{G}_\sigma$ owing to the induction of the so-called Crittenden connection on $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathcal{F}_\sigma$ from $\mathcal{A}_\sigma$. In the case of simple tensorial terms in the action functional, the desired goal is attained through a variant of the classic 'minimal-coupling recipe' – the relevant prescription for the Nambu-Goto functional in Eq. (2.3) was explicited in Ref. [89, Eq. (10.4)]. Topological factors in the DF amplitudes, on the other hand, call for a substantially more involved treatment that develops over the nerve $\mathsf{N}^\bullet(\mathrm{G}_\sigma\ltimes M) \equiv \mathrm{G}_\sigma^{\times\bullet} \times M$ of the action groupoid $\mathrm{G}_\sigma\ltimes M$, that is over the simplicial manifold described in Ref. [89, Def. 2.18]. Its point of departure is the demand that the curvature of the (1-)gerbe satisfy the *strong* $\mathfrak{g}_\sigma$-invariance condition[4] with respect to the fundamental vector fields $\mathcal{K}_\cdot : \mathfrak{g}_\sigma \times M \longrightarrow \mathsf{T}M : (X,m) \longmapsto \frac{\mathsf{d}}{\mathsf{d}t}\restriction_{t=0}\lambda_{\mathrm{e}^{-t\,X}}(m) \equiv \mathcal{K}_X(m)$ induced by $\lambda_\cdot$, that is that the identities

$$\mathcal{K}_X \lrcorner\, \mathrm{curv}\big(\mathcal{G}^{(1)}\big) = -\mathsf{d}\kappa_X \tag{2.7}$$

hold true for some $\kappa_X \in \Omega^1(M)$. Once this is ensured, we pick up a principal G_σ-bundle $\mathsf{P}_{\mathrm{G}_\sigma}$ and define, after Ref. [89, Sec. 10], the $\mathsf{P}_{\mathrm{G}_\sigma}$-extended 1-gerbe

$$\widetilde{\mathcal{G}}^{(1)}_{\mathcal{A}_\sigma} := \mathrm{pr}_2^*\mathcal{G}^{(1)} \otimes \mathcal{I}^{(1)}_{\widetilde{\varrho}_{\mathcal{A}_\sigma}} \tag{2.8}$$

over $\widetilde{M} \equiv \mathsf{P}_{\mathrm{G}_\sigma} \times M$ in terms of the trivial 1-gerbe with the global curving $\widetilde{\varrho}_{\mathcal{A}_\sigma} \in \Omega^2(\widetilde{M})$ explicited in [89, Eq. (10.5)]. In order to formulate conditions of descent of the extended 1-gerbe to the total space of the associated bundle, we need

Definition 2.3. In the hitherto notation, let M be a manifold equipped with an action $\lambda_\cdot$ of a Lie group G_σ with the tangent Lie algebra $\mathfrak{g}_\sigma$ as in Eq. (2.5) and let $\mathcal{G}^{(1)}$ be a 1-gerbe over M whose curvature satisfies the strong invariance condition (2.7). The **small gauge anomaly** for $\mathcal{G}^{(1)}$ is the $\mathbb{R}$-bilinear map $\alpha^{(1)} : \mathfrak{g}_\sigma^{\times 2} \longrightarrow \Omega^1(M) \times C^\infty(M,\mathbb{R}) : (X,Y) \longmapsto (\mathscr{L}_{\mathcal{K}_X}\kappa_Y - \kappa_{[X,Y]_{\mathfrak{g}_\sigma}}, \mathcal{K}_X \lrcorner\, \kappa_Y + \mathcal{K}_Y \lrcorner\, \kappa_X)$. If the latter vanishes, the de Rham 2-cocycles $\lambda_\cdot^*\mathrm{curv}(\mathcal{G}^{(1)})$ and $\mathrm{pr}_2^*\mathrm{curv}(\mathcal{G}^{(1)})$ are cohomologous, and then the **large gauge anomaly** for $\mathcal{G}^{(1)}$ is the obstruction against existence of a 1-isomorphism $\lambda_\cdot^*\mathcal{G}^{(1)} \cong \mathrm{pr}_2^*\mathcal{G}^{(1)} \otimes \mathcal{I}_\varrho$, given in terms of a 2-form $\varrho \in \Omega^2(\mathrm{G}_\sigma \times M)$ determined by the κ_X, and the rest of a G_σ-equivariant structure on $\mathcal{G}^{(1)}$, as described in Ref. [87, Def. 5.1]. 1-Gerbes over M with a G_σ-equivariant structure together with the corresponding 1- and 2-isomorphisms of Ref. [87, Def. 5.1] form a bicategory $\mathfrak{BGrb}_\nabla(M)^{\mathrm{G}_\sigma}_\varrho$ that contains a sub-bicategory $\mathfrak{BGrb}_\nabla(M)^{\mathrm{G}_\sigma}_0$ of 1-gerbes with a G_σ-equivariant structure such that $\lambda_\cdot^*\mathcal{G}^{(1)} \cong \mathrm{pr}_2^*\mathcal{G}^{(1)}$, termed **descendable**.

◇

Remark 2.4. The small gauge anomaly admits a neat interpretation in terms of an obstruction against existence of an explicit realisation, spanned on the basis pairs $(\mathcal{K}_A, \kappa_A) \equiv (\mathcal{K}_{t_A}, \kappa_{t_A})$, $A \in \overline{1, \dim\mathfrak{g}_\sigma}$, of the tangent Lie

[4]The condition is stronger than that of $\mathfrak{g}_\sigma$-invariance, $\mathscr{L}_{\mathcal{K}_X}\mathrm{curv}(\mathcal{G}^{(1)}) = 0$, but weaker than that of $\mathfrak{g}_\sigma$-basicness, requiring also $\mathfrak{g}_\sigma$-horizontality $\mathcal{K}_X \lrcorner\, \mathrm{curv}(\mathcal{G}^{(1)}) = 0$.

algebroid $\mathfrak{g}_\sigma \ltimes M$ of the action groupoid $\mathrm{G}_\sigma \ltimes M$ within Hitchin's generalised tangent bundle $\mathsf{E}^{1,1}M \equiv \mathsf{T}M \oplus \mathsf{T}^*M$ equipped with the Courant bracket (*cp* Refs. [93] and [90]) twisted by $\mathrm{curv}\big(\mathcal{G}^{(1)}\big)$ à la Ševera and Weinstein, *cp* Ref. [167]. The latter itself has an intrinsically gerbe-theoretic nature, *cp* Ref. [159].

The field-theoretic relevance of the above entities is documented in Ref. [87, Prop. 5.5 & Thm. 5.9] which shows that in the presence of a G_σ-equivariant structure on the 1-gerbe $\mathcal{G}^{(1)}$ of the σ-model of Def. 2.2, the $\mathsf{P}_{\mathrm{G}_\sigma}$-extended 1-gerbe canonically defines a unique 1-gerbe $\mathcal{G}^{(1)}_{\mathcal{A}_\sigma}$ over $\mathsf{P}_{\mathrm{G}_\sigma} \times_{\mathrm{G}_\sigma} \mathcal{F}_\sigma$ such that $\pi_\sim^*\mathcal{G}^{(1)}_{\mathcal{A}_\sigma} \cong \widetilde{\mathcal{G}}^{(1)}_{\mathcal{A}_\sigma}$. Together with a metric $\mathrm{g}_{\mathcal{A}_\sigma}$ on the associated bundle determined by g (through the 'minimal coupling', *cp* [89, Eq. (10.4)]), it then defines the **gauged two-dimensional nonlinear σ-model**, which is invariant under the natural action of the gauge group $\Gamma(\mathrm{Ad}\,\mathsf{P}_{\mathrm{G}_\sigma})$. The theorem is a straightforward consequence of the fundamental

THEOREM 2.5 ([87, Thm. 5.3]). *Adopt the hitherto notation. If G_σ acts on M in such a manner that M/G_σ is a smooth manifold and the quotient map $M \longrightarrow M/\mathrm{G}_\sigma$ is a principal G_σ-bundle, there exists a canonical equivalence of bicategories*

$$\mathfrak{BGrb}_\nabla(M)_0^{\mathrm{G}_\sigma} \cong \mathfrak{BGrb}_\nabla(M/\mathrm{G}_\sigma)\,.$$

Thus, crucially from the point of view of subsequent supergeometric considerations, symmetry-equivariant structures on the (1-)gerbe are seen to not only quantify the amenability of the corresponding global symmetry of the σ-model to gauging, but also to identify those (1-)gerbes on a G_σ-manifold which (are 1-isomorphic with those that) pull back from the orbispace. The structures were first introduced in the present context in Refs. [87, 89], where a mixture of geometric and categorial arguments enriched with field-theoretic considerations (partial reduction of the gauged σ-model through integration of the Euler–Lagrange equations for the non-dynamical gauge fields) was invoked to prove that the structures do, indeed, ensure descent of the field theory to the orbispace M/G_σ whenever the latter is a smooth manifold, or transform the original theory into a model of a field theory with the orbispace as the target space otherwise. Their indispensability for the descent was proven in Ref. [159, Sec. 8.3] (*cp* also Refs. [158, 160]), where the G_σ-equivariant structure of Def. 2.3 was demonstrated to provide target-space data for the topological gauge-symmetry defect that implements the gauging through a natural generalisation, along the lines of Ref. [139], of the worldsheet-orbifold construction of Refs. [42, 43] and of its later application in the framework of the TFT quantisation of CFT in Ref. [51] (*cp* also Ref. [103]).

Remark 2.6. It is worth pointing out that the G_σ-equivariant structure has a cohomological interpretation as a completion of the 2-cocycle of sheaf-theoretic data associated with $\mathrm{curv}(\mathcal{G}^{(1)})$ to a 2-cocycle in the tricomplex obtained from the Čech-Deligne bicomplex by extension in the direction of group cohomology (for a suitable choice of the open cover $\mathscr{O}_M$, *cp* Ref. [89, App. I]). The interpretation paves the way to a straightforward generalisation of this useful concept to higher p-gerbes.

The criterion of high symmetry sets apart a class of σ-models with compact Lie groups G and their homogeneous spaces G/H associated with Lie subgroups $\mathrm{H} \subset \mathrm{G}$ as target spaces. These are the celebrated Wess-Zumino-Witten (WZW) σ-models, advanced in Ref. [181] and quantised equivariantly in Ref. [113], and their gauged variants, with the quantum prototype postulated in Ref. [73] in the language of the representation theory of current-algebra extensions of the Virasoro algebra associated with the pair $(\mathfrak{g}, \mathfrak{h} \subset \mathfrak{g})$ of Lie algebras (of G and H, respectively), and with an explicit lagrangean formulation as a WZW σ-model coupled to a non-dynamical connection on a principal H-bundle over the worldsheet and a path integral-based quantisation given in Refs. [72, 71]. In the basic case of G simple and 1-connected, with the tangent Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ equipped with a non-degenerate (negative definite) Killing form $\kappa_{\mathfrak{g}} \ :\ \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R} \ :\ (X, Y) = k\,\mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}_X \circ \mathrm{ad}_Y),\ k \in \mathbb{R}_{>0}$, the former are σ-models of Def. 2.2 (customarily in the Polyakov formulation) with the background $(\mathrm{G}, \mathrm{g}_k, \mathrm{H}_k)$ composed of the Cartan-Killing metric $\mathrm{g}_k = \kappa_{\mathfrak{g}} \circ (\theta_{\mathrm{H}} \otimes \theta_{\mathrm{H}})$, $\mathrm{H} \in \{\mathrm{L}, \mathrm{R}\}$ ($\theta_{\mathrm{L/R}}$ is the left/right-invariant (LI/RI) $\mathfrak{g}$-valued Maurer-Cartan form, with components $\theta^A_{\mathrm{L/R}}$ dual to the LI/RI vector fields L_A/R_A) and the Cartan 3-cocycle

$$\mathrm{H}_k \quad = \quad \tfrac{1}{24\pi}\,\kappa_{\mathfrak{g}} \circ \big([\cdot,\cdot]_{\mathfrak{g}} \otimes \mathrm{id}_{\mathfrak{g}}\big) \circ \big(\theta_{\mathrm{H}} \wedge \theta_{\mathrm{H}} \wedge \theta_{\mathrm{H}}\big)\,. \tag{2.9}$$

The 3-form has $\mathrm{Per}(\mathrm{H}_k) = 2k\pi\,\mathbb{Z}$ and generates the third de Rham cohomology group $H^3(\mathrm{G}, \mathbb{R})$ of G, and so for any integer value of the **level** $k \in \mathbb{N}^\times$, it geometrises as the k-th (Deligne-)tensor power $\mathcal{G}^{(1)}_k = \mathcal{G}_{\mathrm{b}}^{\otimes k}$ of the so-called **basic (1-)gerbe** $\mathcal{G}_{\mathrm{b}} \equiv \mathcal{G}^{(1)}_{k=1}$. Basic gerbes for compact simple 1-connected Lie groups were first explicitly constructed for $\mathrm{G} = \mathrm{SU}(N)$, $N \in \mathbb{N}^\times$ by Gawędzki and Reis in Ref. [80], and subsequently for arbitrary G by Meinrenken in Ref. [123].

The left and right regular actions of the target Lie group G on itself give rise to the respective *chiral* symmetries $\mathsf{L}\mathrm{G} \times [\Sigma, \mathrm{G}] \times \mathsf{L}\mathrm{G} \longrightarrow [\Sigma, \mathrm{G}]\ :\ (h_+, g, h_-) \longmapsto (h_+ \circ \pi_+) \cdot g \cdot (h_- \circ \pi_-)$ (for $\pi_\pm(\sigma^0, \sigma^1) = \sigma^0 \pm \sigma^1 \equiv \sigma^\pm$). The Poisson bracket of the corresponding chiral Noether currents $J_{+/-}(g) = \partial_{+/-} \lrcorner\, g^*\theta_{\mathrm{R/L}}$, written for $g \in [\Sigma, \mathrm{G}]$ and $\partial_{+/-} \equiv \frac{\partial}{\partial \sigma^{+/-}}$ and determined by the presymplectic form (2.4), gives a field-theoretic realisation of the H_k-twisted Courant bracket of the distinguished sections $(H_A, \frac{k}{8\pi}\,\epsilon_{\rm H}\,\theta_{\rm H}^A) \in \Gamma(\mathsf{E}^{1,1}{\rm G})$, $\epsilon_{\rm R/L} = +1/-1$, as first noticed in Ref. [8]. These lift to quantum symmetries and decompose the Hilbert space of the σ-model into modules of the central extension $\mathbf{0} \longrightarrow \mathbb{R} \xrightarrow{\kappa} \widehat{\mathfrak{g}} \longrightarrow \mathsf{L}\mathfrak{g} \longrightarrow \mathbf{0}$ of the loop algebra $\mathsf{L}\mathfrak{g}$ of $\mathfrak{g}$, generated by the harmonic modes of $J_\pm \equiv J_{+/-}$ and by the central element $K \equiv \kappa(1)$. Thus, the Hilbert space takes the form of the direct sum $\mathcal{H}_k = \bigoplus_{\lambda \in {\rm IHW}_k(\widehat{\mathfrak{g}})} \widehat{\mathcal{V}}_{(\lambda,k)} \otimes \overline{\widehat{\mathcal{V}}}_{(\lambda,k)}$ of tensor products of the irreducible chiral modules $\widehat{\mathcal{V}}_{(\lambda,k)}$ of $\widehat{\mathfrak{g}}_k$ with their complex conjugates, labelled by the integrable highest weights of the affine Kač-Moody algebra $\widehat{\mathfrak{g}} \equiv \widehat{\mathfrak{g}}_k$ at level k, with $K \equiv k\,{\rm id}_{\mathcal{H}_k}$. The weights of interest are those associated with the irreducible highest-weight representations of the horizontal algebra $\mathfrak{g} \subset \widehat{\mathfrak{g}}_k$ of the highest weight λ subject to the integrability constraint $-\kappa_{\mathfrak{g}}(\theta,\lambda) \le k^2$ in which θ is the highest root of $\mathfrak{g}$. The chiral current symmetries extend the respective chiral conformal symmetries of the WZW σ-model realised by two copies of the Virasoro algebra Vir – a central extension of the Witt algebra Witt of vector fields on the unit circle, $\mathbf{0} \longrightarrow \mathbb{R} \xrightarrow{v} {\rm Vir} \longrightarrow {\rm Witt} \longrightarrow \mathbf{0}$, whose non-central generators can be identified with the harmonic modes of the chiral components of the energy-momentum tensor constructed from the loop-symmetry currents à la Sugawara: $T_{\pm\pm} = \frac{1}{2k}\,\kappa_{\mathfrak{g}}(J_\pm, J_\pm)$, and whose central generator $C \equiv v(1)$ acts as $c\,{\rm id}_{\mathcal{H}_k}$ on the Hilbert space, returning the **central charge** $c = \frac{k \dim {\rm G}}{k+g^\vee(\mathfrak{g})}$, where $g^\vee(\mathfrak{g})$ is the dual Coxeter number of $\mathfrak{g}$.

Both one-sided symmetries exhibit a small gauge anomaly, $\alpha_{\rm H}^{(1)}(t_A, t_B) = (0, \frac{k}{4\pi}\,\epsilon_{\rm H}\,\delta_{AB})$, and so cannot be gauged. The data of the adjoint action, on the other hand, are non-anomalous. Accordingly, the adjoint action of G on itself, and so also of any subgroup ${\rm H} \subset {\rm G}$, is amenable to gauging under circumstances constrained by the large gauge anomaly. The problem of existence of an Ad(H)-equivariant structure on $\mathcal{G}_k^{(1)}$ was conveniently reformulated in Ref. [87, Sec. 4.2] (and subsequently extended to the WZW σ-model with maximally symmetric defects in Ref. [89, Sec.5]) and solutions, *i.e.*, 1-gerbes $\mathcal{G}_k^{(1)}$ with k for which there exists an Ad(H)-equivariant structure, were found, for a large class of cases, in Ref. [41].

The WZW σ-model with a non-simply connected target Lie group G can be obtained through (discrete) gauging of a suitable subgroup Z of the centre of the simply connected Lie group $\widetilde{\rm G}$ such that $\widetilde{\rm G}/Z \cong {\rm G}$, *cp* Refs. [79, 80, 85, 88, 87], and the gauging procedure can be further generalised, in a manner worked out in Ref. [88] on the basis of the pioneering Ref. [152], to discrete groups Γ of automorphisms of the metric target space mapping epimorphically onto $\mathbb{Z}/2\mathbb{Z}$ as $\epsilon : \Gamma \longrightarrow \mathbb{Z}/2\mathbb{Z}$ so that the generator $[\gamma]$ of the quotient group $\Gamma/{\rm Ker}\,\epsilon \cong \mathbb{Z}/2\mathbb{Z}$ becomes intertwined with the orientation-changing involution $\sigma : \widehat{\Sigma} \circlearrowleft$ on the oriented double $\widehat{\Sigma}$ of an unoriented worldsheet

$\Sigma \equiv \widehat{\Sigma}/\{\mathrm{id}_{\widehat{\Sigma}}, \sigma\}$ by the embedding $\widehat{x} : \widehat{\Sigma} \longrightarrow M/\mathrm{Ker}\,\epsilon$ as $\widehat{x} \circ \sigma = [\gamma] \circ \widehat{x}$, the embedding itself being understood as the lagrangean field of the (orbifold) σ-model on the unoriented worldsheet Σ. Results of such a generalisation in the Lie-group setting of immediate interest, known as orientifold WZW σ-models, were analysed at length in Refs. [85, 88]. Put together with the results on the gauging of continuous group actions mentioned previously, they provide us with a complete descriptive and constructive gerbe-theoretic framework for the σ-model with Lie groups and their homogeneous spaces as target spaces reaching well beyond the classical régime.

3. ELEMENTS OF SUPER-CARTAN GEOMETRY

Successes of the gerbe-theoretic approach to the two-dimensional bosonic σ-model bear witness to the naturality, versatility and efficacy of higher-geometric, -cohomological and -categorial constructions and methods in the setting of the lagrangean field theory in the presence of a topological charge. As such, they provide motivation and guidance for a systematic effort to extend the scope of applicability of the paradigm of geometrisation of gauge fields coupling to dynamical charge distributions that ensues from that approach. An obvious and important direction of such an extension is the lagrangean field theory of extended uniform distributions of bosonic and fermionic charges in equibalance propagating in an external spacetime that goes under the name of the super-σ-model for the super-p-brane. In spite of a by now rather well-substantiated doubt as to the prevalence in nature of a strict pairing of bosonic and fermionic degrees of freedom in the hitherto-conceived form, originally contemplated in Ref. [124] and later rediscovered in Refs. [81, 77, 173, 174, 10, 187, 188], there are sound reasons to pursue this direction. Indeed, physically, it is the supersymmetric string and brane theories with target spaces modelled on a class of homogeneous spaces of Lie supergroups and the associated effective field theories that give us – *via* the conjectural AdS/CFT 'correspondence' of Refs. [120, 74, 185, 119] (*cp* also Ref. [6] for an early account of more realistic applications) – invaluable insights into the quantum dynamics of QCD-type systems outside the perturbative régime such as, *e.g.*, the strongly coupled quark–gluon plasma, and so we may hope that understanding the higher geometry behind the relevant super-σ-models helps to elucidate the deeper nature of the 'correspondence' and to put it on a mathematically firm ground which it still lacks to date. From the mathematical point of view, extension of gerbe theory to the realm of supergeometry, the latter having been introduced in the pioneering papers [17, 65, 11, 144, 175], and in particular to a (super-)Kleinian variant thereof (*cp* the original Refs. [108, 109] and the more recent

Refs. [54, 28]), with its intricacies stemming from the inherent non-compactness of the supersymmetry group, seems to be a natural step in the current robust development of this domain of research. With all that in mind, we lay out, below, the (super)geometric scenography in which our subsequent physical considerations shall be staged.

The point of departure of our supergeometric discussion is the notion of a **supermanifold**, introduced in Ref. [17, Def. 1], *i.e.*, a locally ringed space $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_\mathcal{M})$ with the underlying topological space $|\mathcal{M}|$ (the **body**) and the **structure sheaf** $\mathcal{O}_\mathcal{M}$ of supercommutative $\mathbb{Z}/2\mathbb{Z}$-graded algebras over it. Differential calculus on supermanifolds is rendered managable by the **functor-of-points** approach (*cp* Ref. [28, Thm. 4.1.11]) based on the **Yoneda embedding** $\mathrm{Yon}. : \mathbf{sMan} \hookrightarrow \mathbf{Presh(sMan)}$ of the category **sMan** of supermanifolds in the category of presheaves on **sMan**. It gives rise to sets $\mathrm{Yon}_\mathcal{M}(\mathcal{S}) \equiv \mathrm{Hom}_{\mathbf{sMan}}(\mathcal{S}, \mathcal{M})$ of the so-called **$\mathcal{S}$-points** of $\mathcal{M}$, each containing in particular the **topological points** $\mathrm{Yon}_\mathcal{M}(\mathbb{R}^{0|0}) = \{\,\widehat{x} : \mathbb{R}^{0|0} \longrightarrow \mathcal{M} \mid x \in |\mathcal{M}|\,\}$, and provides us with local coordinate presentations of supermanifold morphisms that imitate similar presentations of smooth maps between manifolds, *cp* Ref. [165, Sec. 2] for a concise overview. Moreover, they yield natural local bases of the **tangent sheaf** $\mathcal{T}\mathcal{M} \equiv \mathrm{sDer}\,\mathcal{O}_\mathcal{M}$ of superderivations of $\mathcal{O}_\mathcal{M}$ (spanned on coordinate derivations) and of the dual **cotangent sheaf** $\mathcal{T}^*\mathcal{M} \equiv \mathrm{Hom}_{\mathcal{O}_\mathcal{M}-\mathrm{Mod}}(\mathcal{T}\mathcal{M}, \mathcal{O}_\mathcal{M})$ (spanned on coordinate differentials). This is the so-called **$\mathcal{S}$-point picture** of $\mathcal{M}$.

A central position in physics-oriented supergeometric considerations is invariably occupied by supermanifolds with a compatible algebraic structure – these are the supersymmetry 'groups' of field theories with bosonic and fermionic degrees of freedom in equibalance neatly encapsulated in the unifying construction of (inner Hom-)functorial sections of superbundles over the (super)spacetime $\mathcal{M}$ of the field theory that admit an 'action' of the supersymmetry 'group'. The relevant supermanifolds are termed **Lie supergroups** and are most concisely defined – after Ref. [17, Def. 6] – as group objects in **sMan**, composing the category **sLieGrp** of Lie supergroups together with the corresponding homomorphisms. It is the alternative definition of a Lie supergroup, due to Kostant [108, Sec. 3.4] (*cp* also Ref. [109, Secs. 1 & 2] and, in particular, Ref. [28, Defs. 7.4.1 & 7.4.2] for a compact formulation), in terms of a **super-Harish-Chandra pair** $(|\mathrm{G}|, \mathfrak{g})$ consisting of a Lie group $|\mathrm{G}|$ with the tangent Lie algebra $|\mathfrak{g}|$ and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)}$ with $\mathfrak{g}^{(0)} \equiv |\mathfrak{g}|$ on which $|\mathrm{G}|$ is realised by Lie-superalgebra homomorphisms $\rho. : |\mathrm{G}| \longrightarrow \mathrm{Aut}_{\mathbf{sLieAlg}}(\mathfrak{g})$ extending its adjoint action on $|\mathfrak{g}|$, that finds an application in superfield theory. Super-Harish-Chandra pairs (to be abbreviated as sHCp's in what follows) together with the associated sHCp morphisms form the category **sHCp** of super-Harish-Chandra pairs, and we have the fundamental

THEOREM 3.1 ([108, Thm. 3.7 & Prop. 3.5.1]). **sHCp** $\cong$ **sLieGrp**.

Remark 3.2. The equivalence exploits a natural structure of a Hopf superalgebra on the enveloping algebra $\mathrm{U}(\mathfrak{g})$ of the Lie superalgebra $\mathfrak{g}$, *cp* Refs. [108, 143], that leads to the definition of the structure sheaf for $(|\mathrm{G}|, \mathfrak{g})$,

$$\mathcal{O}_{\mathrm{G}} := \mathrm{Hom}_{\mathrm{U}(\mathfrak{g}^{(0)})-\mathrm{Mod}}\big(\mathrm{U}(\mathfrak{g}), C^\infty(\cdot,\mathbb{R})\big) \;:\; \mathscr{T}(|\mathrm{G}|) \longrightarrow \mathbf{sAlg}_{\mathbf{scomm}}\,,$$

with a natural interpretation implied by $\mathcal{O}_{\mathrm{G}} \cong C^\infty(\cdot,\mathbb{R})\otimes\bigwedge^\bullet\mathfrak{g}^{(1)\,*}$ (*cp* Ref. [109]).

We illustrate the abstract definitions with two examples of physical relevance.

Example 3.3. **The super-Poincaré group** $\mathrm{sISO}(d,1|D_{d,1})$**.** Consider the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{d,1})$ of the Minkowski space $\mathbb{R}^{d,1} \equiv (\mathbb{R}^{d+1}, \eta \equiv -\mathsf{d}x^0\otimes \mathsf{d}x^0 + \sum_{i=1}^d \mathsf{d}x^i \otimes \mathsf{d}x^i)$ with generators $\{\Gamma_a \equiv \eta_{ab}\,\Gamma^b\}_{a\in\overline{0,d}}$ and its Majorana spinor module $S_{d,1}$ of dimension $D_{d,1}$ endowed with the spinor metric (or charge-conjugation matrix) $C = (C_{\alpha\beta})_{\alpha,\beta\in\overline{1,D_{d,1}}}$, with the symmetry property $(C\,\Gamma_a)^{\mathrm{T}} = C\,\Gamma_a \equiv \overline{\Gamma}{}^a$. The relevant super-Harish-Chandra pair

$$(\mathrm{ISO}(d,1), \mathfrak{siso}(d,1|D_{d,1})) \equiv \mathrm{sISO}(d,1|D_{d,1})$$

consists of the Poincaré group $\mathrm{ISO}(d,1) \equiv \mathbb{R}^{d+1} \rtimes \mathrm{SO}(d,1)$ and of the Lie superalgebra

$$\mathfrak{siso}(d,1|D_{d,1}) = \bigoplus_{\alpha\in\overline{1,D_{d,1}}} \langle Q_\alpha\rangle \oplus \bigoplus_{a\in\overline{0,d}} \langle P_a\rangle \oplus \bigoplus_{a<b\in\overline{0,d}} \langle J_{ab} = -J_{ba}\rangle$$

with Graßmann parities $|Q_\alpha| = 1$, $|P_a| = 0 = |J_{ab}|$ and the structure relations

$$[Q_\alpha, Q_\beta\} = \overline{\Gamma}{}^a_{\alpha\beta}\,P_a\,, \qquad\qquad [P_a, P_b\} = 0 = [Q_\alpha, P_a\}\,,$$

$$[J_{ab}, J_{cd}\} = \eta_{ad}\,J_{bc} - \eta_{ac}\,J_{bd} + \eta_{bc}\,J_{ad} - \eta_{bd}\,J_{ac}\,,$$

$$[J_{ab}, Q_\alpha\} = \tfrac{1}{2}\,\big(\Gamma_{ab}\big)^\beta_{\ \alpha}\,Q_\beta\,, \qquad\qquad [J_{ab}, P_c\} = \eta_{bc}\,P_a - \eta_{ac}\,P_b\,.$$

Upon restriction to $\bigoplus_{\alpha=1}^{D_{d,1}}\langle Q_\alpha\rangle$, the realisation $\rho.$ is given by the spinor representation of $\mathrm{SO}(d,1)$ generated by products $\Gamma_{ab} \equiv \Gamma_{[a}\,\Gamma_{b]}$ of the Dirac matrices.

Example 3.4. **The Lie supergroup** $\mathrm{SU}(2,2|4)$**.** Consider a realisation of the Clifford algebra $\mathrm{Cliff}(\mathbb{R}^{9,1})$ with generators $\{\gamma_{a'} \equiv \Gamma_{a'}\otimes\mathbf{1}_4\otimes\sigma_1, \gamma_{a''} \equiv \mathbf{1}_4\otimes\Gamma_{b''}\otimes\sigma_2\}_{(a',b'')\in\overline{0,4}\times\overline{5,9}}$ (as above) expressed in terms of generators $\{\Gamma_{a'}\}_{a'\in\overline{0,4}}$ of $\mathrm{Cliff}(\mathbb{R}^{4,1})$ in the 4-dimensional spinor representation, generators $\{\Gamma_{a''}\}_{a''\in\overline{5,9}}$ of $\mathrm{Cliff}(\mathbb{R}^{5,0})$ (for the euclidean space $\mathbb{R}^{5,0} \equiv (\mathbb{R}^5, \delta = (\delta_{a''b''})_{a'',b''\in\overline{5,9}}))$ also in the 4-dimensional spinor representation, and the standard Pauli matrices $\sigma_i,\ i \in \{1,2,3\}$, and introduce the spinor metric $\underline{C}$ and the chirality operator $\underline{\gamma}_{11} \equiv -\mathbf{1}_{16} \otimes \sigma_3$ for $\mathrm{Cliff}(\mathbb{R}^{9,1})$ as above. The relevant super-Harish-Chandra pair $(\mathrm{SO}(4,2)\times\mathrm{SO}(6),\ \mathfrak{su}(2,2|4)) \equiv \mathrm{SU}(2,2|4)$ consists of the product Lie group $\mathrm{SO}(4,2)\times\mathrm{SO}(6)$ and of the Lie superalgebra

$$\begin{aligned}\mathfrak{su}(2,2|4) \;=\; & \bigoplus_{(\alpha',\alpha'',I)\in\overline{1,4}\times\overline{1,4}\times\{1,2\}} \langle Q_{\alpha'\alpha''I}\rangle \oplus \bigoplus_{a'\in\overline{0,4}} \langle P_{a'}\rangle \oplus \bigoplus_{a''\in\overline{5,9}} \langle P_{a''}\rangle \\ & \oplus \bigoplus_{a'<b'\in\overline{0,4}} \langle J_{a'b'} = -J_{b'a'}\rangle \oplus \bigoplus_{a''<b''\in\overline{5,9}} \langle J_{a''b''} = -J_{b''a''}\rangle\end{aligned}$$

with $|Q_\alpha| = 1,\ |P_a| = 0 = |J_{ab}|$ and the structure relations (for $a,b,c,d \in \overline{0,9}$)

$$\begin{aligned}[Q_{\alpha'\alpha''I}, Q_{\beta'\beta''J}\} = 2\mathrm{i}\,\Big(& \big(\underline{C}\,\underline{\gamma}^{a'}\underline{\gamma}_{11}\big)_{\alpha'\alpha''I\beta'\beta''J}\,P_{a'} - \big(\underline{C}\,\underline{\gamma}^{a''}\big)_{\alpha'\alpha''I\beta'\beta''J}\,P_{a''}\Big) \\ & + \big(\underline{C}\,\underline{\gamma}^{a'b'}\big)_{\alpha'\alpha''I\beta'\beta''J}\,J_{a'b'} - \big(\underline{C}\,\underline{\gamma}^{a''b''}\big)_{\alpha'\alpha''I\beta'\beta''J}\,J_{a''b''}\,,\end{aligned}$$

$$[P_{a'}, P_{b'}\} = J_{a'b'}\,, \qquad [P_{a''}, P_{b''}\} = -J_{a''b''}\,, \qquad [P_{a'}, P_{b''}\} = 0\,,$$

$$[J_{ab}, J_{cd}\} = \eta_{ad}\,J_{bc} - \eta_{ac}\,J_{bd} + \eta_{bc}\,J_{ad} - \eta_{bd}\,J_{ac}\,, \qquad [P_a, J_{bc}\} = \eta_{ab}\,P_c - \eta_{ac}\,P_b\,,$$

$$[P_{a'}, Q_{\alpha'\alpha''I}\} = -\tfrac{\mathrm{i}}{2}\big(\underline{\gamma}_{a'}\underline{\gamma}_{11}\big)^{\beta'\beta''J}_{\ \ \alpha'\alpha''I}\,Q_{\beta'\beta''J}\,,\ [P_{a''}, Q_{\alpha'\alpha''I}\} = \tfrac{\mathrm{i}}{2}\big(\underline{\gamma}_{a''}\big)^{\beta'\beta''J}_{\ \ \alpha'\alpha''I}\,Q_{\beta'\beta''J}\,,$$

$$[J_{ab}, Q_{\alpha'\alpha''I}\} = \tfrac{1}{2}\big(\underline{\gamma}_{ab}\big)^{\beta'\beta''J}_{\ \ \alpha'\alpha''I}\,Q_{\beta'\beta''J}\,.$$

The last two lines encode the realisation ρ..

The next building block of the physical analysis to come is the supergeometric model of the superfield bundle on which a supersymmetry 'group' is assumed to 'act'. This is an object from the **category of G-manifolds**, as defined in Ref. [108, Sec. 3.4], *i.e.*, a supermanifold $\mathcal{M}$ that comes with a supermanifold morphism $\lambda : \mathrm{G}\times\mathcal{M} \longrightarrow \mathcal{M}$, termed the **left action** of G, which satisfies the usual identities $\lambda\circ(\mu\times\mathrm{id}_{\mathcal{M}}) = \lambda\circ(\mathrm{id}_{\mathrm{G}}\times\lambda)$ and $\lambda\circ(\widehat{e}\times\mathrm{id}_{\mathcal{M}}) = \mathrm{id}_{\mathcal{M}}$ in which μ and $\widehat{e}$ are the binary operation of and the topological unit in G, respectively (a right action ϱ is defined analogously). As in the purely even setting, the action distinguishes sections $\mathcal{K}^{\lambda}_{\mathrm{ev}_e\circ\cdot} : \mathrm{sLie}(\mathrm{G}) \longrightarrow \mathcal{T}\mathcal{M} : L \longmapsto -(L_e\otimes\mathrm{id}_{\mathcal{O}_{\mathcal{M}}})\circ\lambda^*,\ L_e \equiv \mathrm{ev}_e\circ L$ of the tangent sheaf of a G-supermanifold, written for the **evaluation mapping** $\mathrm{ev}_e \equiv \widehat{e}^* : \mathcal{O}_{\mathrm{G}} \longrightarrow \mathcal{O}_{\mathbb{R}^{0|0}} \equiv \mathbb{R}$ and known as the **fundamental vector fields** for λ (those for ϱ are defined analogously). The fundamental vector fields on G induced by the the left regular action $\ell \equiv \mu$ and the right regular one $\wp \equiv \mu$ (with the understanding that the 'acting' group is the second/right cartesian factor) are the RI and LI vector fields, respectively. Note also that the left action of G on itself engenders a left action of the body Lie group $|\mathrm{G}| \ni g$ on G by supermanifold morphisms $l_g := \ell\circ(\widehat{g}\times\mathrm{id}_{\mathrm{G}}) : \mathbb{R}^{0|0}\times\mathrm{G} \equiv \mathrm{G} \longrightarrow \mathrm{G}$, and similarly for the right action, $r_g := \wp\circ\big(\mathrm{id}_{\mathrm{G}}\times\widehat{g}\big) : \mathrm{G}\times\mathbb{R}^{0|0} \equiv \mathrm{G} \longrightarrow \mathrm{G}$. It is straightforward to transcribe the above definitions and statements into the sHCp formalism, *cp* Ref. [28, Sec. 8.3].

Finally, we may pass to the G-supermanifolds of immediate interest to us. These are homogeneous spaces of Lie supergroups, as described in

Definition 3.5. Adopt the hitherto notation. Let G be a Lie supergroup and K its closed sub-supergroup with the Lie superalgebra $\mathrm{sLie}(\mathrm{K}) \subset \mathrm{sLie}(\mathrm{G})$. The **homogeneous space of** G **relative to** K is the supermanifold $\mathrm{G}/\mathrm{K} \equiv (|\mathrm{G}|/|\mathrm{K}|, \mathcal{O}_{\mathrm{G}/\mathrm{K}})$ with – for $\pi_{|\mathrm{G}|/|\mathrm{K}|} : |\mathrm{G}| \longrightarrow |\mathrm{G}|/|\mathrm{K}|$ –

$$\begin{aligned}\mathcal{O}_{\mathrm{G}/\mathrm{K}} &: \mathscr{T}\big(|\mathrm{G}|/|\mathrm{K}|\big) \longrightarrow \mathbf{sAlg_{scomm}} \\ &: |\mathcal{V}| \longmapsto \big\{ f \in \mathcal{O}_{\mathrm{G}}\big(\pi_{|\mathrm{G}|/|\mathrm{K}|}^{-1}(|\mathcal{V}|)\big) \mid \forall_{(L,k)\in\mathrm{sLie}(\mathrm{K})\times|\mathrm{K}|} : L(f) = 0 \ \wedge \ r_k^*(f) = f \big\}.\end{aligned}$$

$\diamond$

The meaningfulness of the above definition follows from [54, Thm. 3.3]. Uniqueness of the supermanifold structure given in it is quantified in

THEOREM 3.6 ([108, Prop. 3.10.1], [28, Thm. 9.3.7]). *The homogeneous space of* G *relative to* K *described in Def. 3.5 is a unique, up to a unique supermanifold isomorphism, structure of a supermanifold over the body* $|\mathrm{G}|/|\mathrm{K}|$ *for which there exists a submersive extension of the smooth quotient map* $\pi_{|\mathrm{G}|/|\mathrm{K}|} : |\mathrm{G}| \longrightarrow |\mathrm{G}|/|\mathrm{K}|$ *to a supermanifold morphism* $\pi_{\mathrm{G}/\mathrm{K}} : \mathrm{G} \longrightarrow \mathrm{G}/\mathrm{K}$ *(such that* $|\pi_{\mathrm{G}/\mathrm{K}}| = \pi_{|\mathrm{G}|/|\mathrm{K}|}$*) and a left* G*-action* $[\ell]^{\mathrm{K}} : \mathrm{G} \times \mathrm{G}/\mathrm{K} \longrightarrow \mathrm{G}/\mathrm{K}$ *on that supermanifold induced from the left regular action* ℓ *of* G *on itself along the submersion,*

$$\pi_{\mathrm{G}/\mathrm{K}} \circ \ell = [\ell]^{\mathrm{K}} \circ \big(\mathrm{id}_{\mathrm{G}} \times \pi_{\mathrm{G}/\mathrm{K}}\big)\,. \tag{3.1}$$

The induced action descends to an action of $|\mathrm{G}|$ *on* G/K *by supermanifold morphisms* $[l]_g^{\mathrm{K}} := [\ell]^{\mathrm{K}} \circ \big(\widehat{g} \times \mathrm{id}_{\mathrm{G}/\mathrm{K}}\big) : \mathbb{R}^{0|0} \times \mathrm{G}/\mathrm{K} \equiv \mathrm{G}/\mathrm{K} \longrightarrow \mathrm{G}/\mathrm{K}$, $g \in |\mathrm{G}|$.

The last theorem, in conjunction with the Local Frobenius Theorem for supermanifolds [28, Thms. 6.1.12], paves the way to the identification of the structure of a principal K-(super)bundle in the surjective submersion $\pi_{\mathrm{G}/\mathrm{K}} : \mathrm{G} \longrightarrow \mathrm{G}/\mathrm{K}$, in full analogy with its purely even counterpart known to exist over the body. Indeed, the theorems ensure existence of an open superdomain $\mathcal{U}_0^{\mathrm{K}}$ in G/K with the body $|\mathcal{U}_0^{\mathrm{K}}|$ given by a neighbourhood of $e\,|\mathrm{K}| \subset |\mathrm{G}|/|\mathrm{K}|$ and, on it, of a local (super)trivialisation $\tau_0^{\mathrm{K}} : \pi_{\mathrm{G}/\mathrm{K}}^{-1}(\mathcal{U}_0^{\mathrm{K}}) \xrightarrow{\cong} \mathcal{U}_0^{\mathrm{K}} \times \mathrm{K}$, and so also of a local (super)section $\sigma_0^{\mathrm{K}} \equiv \tau_0^{\mathrm{K}\,-1} \circ (\mathrm{id}_{\mathcal{U}_0^{\mathrm{K}}} \times \widehat{e}) : \mathcal{U}_0^{\mathrm{K}} \times \mathbb{R}^{0|0} \equiv \mathcal{U}_0^{\mathrm{K}} \longrightarrow \mathrm{G}$ of the principal K-(super)bundle

$$\mathrm{K} \longrightarrow \mathrm{G} \xrightarrow{\pi_{\mathrm{G}/\mathrm{K}}} \mathrm{G}/\mathrm{K}\,. \tag{3.2}$$

The above local structure may subsequently be propagated all over the body $|\mathrm{G}|/|\mathrm{K}|$ of the base of the (super)bundle with the help of the morphisms l_{g_i} and $[l]^{\mathrm{K}}_{g_i}$ defined for a family $\{g_i\}_{i\in I_{\mathrm{K}}} \ni e \equiv g_0$ of pairwise distinct topological points $g_i \in |\mathrm{G}|$ chosen such that the family $\{|[l]^{\mathrm{K}}_{g_i}|(|\mathcal{U}^{\mathrm{K}}_0|) \equiv |\mathcal{U}^{\mathrm{K}}_i|\}_{i\in I_{\mathrm{K}}}$ is an open cover of $|\mathrm{G}|/|\mathrm{K}|$. Indeed, the corresponding 'cover' of G/K composed of the superdomains $(|\mathcal{U}^{\mathrm{K}}_i|, \mathcal{O}_{\mathrm{G}/\mathrm{K}}\restriction_{|\mathcal{U}^{\mathrm{K}}_i|}) \equiv \mathcal{U}^{\mathrm{K}}_i$ supports the family of local sections (*cp* Ref. [165, Sec. 2])

$$\sigma^{\mathrm{K}}_i \equiv l_{g_i} \circ \sigma^{\mathrm{K}}_0 \circ ([l]^{\mathrm{K}}_{g_i})^{-1} \; : \; \mathcal{U}^{\mathrm{K}}_i \longrightarrow \mathrm{G}\,, \qquad i \in I_{\mathrm{K}}\,.$$

The existence of local sections, and hence of the (super)bundle structure on (3.2), was indicated already in Ref. [108], but the punchline got lost in the somewhat unwieldy sheaf-theoretic and superalgebraic formalism adopted by the author. A very lucid and detailed treatment was given only in Ref. [54] (*cp* also Ref. [28]), and the construction was completed in Ref. [165], where, moreover, the choice of σ^{K}_0 was fixed in a physically motivated manner, to be recapitulated in the next section.

A direct consequence of existence of the local sections σ^{K}_i, to be exploited in field-theoretic model-building, is the possibility to explicitly model the tensor calculus on G/K in terms of the (super-)Cartan differential calculus on G. The idea boils down to pulling back, along the σ^{K}_i, those covariant tensors $\mathrm{T} \in \Gamma(\mathcal{T}^*\mathrm{G}^{\otimes N})$, $N \in \mathbb{N}^\times$ on G which are **right-K-basic**, *i.e.*, right-K-invariant and sLie(K)-horizontal. The former property is an adaptation of the standard notion expressed by the conditions (*cp* Def. 3.5): $\mathscr{L}_L\mathrm{T} = 0$ and $r^*_k\mathrm{T} = \mathrm{T}$, written for arbitrary $(L,k) \in \mathrm{sLie}(\mathrm{K}) \times |\mathrm{K}|$, whereas the latter one is defined by the implication

$$\forall_{\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_N \in \Gamma(\mathcal{T}\mathrm{G})} \; : \; \big(\; \exists_{k\in\overline{1,N}} \; : \; \mathcal{V}_k \in \mathrm{sLie}(\mathrm{K}) \quad \Longrightarrow \quad \mathrm{T}(\mathcal{V}_1,\mathcal{V}_2,\ldots,\mathcal{V}_N) = 0 \; \big)\,.$$

The pullbacks are independent of the arbitrary choice of the local sections. We may be much more specific upon making further assumptions with regard to the pair (G,K) that are satisfied in the physical setting of interest. Thus, we presuppose $\mathrm{K} \subset |\mathrm{G}|$ to be a standard Lie group of $\dim \mathrm{K} \equiv K$ and the direct-sum complement $\mathcal{L}$ of $\mathrm{sLie}(\mathrm{K}) \equiv \mathrm{Lie}(\mathrm{K})$ in $\mathrm{sLie}(\mathrm{G})$, spanned on the LI vector fields $L_\zeta \equiv (\mathrm{id}_{\mathcal{O}_{\mathrm{G}}} \otimes T_\zeta) \circ \mu^*$, $\zeta \in \overline{0, D-K}$, $D \equiv \dim \mathrm{sLie}(\mathrm{G}) - 1$ engendered by a basis $\{T_\zeta\}_{\zeta\in\overline{0,D-K}} \subset \mathrm{ev}_e \circ \mathrm{sLie}(\mathrm{G}) \equiv \mathcal{T}_e\mathrm{G}$ of some fixed direct-sum (supervector-space) completion of $\mathrm{ev}_e \circ \mathrm{Lie}(\mathrm{K}) \equiv \bigoplus_{Z\in\overline{1,K}} \langle J_Z \rangle$ in $\mathcal{T}_e\mathrm{G}$, to furnish an $\mathrm{ad}_{\mathrm{Lie}(\mathrm{K})}$-module, *i.e.*, $[\mathrm{Lie}(\mathrm{K}), \mathcal{L}\} \subset \mathcal{L}$. We then speak of a **reductive decomposition** $\mathrm{sLie}(\mathrm{G}) = \mathcal{L} \oplus \mathrm{Lie}(\mathrm{K})$. Consider, next, the space of LI 1-forms on G, *i.e.*, those $\omega \in \Gamma(\mathcal{T}^*\mathrm{G}) \equiv \Omega^1(\mathrm{G})$ that satisfy, for every RI vector field R and $g \in |\mathrm{G}|$, the conditions: $\mathscr{L}_R\omega = 0$ and $l^*_g\omega = \omega$. The basis $\{L_A \equiv (\mathrm{id}_{\mathcal{O}_{\mathrm{G}}} \otimes t_A) \circ \mu^*\}_{A\in\overline{0,D}}$ of $\mathrm{sLie}(\mathrm{G})$ induced by the adapted basis $\{t_A\}_{A\in\overline{0,D}} \equiv \{T_\zeta\}_{\zeta\in\overline{0,D-K}} \cup \{J_Z\}_{Z\in\overline{1,K}}$

of $\mathcal{T}_e\mathrm{G}$ distinguishes the dual basis $\{\theta_{\mathrm{L}}^A\}^{A\in\overline{0,D}}$ in the space of LI 1-forms, and we define the Maurer–Cartan super-form $\theta_{\mathrm{L}} = \theta_{\mathrm{L}}^A \otimes t_A \in \Omega^1(\mathrm{G}) \otimes \mathcal{T}_e\mathrm{G}$. In the presence of a reductive decomposition, the super-1-form splits into parts: $\theta_{\mathrm{L}}^\zeta \otimes T_\zeta$ and $\theta_{\mathrm{L}}^Z \otimes J_Z \equiv \Theta_{\mathrm{K}}$ with a qualitatively different behaviour under the *right* action of K. Indeed, while Θ_{K} behaves like (and so can be chosen as) a principal K-connection super-1-form on (3.2), the Lie(K)-horizontal $\theta_{\mathrm{L}}^\zeta$ transform as (ρ_{K}-)tensors. Accordingly, linear combinations $\mathrm{T} = \mathrm{T}_{\zeta_1\zeta_2\ldots\zeta_N}\,\theta_{\mathrm{L}}^{\zeta_1} \otimes \theta_{\mathrm{L}}^{\zeta_2} \otimes \cdots \otimes \theta_{\mathrm{L}}^{\zeta_N}$ of (super)tensor products of these super-1-forms with constant ρ_{K}-invariant tensors as coefficients are K-basic and descend to G/K. Particular such tensors shall be used to define the supersymmetric field theories of interest in the next section. We conclude the present section with examples of homogeneous spaces of the previously introduced Lie supergroups and physically relevant tensors on them.

Example 3.7. **The super-Minkowski space.** Adopt the notation of Example 3.3 and assume $C^{\mathrm{T}} = -C$. The relevant homogeneous space

$$\mathrm{sMink}(d,1\,|\,D_{d,1}) \equiv \mathrm{sISO}(d,1\,|\,D_{d,1})/\mathrm{SO}(d,1)$$

has the body $|\mathrm{sMink}(d,1\,|D_{d,1})| = \mathbb{R}^{\times d+1} \equiv \mathrm{Mink}(d,1)$ and supports the $\mathrm{SO}(d,1)$-basic tensors: $\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b$ and $\chi_{p+2}^{\mathrm{GS}} = \theta_{\mathrm{L}}^\alpha \wedge (\overline{\Gamma}_{a_1a_2\ldots a_p})_{\alpha\beta}\,\theta_{\mathrm{L}}^\beta \wedge \theta_{\mathrm{L}}^{a_1} \wedge \theta_{\mathrm{L}}^{a_1} \wedge \cdots \wedge \theta_{\mathrm{L}}^{a_p}$ (we impose $\overline{\Gamma}^{\mathrm{T}}_{a_1a_2\ldots a_p} = \overline{\Gamma}_{a_1a_2\ldots a_p} \equiv C\,\Gamma_{[a_1}\,\Gamma_{a_2}\cdots\Gamma_{a_p]}$), and – for $d = 9$ and $D_{d,1} = 32$ – $\chi_2^{\mathrm{GS}} = \theta_{\mathrm{L}}^\alpha \wedge (\overline{\Gamma}_{11})_{\alpha\beta}\,\theta_{\mathrm{L}}^\beta$ with $\overline{\Gamma}_{11} \equiv C\,\Gamma_{11} = -C\,\Gamma_0\,\Gamma_1\cdots\Gamma_9$. The super-$q$-form χ_q^{GS} ($q \geq 3$) is closed if the following Fierz identity obtains:

$$\eta_{ab}\,\overline{\Gamma}^a_{(\alpha\beta}\,\overline{\Gamma}^{ba_1a_2\ldots a_{q-3}}_{\gamma\delta)} = 0\,. \tag{3.3}$$

The homogeneous space is a Lie supergroup with the binary operation (written in the coordinate $\mathcal{S}$-point picture): $\mu((\theta_1^\alpha, x_1^a), (\theta_2^\alpha, x_2^a)) = (\theta_1^\alpha + \theta_2^\alpha, x_1^a + x_2^a - \frac{1}{2}\,\theta_1\,\overline{\Gamma}^a\,\theta_2)$.

Example 3.8. **The super-$\mathrm{AdS}_5\times\mathbb{S}^5$ space.** Adopt the notation of Example 3.4. The relevant homogeneous space $\mathrm{s}(\mathrm{AdS}_5\times\mathbb{S}^5) = \mathrm{SU}(2,2\,|\,4)/(\mathrm{SO}(4,1)\times \mathrm{SO}(5))$ has the body $|\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5)| = \mathrm{SO}(4,2)/\mathrm{SO}(4,1) \times \mathrm{SO}(6)/\mathrm{SO}(5) \equiv \mathrm{AdS}_5 \times \mathbb{S}^5$ and supports the $(\mathrm{SO}(4,1) \times \mathrm{SO}(5))$-basic tensors: $\mathrm{g} = \eta_{ab}\,\theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b$ and $\chi_3^{\mathrm{MT}} = -\mathrm{i}\,\theta_{\mathrm{L}}^{\alpha'\alpha''I} \wedge ((\underline{C}\,\underline{\gamma}_{a'})_{\alpha'\alpha''I\beta'\beta''J}\,\theta_{\mathrm{L}}^{a'} - (\underline{C}\,\underline{\gamma}_{a''}\underline{\gamma}_{11})_{\alpha'\alpha''I\beta'\beta''J}\,\theta_{\mathrm{L}}^{a''}) \wedge \theta_{\mathrm{L}}^{\beta'\beta''J}$.

4. A DUAL PAIR OF PHYSICAL MODELS OF SUPER-EMBEDDINGS

A strict symmetry-mediated pairing between fields obeying the two standard types of quantum statistics: the Bose–Einstein and the Fermi–Dirac has long been considered a conceptually appealing possibility, and the logical capstone of the long success story of unification of theoretical descriptions of various elementary-particle species and fundamental interactions between them, founded on the principle of symmetry. Alas, overwhelming phenomenological evidence to date seems to have rendered the idea of the thus understood **supersymmetry** in Nature inviable, at least in its simplest form. Nevertheless, field theories with manifest supersymmetry, and in particular those related to or inspired by superstring theory, do remain, as argued in the previous section, theoretically interesting and potentially useful as repositories of novel concepts and intuitions as well as powerful computational tools that give hope for a paradigm shift in a predictive modelling of strongly coupled systems with a gauge symmetry. Hence, it is not unreasonable to pursue their mathematical study, also from the lagrangean perspective. Such a study is bound to land us in the category of supermanifolds with actions of Lie supergroups. The former are to be thought of as natural models of field bundles on which the latter act as supersymmetry groups. But then a difficulty arises immediately with an appropriate generalisation and formalisation of the elementary notion of a lagrangean field – indeed, the naïve choice, that is a supermanifold morphism between the spacetime (super)manifold X of the would-be field theory and its non-even field (super)bundle $\mathcal{F} \xrightarrow{\pi_{\mathcal{F}}} X$, fails to probe the non-bosonic degrees of freedom of the field bundle in the realistic setting with a purely Graßmann-even spacetime. The difficulty can be overcome by replacing the naïve definition $\{ \sigma \in \mathrm{Hom}_{\mathbf{sMan}}(X,\mathcal{F}) \mid \pi_{\mathcal{F}} \circ \sigma = \mathrm{id}_X \}$ of the space of lagrangean fields with the functorial one:

Definition 4.1. Let X and $\mathcal{F}$ be supermanifolds. The **mapping supermanifold** for the pair $(X,\mathcal{F})$ is the inner-Hom functor

$$[X,\mathcal{F}] \equiv \underline{\mathrm{Hom}}_{\mathbf{sMan}}(X,\mathcal{F}) := \mathrm{Hom}_{\mathbf{sMan}}(X \times -,\mathcal{F}) \;:\; \mathbf{sMan} \longrightarrow \mathbf{Set}\,.$$

Whenever $\mathcal{F}$ is a fibre bundle over X so that there exists a surjective submersion $\pi_{\mathcal{F}} \in \mathrm{Hom}_{\mathbf{sMan}}(\mathcal{F},X)$, the **supermanifold of global sections of** $\pi_{\mathcal{F}}$ is the subfunctor $\Gamma(\mathcal{F}) \hookrightarrow [X,\mathcal{F}]$ with the object component

$$\Gamma(\mathcal{F}) \;:\; \mathrm{Ob}\,\mathbf{sMan} \longrightarrow \mathrm{Ob}\,\mathbf{Set} \;:\; \mathcal{S} \longmapsto \{ \varphi \in [X,\mathcal{F}](\mathcal{S}) \mid \pi_{\mathcal{F}} \circ \varphi = \mathrm{pr}_1 \}\,.$$

Elements of the set $\Gamma(\mathcal{F})(\mathcal{S})$ are called ($\mathcal{F}$**-valued**) $\mathcal{S}$**-superfields on** X.

◇

The idea of a superfield can be traced back to the early works on supersymmetry by Salam and Strathdee, *cp* Refs. [150, 151], and by Ferrara, Zumino and Wess, *cp* Ref. [62], where the notion was introduced from a pragmatic standpoint. The use of the inner-Hom construction comes from [58] (*cp* also Ref. [39]).

In what may be regarded as a super-variant of Klein's Erlangen approach, supergeometry endowed with an action of a Lie supergroup, central to physical applications of the general theory, naturally distinguishes homogeneous spaces of Lie supergroups. Supersymmetry is realised on these supermanifolds by means of the induced action $[\ell]^{\mathrm{K}}$ of (3.1) and is customarily lifted, in the form of (right-)K-corrected left translations, to the local sections σ_i^{K} that model G/K patchwise in G. This puts us in the physical setting of the so-called non-linear realisations of supersymmetry, originally proposed by Akulov and Volkov *et al.* in Refs. [173, 174, 99, 116, 172, 100, 166, 56, 26] in an adaptation of a general scheme advanced by Schwinger and Weinberg in Refs. [142, 179] in the context of effective field theories with chiral symmetries, and subsequently elaborated in Refs. [36, 29, 148], only to be adapted to the study of spacetime symmetries by Salam, Strathdee and Isham in Refs. [149, 102]. Nonlinear realisations of supersymmetry were extensively employed in the guise encountered below in the field-theoretic circumstances of interest by West *et al.* in Refs. [180, 75] and by McArthur in Refs. [121, 122]. In this very broad field, we shall be interested in theories mimicking structurally the bosonic σ-model of Sec. 2, that is, theories of generalised (charge force-deformed) minimal functorial embeddings of purely even worl*volumes* of uniform extended distributions of mass and super-charge (of a spatial dimension constrained solely by the internal consistency of the underlying superstring theory) in an ambient homogeneous space of a supersymmetry Lie supergroup. Here, the choice of the *even* isotropy group K is determined either by purely geometric considerations (we want the extended object to propagate in a specific body geometry $|\mathrm{G}|/\mathrm{K}$) or by the nature of the classical vacuum of the field theory (the 'embedded' worldvolume determines the scheme of a spontaneous partial supersymmetry breakdown). We shall now introduce the field theories of interest in a dual pair of formulations, paving the way to their convenient geometrisation.

Definition 4.2. Adopt the hitherto notation. Let G be a Lie supergroup and $\mathrm{H} \subset |\mathrm{G}|$ a closed subgroup of its body whose tangent Lie algebra $\mathrm{Lie}(\mathrm{H})$ defines a reductive decomposition $\mathrm{sLie}(\mathrm{G}) = \mathcal{L} \oplus \mathrm{Lie}(\mathrm{H})$ of the tangent Lie superalgebra $\mathrm{sLie}(\mathrm{G})$ of G. Assume given an H-basic LI metric tensor $\mathrm{g} = \mathrm{g}_{ab}\,\theta_{\mathrm{L}}^a \otimes \theta_{\mathrm{L}}^b$ on $\mathcal{L}^{(0)} \equiv \bigoplus_{a=0}^d \langle L_a \rangle \subset \bigoplus_{\underline{A}=0}^{\delta} \langle L_{\underline{A}} \rangle \equiv \mathcal{L}$, $d < \delta \in \mathbb{N}^\times$, and an H-basic LI de Rham super-$(p+2)$-cocycle, to be termed the **Green–Schwarz super-$(p+2)$-cocycle**,

$$\chi_{p+2} = \chi_{\underline{A}_1 \underline{A}_2 \ldots \underline{A}_{p+2}}\,\theta_{\mathrm{L}}^{\underline{A}_1} \wedge \theta_{\mathrm{L}}^{\underline{A}_2} \wedge \cdots \wedge \theta_{\mathrm{L}}^{\underline{A}_{p+2}} \in \Omega^{p+2}(\mathrm{G})$$

with a global H-basic primitive $\beta_{p+1} \in \Omega^{p+1}(\mathrm{G})$, *i.e.*, $\mathsf{d}\beta_{p+1} = \chi_{p+2}$, that we presuppose to be *quasi*-supersymmetric, by which we mean that there exist – for any $g \in |\mathrm{G}|$ and any $X \in \mathcal{T}_e\mathrm{G}$, with the associated $R_X \equiv (X \otimes \mathrm{id}_{\mathcal{O}_{\mathrm{G}}}) \circ \mu^*$ – corresponding p-forms $\alpha_g, \alpha_X \in \Omega^p(\mathrm{G})$ satisfying the relations $l_g^*\beta_{p+1} = \beta_{p+1} + \mathsf{d}\alpha_g$ and $\mathscr{L}_{R_X}\beta_{p+1} = \mathsf{d}\alpha_X$. Consider a $(p+1)$-dimensional closed oriented manifold Ω_p, termed the **worldvolume**, and the associated mapping supermanifold $[\Omega_p, \mathrm{G}/\mathrm{H}]$, as introduced in Def. 4.1 and to be evaluated – after Ref. [58] – on odd hyperplanes $\mathbb{R}^{0|N}$ of an arbitrary superdimension $(0|N)$, $N \in \mathbb{N}^\times$. Fix a trivialising open cover $\{\mathcal{U}_i^{\mathrm{H}}\}_{i\in I} \equiv \mathcal{U}^{\mathrm{H}}$ of G/H by the superdomains $\mathcal{U}_i^{\mathrm{H}}$ introduced previously, coming with the respective local sections $\sigma_i^{\mathrm{H}} : \mathcal{U}_i^{\mathrm{H}} \longrightarrow \mathrm{G}$, and let $\triangle(\Omega_p)$ be an arbitrary tessellation of Ω_p, composed of sets of k-cells $\mathfrak{T}_k$ with $k \in \overline{0, p+1}$, which is **subordinate** to $\mathcal{U}^{\mathrm{H}}$ for a given $\xi \in [\Omega_p, \mathrm{G}/\mathrm{H}](\mathbb{R}^{0|N})$, so that there exists a map $\imath : \triangle(\Omega_p) \longrightarrow I$ with the property $\xi(\widetilde{\tau}) \subset \mathcal{U}_{\imath_\tau}^{\mathrm{H}}$ for $\widetilde{\tau} \subset \Omega_p \times \mathbb{R}^{0|N}$ such that $|\widetilde{\tau}| \equiv \tau$. The **Green–Schwarz super-σ-model in the Nambu–Goto formulation for the super-p-brane** in the **Green–Schwarz superbackground**

$$\mathfrak{sB}_p^{(\mathrm{NG})} \equiv \big(\mathrm{G}/\mathrm{H}, \mathrm{g}, \chi_{p+2}\big)$$

over the **supertarget** G/H is the lagrangean theory of (functorial) mappings from $[\Omega_p, \mathrm{G}/\mathrm{H}]$ determined by the principle of least action applied to the DF amplitude

$$\mathcal{A}_{\mathrm{DF}}^{(\mathrm{NG}),p,\mu_p}[\xi] := \mathrm{e}^{\mathrm{i}\, S_{\mathrm{GS},p}^{(\mathrm{NG}),\mu_p}[\xi]}$$

written in terms of the action functional ($\xi_{\widetilde{\tau}} \equiv \xi \restriction_{\widetilde{\tau}}$ and $\mu_p \in \mathbb{R}^\times$ is a parameter)

$$S_{\mathrm{GS},p}^{(\mathrm{NG}),\mu_p}[\xi] = \sum_{\tau \in \mathfrak{T}_{p+1}} \left(\mu_p \int_\tau \sqrt{\det{}_{(p)}\big(\big(\sigma_{\imath_\tau}^{\mathrm{H}} \circ \xi_\tau\big)^* \mathrm{g}\big)} + \int_\tau \big(\sigma_{\imath_\tau}^{\mathrm{H}} \circ \xi_\tau\big)^* \beta_{p+1} \right).$$

◇

The first models from the above class were proposed in the breakthrough works on the dynamics of the superparticle and of the superstring in the super-Minkowski space by de Azcárraga and Lukierski, *cp* Ref. [37] (building on the earlier attempts by Casalbuoni, *cp* Ref. [27], and Brink and Schwarz, *cp* Ref. [20]) and by Green and Schwarz [82, 83], respectively. Their higher-dimensional analogons for super-p-branes appeared in Ref. [3]. These were followed by models of the superstring in supertargets with the body of the type $\mathrm{AdS}_m \times \mathbb{S}^n$ for distinguished values of $m, n \in \mathbb{N}^\times$, constructed in Refs. [131, 189, 4, 86, 52, 40], and the M-brane models of Refs. [18, 47, 35], as well as the superstring [21] and supermembrane [22] theories in curved supergravity backgrounds, all written for the restricted class of embeddings with $\xi(\Omega_p \times \mathbb{R}^{0|N}) \subset \mathcal{U}_0^{\mathrm{H}}$. The above global formulation was first given in Ref. [162] and employed in Ref. [164]. Rigid supersymmetry is built into it through the assumption of quasi-supersymmetry of the super-$(p+1)$-form β_{p+1}.

It ought to be noted that the definition of the DF amplitude depends neither on the choice of the local sections σ_i^{H} nor on the tessellation, all that owing to the assumed H-basicness of g and β_{p+1}. In fact, we could write it in terms of the respective descendants of the two tensors to G/H, and the only reason why we have not done so is that distinguished sections σ_i^{H} are going to play a rôle in an explicit description of a correspondence between the above Nambu–Goto (NG) formulation and another one that we discuss presently. As for the background itself, the assumption of the triviality of the de Rham class of the Green–Schwarz (GS) super-$(p+2)$-cocycle might appear too stringent, so much so that – indeed – there is no interesting cohomology behind the Wess–Zumino term. However, first of all, essentially all superbackgrounds originally considered in the literature are of this kind (*cp* Ref. [165, Sec. 3] and below), and, secondly, the said cohomology is actually far from trivial, as we are about to demonstrate. Before that, though, let us make another comment that leads us directly to an alternative formulation of the super-σ-model.

A critical (super)field configuration of the purely (super)geometric super-σ-model that minimises $\mathcal{A}_{\mathrm{DF}}^{(\mathrm{NG}),p,\mu_p}$, to be referred to as the **vacuum** in what follows, is a super-embedded (odd-extended) worldvolume. Choosing the vacuum effects a spontaneous partial breakdown of the supersymmetry of the field theory, with the generators of vacuum-preserved supersymmetries contained in the tangent sheaf of the embedded super-p-brane. While the dimension of the body of the latter and so also the number of preserved generators of even translation is naturally fixed by the dimension $p+1$ of the worldvolume (as most straightforwardly seen in the so-called static gauge), there is no built-in constraint in the definition of the super-σ-model that would enforce the accompanying reduction of odd degrees of freedom in the vacuum, necessary for the fundamental balance between them and their even counterparts and transmitted from the even sector to the odd sector through the superalgebra $[\mathrm{sLie}(\mathrm{G})^{(1)}, \mathrm{sLie}(\mathrm{G})^{(1)}\} \subset \mathrm{sLie}(\mathrm{G})^{(0)}$. In the light of our remarks from p. 11, a natural mechanism of reduction is a gauge symmetry, which should, therefore, have a purely odd component in the case in hand. Such a gauge symmetry was identified in the superparticle model by de Azcárraga and Lukierski in Ref. [38], and subsequently rediscovered in the superstring model and elaborated by Siegel in Refs. [146, 147]. Since, as any gauge symmetry, it belongs to the kernel of the presymplectic form of the super-σ-model, and yet does not come from an isometry of the supertarget from the kernel of the GS super-$(p+2)$-cocycle, it perturbs *both* terms in the action functional: the metric term and the WZ term, and requires a precise matching of the respective charges: the mass and the topological charge in front of them. Consequently, it has been used, since its discovery, as an effective model-building tool that fixes the structure (to a large extent) and the normalisation of the WZ term relative to the canonical metric term, *cp* Refs. [131, 134, 189]. Given its function, which is restoration of a supersymmetric balance in the localised vacuum of the super-σ-model through identification of some (typically a half) of the odd degrees of freedom as pure gauge and which, for a long time, has been invoked as its only definition, the odd gauge supersymmetry is not expected to engender, through the supercommutator, an *off-shell*-closed gauge-symmetry superalgebra, and – indeed – it was established fairly generally in Ref. [121] that the superalgebra closes only on-shell, *i.e.*, upon imposition of the field equations of the super-σ-model, and that to a hybrid structure spanned additionally by worldvolume diffeomorphisms and sometimes also transformations from the 'hidden' gauge algebra Lie(H) (some of which do *not* preserve the vacuum). Various attempts at elucidating the deeper geometric nature of the peculiar *infinitesimal* odd gauge symmetry of the super-σ-model have served to link it to *right* translations on the Lie supergroup (explaining its 'wrong' sign – *cp* Refs. [121, 75]) and to realise it as a supertarget manifestation of odd superdiffeomorphisms of a *fixed* odd-hyperplane super-extension of the even worldvolume Ω_p, to be embedded in the supertarget in a rigidly determined manner. The latter construction, postulated by Sorokin *et al.* in Ref. [156] and developed in subsequent studies under the name of the 'superembedding formalism', achieved the nontrivial goal of putting all components of the gauge-supersymmetry algebra on the same footing, however, it did that at the very high cost of losing the functoriality of the super-σ-model. The apparent conflict between a uniform supergeometric treatment of the gauge-supersymmetry superalgebra of the super-σ-model and functoriality of the latter was resolved in Ref. [165] where the former was reinterpreted entirely in terms of the calculus of physically distinguished superdistributions over the supertarget, the analysis developing in parallel with a natural and simple geometrisation of the field equations. The idea behind the supergeometrisation of the canonical analysis of the super-σ-model, to be recapitulated in the next section, is a purely topological reformulation of the original field theory that we present now.

To begin with, let us transfer our considerations to the superalgebraic model of the supertarget based on the relevant super-Harish-Chandra pairs $\mathrm{G} \equiv (|\mathrm{G}|, \mathfrak{g})$ and $(\mathrm{H}, \mathfrak{h})$, with $\mathfrak{g} \equiv \bigoplus_{A=0}^{D} \langle t_A \rangle$, $\mathfrak{h} \equiv \bigoplus_{S=1}^{D-\delta} \langle J_S \rangle$ and $(\mathcal{L} \cong)\mathfrak{t} \equiv \bigoplus_{\underline{A}=0}^{\delta} \langle t_{\underline{A}} \rangle \subset \mathfrak{g}$, so that

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{h}\,, \qquad [\mathfrak{h}, \mathfrak{t}\}_{\mathfrak{g}} \subset \mathfrak{t}\,.$$

We denote the structure constants of $\mathfrak{g}$ in the above homogeneous basis as

$$[t_A, t_B\}_{\mathfrak{g}} =: f_{AB}{}^{C}\, t_C\,, \qquad f_{BA}{}^{C} = (-1)^{|A|\cdot|B|+1} f_{AB}{}^{C}\,.$$

The direct-sum complement $\mathfrak{t}$ of the **isotropy algebra** $\mathfrak{h}$ in the **supersym-**

metry algebra $\mathfrak{g}$ inherits a $\mathbb{Z}/2\mathbb{Z}$-gradation,

$$\mathfrak{t} = \mathfrak{t}^{(0)} \oplus \mathfrak{t}^{(1)}\,, \qquad \mathfrak{t}^{(0)} \equiv \bigoplus_{a\in\overline{0,d}} \langle P_a \rangle\,, \qquad \mathfrak{t}^{(1)} \equiv \bigoplus_{\alpha\in\overline{1,\delta-d}} \langle Q_\alpha \rangle\,,$$

and we fix subspaces ($p \leq d$)

$$\mathfrak{t}^{(0)}_{\rm vac} \equiv \bigoplus_{\underline{a}\in\overline{0,p}} \langle P_{\underline{a}} \rangle \subset \mathfrak{t}^{(0)} \supset \bigoplus_{\widehat{a}\in\overline{p+1,d}} \langle P_{\widehat{a}} \rangle \equiv \mathfrak{e}^{(0)}\,,$$

of which the former is to be thought of as an algebraic model of the tangent sheaf of the body of the vacuum, alongside its ad.-stabiliser within $\mathfrak{h}$, to be denoted as $\mathfrak{h}_{\rm vac} \equiv \bigoplus_{\underline{S}=1}^{\delta} \langle J_{\underline{S}} \rangle$ and termed the **vacuum isotropy algebra**. The latter, itself the Lie algebra of a Lie subgroup $\mathrm{H}_{\rm vac} \subset \mathrm{H}$ whose adjoint action on $\mathfrak{t}^{(0)}_{\rm vac}$ is assumed **unimodular**,

$$\forall_{h\in\mathrm{H}_{\rm vac}}\ :\ \det\left(\mathsf{T}_e\mathrm{Ad}_h\restriction_{\mathfrak{t}^{(0)}_{\rm vac}}\right) \stackrel{!}{=} 1\,,$$

has a direct-sum complement $\mathfrak{d} \equiv \bigoplus_{\widehat{S}=\underline{\delta}+1}^{D-\delta} \langle J_{\widehat{S}} \rangle$ in $\mathfrak{h}$ which we assume to be an ad.-module of $\mathfrak{h}_{\rm vac}$, so that the new decomposition, with $\mathfrak{t}\oplus\mathfrak{d} \equiv \bigoplus_{\mu=0}^{D-\underline{\delta}} \langle t_\mu \rangle$,

$$\mathfrak{g} = \left(\mathfrak{t}\oplus\mathfrak{d}\right) \oplus \mathfrak{h}_{\rm vac}\,, \qquad [\mathfrak{h}_{\rm vac},\mathfrak{d}\}_{\mathfrak{g}} \subset \mathfrak{d}\,,$$

is also reductive. Furthermore, we impose the **Even Effective-Mixing Constraints**

$$\begin{aligned} &[\mathfrak{h}_{\rm vac},\mathfrak{t}^{(0)}_{\rm vac}\}_{\mathfrak{g}} \stackrel{!}{\subset} \mathfrak{t}^{(0)}_{\rm vac}\,, && [\mathfrak{h}_{\rm vac},\mathfrak{e}^{(0)}\}_{\mathfrak{g}} \stackrel{!}{\subset} \mathfrak{e}^{(0)}\,, \\ &[\mathfrak{d},\mathfrak{t}^{(0)}_{\rm vac}\}_{\mathfrak{g}} \stackrel{!}{\subset} \mathfrak{e}^{(0)}\,, && [\mathfrak{d},\mathfrak{e}^{(0)}\}_{\mathfrak{g}} \stackrel{!}{\subset} \mathfrak{t}^{(0)}_{\rm vac}\,. \end{aligned} \tag{4.1}$$

For these, we give

Definition 4.3. Adopt the hitherto notation. Let $\mathrm{G} \equiv (|\mathrm{G}|,\mathfrak{g}) \supset (\mathrm{H},\mathfrak{h}) \supset (\mathrm{H}_{\rm vac},\mathfrak{h}_{\rm vac})$ be super-Harish-Chandra pairs as defined and constrained above. Assume given a Green-Schwarz super-$(p+2)$-cocycle as in Def. 4.2 alongside the volume form on $\mathfrak{t}^{(0)}_{\rm vac}$,

$$\beta^{(\mathrm{HP})} = \theta^0_{\mathrm{L}} \wedge \theta^1_{\mathrm{L}} \wedge \cdots \wedge \theta^p_{\mathrm{L}}\,,$$

called the **Hughes-Polchinski super-**$(p+1)$**-form**, and consider a worldvolume Ω_p together with the associated mapping supermanifold $[\Omega_p,\mathrm{G}/\mathrm{H}_{\rm vac}]$ as described *ibid.* Fix a trivialising open cover $\{\mathcal{U}^{\mathrm{H}_{\rm vac}}_i\}_{i\in I} \equiv \mathcal{U}^{\mathrm{H}_{\rm vac}}$ of $\mathrm{G}/\mathrm{H}_{\rm vac}$ by the superdomains $\mathcal{U}^{\mathrm{H}_{\rm vac}}_i$ introduced previously, coming with the respective local sections $\sigma^{\mathrm{H}_{\rm vac}}_i\ :\ \mathcal{U}^{\mathrm{H}_{\rm vac}}_i \longrightarrow \mathrm{G}$, and let $\triangle(\Omega_p)$ be an arbitrary tessellation of Ω_p, composed of sets of k-cells $\mathfrak{T}_k$ with $k \in \overline{0,p+1}$, which is **subordinate** to $\mathcal{U}^{\mathrm{H}_{\rm vac}}$ for a given $\widehat{\xi} \in [\Omega_p,\mathrm{G}/\mathrm{H}_{\rm vac}](\mathbb{R}^{0|N})$, so that there exists a map

$\imath_\cdot : \triangle(\Omega_p) \longrightarrow I$ with the property $\widehat{\xi}(\widetilde{\tau}) \subset \mathcal{U}^{\mathrm{H_{vac}}}_{\imath_\tau}$ for $\widetilde{\tau} \subset \Omega_p \times \mathbb{R}^{0|N}$ such that $|\widetilde{\tau}| \equiv \tau$. The **Green-Schwarz super-σ-model in the Hughes-Polchinski formulation for the super-p-brane** in the **Hughes-Polchinski super-background** ($\lambda_p \in \mathbb{R}^\times$ is a parameter)

$$\mathfrak{sB}^{\mathrm{(HP)}}_{p,\lambda_p} \equiv \big(\mathrm{G}/\mathrm{H_{vac}}, \chi_{p+2} + \lambda_p\, \mathsf{d}\beta^{\mathrm{(HP)}} \equiv \widehat{\chi}_{p+2}\big)$$

over the **supertarget** $\mathrm{G}/\mathrm{H_{vac}}$ is the lagrangean theory of (functorial) mappings from $[\Omega_p, \mathrm{G}/\mathrm{H_{vac}}]$ determined by the principle of least action applied to the DF amplitude

$$\mathcal{A}^{\mathrm{(HP)},p,\lambda_p}_{\mathrm{DF}}[\widehat{\xi}] := \mathrm{e}^{\mathrm{i}\, S^{\mathrm{(HP)},\lambda_p}_{\mathrm{GS},p}[\widehat{\xi}]}$$

written in terms of the action functional ($\widehat{\xi}_{\widetilde{\tau}} \equiv \widehat{\xi}\restriction_{\widetilde{\tau}}$)

$$S^{\mathrm{(HP)},\lambda_p}_{\mathrm{GS},p}[\widehat{\xi}] = \sum_{\tau \in \mathfrak{T}_{p+1}} \int_\tau \big(\sigma^{\mathrm{H_{vac}}}_{\imath_\tau} \circ \widehat{\xi}_\tau\big)^* \big(\beta_{p+1} + \lambda_p\, \beta^{\mathrm{(HP)}}\big)\,.$$

$\diamond$

Concrete field-theoretic considerations, some of which shall be presented below, call for an explicit choice of the local sections σ^{K}_i, $\mathrm{K} \in \{\mathrm{H}, \mathrm{H_{vac}}\}$ entering the above definitions. As argued on p. 22, this choice may be fixed by picking up an explicit form of the section σ^{K}_0 over the unital coset K. Such sections have been used in the physics literature for a long time without an easily tractable statement as to their mathematical status, basing on formal affinities between (the vector calculus on) Lie groups and their super-counterparts, and – seldom, and vaguely – on an alternative approach to the latter due to Berezin and Kač, *cp* Ref. [16]. The situation was clarified in Ref. [165], where a simple sheaf-theoretic analysis was carried out to justify the expression

$$\mathrm{e}^{\theta^\alpha \otimes Q_\alpha} \cdot \mathrm{e}^{x^a \otimes P_a} \cdot \mathrm{e}^{\varepsilon_{\mathrm{K}}\, \phi^{\widehat{S}} \otimes J_{\widehat{S}}} \equiv \sigma^{\mathrm{K}}_0 = \big(|\sigma^{\mathrm{K}}_0|, \sigma^{\mathrm{K}\,*}_0\big) : \xi^{\mathrm{K}}_e\big(\mathcal{U}^{\mathrm{K}}_0\big) \longrightarrow \sigma^{\mathrm{K}}_0\big(\mathcal{U}^{\mathrm{K}}_0\big) \equiv \mathcal{V}^{\mathrm{K}}_0, \tag{4.2}$$

dubbed the **exponential superparametrisation**, with the sheaf component

$$\sigma^{\mathrm{K}\,*}_0 := \big(\mathrm{id}_{\mathcal{O}_{\mathcal{U}^{\mathrm{K}}_0}} \otimes \mathrm{ev}_e\big) \circ \Big(\sum_{k=0}^{\delta-d} \tfrac{1}{k!} \big(\theta^{\alpha_1} \otimes Q_{\alpha_1}\big) \circ \big(\theta^{\alpha_2} \otimes Q_{\alpha_2}\big) \circ \cdots$$
$$\cdots \circ \big(\theta^{\alpha_k} \otimes Q_{\alpha_k}\big)\Big) \circ r^*_{\mathrm{e}^{x^a \otimes P_a}} \circ r^*_{\mathrm{e}^{\varepsilon_{\mathrm{K}}\, \phi^{\widehat{S}} \otimes J_{\widehat{S}}}}\,,$$

written, for $\varepsilon_{\mathrm{K}} \in \{0 \equiv \varepsilon_{\mathrm{H}}, 1 \equiv \varepsilon_{\mathrm{H_{vac}}}\}$, in terms of the coordinate sections $\{\chi^\zeta\}^{\zeta \in \overline{0, D-K}} \equiv \{\chi^a \equiv x^a\}^{a \in \overline{0,d}} \cup \{\chi^\alpha \equiv \theta^\alpha\}^{\alpha \in \overline{1, \delta-d}} \cup \{\chi^{\widehat{S}} \equiv \varepsilon_{\mathrm{K}}\, \phi^{\widehat{S}}\}^{\widehat{S} \in \overline{\underline{\delta}+1, D-\delta}}$ of the local chart $\xi^{\mathrm{K}}_e(\mathcal{U}^{\mathrm{K}}_0)$ over $\mathcal{U}^{\mathrm{K}}_0$. In its body part (involving the even x^a and $\phi^{\widehat{S}}$), we recognise a standard local section $|\sigma^{\mathrm{K}}_0|$ of the principal K-bundle $|\mathrm{G}| \longrightarrow |\mathrm{G}|/\mathrm{K}$ near e.

Astonishingly, there turns out to be a tight correspondence between the two seemingly unrelated field-theoretic constructs introduced above, to wit, THEOREM 4.4 ([164, Thms. 5.1 & 5.2], [165, Thm. 3.4]). *Adopt the hitherto notation. Let the super-Harish–Chandra pairs* $\mathrm{G} \equiv (|\mathrm{G}|, \mathfrak{g}) \supset (\mathrm{H}, \mathfrak{h}) \supset (\mathrm{H}_{\mathrm{vac}}, \mathfrak{h}_{\mathrm{vac}})$ *and the superbackgrounds* $\mathfrak{s}\mathfrak{B}_p^{(\mathrm{NG})}$ *and* $\mathfrak{s}\mathfrak{B}_{p,\lambda_p}^{(\mathrm{HP})}$ *be as defined and constrained above. Write* $\mathfrak{d}^{-1}\mathfrak{t}_{\mathrm{vac}}^{(0)} := \left\langle P_{\widehat{a}} \,|\, \exists_{(\underline{b},\widehat{S})\in\overline{0,p}\times\overline{D-\underline{\delta}+1,D-\delta}} \;:\; f_{\widehat{S}\widehat{a}}{}^{\underline{b}} \neq 0 \right\rangle \equiv \bigoplus_{\underline{\widehat{a}}=p+1}^{d-L} \left\langle P_{\underline{\widehat{a}}} \right\rangle$ *for some* $0 \leq L \leq d-p$ *and subsequently decompose* $\mathfrak{e}^{(0)} \equiv \mathfrak{d}^{-1}\mathfrak{t}_{\mathrm{vac}}^{(0)} \oplus \mathfrak{l}^{(0)}$*, assuming that both* $\mathfrak{d}^{-1}\mathfrak{t}^{(0)} \equiv \mathfrak{t}_{\mathrm{vac}}^{(0)} \oplus \mathfrak{d}^{-1}\mathfrak{t}_{\mathrm{vac}}^{(0)}$ *and* $\mathfrak{l}^{(0)}$ *are* $\mathsf{T}_e\mathrm{Ad}_{\mathrm{H}}$*-invariant, and – finally – let* $\mathrm{g}_{\mathrm{vac}}$ *be a* $\mathsf{T}_e\mathrm{Ad}_{\mathrm{H}}$*-invariant scalar product on* $\mathfrak{d}^{-1}\mathfrak{t}^{(0)}$ *such that* $\mathfrak{t}_{\mathrm{vac}}^{(0)} \perp_{\mathrm{g}_{\mathrm{vac}}} \mathfrak{d}^{-1}\mathfrak{t}_{\mathrm{vac}}^{(0)}$*. Then, the GS super-$\sigma$-model in the HP formulation restricted to field configurations obeying the* ***Body-Localisation Constraints*** *(BLC)*

$$\left(\sigma_{\imath_\tau}^{\mathrm{H}_{\mathrm{vac}}} \circ \widehat{\xi}\right)^* \theta_{\mathrm{L}}^{\widehat{a}} \stackrel{!}{=} 0\,, \qquad \widehat{a} \in \overline{d-L+1,d} \tag{4.3}$$

and partially reduced through imposition of the ***Inverse Higgs Constraints*** *(IHC)*

$$\left(\sigma_{\imath_\tau}^{\mathrm{H}_{\mathrm{vac}}} \circ \widehat{\xi}\right)^* \theta_{\mathrm{L}}^{\underline{\widehat{a}}} \stackrel{!}{=} 0\,, \qquad \underline{\widehat{a}} \in \overline{p+1,d-L} \tag{4.4}$$

is (classically) equivalent to the GS super-σ-model in the NG formulation for a unique value μ_p^* *of its parameter* μ_p *and for* g *such that* $\mathrm{g}\restriction_{\mathfrak{d}^{-1}\mathfrak{t}^{(0)}} \equiv \mathrm{g}_{\mathrm{vac}}$*, restricted to field configurations subject to the (same) BLC*

$$\left(\sigma_{\imath_\tau}^{\mathrm{H}} \circ \xi\right)^* \theta_{\mathrm{L}}^{\widehat{a}} \stackrel{!}{=} 0\,, \qquad \widehat{a} \in \overline{d-L+1,d}\,,$$

where it is to be understood that the DF amplitude $\mathcal{A}_{\mathrm{DF}}^{(\mathrm{HP}),\mathrm{p},\lambda_p}$ *written in the* gauge $\sigma_i^{\mathrm{H}_{\mathrm{vac}}} \equiv \sigma_i^{\mathrm{vac}}$*,* $i \in I_{\mathrm{H}_{\mathrm{vac}}}$ *of Eq.* (4.2) *(for* $\mathrm{K} \equiv \mathrm{H}_{\mathrm{vac}}$*) reproduces the DF amplitude* $\mathcal{A}_{\mathrm{DF}}^{(\mathrm{NG}),\mathrm{p},\mu_{\mathrm{p}}^*}$ *written in the* gauge $\sigma_i^{\mathrm{H}} \equiv \sigma_i^{\cancel{\mathrm{vac}}}$*,* $i \in I_{\mathrm{H}_{\mathrm{vac}}}$ *of Eq.* (4.2) *(for* $\mathrm{K} \equiv \mathrm{H}$*).*

Concrete instances of correspondences of the above type were first contemplated by Hughes and Polchinski in Ref. [96] in the context of spontaneous partial breakdown of global supersymmetry, and were later elaborated significantly by Gauntlett *et al.* in Ref. [70]. More recently, they were used by McArthur (*cp* Refs. [121, 122]) and West *et al.* (*cp* Refs. [180, 76, 75]) in the construction of (super-)σ-models for (super-)p-branes in the broader context of nonlinear realisations of (super)symmetries. The general conditions to be satisfied by the triple $(\mathrm{G}, \mathrm{H}, \mathrm{H}_{\mathrm{vac}})$ for the duality to obtain were identified in Refs. [164, 165].

Remark 4.5. The judicious choice of the exponential parametrisations enables us to keep track of the purely field-theoretic aspect of the correspondence that boils down to the inegration of the non-dynamical goldstone fields $\phi^{\widehat{S}}$ associated with the reduction $\mathrm{H} \searrow \mathrm{H}_{\mathrm{vac}}$ of the hidden gauge-symmetry group in the vacuum and hence parametrised by $\mathfrak{h}/\mathfrak{h}_{\rm vac}$. Upon integration, these fields become 'functionals' of the remaining lagrangean fields x^a and θ^α that coordinatise the 'physical' coset $\mathrm{G/H}$ locally. This is (the gist of) the **inverse Higgs effect**, first noted by Ivanov and Ogievetsky in Ref. [101] in the context of nonlinear realisations of symmetries. The knowledge of the structure of the parametrisations also gives us better control of the mechanism of gauge-symmetry enhancement that we discuss – after Refs. [164, 165] – in the next section. Lastly, it paves the way to a hands-on asymptotic analysis, in the spirit of Refs. [162, 163], of the super-σ-models on curved supertargets and the attendant higher-geometric objects in the (local) limit of a vanishing curvature that we mention in Section 6.

What makes the **HP/NG correspondence** established in the above theorem truly remarkable is the effective transcription of the metric degrees of freedom of the super-σ-model in the NG formulation into the purely topological ones in its HP formulation at the expense of turning on the goldstone fields, or – indeed – the goldstone background that encodes, in its critical configurations, complete information on the *dynamics* of the field theory under consideration. This fact has far-reaching (higher-)geometric consequences that shall be discussed at length in the remaining sections of the present review. Meanwhile, we give a foretaste of the phenomena to be encountered by presenting a convenient reinterpretation of the BLC and IHC that implement the correspondence.

The reformulation of the GS super-σ-model leads to the complete geometrisation of its canonical analysis. This is readily seen on the (redundant) model of the HP supertarget $\mathrm{G/H_{vac}}$ defined as the disjoint union of superdomains

$$(4.5)\qquad \Sigma^{\rm HP} := \bigsqcup_{i\in I_{\rm H_{vac}}} \mathcal{V}_i\,, \qquad\qquad \mathcal{V}_i := l_{g_i}\big(\mathcal{V}_e^{\rm H_{vac}}\big) \equiv \sigma_i^{\rm vac}\big(\mathcal{U}_i^{\rm H_{vac}}\big) \subset \mathrm{G}$$

faithfully representing $\mathrm{G/H_{vac}}$ within G *patchwise*. The physically relevant supermanifold $\Sigma^{\rm HP}$ was dubbed the **Hughes-Polchinski section** in Ref. [165]. An explicit description of its tangent sheaf $\mathcal{T}\Sigma^{\rm HP} \equiv \bigsqcup_{i\in I_{\rm H_{vac}}} \mathcal{T}\mathcal{V}_i$ as a free $\mathcal{O}_{\Sigma^{\rm HP}}$-module generated by vector fields $\mathcal{T}_\mu,\ \mu \in \overline{0, D-\underline{\delta}}$ with restrictions

$$(4.6)\qquad \mathcal{T}_\mu\restriction_{\mathcal{V}_i} \equiv \mathcal{T}_{\mu\,i} = L_\mu\restriction_{\mathcal{V}_i} + T_{\mu\,i}{}^{\underline{S}}\, L_{\underline{S}}\,,$$

with the $T_{\mu\,i}{}^{\underline{S}} \in \mathcal{O}_{\mathrm{G}}(\mathcal{V}_i)$ uniquely fixed by the tangency condition $\mathcal{T}_{\mu\,i} \overset{!}{\in} \Gamma(\mathcal{T}\mathcal{V}_i)$ in the form given in [165, Prop. 3.6], permits us to formulate

Definition 4.6. Adopt the hitherto notation. The **correspondence superdistribution of** $\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p}$ is the superdistribution

$$\mathrm{Corr}_{\mathrm{HP/NG}}\big(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p}\big) := \mathrm{Ker}\,\big(\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}^{(0)}} \circ \theta_{\mathrm{L}} \restriction_{\mathcal{T}\Sigma^{\mathrm{HP}}}\big) \subset \mathcal{T}\Sigma^{\mathrm{HP}} \tag{4.7}$$

written in terms of the projector $\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}^{(0)}} : \mathfrak{g} \circlearrowleft$ onto $\mathfrak{e}^{(0)}$, with the kernel $\mathrm{Ker}\,\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}^{(0)}} = \mathfrak{t}^{(1)} \oplus \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{h}$ whose proper subspace $\mathfrak{t}^{(1)} \oplus \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{d}$ models $\mathrm{Corr}_{\mathrm{HP/NG}}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p})$ locally.

◇

We may now conclude the section with a purely geometric restatement of Thm. 4.4. Thus, the HP/NG correspondence obtains for those mappings $\widehat{\xi} \in [\Omega_p, \mathrm{G}/\mathrm{H}_{\mathrm{vac}}]$ for which the images of the tangents of the superpositions $\sigma_i^{\mathrm{vac}} \circ \widehat{\xi}$ are contained in the correspondence superdistribution – we shall refer to the entirety of such mappings as the **HP/NG correspondence sector**. While phrased in terms of the specific local sections σ_i^{vac} and for a fixed tessellation of Ω_p, the tangent localisation constraints can be continued across the trivialising patches $\mathcal{U}_i^{\mathrm{H}_{\mathrm{vac}}}$ and do not depend on the arbitrary choices made. The definition of the correspondence superdistribution sets the stage for the canonical analysis of the super-σ-model that we turn to next.

5. THE ODD SQUARE ROOT OF THE TOPOLOGICAL VACUUM

The fundamental consequence of the correspondence stated in Thm. 4.4 is the full-blown topologisation of the field theory of the super-charged extended objects that bases on an implicit partial localisation of the vacuum $\mathrm{Vac}_p \equiv \widehat{\xi}(\Omega_p \times \mathbb{R}^{0|N})$ (for $\widehat{\xi}$ critical), best expressed as

$$\beta^{(\mathrm{HP})} \restriction_{\bigsqcup_{i \in I_{\mathrm{H}_{\mathrm{vac}}}} \sigma_i^{\mathrm{vac}}(\mathrm{Vac}_p)} \sim \pi^*_{\mathrm{G}/\mathrm{H}_{\mathrm{vac}}} \mathrm{Vol}\big(\mathrm{Vac}_p\big) \neq 0\,.$$

The merit of working with the dual HP formulation of the super-σ-model turns out to be a complete geometrisation of its canonical data, *i.e.*, of the description of the vacuum and its supersymmetries. Such a possibility is hinted at already by our reinterpretation of the constraints (4.3) and (4.4) enforcing the correspondence as a definition of the correspondence superdistribution $\mathrm{Corr}_{\mathrm{HP/NG}}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p}) \subset \mathcal{T}\Sigma^{\mathrm{HP}}$ within the tangent sheaf of the HP section (4.5). Indeed, the constraints constitute a subset of the Euler-Lagrange equations of the super-σ-model of Def. 4.3. The geometrisation becomes effective upon subjecting the triple $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_{\mathrm{vac}})$ to a number of natural constraints among which a distinguished place is occupied by the requirement of existence of the gauge supersymmetry, alluded to formerly, that achieves the necessary reduction of the redundant odd field excitations in the vacuum. Altogether, we land in the setting of the topological gauge field theory, with all its peculiarities anticipated to manifest themselves in the present setting. These are instantiated amply in the three-dimensional Chern-Simons theory of Refs. [46, 184, 53] in which we observe absence of local dynamical degrees of freedom and replacement of propagation of field configurations localised on Cauchy hypersurfaces of the underlying spacetime by gauge transformations. In view of the intrinsically geometric nature of the field theory in hand, whose fundamental objects (charged and massive pointlike particles, loops, membranes *etc.*) sweep minimal hypervolumes perturbed by the Lorentz-type forces in the supertarget, these expectations take on a concrete form: The vacuum of the theory should be an integral leaf of a gauge-supersymmetry superdistribution. An appropriate formalisation of this intuition calls for an adaptation, to the supergeometric environment, of Tanaka's treatment of non-involutive distributions over differentiable manifolds, proposed and employed in Ref. [165] (*cp* Ref. [165, Def. 4.9] for the relevant constructions and nomenclature).

The canonical study of the GS super-σ-model in the HP formulation begins with the derivation of its Euler-Lagrange equations. These are readily extracted from Def. 4.3 upon imposition of a number of constraints on the superbackground of the theory. The first of these enforces the presence of a *single* topological charge on the super-p-brane, as reflected by the uniformity of the linear dimension of all components of the GS super-$(p+2)$-cocycle computed for the natural assignment: $[x^a] \equiv [\theta^a_{\rm L}] = 1{\rm m} = [\theta^\alpha_{\rm L}]^2 = [\theta^\alpha]^2$, and the charge is assumed to have the same dimension (essentially of a mass/energy density) as the HP dual of the NG metric term. That is, upon restoring the dimensionful coefficients μ and q in $\mu\,\beta^{(\rm HP)}$ and $q\,\chi_{\underline{A}_1\underline{A}_2\cdots\underline{A}_{p+2}}\,\theta_{\rm L}^{\underline{A}_1}\wedge\theta_{\rm L}^{\underline{A}_2}\wedge\cdots\wedge\theta_{\rm L}^{\underline{A}_{p+2}}$ for $[\chi_{\underline{A}_1\underline{A}_2\ldots\underline{A}_{p+2}}] = 1$ (dimensionless), we demand that $[q] = {\rm kg}\cdot{\rm m}^{1-p}\cdot{\rm s}^{-2} \equiv [\mu]$, which yields the **Dimensional Constraint** of Ref. [165], fixing the structure of the super-$(p+2)$-cocycle as

$$(5.1)\qquad \chi_{p+2} \equiv \tfrac{1}{2p!}\,\chi_{\alpha\beta a_1 a_2\ldots a_p}\,\theta_{\rm L}^\alpha\wedge\theta_{\rm L}^\beta\wedge\theta_{\rm L}^{a_1}\wedge\theta_{\rm L}^{a_2}\wedge\cdots\wedge\theta_{\rm L}^{a_p}\,.$$

Among the most-studied superbackgrounds of the type discussed, the (technical) constraint excludes[5] the Zhou super-0-brane in super-$\mathrm{AdS}_2\times\mathbb{S}^2$ of [189], the Metsaev-Tseytlin D3-brane in super-$\mathrm{AdS}_5\times\mathbb{S}^5$ of [132] and the M-branes: the M2-brane of [47] and the M5-brane of [35] in super-$\mathrm{AdS}_4\times\mathbb{S}^7$ and super-$\mathrm{AdS}_7\times\mathbb{S}^4$, leaving us with the old brane scan for the super-Minkowski space as well as the super-1-branes (or superstrings): in super-$\mathrm{AdS}_2\times\mathbb{S}^2$ (*cp* [189]), in

[5]The Zhou super-0-brane was investigated separately in Ref. [165, App. A].

super- $\mathrm{AdS}_3 \times \mathbb{S}^3$ (*cp* [134]) and in super- $\mathrm{AdS}_5 \times \mathbb{S}^5$ (*cp* [131]) as a rich reference pool for our abstract considerations.

The next, fundamental set of **κ-Symmetry Constraints** provides a formalisation of our earlier argument in favour of an odd gauge supersymmetry that leads us to postulate the existence of a projector $\mathsf{P}^{(1)} \in \mathrm{End}\,\mathfrak{t}^{(1)}$, with the image

$$\mathfrak{t}^{(1)}_{\mathrm{vac}} \equiv \mathrm{Im}\,\mathsf{P}^{(1)} \equiv \bigoplus_{\underline{\alpha}\in\overline{1,q}} \left\langle Q_{\underline{\alpha}} \equiv \underline{\Lambda}^{\beta}{}_{\underline{\alpha}}\, Q_\beta \right\rangle\,, \qquad\qquad \underline{\Lambda}^{\beta}{}_{\underline{\alpha}} \in \mathbb{C}\,,$$

to be interpreted as a model of the spinorial sector of the vacuum, and with the kernel $\mathfrak{e}^{(1)} \equiv \mathrm{Ker}\,\mathsf{P}^{(1)}$, whence also the first subset of constraints:

$$[\mathfrak{t}^{(1)}_{\mathrm{vac}}, \mathfrak{t}^{(1)}_{\mathrm{vac}}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{h}\,. \tag{5.2}$$

Upon inspection of the variation of the DF amplitude of the super-σ-model of Def. 4.3, these are augmented with the requirement that the identity

$$\chi_{\alpha\beta\underline{a}_1\underline{a}_2\ldots\underline{a}_p} + \lambda^*_p\, f_{\alpha\beta}{}^{\underline{a}_0}\, \epsilon_{\underline{a}_0\underline{a}_1\underline{a}_2\ldots\underline{a}_p} \overset{!}{=} \left(\mathbf{1}_{\delta-d} - \mathsf{P}^{(1)}\right)^{\gamma}{}_{\alpha}\, \Delta_{\beta\gamma\underline{a}_1\underline{a}_2\ldots\underline{a}_p} \tag{5.3}$$

hold for some $\Delta_{\beta\gamma\underline{a}_1\underline{a}_2\ldots\underline{a}_p}$ such that for any p-tuple $\underline{a}_k \in \overline{0,p}$, $k \in \overline{1,p}$ the matrix $(\Delta_{\alpha\beta\underline{a}_1\underline{a}_2\ldots\underline{a}_p})_{\alpha,\beta\in\overline{1,\delta-d}}$ is invertible, and for a *unique* (up to a sign) value $\lambda_p \equiv \lambda^*_p \in \mathbb{R}^\times$. The remaining constraints are listed in the following

PROPOSITION 5.1 ([165, Prop. 4.2]). *Adopt the notation of Def. 4.3. If the HP superbackground $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda^*_p}$ satisfies the Even Effective-Mixing Constraints* (4.1)*, the Dimensional Constraint* (5.1)*, the κ-Symmetry Constraints* (5.2) *and* (5.3)*, and*

$$[\mathfrak{t}^{(0)}_{\mathrm{vac}}, \mathfrak{t}^{(0)}_{\mathrm{vac}}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{h}_{\mathrm{vac}} \overset{!}{\supset} [\mathfrak{e}^{(0)}, \mathfrak{e}^{(0)}\}_{\mathfrak{g}}\,, \qquad\qquad [\mathfrak{t}^{(0)}_{\mathrm{vac}}, \mathfrak{e}^{(0)}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{d}\,,$$

$$\forall_{(\alpha,\beta,\underline{a}_k,\widehat{a})\in\overline{1,\delta-d}^{\times 2}\times\overline{0,p}\times\overline{p+1,d},\ k\in\overline{1,p-1}}\ :$$

$$\chi_{\alpha\gamma\widehat{a}\underline{a}_1\underline{a}_2\ldots\underline{a}_{p-1}}\, \mathsf{P}^{(1)\,\gamma}{}_{\beta} = \chi_{\gamma\beta\widehat{a}\underline{a}_1\underline{a}_2\ldots\underline{a}_{p-1}} \left(\mathbf{1}_{\delta-d} - \mathsf{P}^{(1)}\right)^{\gamma}{}_{\alpha}\,,$$

then the Euler-Lagrange equations of the corresponding GS super-σ-model in the HP formulation for the super-p-brane in $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda^*_p}$ restricted by the BLC (4.3) *are linear constraints on the tangents of the embedding field $\widehat{\xi}$ post-composed with the σ^{vac}_i that further restrict the tangents to the **Hughes-Polchinski vacuum superdistribution***

$$\mathrm{Vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda^*_p}\big) := \mathrm{Ker}\left(\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}\oplus\mathfrak{d}} \circ \theta_{\mathrm{L}} \restriction_{\mathcal{T}\Sigma^{\mathrm{HP}}}\right) \subset \mathrm{Corr}_{\mathrm{HP/NG}}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda^*_p}\big) \subset \mathcal{T}\Sigma^{\mathrm{HP}}\,, \tag{5.4}$$

defined in terms of the projector $\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}\oplus\mathfrak{d}} : \mathfrak{g} \circlearrowleft$ onto $\mathfrak{e}\oplus\mathfrak{d}$ for $\mathfrak{e} = \mathfrak{e}^{(0)} \oplus \mathfrak{e}^{(1)}$ with $\mathrm{Ker}\,\mathsf{P}^{\mathfrak{g}}{}_{\mathfrak{e}\oplus\mathfrak{d}} = \mathfrak{t}_{\mathrm{vac}} \oplus \mathfrak{h}_{\mathrm{vac}}$ *and hence modelled on $\mathfrak{t}_{\mathrm{vac}} \equiv \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{t}^{(1)}_{\mathrm{vac}} \equiv \bigoplus_{\underline{\underline{A}}=0}^{p+q} \left\langle t_{\underline{\underline{A}}} \right\rangle$. In particular, the IHC* (4.4) *are among the field equations.*

The ratio $\frac{q}{\delta-d} \equiv \mathrm{BPS}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ features as the **BPS fraction of the vacuum of $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$** in the physics literature.

From the field-theoretic vantage point, it is natural to enquire as to circumstances in which the HP vacuum superdistribution defines a foliation of the HP section that descends to a foliation of the supertarget $\mathrm{G}/\mathrm{H}_{\mathrm{vac}}$ whose leaves we are bound to identify as inequivalent vacua. The answer to this question is contained in

PROPOSITION 5.2 ([165, Props. 4.5 & 4.6]). *Adopt the notation of Prop. 5.1 and assume the constraints listed therein to be satisfied. The HP vacuum superdistribution is involutive and hence determines a foliation (the disjoint union of integral leaves)*

$$\iota_{\mathrm{vac}} \;:\; \Sigma^{\mathrm{HP}}_{\mathrm{vac}} \hookrightarrow \Sigma^{\mathrm{HP}}\,, \tag{5.5}$$

termed the ***Hughes-Polchinski vacuum foliation of*** Σ^{HP}, *iff the* ***Vacuum-Superalgebra Constraints***

$$[\mathfrak{t}^{(1)}_{\mathrm{vac}}, \mathfrak{t}^{(1)}_{\mathrm{vac}}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{t}^{(0)}_{\mathrm{vac}} \oplus \mathfrak{h}_{\mathrm{vac}}\,, \qquad [\mathfrak{t}^{(0)}_{\mathrm{vac}}, \mathfrak{t}^{(1)}_{\mathrm{vac}}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{t}^{(1)}_{\mathrm{vac}}\,, \qquad [\mathfrak{h}_{\mathrm{vac}}, \mathfrak{t}^{(1)}_{\mathrm{vac}}\}_{\mathfrak{g}} \overset{!}{\subset} \mathfrak{t}^{(1)}_{\mathrm{vac}}$$

are satisfied. The foliation then descends to $\mathrm{G}/\mathrm{H}_{\mathrm{vac}}$.

The geometric structure has an algebraic counterpart indicated in

PROPOSITION 5.3 ([165, Prop. 4.6]). *Adopt the notation of Prop. 5.2. The existence of the HP vacuum foliation of* Σ^{HP} *is equivalent to the closure of the Lie superbracket* $[\cdot,\cdot\}_{\mathfrak{g}}$ *of* $\mathfrak{g}$ *on the* ***vacuum supervector space of*** $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$

$$\mathfrak{vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \equiv \mathfrak{t}_{\mathrm{vac}} \oplus \mathfrak{h}_{\mathrm{vac}} \subset \mathfrak{g}\,,$$

giving rise to the ***vacuum superalgebra of*** $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$.

The situation in which the vacuum superdistribution is non-involutive and hence – in virtue of the Global Frobenius Theorem, *cp* Ref. [28, Thm. 6.2.1] – non-integrable does not seem to have a natural physical interpretation. We may, nevertheless, ask the purely geometric question regarding the nature of the inclusion of the **limit** of the weak derived flag of $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ in $\mathcal{T}\Sigma^{\mathrm{HP}}$ (μ is the height of the regular vacuum superdistribution)

$$\mathrm{Vac}^{-\infty}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \equiv \mathrm{Vac}^{-\mu}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \subseteq \mathcal{T}\Sigma^{\mathrm{HP}}\,.$$

There seems to be a strong positive correlation between non-involutivity of the vacuum superdistribution and bracket-generation of the tangent sheaf of the

HP section by it, as exemplified by the models from the list on p. 35. Below, we illustrate the typical behaviour of $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda^*_p})$ on three examples taken from Ref. [165].

Example 5.4. **The Green-Schwarz super-p-brane in**

$$\mathscr{T}_p^{\mathrm{GS}} \equiv \mathrm{sISO}(d,1|D_{d,1})/(\mathrm{SO}(p,1)\times\mathrm{SO}(d-p))$$

for $p \equiv 1$ **or** $2 \bmod 4$**.** Adopt the notation of Example 3.7 and denote $\mathfrak{sB}^{(\mathrm{NG})}_{p,2p!,\mathrm{GS}} \equiv (\mathscr{T}_p^{\mathrm{GS}}, \chi^{\mathrm{GS}}_{p+2} + 2p!\,\mathsf{d}(\theta^0_{\mathrm{L}}\wedge\theta^1_{\mathrm{L}}\wedge\cdots\wedge\theta^p_{\mathrm{L}}))$. For $\mathfrak{t}^{(0)}_{\mathrm{vac}} = \bigoplus_{\underline{a}\in\overline{0,p}} \langle P_{\underline{a}}\rangle$ and $\mathfrak{h}_{\mathrm{vac}} = \bigoplus_{\underline{a}<\underline{b}\in\overline{0,p}} \langle J_{\underline{ab}}\rangle \oplus \bigoplus_{\widehat{a}<\widehat{b}\in\overline{p+1,d}} \langle J_{\widehat{a}\widehat{b}}\rangle$, we find

$$\mathfrak{d}^{-1}\mathfrak{t}^{(0)}_{\mathrm{vac}} \equiv \mathfrak{e}^{(0)}, \qquad \mathsf{P}^{(1)} = \frac{\mathbf{1}_{D_{d,1}} + \Gamma^0\,\Gamma^1\cdots\Gamma^p}{2}$$

with $\mathrm{tr}_{\mathfrak{t}^{(1)}}\,\mathsf{P}^{(1)} = \frac{D_{d,1}}{2}$, and the involutive HP vacuum superdistribution

$$\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{NG})}_{p,2p!,\mathrm{GS}}) = \bigoplus_{\underline{\alpha}\in\overline{1,\frac{D_{d,1}}{2}}} \langle \mathcal{T}_{\underline{\alpha}}\rangle \oplus \bigoplus_{\underline{a}\in\overline{0,p}} \langle \mathcal{T}_{\underline{a}}\rangle$$

with the corresponding vacuum superalgebra

$$\mathfrak{vac}(\mathfrak{sB}^{(\mathrm{NG})}_{p,2p!,\mathrm{GS}}) = \bigoplus_{\underline{\alpha}\in\overline{1,\frac{D_{d,1}}{2}}} \langle Q_{\underline{\alpha}}\rangle \oplus \bigoplus_{\underline{a}\in\overline{0,p}} \langle P_{\underline{a}}\rangle \oplus \bigoplus_{\underline{a}<\underline{b}\in\overline{0,p}} \langle J_{\underline{ab}}\rangle \oplus \bigoplus_{\widehat{a}<\widehat{b}\in\overline{p+1,d}} \langle J_{\widehat{a}\widehat{b}}\rangle .$$

Example 5.5. **The Metsaev-Tseytlin super-1-brane in**

$$\mathscr{T}^{\mathrm{MT}}_{1(1)} \equiv \mathrm{SU}(2,2\,|\,4)/(\mathrm{SO}(1,1)\times\mathrm{SO}(3)\times\mathrm{SO}(5)).$$

Adopt the notation of Example 3.8 and denote

$$\mathfrak{sB}^{(\mathrm{HP})}_{1,1,\mathrm{MT}(1)} \equiv (\mathscr{T}^{\mathrm{MT}}_{1(1)}, \chi^{\mathrm{MT}}_3 + \mathsf{d}(\theta^0_{\mathrm{L}}\wedge\theta^1_{\mathrm{L}})).$$

For $\mathfrak{t}^{(0)}_{\mathrm{vac}} = \langle P_0, P_1\rangle$ and $\mathfrak{h}_{\mathrm{vac}} = \langle J_{01}\rangle \oplus \bigoplus_{a'<b'\in\{2,3,4\}} \langle J_{a'b'}\rangle \oplus \bigoplus_{a''<b''\in\overline{5,9}} \langle J_{a''b''}\rangle$, we find $\mathfrak{d}^{-1}\mathfrak{t}^{(0)}_{\mathrm{vac}} = \langle P_2, P_3, P_4\rangle \subsetneq \mathfrak{e}^{(0)}$, $\mathsf{P}^{(1)} = \frac{\mathbf{1}_{32}+\underline{\gamma}^0\,\underline{\gamma}^1\,\underline{\gamma}_{11}}{2}$ with $\mathrm{tr}_{\mathfrak{t}^{(1)}}\,\mathsf{P}^{(1)} = \frac{32}{2}$, and the involutive HP vacuum superdistribution $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{1,1,\mathrm{MT}(1)}) = \bigoplus_{\underline{\alpha}\in\overline{1,16}} \langle \mathcal{T}_{\underline{\alpha}}\rangle \oplus \langle \mathcal{T}_0, \mathcal{T}_1\rangle$ with the corresponding vacuum superalgebra

$$\mathfrak{vac}(\mathfrak{sB}^{(\mathrm{HP})}_{1,1,\mathrm{MT}(1)}) = \bigoplus_{\underline{\alpha}\in\overline{1,16}} \langle Q_{\underline{\alpha}}\rangle \oplus \langle P_0, P_1\rangle \oplus \langle J_{01}\rangle \oplus \bigoplus_{a'<b'\in\{2,3,4\}} \langle J_{a'b'}\rangle \oplus \bigoplus_{a''<b''\in\overline{5,9}} \langle J_{a''b''}\rangle .$$

Example 5.6. **The Metsaev-Tseytlin super-1-brane in**

$$\mathscr{T}^{\mathrm{MT}}_{1(2)} \equiv \mathrm{SU}(2,2\,|\,4)/(\mathrm{SO}(4)\times\mathrm{SO}(4)).$$

Adopt the notation of Example 3.8 and denote

$$\mathfrak{sB}^{(\mathrm{HP})}_{1,1,\mathrm{MT}(2)} \equiv (\mathscr{T}^{\mathrm{MT}}_{1(2)}, \chi^{\mathrm{MT}}_3 - \mathsf{d}(\theta^0_{\mathrm{L}}\wedge\theta^5_{\mathrm{L}})).$$

For $\mathfrak{t}_{\rm vac}^{(0)} = \langle P_0, P_5 \rangle$ and $\mathfrak{h}_{\rm vac} = \bigoplus_{a'<b' \in \overline{1,4}} \langle J_{a'b'} \rangle \oplus \bigoplus_{a''<b'' \in \overline{6,9}} \langle J_{a''b''} \rangle$, we find $\eth^{-1}\mathfrak{t}_{\rm vac}^{(0)} = \bigoplus_{a'=1}^{4} \langle P_{a'} \rangle \oplus \bigoplus_{a''=6}^{9} \langle P_{a''} \rangle \equiv \mathfrak{e}^{(0)}$, $\mathsf{P}^{(1)} = \frac{\mathbf{1}_{32} + \underline{\gamma}^0 \underline{\gamma}^5}{2}$ with $\mathrm{tr}_{\mathfrak{t}^{(1)}} \mathsf{P}^{(1)} = \frac{32}{2}$, and the *non*-involutive HP vacuum superdistribution $\mathrm{Vac}(\mathfrak{sB}^{(\rm HP)}_{1,1,\rm MT(2)}) = \bigoplus_{\underline{\alpha} \in \overline{1,16}} \langle \mathcal{T}_{\underline{\alpha}} \rangle \oplus \langle \mathcal{T}_0, \mathcal{T}_5 \rangle$ which is bracket-generating for $\mathcal{T}\Sigma^{\rm HP}$, *i.e.*,

$$\mathrm{Vac}^{-\infty}(\mathfrak{sB}^{(\rm HP)}_{1,1,\rm MT(2)}) \equiv \mathcal{T}\Sigma^{\rm HP}.$$

Geometrisation of the canonical data in the HP formulation encompasses also symmetries of the super-σ-model. Upon restriction to the HP section, their geometric realisation becomes largely determined by the inherent redundancy of the geometric model of $\mathrm{G}/\mathrm{H}_{\rm vac}$ within G over the $|\mathcal{U}_i^{\rm H_{vac}}| \cap |\mathcal{U}_j^{\rm H_{vac}}|$, $i, j \in I_{\rm H_{vac}}$. In the case of the global supersymmetry described by left translations on G, the redundancy affects merely the *description* of the field-theoretic entity: The ambiguity of the choice of the index $j \in I$ to be assigned to a supersymmetric translate of a given $\mathcal{V}_i$ leaves us with a canonical definition of the vector-field germ of the action only (in the field theory, this ambiguity is masked by the $\mathrm{H_{vac}}$-basicness of the tensors entering the definition of the action functional). The local tangent lifts $\mathcal{K}_{A\,i}(\sigma_i^{\rm vac}(\underline{\chi_i})) \equiv \mathsf{T}_{\underline{\chi_i}} \sigma_i^{\rm vac}\big(\underline{\mathcal{K}}_A(\underline{\chi_i})\big)$, to $\mathcal{V}_i$, $i \in I_{\rm H_{vac}}$, of the fundamental vector fields $\underline{\mathcal{K}}_A$ for $[\ell]^{\rm H_{vac}}$ take the vertically corrected form

$$\mathcal{K}_{A\,i} = R_A \restriction_{\mathcal{V}_i} + \Xi_{A\,i}{}^{\underline{S}} L_{\underline{S}}\,, \qquad A \in \overline{0, D}\,, \tag{5.6}$$

for $\Xi_{A\,i}{}^{\underline{S}} \in \mathcal{O}_{\rm G}(\mathcal{V}_i)$ derived in Ref. [165, Prop. 5.1]. We call the $\mathbb{R}$-linear span of the vector fields $\mathcal{K}_A \equiv \mathcal{K}_{t_A} \in \Gamma\big(\mathcal{T}\Sigma^{\rm HP}\big)$, $\mathcal{K}_A \restriction_{\mathcal{V}_i} = \mathcal{K}_{A\,i}$ the **global-supersymmetry subspace of** $\mathcal{T}\Sigma^{\rm HP}$ and denote it as

$$S_{\rm G}^{\rm HP} = \big\langle\ \mathcal{K}_A \ \big|\ A \in \overline{0, D}\ \big\rangle \subset \Gamma\big(\mathcal{T}\Sigma^{\rm HP}\big)\,. \tag{5.7}$$

Its subspace aligned with the vacuum is termed the **residual global-supersymmetry subspace of** $\mathrm{Vac}(\mathfrak{sB}^{(\rm HP)}_{p,\lambda_p^*})$ and denoted as

$$S_{\rm G}^{\rm HP,vac} \equiv S_{\rm G}^{\rm HP} \cap \mathrm{Vac}\big(\mathfrak{sB}^{(\rm HP)}_{p,\lambda_p^*}\big)\,. \tag{5.8}$$

As for local supersymmetry, to be identified hereunder, the redundancy may actually be employed to encode the very *structure* of the symmetry transformations if we judiciously choose to model them with *right* translations on G. Indeed, such translations do *not*, in general, descend to $\mathrm{G}/\mathrm{H}_{\rm vac}$ in a meaningful manner since changing the arbitrary choice of the local sections $\sigma_i^{\rm H_{vac}}$ on which we define the *right* action leads to a $\rho_{\rm H_{vac}}$-transformation of the parameters of the translation and thus enforces locality of the latter, *cp* Ref. [165, Sec. 5.2]. Taking the translations from a $\rho_{\rm H_{vac}}$-invariant subgroup of G engendered by elements of an $\mathrm{ad}_{\mathfrak{h}_{\rm vac}}$-stabilised Lie sub-superalgebra of $\mathfrak{g}$ with the associated *left*-invariant vector fields from the kernel of the extended super-$(p+2)$-cocycle $\widehat{\chi}_{p+2}$ (as required by the form of a variation of the DF amplitude in the HP formulation, which is that of a differential character), we obtain a natural geometric model of a (tangential) local supersymmetry of the GS super-σ-model. This identification is in keeping with the standard definition of (infinitesimal) gauge symmetries as generators of the kernel of the presymplectic form of the lagrangean field theory, *cp* Ref. [64] and p. 11, as the said (2-)form can be written, with the help of the first-oder formalism referred to in footnote 3, as $\Omega_\sigma[\widehat{\xi}_*] = \sum_{\tau\in\mathfrak{T}_{p+1}} \int_{\mathscr{C}_p\cap\tau} \big(\sigma^{\mathrm{H}_{\mathrm{vac}}}_{\imath_\tau} \circ \widehat{\xi}_*\big)^* \widehat{\chi}_{p+2}$ in terms of the restriction $\widehat{\xi}_* \equiv \widehat{\xi}\restriction_{\mathscr{C}_p}$ of a classical field configuration[6] $\widehat{\xi}$ to the Cauchy hypersurface $\mathscr{C}_p \subset \Omega_p$, *cp* Eq. (2.4). Given that we are ultimately interested in symmetries of the GS super-σ-model in the original NG formulation, we should further restrict $\widehat{\chi}_{p+2}$ to the correspondence superdistribution, and – in particular – take generators of the admissible gauge transformations to come from the latter. Whenever the extra **Even Achirality Constraints** (EAC)

$$(5.9)\qquad \Pi_{(\underline{a}_0,\underline{a}_1|\underline{a}_2,\underline{a}_3,\ldots,\underline{a}_p)} \equiv \mathrm{tr}_{\mathfrak{t}^{(1)}}\big(\Delta^{-1}_{\underline{a}_0\underline{a}_2\underline{a}_3\ldots\underline{a}_p}\,\Delta_{\underline{a}_1\underline{a}_2\underline{a}_3\ldots\underline{a}_p}\,\big(\mathbf{1}_{\delta-d} - \mathsf{P}^{(1)}\big)\big) \stackrel{!}{=} 0$$

are satisfied for all $(p+1)$-tuples $\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_p \in \overline{0,p}$ (and for the $\Delta_{\underline{a}_0\underline{a}_2\underline{a}_3\ldots\underline{a}_p}$ naturally viewed as endomorphisms of $\mathfrak{t}^{(1)}$), our search for *generic* gauge symmetries of the HP/NG correspondence sector leads to the definition of the **enhanced gauge-symmetry superdistribution of** $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$,

$$\mathcal{GS}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \coloneqq \bigoplus_{\underline{\alpha}\in\overline{1,q}} \big\langle \mathcal{T}_{\underline{\alpha}} \equiv \underline{\Lambda}^{\beta}_{\ \underline{\alpha}}\,\mathcal{T}_\beta \big\rangle \oplus \bigoplus_{\widehat{A}\in\overline{\underline{\delta}+1,D-\delta}} \big\langle \mathcal{T}_{\widehat{S}} \big\rangle \subset \mathrm{Corr}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big)\,,$$

modelled on the supervector space $\mathfrak{gs}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \equiv \mathfrak{t}^{(1)}_{\mathrm{vac}} \oplus \mathfrak{d}$. If the constraints are violated, the set of generators of the superdistribution is augmented with vector fields

$$(5.10)\,\mathcal{T}_{\underline{a}_0\underline{a}_1} = \mathcal{T}_{\underline{a}_0} - \tfrac{1}{\delta-d-q}\,\Pi_{(\underline{a}_1,\underline{a}_0|\underline{a}_2,\underline{a}_3,\ldots,\underline{a}_p)}\,\mathcal{T}_{\underline{a}_1}\,, \qquad\qquad \underline{a}_0 < \underline{a}_1 \in \overline{0,p}\,.$$

The inspection of those from our list of the most-studied super-σ-models which admit an integrable vacuum superdistribution, carried out in Ref. [165], reveals that the EAC are satisfied in all superbackgrounds except one, to wit, the GS super-1-brane in $\mathrm{sMink}(d,1|D_{d,1})$, which we detail below.

The appearance of the above tangential gauge symmetries in the correspondence sector leads to a natural enhancement of the invisible gauge-symmetry algebra $\mathfrak{h}_{\mathrm{vac}}$ of the vacuum to the full invisible gauge-symmetry algebra $\mathfrak{h}$ of the super-σ-model in the NG formulation. On the other hand,

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[6]The field theory being topological, there are no kinetic momenta in the canonical description.

the gauge transformations engendered by vectors from $\mathfrak{d}$ do *not* preserve (the gauge of) the vacuum. Our search for gauge supersymmetries of the vacuum requires that we project $\mathcal{GS}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ onto $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$. We thus arrive at the fundamental

Definition 5.7. Adopt the hitherto notation. The **κ-symmetry superdistribution of $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$** is the vacuum component $\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ of the enhanced gauge-symmetry superdistribution of that superbackground, given by

$$\kappa\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) := \mathcal{GS}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \cap \mathrm{Vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big)\,.$$

Thus, if the Even Achirality Constraints (5.9) are satisfied, it takes the form

$$\kappa\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \equiv \bigoplus_{\underline{\alpha}\in\overline{1,q}} \big\langle \mathcal{T}_{\underline{\alpha}} \equiv \underline{\Lambda}^{\beta}{}_{\underline{\alpha}}\,\mathcal{T}_{\beta} \big\rangle\,,$$

and otherwise its set of generators contains additionally the 'chiral' vector fields (5.10).

◇

The postulated status of κ-symmetry – that of a vacuum (gauge) supersymmetry – can be maintained iff the integrable superdistribution engendered by $\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$, *i.e.*, the limit of its weak derived flag, stays within $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$, in which case we call the modelling Lie superalgebra of the limit,

$$(\mathfrak{h}_{\mathrm{vac}} \subset)\,\mathfrak{gs}_{\mathrm{vac}}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big) \subseteq \mathfrak{vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}\big)$$

the **κ-symmetry superalgebra**. Under these circumstances, the vacuum becomes foliated by gauge orbits. The topological field-theoretic intuition invoked earlier, in conjunction with the purely geometric nature of the theory in hand, lead us to envisage the physically most natural scenario in which the *entire* vacuum is a single orbit of a configuration under gauge transformations generated by $\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ and $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ is bracket-generated by $\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$ whose weak derived flag envelops the embedded worldvolume, winning the κ-symmetry superdistribution its name: the **square root of the vacuum**, given in Ref. [165]. We have

PROPOSITION 5.8 ([165, Thm. 5.10]). *Adopt the hitherto notation and assume the constraints listed in Prop. 5.1, as well as the Even Achirality Constraints* (5.9) *to be satisfied. The relation* $\kappa^{-\infty}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}) \subseteq \mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ *holds true if also the Vacuum-Superalgebra Constraints of Prop. 5.2 are satisfied. If,* *in addition, the endomorphisms* $f_{\underline{a}}$ *of* $\mathfrak{t}^{(1)}$ *natually associated with the respective matrices* $(f_{\alpha\beta}{}^{\underline{a}})_{\alpha,\beta\in\overline{1,\delta-d}}$ *satisfy the* ***Odd Achirality Constraints*** *(OAC)*

$$\forall_{\underline{a}\in\overline{0,p}}\ \exists_{f_{\underline{a}}^{-1}\in\mathrm{End}(\mathfrak{t}^{(1)})}\ :\ f_{\underline{a}}^{-1}\circ f_{\underline{a}}=\mathrm{id}_{\mathfrak{t}^{(1)}}\,,$$

$$\forall_{\underline{a},\underline{b}\in\overline{0,p}}\ \exists_{\lambda_{\underline{a}}\in\mathbb{R}^\times}\ :\ \Pi_{\underline{ab}}\equiv\mathrm{tr}_{\mathfrak{t}^{(1)}}\big(f_{\underline{a}}^{-1}\circ\mathsf{P}^{(1)\,\mathrm{T}}\circ f_{\underline{b}}\circ\mathsf{P}^{(1)}\big)\stackrel{!}{=}\lambda_{\underline{a}}\,\delta_{\underline{ab}}\,,$$

$\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ *is bracket-generating for the integrable HP vacuum superdistribution,*

$$\kappa^{-\infty}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}\big)=\mathrm{Vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}\big)\,,$$

and we have

$$\mathfrak{gs}_{\mathrm{vac}}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}\big)=\mathfrak{vac}\big(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}\big)\,.$$

Both $\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ *and* $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ *then descend to* $\mathrm{G}/\mathrm{H}_{\mathrm{vac}}$ *in a manner compatible with global residual supersymmetry as* $[S_{\mathrm{G}}^{\mathrm{HP,vac}},\mathcal{D}\}\subset\mathcal{D}$ *for*

$$\mathcal{D}\in\{\kappa(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p}),\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})\}.$$

As demonstrated in Ref. [165], in all classic super-σ-models listed earlier except the one for the GS super-1-brane in $\mathrm{sISO}(d,1|D_{d,1})/(\mathrm{SO}(1,1)\times\mathrm{SO}(d-1))$ the EAC and the OAC are satisfied, and so the κ-symmetry superdistribution bracket-generates $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p})$ in all of them with the latter involutive – this happens for: the Green–Schwarz (GS) super-0-brane in $\mathrm{sISO}(9,1\,|\,32)/\mathrm{SO}(9)$, the GS super-5-brane in $\mathrm{sISO}(d,1|D_{d,1})/(\mathrm{SO}(5,1)\times\mathrm{SO}(d-5))$, the GS super-2-brane in $\mathrm{sISO}(d,1|D_{d,1})/(\mathrm{SO}(2,1)\times\mathrm{SO}(d-2))$, the Zhou super-1-brane in $\mathrm{SU}(1,1\,|\,2)_2/(\mathrm{SO}(1,1)\times\mathrm{SO}(2))$, the Park–Rey super-1-brane in $(\mathrm{SU}(1,1\,|\,2)\times\mathrm{SU}(1,1\,|\,2))_2/(\mathrm{SO}(1,1)\times\mathrm{SO}(3))$ and the Metsaev–Tseytlin super-1-brane in $\mathrm{SU}(2,2\,|\,4)/(\mathrm{SO}(1,1)\times\mathrm{SO}(3)\times\mathrm{SO}(5))$. The missing super-$\sigma$-model is described in

Example 5.9. **The square root of the Green-Schwarz super-1-brane in** $\mathscr{T}_1^{\mathrm{GS}}\equiv\mathrm{sISO}(d,1\,|\,D_{d,1})/(\mathrm{SO}(1,1)\times\mathrm{SO}(d-1))$**.** Adopt the notation of Example 3.7. The tensors $\Delta_0\equiv-4\overline{\Gamma}^1$, $\Delta_1\equiv4\overline{\Gamma}^0$ and $f_{\underline{a}}\equiv\overline{\Gamma}^{\underline{a}}$ satisfy the identities $\Pi_{(0,1|-)}=-\frac{D_{d,1}}{2}$ and $\Pi_{\underline{ab}}=\frac{D_{d,1}}{2}\big(\delta_{\underline{a}}{}^{\underline{b}}-\eta_{\underline{bb}}\,\epsilon_{\underline{ab}}\big)$, and so manifestly violate the EAC and the OAC. In consequence, the correspondence sector of the super-σ-model exhibits an additional even symmetry generated by $\mathcal{T}_+\equiv\mathcal{T}_0+\mathcal{T}_1$, whence $\mathcal{GS}(\mathscr{T}_1^{\mathrm{GS}})=\bigoplus_{\underline{\alpha}\in\overline{1,\frac{D_{d,1}}{2}}}\langle\mathcal{T}_{\underline{\alpha}}\rangle\oplus\langle\mathcal{T}_+\rangle\oplus\bigoplus_{(\underline{a},\widehat{b})\in\{0,1\}\times\overline{2,d}}\big\langle\mathcal{T}_{\underline{a}\widehat{b}}\big\rangle$. The limit of the weak derived flag of the odd component $\kappa_{\mathrm{odd}}(\mathscr{T}_1^{\mathrm{GS}})\equiv\bigoplus_{\underline{\alpha}=1}^{\frac{D_{d,1}}{2}}\langle\mathcal{T}_{\underline{\alpha}}\rangle$ of the κ-symmetry superdistribution is given by $\kappa^{-\infty}_{\rm odd}(\mathscr{T}_1^{\rm GS}) = \bigoplus_{\underline{\alpha}\in\overline{1,\frac{D_{d,1}}{2}}} \langle\mathcal{T}_{\underline{\alpha}}\rangle \oplus \langle\mathcal{T}_- \equiv \mathcal{T}_0 - \mathcal{T}_1\rangle \subsetneq {\rm Vac}(\mathscr{T}_1^{\rm GS})$. The two vector fields: $\mathcal{T}_\pm$ are complementary, and it is natural to think of them as target-superspace counterparts of the chiral worldsheet diffeomorphisms. Thus, $\kappa_{\rm odd}(\mathscr{T}_1^{\rm GS})$ alone generates one chiral half of the integrable vacuum of the superstring. The appearance of the chiral field $\mathcal{T}_-$ in $\kappa^{-\infty}_{\rm odd}(\mathscr{T}_1^{\rm GS})$ can be understood as a variant of the chiral Sugawara mechanism, *cp* Ref. [164, Remark 6.2]. Upon augmenting $\kappa_{\rm odd}(\mathscr{T}_1^{\rm GS})$ with $\mathcal{T}_+$, that is by taking $\kappa(\mathscr{T}_1^{\rm GS}) \equiv \bigoplus_{\underline{\alpha}=1}^{\frac{D_{d,1}}{2}} \langle\mathcal{T}_{\underline{\alpha}}\rangle \oplus \langle\mathcal{T}_+\rangle$, we obtain $\kappa^{-\infty}(\mathscr{T}_1^{\rm GS}) \equiv {\rm Vac}(\mathscr{T}_1^{\rm GS})$.

Having reconstructed and elucidated in detail the geometry of the classical vacuum of the super-σ-model consistently with the supersymmetries present, we may, finally, pass to the discussion of the higher (super)geometry and cohomology behind it.

6. SUPERGERBES – CONSTRUCTION, DESCENT, CONTRACTION

The fundamental principle of the field theories of interest is invariance of the dynamics under an action of the supersymmetry Lie supergroup ${\rm G}$ on the fibre $\mathcal{M}$ of the field bundle. Consistent imposition of the principle entails existence of a gauge-reduction mechanism that freezes the redundant (super)goldstone degrees of freedom in the vacuum of the theory. Thus, in the distinguished class of supersymmetric field theories considered heretofore, the mechanism makes the incorporation of a nontrivial topological term in the action functional a *necessity*. In this manner, supersymmetry and cohomology become tightly entangled, and the basic question concerning the adequate choice of the cohomology in which to place the super-$(p+2)$-cocycle defining the topological term acquires a straightforward answer, to wit, the **G-invariant de Rham cohomology** $H^\bullet_{\rm dR}(\mathcal{M})^{\rm G}$. In consequence of the inherent non-compactness of ${\rm G}$, the latter choice is material as the Chevalley–Eilenberg Theorem [31, Thm. 2.3] cannot be invoked and there are known examples of de Rham super-coboundaries with *non*trivial classes in $H^\bullet_{\rm dR}(\mathcal{M})^{\rm G}$ – indeed, many of the physically relevant GS super-$(p+2)$-cocycles are of that kind. Therefore, in keeping with the universal argumentation laid out in Sec. 2, we should seek to geometrise the supersymmetric refinement of the de Rham cohomology on the homogeneous spaces $\mathcal{M} \equiv {\rm G}/{\rm K}$. In so doing, the natural strategy is to first internalise the notion of an abelian p-gerbe introduced formerly in the category of supermanifolds and subsequently demand that it be G-invariant in an appropriate sense. The criterion of 'appropriateness' is provided by the recursive definition of a $(p+1)$-gerbe $\mathcal{G}^{(p+1)}$ itself in which the trivial p-gerbe plays the pivotal rôle of a bridge between p-gerbes, assumed known, and the new object $\mathcal{G}^{(p+1)}$ under construction. Thus, it suffices to understand compatibility of the higher-geometric structure with supersymmetry for a trivial $(p+1)$-gerbe and then 'propagate' it consistently all over the lower-rank substructure that resolves $\mathcal{G}^{(p+1)}$. But a trivial $(p+1)$-gerbe is a de Rham primitive of the pullback of the curvature $(p+3)$-cocycle, and so we are naturally led to

Definition 6.1 ([165, Def. 6.2]). Let G be a Lie supergroup with a non-zero odd component of $\mathcal{O}_{\mathrm{G}}$ and $\mathcal{M}$ a supermanifold endowed with an action $\lambda : \mathrm{G} \times \mathcal{M} \longrightarrow \mathcal{M}$ of G. A **super-p-gerbe**, or a **supersymmetric p-gerbe**, is an abelian (bundle) p-gerbe, in the sense of Ref. [63], with total spaces of all surjective submersions entering its definition endowed with the respective (projective) lifts of λ, commuting with the defining $\mathbb{C}^\times$-actions on the total spaces of all the principal $\mathbb{C}^\times$-bundles present, with respect to which the submersions are equivariant, all connections are invariant and all (connection-preserving) principal $\mathbb{C}^\times$-bundle isomorphisms are equivariant. **Super-p-gerbe k-isomorphisms**, or **supersymmetric p-gerbe k-isomorphisms**, for $k \in \overline{1, p+1}$ are defined analogously. $\diamond$

Remark 6.2. Using the structure of a super-p-gerbe, we may readily construct super-p-gerbe 1-isomorphisms generalising (2.6) that lift the action of G to the geometric object $\mathcal{G}^{(p)}$ from its base. This implication was illustrated in [164, Secs. 4.1 & 4.2] where definitions of super-0- and super-1-gerbes were written out in detail. The feature is central to any application of super-p-gerbes in a field theory whose DF amplitude is determined by the differential character associated with $\mathcal{G}^{(p)}$. It is preserved in a more general situation in which instead of the lifts of the action of G on the base $\mathcal{M}$ of the gerbe to its various surjective submersions we consider actions, on these total spaces, of respective *compatible extensions* of G, by which we mean that whenever there is an action λ of G as above on the base of a surjective submersion $\pi_{\mathsf{Y}\mathcal{M}} : \mathsf{Y}\mathcal{M} \longrightarrow \mathcal{M}$, that action is covered by an action $\mathsf{Y}\lambda : \mathsf{Y}\mathrm{G} \times \mathsf{Y}\mathcal{M} \longrightarrow \mathsf{Y}\mathcal{M}$ of an extension $\pi_{\mathsf{Y}\mathrm{G}} : \mathsf{Y}\mathrm{G} \longrightarrow \mathrm{G}$ of G on its total space, that is $\pi_{\mathsf{Y}\mathcal{M}} \circ \mathsf{Y}\lambda = \lambda \circ (\pi_{\mathsf{Y}\mathrm{G}} \times \pi_{\mathsf{Y}\mathcal{M}})$. It is this more general scenario that is naturally encountered in the physical setting, and we shall refer to it by the name of a **generalised super-p-gerbe**, or a **generalised supersymmetric p-gerbe**, with the corresponding k-isomorphisms to be called **generalised supersymmetric p-gerbe k-isomorphisms**.

The cohomological considerations motivating the above definition raise a natural question as to the character of the extra topological information – if any

– contained in $H^\bullet_{\rm dR}(\mathcal{M})^{\rm G}$. The question was first addressed in the convenient setting of the GS super-σ-model for the super-1-brane in $\mathcal{M} \equiv {\rm sMink}(d,1|D_{d,1})$ (with $H^\bullet_{\rm dR}(\mathcal{M}) \equiv 0$) by Rabin and Crane in Refs. [136, 135], in which the authors looked for a super-orbifold $\mathcal{M}/\Gamma$ of the original supertarget $\mathcal{M}$ relative to a proper sub-supergroup $\Gamma \subset \underline{\rm G}$ of the (restricted) supersymmetry group $\underline{\rm G} \equiv {\rm sMink}(d,1|D_{d,1})$ such that $\mathcal{M}/\Gamma \cong_{\rm loc.} \mathcal{M}$ and – crucially – $H^\bullet_{\rm dR}(\mathcal{M}/\Gamma) \cong H^\bullet_{\rm dR}(\mathcal{M})^{\underline{\rm G}}$, *i.e.*, with the homology dual to and so carrying the information on $H^\bullet_{\rm dR}(\mathcal{M})^{\underline{\rm G}}$. The orbifold supergroup Γ was identified as the discrete Kostelecký–Rabin supersymmetry group generated by integral odd translations (in the DeWitt description of $\mathcal{M}$), *cp* Ref. [136] (and Ref. [161, Rem. 4.1]) and known from the previous studies of supersymmetric lattice field theories reported in Ref. [110]. The appearance of *compact* (periodic) odd fibres in the super-orbifold led the Author to consider, in the analysis of the Poisson algebra of Noether charges of global supersymmetry in the same super-σ-model, carried out in Ref. [162, Sec. 4], the eventuality of having a non-vanishing monodromy of the odd components θ^α, $\alpha \in \overline{1, D_{d,1}}$ of the embedding field along the Cauchy contour $\mathscr{C}_1 \cong \mathbb{S}^1$ in the worldsheet. The analysis yielded the Green supercentral extension of the Lie superalgebra $\mathfrak{smink}(d,1|D_{d,1})$ of Example 3.3, and together with analogous results obtained for several other super-σ-models lent weight to the Rabin–Crane idea of encoding the discrepancy between $H^\bullet_{\rm dR}(\mathcal{M})^{\rm G}$ and $H^\bullet_{\rm dR}(\mathcal{M})$ in extensions of the supersymmetry algebra $\mathfrak{g}$ by super-p-brane **wrapping charges** sourced by monodromies around, *i.a.*, odd cycles in the supertarget (to be). The field-theoretic observation can be formalised upon lifting the de Rham-cohomological analysis to the supermanifold[7] G and invoking simple cohomological correspondences that we recapitulate below.

Let us begin with the Lie-superalgebra cohomology, first introduced in the $\mathbb{Z}/2\mathbb{Z}$-graded setting by Leïtes in Ref. [115] (*cp* also Ref. [161, App. C] for the relevant results), which permits us to state the classic isomorphism

$$\text{(6.1)} \qquad {\rm CaE}^\bullet({\rm G}) := H^\bullet_{\rm dR}({\rm G})^{\rm G} \cong H^\bullet(\mathfrak{g},\mathbb{R}) =: {\rm CE}^\bullet(\mathfrak{g})$$

between the Cartan–Eilenberg (CaE) cohomology of G and the Chevalley–Eilenberg (CE) cohomology of its tangent Lie superalgebra $\mathfrak{g}$ with values in the trivial module $\mathbb{R}$ of the latter, and use it to rewrite the differential-geometric problem in hand as a superalgebraic one. Indeed, the isomorphism sets (the classes of) the GS super-$(p+2)$-cocycles in correspondence with (equivalence classes of) Lie-superalgebraic structures based on $\mathfrak{g}$ and enumerated by elements of ${\rm CE}^\bullet(\mathfrak{g})$. In general, the correspondence associates with a class in

[7] As the argument bases on the Lie-supergroup structure, we may stay on $\underline{\rm G}$ in the super-Minkowskian case.

$\mathrm{CaE}^{p+2}(\mathrm{G})$ a so-called **slim Lie $(p+1)$-superalgebra** of Baez and Huerta, introduced in Refs. [15, 98] after the pioneering work [12], due to Baez and Crans, on the non-$\mathbb{Z}/2\mathbb{Z}$-graded precursor of the correspondence. The latter are special examples of the L_∞-superalgebras of Stasheff and Lada, having appeared naturally in the setting of closed string field theory, *cp* Refs. [153, 117]. There exist some preliminary ideas about their integration to the corresponding **Lie $(p+1)$-supergroups**, *cp* Ref. [98, Chap. 7], that constitute the basis of the formal constructions of Fiorenza, Sati and Schreiber postulated in Ref. [60], but the range of applicability of the known integration method remains an open question and no *concrete* examples of physical (superstring-theoretic) relevance have been constructed explicitly to date, so that, in particular, the all-important equivariance properties of the formal constructs are unknown. Luckily, in a large class of super-σ-models, a non-algorithmic yet constructive and sufficiently robust alternative to the integration of the L_∞-superalgebras seems to exist, first advanced by de Azcárraga *et al.* in Ref. [30] and subsequently elaborated and employed in a construction of super-p-gerbes and the attendant morphisms by the Author in a series of papers [161, 164, 162, 163]. It employs the notion of a **supercentral extension** $(\widetilde{\mathfrak{g}},[\cdot,\cdot\}_{\widetilde{\mathfrak{g}}})$ of a Lie superalgebra $(\mathfrak{g},[\cdot,\cdot\}_{\mathfrak{g}})$ by a supercommutative one $(\mathfrak{a},[\cdot,\cdot\}_{\mathfrak{a}}\equiv 0)$, encoded in a short exact sequence of Lie superalgebras

$$\mathbf{0}\longrightarrow\mathfrak{a}\xrightarrow{\jmath_{\mathfrak{a}}}\widetilde{\mathfrak{g}}\xrightarrow{\pi_{\widetilde{\mathfrak{g}}}}\mathfrak{g}\longrightarrow\mathbf{0}\,,$$

with **equivalences** of such supercentral extensions $(\widetilde{\mathfrak{g}}_A,[\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_A}),A\in\{1,2\}$ represented by commutative diagrams (in **sLieAlg**)

$$\begin{tikzcd} & & \widetilde{\mathfrak{g}}_1 \arrow[dd, "\cong"] \arrow[dr, "\pi_{\widetilde{\mathfrak{g}}_1}"] & & \\ \mathbf{0} \arrow[r] & \mathfrak{a} \arrow[ur, "\jmath^1_{\mathfrak{a}}"] \arrow[dr, "\jmath^2_{\mathfrak{a}}"'] & & \mathfrak{g} \arrow[r] & \mathbf{0} \\ & & \widetilde{\mathfrak{g}}_2 \arrow[ur, "\pi_{\widetilde{\mathfrak{g}}_2}"'] & & \end{tikzcd}\,,$$

and hinges upon

THEOREM 6.3 ([161, Props. C.4 & C.5]). *Adopt the hitherto notation and let $(\mathfrak{g},[\cdot,\cdot\}_{\mathfrak{g}})$ and $(\mathfrak{a},[\cdot,\cdot\}_{\mathfrak{a}}\equiv 0)$ be two Lie superalgebras over a field $\mathbb{K}$, with $\mathfrak{a}$ regarded as a trivial $\mathfrak{g}$-module. There exists a one-to-one correspondence between classes in $H^2_0(\mathfrak{g},\mathfrak{a})$ and equivalence classes of supercentral extensions of $\mathfrak{g}$ by $\mathfrak{a}$.*

Taking all the above into account, suppose there exists, for a given Lie superalgebra $(\mathfrak{g},[\cdot,\cdot\}_{\mathfrak{g}})\equiv(\widetilde{\mathfrak{g}}_0,[\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_0})$ coming with a $(p+2)$-cocycle $\Omega\equiv\widetilde{\Omega}_0\in C^{p+2}(\mathfrak{g},\mathbb{R})$ with values in the trivial module $\mathbb{R}$ that represents the class $[\Omega]_{\mathfrak{g}} \in \mathrm{CE}^{p+2}(\mathfrak{g})$ corresponding to the class $[\chi_{p+2}]_{\mathrm{CaE}} \in \mathrm{CaE}^{p+2}(\mathrm{G})$ of a super-$(p+2)$-cocycle $\chi_{p+2} \in \Omega^{p+2}(\mathrm{G})^{\mathrm{G}}$ and for some $N \in \mathbb{N}$, a collection $\{(\mathfrak{a}_j, [\cdot,\cdot\}_{\mathfrak{a}_j} \equiv 0)\}_{j\in\overline{1,N}}$ of supercommutative Lie superalgebras and a collection $\{(\widetilde{\mathfrak{g}}_j, [\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_j})\}_{j\in\overline{1,N}}$ of supercentral Lie-superalgebra extensions

$$\mathbf{0} \longrightarrow \mathfrak{a}_{k+1} \xrightarrow{\jmath_{\mathfrak{a}_{k+1}}} \widetilde{\mathfrak{g}}_{k+1} \xrightarrow{\pi_{\widetilde{\mathfrak{g}}_{k+1}} \equiv \widetilde{\pi}_{k+1}} \widetilde{\mathfrak{g}}_k \longrightarrow \mathbf{0}\,, \qquad k \in \overline{0, N-1}$$

determined – *via* Thm. 6.3 – by respective super-2-cocycles $\widetilde{\Theta}_k \in Z^2(\widetilde{\mathfrak{g}}_k, \mathfrak{a}_{k+1})$ and endowed with respective $(\widetilde{\Omega}_j, \widetilde{\eta}_j) \in Z^{p+2}(\widetilde{\mathfrak{g}}_j, \mathbb{R}) \times C^{p+1}(\widetilde{\mathfrak{g}}_j, \mathbb{R})$ such that

$$\widetilde{\pi}_{k+1}^*\widetilde{\Omega}_k = \widetilde{\Omega}_{k+1} + \delta^{(p+1)}_{\widetilde{\mathfrak{g}}_{k+1}}\widetilde{\eta}_{k+1}\,, \qquad k \in \overline{0, N-1}\,,$$

where either $[\Omega = \delta^{(p+1)}_{\mathfrak{g}}\widetilde{\eta}_1]_{\mathfrak{g}} = 0 \in \mathrm{CE}^{p+2}(\mathfrak{g})$ with $N = 1$, $\mathfrak{a}_1 = \mathbf{0}$ and $\widetilde{\pi}_1 \equiv \mathrm{id}_{\mathfrak{g}}$, or $[\widetilde{\Omega}_k]_{\widetilde{\mathfrak{g}}_k} \neq 0 \in \mathrm{CE}^{p+2}(\widetilde{\mathfrak{g}}_k)$ for all $k \in \overline{0, N-1}$ and $\widetilde{\Omega}_N \equiv 0$ with $N \geq 1$. The above yields, for $\widetilde{\pi}_{N,j} \equiv \widetilde{\pi}_{N+1-j} \circ \widetilde{\pi}_{N+2-j} \circ \cdots \circ \widetilde{\pi}_N$, $j \in \overline{1,N}$ and $\widetilde{\pi}_{N,0} \equiv \mathrm{id}_{\widetilde{\mathfrak{g}}_N}$, the identity

$$\widetilde{\pi}_{N,N}^*\Omega = \delta^{(p+1)}_{\widetilde{\mathfrak{g}}_N}\widetilde{\eta}\,, \qquad \widetilde{\eta} = \sum_{k\in\overline{0,N-1}} \widetilde{\pi}_{N,k}^*\widetilde{\eta}_{N-k}\,.$$

Suppose, furthermore, that each of the Lie-superalgebra extensions integrates to a corresponding Lie-supergroup extension described by a short exact sequence

$$\mathbf{1} \longrightarrow A_{k+1} \xrightarrow{\jmath_{A_{k+1}}} \widetilde{\mathrm{G}}_{k+1} \xrightarrow{\pi_{\widetilde{\mathrm{G}}_{k+1}} \equiv \widetilde{\varpi}_{k+1}} \widetilde{\mathrm{G}}_k \longrightarrow \mathbf{1}$$

of Lie supergroups, with $\mathrm{sLie}(A_{k+1}) \equiv \mathfrak{a}_{k+1}$ and $\mathrm{sLie}(\widetilde{\mathrm{G}}_k) \equiv \widetilde{\mathfrak{g}}_k$, a process controlled by a super-variant of the Tuynman-Wiegerinck Criterion of Ref. [171, Thm. 5.4]. We then obtain – in virtue of relation (6.1) – a collection $\{\widetilde{\beta}_j\}_{j\in\overline{1,N}}$ of LI super-$(p+1)$-forms $\widetilde{\beta}_j \in \Omega^{p+1}(\widetilde{\mathrm{G}}_j)^{\widetilde{\mathrm{G}}_j}$ which collectively trivialise the pullback of χ_{p+2} to $\widetilde{\mathrm{G}}_N$ in the corresponding CaE cohomology as

$$\widetilde{\varpi}_{N,N}^*\,\chi_{p+2} = \mathsf{d}\widetilde{\beta}\,, \qquad \widetilde{\beta} = \sum_{k\in\overline{0,N-1}} \widetilde{\varpi}_{N,k}^*\widetilde{\beta}_{N-k}\,,$$

where $\widetilde{\varpi}_{N,j} \equiv \widetilde{\varpi}_{N+1-j} \circ \widetilde{\varpi}_{N+2-j} \circ \cdots \circ \widetilde{\varpi}_N$, $j \in \overline{1,N}$ and $\widetilde{\varpi}_{N,0} \equiv \mathrm{id}_{\widetilde{\mathrm{G}}_N}$. It is, then, natural to take $\pi_{\mathsf{Y}\mathrm{G}} \equiv \widetilde{\varpi}_{N,N} \,:\, \mathsf{Y}\mathrm{G} \equiv \widetilde{\mathrm{G}}_N \longrightarrow \mathrm{G}$ and $\widetilde{\beta}$ as the surjective submersion and the curving, respectively, of a p-gerbe over G that we associate with the super-$(p+2)$-form χ_{p+2}, fixing the latter as its curvature. At this stage, we must assume that the stepwise procedure of trivialisation of the super-$(p+2)$-cocycle can be successfully applied to all the remaining super-$(p+1-l)$-forms, with $l \in \overline{0,p}$, encountered along the way in the reconstruction of the super-p-gerbe with the formerly established surjective submersion and curving, whereby we arrive at

Definition 6.4. A **Cartan–Eilenberg super-p-gerbe** is a generalised super-p-gerbe object in **sLieGrp**, with all constitutive surjective submersions given by Lie-supergroup extensions, all connection-preserving principal $\mathbb{C}^\times$-bundle isomorphisms lifting to Lie-supergroup isomorphisms, all Lie-supergroup actions induced by respective products on their total spaces and all structural differential forms on the latter (left-)invariant.

◇

We exemplify the structure described in the general definition with the physically much relevant specialisation contained in

Definition 6.5 ([165, Def. 6.10]). A **Cartan-Eilenberg super-1-gerbe** is a generalised super-1-gerbe in the sense of Def. 6.1 and Rem. 6.2, such that

- $\mathcal{M} \equiv \mathrm{G}$, taken with the left action induced by the multiplication of G;
- $\mathsf{Y}\mathcal{M} \equiv \mathsf{Y}\mathrm{G}$ resp. L are endowed with a Lie-supergroup (sLieGrp) structure extending that on G resp. on $\mathsf{Y}^{[2]}\mathcal{M}$;
- $\mathcal{B}$ is left-$\mathsf{Y}\mathrm{G}$-invariant and $\mathcal{A}_L$ is left-L-invariant;
- $\pi_{\mathsf{Y}\mathrm{G}}$, π_L and μ_L are sLieGrp homomorphisms.

A **1-isomorphism between Cartan-Eilenberg super-1-gerbes**

$$(\mathsf{Y}_A\mathrm{G}, \pi_{\mathsf{Y}_A\mathrm{G}}, \mathcal{B}_A, L_A, \pi_{L_A}, \mathcal{A}_{L_A}, \mu_{L_A}), \ A \in \{1,2\}$$

is a super-1-gerbe isomorphism $\Phi = (\mathsf{Y}\mathsf{Y}_{1,2}\mathrm{G},\ \pi_{\mathsf{Y}\mathsf{Y}_{1,2}\mathrm{G}}, E, \pi_E, \mathcal{A}_E, \alpha_E)$ with the following properties:

- $\mathsf{Y}\mathsf{Y}_{1,2}\mathrm{G}$ is endowed with a sLieGrp structure that extends the one on $\mathsf{Y}_{1,2}\mathrm{G}$;
- E is endowed with a sLieGrp structure that extends that on $\mathsf{Y}\mathsf{Y}_{1,2}\mathrm{G}$;
- $\mathcal{A}_E$ is left-E-invariant;
- $\pi_{\mathsf{Y}\mathsf{Y}_{1,2}\mathrm{G}}$, π_E and α_E are sLieGrp homomorphisms.

A **2-isomorphism between 1-isomorphisms**

$$\Phi_A = (\mathsf{Y}^A\mathsf{Y}_{1,2}\mathrm{G}, \pi_{\mathsf{Y}^A\mathsf{Y}_{1,2}\mathrm{G}}, E_A, \pi_{E_A}, \mathcal{A}_{E_A}, \alpha_{E_A}), \ A \in \{1,2\}$$

between Cartan-Eilenberg super-1-gerbes

$$(\mathsf{Y}_B\mathrm{G}, \pi_{\mathsf{Y}_B\mathrm{G}}, \mathcal{B}_B, L_B, \pi_{L_B}, \mathcal{A}_{L_B}, \mu_{L_B}), \ B \in \{1,2\}$$

is a super-1-gerbe 2-isomorphism $\varphi = (\mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M,\ \pi_{\mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M}, \beta)$ with the following properties:

• $\mathsf{YY}^{1,2}\mathsf{Y}_{1,2}M$ is endowed with a sLieGrp structure that extends that on $\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}\mathrm{G}$;

• β is a sLieGrp homomorphism.

$\diamond$

If the homogeneous space of physical interest is itself a Lie supergroup, as is the case, *e.g.*, for $\mathrm{sISO}(d,1|D_{d,1})/\mathrm{SO}(d,1) \equiv \mathrm{sMink}(d,1|D_{d,1})$, and we may consistently restrict to the sub-supersymmetry engendered by it, the above constructions, whenever effective, give us a desired supersymmetric geometrisation of the GS super-$(p+2)$-cocycle whose adequacy may subsequently be tested, *cp* the next section. If, on the other hand, there is no obvious Lie-supergroup structure on the supertarget G/H, we must ensure that the higher-geometric object descends to the quotient from the supermanifold G over which it has been erected, and that in a manner compatible with the supersymmetry present. In the light of Thm. 2.5, this is achieved by putting a *descendable* H-equivariant structure on the CaE super-p-gerbe over G, and we ought to further assume the structure to be generalised supersymmetric, in the sense of Def. 6.1 (and Rem. 6.2), with respect to the natural lifts $\ell \times \mathrm{id}_{\mathrm{H}^{\times n}}$ of the left regular G-action ℓ to the components $\mathrm{G} \times \mathrm{H}^{\times n}$, $n \in \mathbb{N}^\times$ of the nerve $\mathsf{N}^\bullet(\mathrm{G} \rtimes \mathrm{H}) \equiv \mathrm{G} \times \mathrm{H}^{\times \bullet}$ of the (right-)action groupoid $\mathrm{G} \rtimes \mathrm{H}$. These lifts exemplify the general construction of an extension of an action of the global-symmetry group G_0 of a (super-)σ-model with a (super)target $\mathcal{M}$ to the nerve $\mathsf{N}^\bullet(\mathrm{G}.\ltimes\mathcal{M})$ of the action groupoid $\mathrm{G}.\ltimes\mathcal{M}$ whose relevance to the gauging of a global symmetry modelled by $\mathrm{G}. \equiv \mathrm{G}_\sigma$ was indicated in Sec. 2. Given actions $\ell : \mathrm{G}_0 \times \mathcal{M} \longrightarrow \mathcal{M}$ and $\alpha : \mathrm{G}_0 \times \mathrm{G}. \longrightarrow \mathrm{G}.$, a natural constraint to be imposed upon them is the requirement of G_0-equivariance of the face maps of $\mathsf{N}^\bullet(\mathrm{G}.\ltimes\mathcal{M})$ relative to the extension. This readily yields – upon inspection of the face maps $d^{(\bullet)}_{i_\bullet}$ of the nerve – the compatibility conditions ($\tau : \mathrm{G}_0 \times \mathrm{G}. \longrightarrow \mathrm{G}. \times \mathrm{G}_0$ is the standard transposition, and $\mu_{\mathrm{G}.}$ is the binary operation on $\mathrm{G}.$)

(6.2)
$$\begin{aligned} &\lambda \circ \big(\alpha \times \ell\big) \circ \Big(\big(\mathrm{id}_{\mathrm{G}_0} \times \tau\big) \circ \Big(\big(\mathrm{id}_{\mathrm{G}_0}, \mathrm{id}_{\mathrm{G}_0}\big) \times \mathrm{id}_{\mathrm{G}.}\Big) \times \mathrm{id}_{\mathcal{M}}\Big) \overset{!}{=} \ell \circ \big(\mathrm{id}_{\mathrm{G}_0} \times \lambda\big)\,, \\ &\mu_{\mathrm{G}.} \circ \big(\alpha \times \alpha\big) \circ \big(\mathrm{id}_{\mathrm{G}_0} \times \tau \times \mathrm{id}_{\mathrm{G}.}\big) \circ \Big(\big(\mathrm{id}_{\mathrm{G}_0}, \mathrm{id}_{\mathrm{G}_0}\big) \times \mathrm{id}_{\mathrm{G}.\times\mathrm{G}.}\Big) \overset{!}{=} \alpha \circ \big(\mathrm{id}_{\mathrm{G}_0} \times \mu_{\mathrm{G}.}\big)\,. \end{aligned}$$

The structure is then to be thought of as a geometrisation of a class in the refinement $\mathrm{CaE}^\bullet(\mathrm{G})_{\mathrm{H-basic}}$ of $\mathrm{CaE}^\bullet(\mathrm{G})$ in which we work with H-basic (and G-invariant) forms on G. We thus wind up with the all-important

Definition 6.6. Adopt the hitherto notation. A **descended Cartan–Eilenberg super-p-gerbe for $\mathfrak{sB}_p^{(\mathrm{NG})}$** is a generalised super-p-gerbe $\mathcal{G}^{(p)}$ over G/H determined by a CaE super-p-gerbe $\widetilde{\mathcal{G}}^{(p)}$ over G of curvature

χ_{p+2} with a generalised supersymmetric descendable H-equivariant structure as the p-gerbe whose pullback $\pi_{\mathrm{G/H}}^*\mathcal{G}^{(p)}$ is (generalised-)supersymmetrically 1-isomorphic with $\widetilde{\mathcal{G}}^{(p)}$.

$\diamond$

We illustrate the above abstract notions on two physically relevant examples.

Example 6.7. **The CaE super-1-gerbe for**

$$\mathfrak{sB}^{(\mathrm{NG})}_{1,\mathrm{GS}} \equiv (\mathrm{sMink}(d,1|D_{d,1}),\eta,\chi_3^{\mathrm{GS}}).$$

Adopt the notation of Example 3.7 and let $\{q^\alpha\}^{\alpha\in\overline{1,D_{d,1}}} \cup \{p^a\}^{a\in\overline{0,d}}$ be the basis of $\mathfrak{smink}(d,1|D_{d,1})^*$ dual to $\{Q_\alpha\}_{\alpha\in\overline{1,D_{d,1}}} \cup \{P_a\}_{a\in\overline{0,d}}$, with $q^\alpha(Q_\beta) = \delta^\alpha{}_\beta$ and $p^a(P_b) = \delta^a{}_b$. The CaE 2-cocycles $\widetilde{\Theta}_{\widehat{\alpha}} = 2\overline{\Gamma}_{a\,\widehat{\alpha}\beta}\, q^\beta \wedge p^a$, $\widehat{\alpha} \in \overline{1,D_{d,1}}$ (whose closedness is ensured by (3.3) for $q=3$) give rise to a supercentral extension

$$\mathbf{0} \longrightarrow \bigoplus_{\widehat{\alpha}=1}^{D_{d,1}} \big\langle Z^{\widehat{\alpha}}\big\rangle \equiv \mathfrak{a}_1 \xrightarrow{0\oplus \mathrm{id}_{\mathfrak{a}_1}} \mathfrak{smink}\big(d,1|D_{d,1}\big) \oplus \mathfrak{a}_1 \equiv \widetilde{\mathfrak{g}}_1$$

$$\xrightarrow{\widetilde{\pi}_1\equiv \mathrm{pr}_1} \mathfrak{smink}\big(d,1|D_{d,1}\big) \equiv \mathfrak{g} \longrightarrow \mathbf{0}$$

with the Lie superbracket $[\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_1}$ on $\widetilde{\mathfrak{g}}_1 \ni X_1, X_2$ given by

$$[X_1,X_2\}_{\widetilde{\mathfrak{g}}_1} = [\mathrm{pr}_1(X_1),\mathrm{pr}_1(X_2)\}_{\mathfrak{g}} + \widetilde{\Theta}_{\widehat{\alpha}}\big(\mathrm{pr}_1(X_1),\mathrm{pr}_1(X_2)\big)\, Z^{\widehat{\alpha}}.$$

The Lie-superalgebra extension integrates to a Lie-supergroup extension

$$\mathbf{1} \longrightarrow \mathbb{R}^{0\,|\,D_{d,1}} \xrightarrow{(0,\mathrm{id}_{\mathbb{R}^{0\,|\,D_{d,1}}})} \mathrm{sMink}(d,1\,|\,D_{d,1}) \ltimes \mathbb{R}^{0\,|\,D_{d,1}} \equiv \widetilde{\mathrm{G}}_1$$

$$\xrightarrow{\widetilde{\varpi}_1\equiv \mathrm{pr}_1} \mathrm{sMink}(d,1\,|\,D_{d,1}) \longrightarrow \mathbf{1}\,,$$

with the binary supergroup operation on $\widetilde{\mathrm{G}}_1$ (written, in the $\mathcal{S}$-point picture, in the coordinates: $X_A \equiv (\theta^\alpha_A, x^a_A)$, $A \in \{1,2\}$ on $\mathrm{sMink}(d,1\,|\,D_{d,1})$ and $\xi_{A\widehat{\alpha}}$ on $\mathbb{R}^{0\,|\,D_{d,1}}$)

$$\widetilde{\mu}_1\big(\big(X_1,\xi_{1\widehat{\alpha}}\big),\big(X_2,\xi_{2\widehat{\alpha}}\big)\big) = \big(\mu(X_1,X_2),\xi_{1\widehat{\alpha}} + \xi_{2\widehat{\alpha}} + \overline{\Gamma}_{a\,\widehat{\alpha}\beta}\,\theta^\beta_1\, x^a_2 \\ -\tfrac{1}{6}\left(\theta_1\,\overline{\Gamma}_a\,\theta_2\right)\overline{\Gamma}^a_{\widehat{\alpha}\beta}\left(2\theta^\beta_1 + \theta^\beta_2\right)\big)\,.$$

Next, upon equipping the supervector space

$$\widetilde{\mathfrak{g}}_1^{[2]} \equiv \widetilde{\mathfrak{g}}_1 \oplus_{\mathfrak{smink}(d,1|D_{d,1})} \widetilde{\mathfrak{g}}_1 = \big\{\ (X_1,X_2) \in \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_1 \mid \mathrm{pr}_1(X_1) = \mathrm{pr}_1(X_2)\ \big\}\,,$$

$$\big(\widetilde{\mathfrak{g}}_1^{[2]}\big)^{(0)} = \bigoplus_{a=0}^{d} \langle (P_a,P_a)\rangle\ ,$$

$$\big(\widetilde{\mathfrak{g}}_1^{[2]}\big)^{(1)} = \bigoplus_{\alpha=1}^{D_{d,1}} \langle (Q_\alpha,Q_\alpha)\rangle \oplus \bigoplus_{\widehat{\alpha}=1}^{D_{d,1}} \big\langle (Z^{\widehat{\alpha}},0)\big\rangle \oplus \bigoplus_{\widehat{\beta}=1}^{D_{d,1}} \big\langle (0,Z^{\widehat{\beta}})\big\rangle$$

with the direct-sum Lie superbracket $[\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_1^{[2]}} = ([\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_1} \circ (\mathrm{pr}_1 \otimes \mathrm{pr}_1), [\cdot,\cdot\}_{\widetilde{\mathfrak{g}}_1} \circ (\mathrm{pr}_2 \otimes \mathrm{pr}_2))$, we find, on the resulting Lie superalgebra, the CaE 2-cocycle $\widetilde{\Theta} = \delta_\alpha^{\widehat{\alpha}}\, q^\alpha \wedge z_{\widehat{\alpha}} \circ (\mathrm{pr}_2 \otimes \mathrm{pr}_2 - \mathrm{pr}_1 \otimes \mathrm{pr}_1)$, defined in terms of elements of the basis $\{z_{\widehat{\alpha}}\}_{\widehat{\alpha}\in\overline{1,D_{d,1}}}$ of $\mathfrak{a}_1^*$ dual to $\{Z^{\widehat{\alpha}}\}^{\widehat{\alpha}\in\overline{1,D_{d,1}}}$. The 2-cocycle induces a central extension

$$\mathbf{0} \longrightarrow \langle Z \rangle \equiv \mathfrak{a} \xrightarrow{0\oplus\mathrm{id}_{\mathfrak{a}}} \widetilde{\mathfrak{g}}_1^{[2]} \oplus \mathfrak{a} \equiv \widetilde{\mathfrak{l}} \xrightarrow{\mathrm{pr}_1} \widetilde{\mathfrak{g}}_1^{[2]} \longrightarrow \mathbf{0}$$

with the Lie superbracket $[\cdot,\cdot\}_{\widetilde{\mathfrak{l}}}$ on $\widetilde{\mathfrak{l}} \ni \widetilde{X}_1, \widetilde{X}_2$ given by

$$[\widetilde{X}_1, \widetilde{X}_2\}_{\widetilde{\mathfrak{l}}} = [\mathrm{pr}_1(\widetilde{X}_1), \mathrm{pr}_1(\widetilde{X}_2)\}_{\widetilde{\mathfrak{g}}_1^{[2]}} + \widetilde{\Theta}\big(\mathrm{pr}_1(\widetilde{X}_1), \mathrm{pr}_1(\widetilde{X}_2)\big)\, Z.$$

The Lie-superalgebra extension integrates to a central Lie-supergroup extension

$$\mathbf{1} \longrightarrow \mathbb{C}^\times \xrightarrow{(0,\mathrm{id}_{\mathbb{C}^\times})} \widetilde{\mathrm{G}}_1^{[2]} \ltimes \mathbb{C}^\times \equiv L \xrightarrow{\pi_L \equiv \mathrm{pr}_1} \widetilde{\mathrm{G}}_1^{[2]} \longrightarrow \mathbf{1}\,,$$

of the Lie sub-supergroup $\widetilde{\mathrm{G}}_1^{[2]} \equiv \widetilde{\mathrm{G}}_1 \times_{\mathrm{sMink}(d,1|D_{d,1})} \widetilde{\mathrm{G}}_1 \subset \widetilde{\mathrm{G}}_1^{\times 2}$ with the binary supergroup operation (written, in the $\mathcal{S}$-point picture, in the coordinates: $\widetilde{X}_A \equiv (\theta^\alpha, x^a, \xi_{A\widehat{\beta}})$, $\widetilde{Y}_A \equiv (\varepsilon^\alpha, y^a, \zeta_{A\widehat{\beta}})$, $A \in \{1,2\}$ on $\widetilde{\mathrm{G}}_1$ and z on $\mathbb{C}^\times$)

$$\widetilde{\mu}\big((X_1, X_2, z_1), (Y_1, Y_2, z_2)\big) = \big(\widetilde{\mu}_1(X_1, Y_1), \widetilde{\mu}_1(X_2, Y_2), \mathrm{e}^{\mathrm{i}\,\delta_\alpha^{\widehat{\alpha}}\,\theta^\alpha\,(\zeta_{2\widehat{\alpha}} - \zeta_{1\widehat{\alpha}})} \cdot z_1 \cdot z_2\big)\,.$$

The Green-Schwarz (CaE) super-1-gerbe of Ref. [161, Def. 5.9] is the 1-gerbe $\mathcal{G}_{\mathrm{GS}}^{(1)} := (\widetilde{\mathrm{G}}_1, \widetilde{\varpi}_1, \beta, L, \pi_L, \mathcal{A}_L, \mu_L)$ with $\widetilde{\mathrm{G}}_1, \widetilde{\varpi}_1, L$ and π_L as above and with the global curving $\beta = \delta_\alpha^{\widehat{\alpha}}\, \mathrm{pr}_1^*\theta_{\mathrm{L}}^\alpha \wedge \theta_{\mathrm{L}\,\widehat{\alpha}}$ given in terms of the components $\theta_{\mathrm{L}\,\widehat{\alpha}}$ of the LI Maurer-Cartan super-form on $\widetilde{\mathrm{G}}_1$ dual to the LI vector fields associated with the respective $Z^{\widehat{\alpha}}$, with the principal $\mathbb{C}^\times$-connection 1-form $\mathcal{A}_L$ given by the component of the LI Maurer-Cartan super-form on L dual to the LI vector field associated with Z, and – finally – with a trivial (product) fibrewise groupoid structure μ_L.

Example 6.8. **The trivial descended CaE super-1-gerbe for $\mathfrak{sB}_{1,\mathrm{MT}}^{(\mathrm{NG})} \equiv (\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5), \eta, \chi_3^{\mathrm{MT}})$.**

Adopt the notation of Example 3.8. The CaE super-3-coboundary χ_3^{MT} admits a $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$-basic primitive $\beta_2^{\mathrm{MT}} = -\theta_{\mathrm{L}}^{\alpha'\alpha''I} \wedge (\underline{C}\,\underline{\gamma}_{11})_{\alpha'\alpha''I\beta'\beta''J}\, \theta_{\mathrm{L}}^{\beta'\beta''J}$, and so there exist unique super-forms $\mathrm{H}_3^{\mathrm{MT}} \in \Omega^3(\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5))^{\mathrm{SU}(2,2|4)}$ and $\mathrm{B}_2^{\mathrm{MT}} \in \Omega^2(\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5))^{\mathrm{SU}(2,2|4)}$ such that $\chi_3^{\mathrm{MT}} = \pi_{\mathrm{s}(\mathrm{AdS}_5\times\mathbb{S}^5)}^*\mathrm{H}_3^{\mathrm{MT}}$ and $\beta_2^{\mathrm{MT}} = \pi_{\mathrm{s}(\mathrm{AdS}_5\times\mathbb{S}^5)}^*\mathrm{B}_2^{\mathrm{MT}}$. Consequently, the trivial CaE super-1-gerbe

$$\widetilde{\mathcal{G}}_{\mathrm{MT}}^{(1)} = (\mathrm{SU}(2,2|4), \mathrm{id}_{\mathrm{SU}(2,2|4)}, \beta_2^{\mathrm{MT}}, \mathrm{SU}(2,2|4) \times \mathbb{C}^\times, \mathrm{pr}_1, 0, \mu_0)$$

for $\mathfrak{sB}_{1,\mathrm{MT}}^{(\mathrm{NG})}$ gives the descended CaE super-1-gerbe

$$\mathcal{G}_{\mathrm{MT}}^{(1)} = (\mathrm{s}\big(\mathrm{AdS}_5 \times \mathbb{S}^5\big), \mathrm{id}_{\mathrm{s}(\mathrm{AdS}_5\times\mathbb{S}^5)}, \mathrm{B}_2^{\mathrm{MT}}, \mathrm{s}\big(\mathrm{AdS}_5 \times \mathbb{S}^5\big) \times \mathbb{C}^\times, \mathrm{pr}_1, 0, \mu_0).$$

Existence of working models for the higher-geometric objects defined in the preceding paragraphs is strongly predetermined by the structure of the known (or, at any rate, most studied) supersymmetric field theories with topological (super)charges, and these very clearly seem to favour nontrivial CaE super-p-gerbes over the homogeneous space $\mathrm{sMink}(d,1|D_{d,1})$ endowed with the structure of a Lie supergroup (Def. 6.4) and trivial CaE super-p-gerbes over Lie supergroups whose associated supertargets, to which the trivial super-p-gerbes descend automatically, are *not* Lie supergroups (Def. 6.6), *cp* the list of primitives of the GS super-$(p+2)$-cocycles in Ref. [165, Examples 3.11–3.16]. In the former case, an extensive study carried out in Ref. [161], in which a semi-algorithmic procedure[8], devised in Ref. [30], of associating non-trivial CaE super-2-cocycles $\widetilde{\Theta}_l$ of the type discussed on p. 46 with the relevant GS super-$(p+2)$-cocycles of Example 3.7, was employed, for $p \in \{0,1,2\}$, to obtain the distinguished surjective submersions $\mathsf{Y}\mathrm{G} \longrightarrow \mathrm{G}$ and subsequently extended to the remaining levels of the gerbe-theoretic construction, provides strong evidence in favour of the claim that the general geometrisation scheme using stepwise supercentral extensions that was sketched on pp. 46-46 is *universally* applicable in the super-Minkowskian setting. As demonstrated in Ref. [164] and recapitulated in the next section, the ensuing CaE super-p-gerbes (dubbed the **Green–Schwarz super-p-gerbes** in the original paper) possess the expected equivariance properties, which lends further support to the postulated geometrisation scheme. Over (classical) curved supertargets, on the other hand, the situation is much subtler – in order to appreciate it, we need to take a closer look at the guiding principles of the construction of the relevant super-σ-models, laid out in Ref. [131], with view to lifting them to the higher supergeometry behind these field theories.

Thus, suppose there exists a super-σ-model for the super-p-brane in a superbackground $\mathfrak{sB}_*^{(\mathrm{NG})} \equiv (\mathcal{M}_*, \mathrm{g}_*, \chi_{p+2\,*})$ and we seek to derive a similar super-σ-model for a supertarget with a predetermined body $|\mathcal{M}_R|$ that is known to asymptote to $|\mathcal{M}_*|$ in some limit R_* of the geometric parameter(s) R (a radius *etc.*), that is, symbolically, $|\mathcal{M}_R| \xrightarrow{R\to R_*} |\mathcal{M}_*|$. We may, then, (choose to) distinguish among consistent superbackgrounds $\mathfrak{sB}_R^{(\mathrm{NG})} \equiv (\mathcal{M}_R, \mathrm{g}_R, \chi_{p+2\,R})$ (with the above body of the supertarget $\mathcal{M}_R$) those which, in some well-defined manner, satisfy the asymptotic relation $(\mathcal{M}_R, \Gamma(R)\cdot \mathrm{g}_R, \Gamma(R)^{\frac{p+1}{2}}\cdot\chi_{p+2\,R}) \xrightarrow{R\to R_*} \mathfrak{sB}_*^{(\mathrm{NG})}$ for a continuous function Γ of the parameter(s), so that the super-σ-model for $\mathfrak{sB}_R^{(\mathrm{NG})}$ goes over to the reference one for

[8]The $\widetilde{\Theta}_l$ were invariably obtained by contracting collections of LI vector fields with (factors within) the CaE super-$(p+2)$-cocycles corresponding to the $\widetilde{\Omega}_l$ by Thm. 6.3.

$\mathfrak{s}\mathfrak{B}_*^{(\mathrm{NG})}$ (the presence of several species of the topological charge might call for additional relative R-dependent renormalisations of the various components of the super-$(p+2)$-cocycle). These very general considerations may be rendered fairly concrete and acquire a natural and tractable structure in our preferred Lie-superalgebraic setting. Indeed, assume given a Lie group $|\mathrm{G}|$ with the tangent Lie algebra $|\mathfrak{g}|$ and its distinguished Lie subgroup H with the tangent Lie algebra $\mathfrak{h}$, defining a reductive decomposition $|\mathfrak{g}| = |\mathfrak{t}|\oplus\mathfrak{h}$, and attempt to formulate a super-σ-model with the supertarget given by a homogeneous space G/H of *a* Lie supergroup G with the body $|\mathrm{G}|$ in a situation in which there exists an İnönü-Wigner (İW) contraction $|\mathfrak{g}| \xrightarrow{R\text{-dep. rescaling}} |\mathfrak{g}|_R = |\mathfrak{t}|_R \oplus \mathfrak{h} \xrightarrow{R\to R_*} \mathfrak{g}_*^{(0)}$ to the even part $\mathfrak{g}_*^{(0)}$ of the tangent Lie superalgebra $\mathfrak{g}_*$ of a predetermined Lie supergroup G_* with the property that the dual limit (taken in the structure sheaf upon rescaling the natural coordinates associated with $|\mathfrak{g}|^*$) sends $|\mathrm{G}|_R/\mathrm{H} \equiv |\mathrm{G}_R/\mathrm{H}|$ to the body $|\mathrm{G}_*|$ of G_* over which there is a well-defined super-σ-model. We may then, after Ref. [131, Sec. 3], constrain the choice of the sought-after superbackground $\mathfrak{s}\mathfrak{B}_p^{(\mathrm{NG})} \equiv (\mathrm{G}/\mathrm{H}, \mathrm{g}, \chi_{p+2})$ of the super-σ-model for $[\Omega_p, \mathrm{G}/\mathrm{H}]$ through imposition of the algebraic requirements:

- existence of an extension $\mathfrak{g} \xrightarrow{R\text{-dep. rescaling}} \mathfrak{g}_R = \mathfrak{t}_R \oplus \mathfrak{h} \xrightarrow{R\to R_*} \mathfrak{g}_*$ of the İW contraction to the entire tangent Lie superalgebra $\mathfrak{g}$ of G;

- existence of a power function $\Gamma(R) \equiv R^N$, $N \in \mathbb{N}$ of the scaling parameter R such that the LI tensors on $\mathfrak{g}$: a bilinear symmetric one $\gamma \in \mathfrak{t}^{(0)*}\widehat{\otimes}\mathfrak{t}^{(0)*}$ and a (H-basic) CE super-$(p+2)$-cocycle $\Omega \in Z^{p+2}(\mathfrak{g}, \mathbb{R})$ determined canonically by the metric g and the GS super-$(p+2)$-cocycle χ_{p+2}, respectively, asymptote to their counterparts γ_* and Ω_*, induced by g_* and $\chi_{p+2\,*}$, as $R \to R_*$ upon a *dual* rescaling of the generators of $\mathfrak{t}^*$ and an overall renormalisation by Γ;

- existence of a pair of projectors $\mathsf{P}^{(n)} : \mathfrak{t}^{(n)} \circlearrowleft$, $n \in \{0,1\}$ giving an algebraic model $\mathrm{Im}\,\mathsf{P}^{(0)} \oplus \mathrm{Im}\,\mathsf{P}^{(1)}$ of the vacuum of the super-σ-model and (consequently) codetermined by $\mathfrak{s}\mathfrak{B}_p^{(\mathrm{NG})}$, as detailed in Sec. 5, with the additional property that the latter model reproduces the one within $\mathfrak{g}_*$ in the limit – in short, existence of a κ-symmetry compatible with the İW contraction.

Consistently with the general philosophy advocated in the present review, we are then led to extend the above requirements of compatibility with the underlying İW contraction to the geometrisation of Ω, and the particular method leading to Def. 6.4 provides us with hands-on means to this end. The anchor point here is the extraction, from the geometrisations of both GS super-$(p+2)$-cocycles in the İW correspondence, of a collection of Lie-superalgebra extensions corresponding (in, say, Kostant's picture) to the complete set of

Lie-supergroup extensions defining the total spaces of the structural surjective submersions of the respective CaE super-p-gerbes together with the associated CE cochains determined by the structural superdifferential forms (*i.e.*, the connective structure) on them. Thus prepared, we may, then, formulate

Definition 6.9. Under the circumstances defined above, and in the hitherto notation, we call a CaE super-p-gerbe **İnönü-Wigner-contractible** if the İW contraction can be consistently extended to all the tangent Lie superalgebras $\{\mathfrak{g}_n\}_{n\in\overline{1,N}}$ of the total spaces of its structural surjective submersions, whereupon the associated (structural) CE cochains of the super-p-gerbe asymptote *uniformly* (as quantified by the single power function Γ) to their counterparts in the definition of the reference CaE super-p-gerbe over G_* upon dual rescaling of the generators of the $\mathfrak{g}_n^*$.

◇

The concept of İnönü-Wigner-contractibility was introduced in Refs. [162, 163], where detailed asymptotic analyses of the Metsaev–Tseytlin super-σ-model for the super-1-brane in $\mathrm{s}(\mathrm{AdS}_5 \times \mathbb{S}^5)$ and of the Zhou super-σ-model for the super-0-brane in $\mathrm{s}(\mathrm{AdS}_2 \times \mathbb{S}^2)$, respectively, were carried out in the régime of a large *common* radius of both curved factors of (the body of) the supertarget: the AdS space and the sphere, in which the homogeneous spaces flatten out towards $\mathrm{sMink}(9,1|32)$ and $\mathrm{sMink}(3,1|2\cdot 4)$, respectively. In the latter superbackground, the analysis of Ref. [163] yielded an İW-contractible CaE super-0-gerbe over the relevant supersymmetry group $\mathrm{SU}(1,1|2)_2$ with manifest κ-symmetry. In the former case, it confirmed the earlier result of Ref. [97] on *non*-contractibility of the global primitive β_2^{MT} of the GS super-3-cocycle χ_3^{MT}, implying *non*-contractibility of the trivial descended CaE super-1-gerbe of Example 6.8 to its flat-superspace counterpart of Example 6.7 (with $d = 9$). It was shown that the flaw can be amended through subtraction of the ('wrong') leading de Rham-exact term in the asymptotic expansion of the primitive, resulting, however, in the loss of sypersymmetry, and so attempts were undertaken, taking guidance from the asymptotic analysis of the wrapping charges in the MT super-σ-model for $\mathfrak{sB}^{(\mathrm{NG})}_{1,\mathrm{MT}}$ whose intimate relation to the extension scheme was indicated previously, to associate with χ_3^{MT} a nontrivial extension of the supersymmetry group $\mathrm{SU}(2,2|4)$ supporting an İW-contractible curving. These revealed a high degree of rigidity of the superalgebraic structure behind $\mathfrak{sB}^{(\mathrm{NG})}_{1,\mathrm{MT}}$, as reflected by a number of no-go theorems proven in Ref. [162] that preclude natural extensions of $\mathfrak{su}(2,2|4)$ trivialising χ_3^{MT}, and so it was postulated that an alternative superisation of the body of $\mathfrak{sB}^{(\mathrm{NG})}_{1,\mathrm{MT}}$ should be sought on the basis of the asymptotic correpondence with $\mathfrak{sB}^{(\mathrm{NG})}_{1,\mathrm{GS}}$ (with Ref. [94] as a potential source of intuition and guidance).

Thus, the incorporation of the principle of $\dot{\rm I}$W contractibility is seen to have opened a new direction in the physics-oriented study of supersymmetric higher supergeometry over homogeneous spaces of Lie supergroups.

7. TOWARDS SUPER-EQUIVARIANCE – A SELF-CONSISTENCY CHECK

Our hitherto considerations, guided and organised by the principles of invariance with respect to the (global-)supersymmetry group G and equivariance with respect to the 'hidden' gauge-symmetry group H, have given us a concrete higher-supergeometric object: the (descended CaE) super-p-gerbe. In the light of our extensive experience with the higher geometry behind σ-models with non-$\mathbb{Z}/2\mathbb{Z}$-graded targets, recapitulated in Sec. 2, we are led to demand that the super-object encode information not just on the *global* symmetries of the underlying superfield theory (which it does, by construction), but also on the *local* ones, the latter to manifest themselves through existence of an equivariant structure compatible with global supersymmetry. Its verification is a crucial self-consistency check for the entire framework advocated herein, all the more so because the postulated geometrisation scheme involves *choices* (indeed, it employs integrable Lie-superalgebra extensions) whose fully fledged systematisation is yet to be established. The prospective gain, on the other hand, is a deep insight into reductions of the field-theoretic degrees of freedom encoded by the gauge symmetries.

Luckily, we *do* have a canonical instantiation of a gauge (super)symmetry at our disposal, to wit, the κ-symmetry of Sec. 5, which, however, has its subtleties: it geometrises over $\Sigma^{\rm HP}$, and that only in the correspondence sector; it is modelled by a superdistribution in general, and by an involutive one (and hence also by a Lie superalgebra) upon further restriction to $\Sigma^{\rm HP}_{\rm vac}$. In consequence thereof, it calls for a linearisation of the standard equivariant structure on a (super-)p-gerbe to be associated with the purely topological HP action functional restricted to $\Sigma^{\rm HP}_{\rm vac}$, compatible with the residual global supersymmetry (if any is present, and then also in a linearised form). Besides the canonical example, the gerbe theory of σ-models with Lie-group targets reviewed briefly on pp. 15-17 suggests yet another supersymmetry that can potentially be lifted to the super-p-gerbe – the adjoint action of the supersymmetry group $\mathrm{sMink}(d,1|D_{d,1})$ on the Lie-supergroup supertarget $\mathrm{sMink}(d,1|D_{d,1})$ of Example 3.7. The idea is reinforced by the observation, originally due to Henneaux *et al.*, *cp* Ref. [95], that the GS super-σ-model for the super-1-brane is a supervariant of the WZW σ-model, albeit with a degenerate and non-bi-invariant metric, the latter fact effectively precluding amenability of the adjoint action to gauging outside the topological sector of the super-σ-model but leaving the question of existence of an $\mathrm{Ad}_{\mathrm{sMink}(d,1|D_{d,1})}$-equivariant structure on $\mathcal{G}^{(1)}_{\mathrm{GS}}$ open. Both issues were addressed in the recent studies [164, 165] whose results, which we summarise briefly below, corroborate the postulated geometrisation scheme.

The gerbe-theoretic investigation of the adjoint supersymmetry on $\mathrm{sMink}(d,1|D_{d,1})$ (*i.e.*, of $\lambda \equiv \mathrm{Ad}$ for which the compatible global supersymmetry is also $\ell \equiv \mathrm{Ad}$, and $\alpha \equiv \mathrm{Ad}$) undertaken in Ref. [164], while restricted to a very special class of super-σ-models and hence potentially non-universal, is also the source of the farthest-reaching structural evidence so far in support of the higher-supergeometric proposal reviewed herein. It bases upon the identification of χ^{GS}_3 as the super-counterpart of the Cartan 3-cocycle (2.9) on the Lie group for the given structure (constants) of the Lie superalgebra $\mathfrak{smink}(d,1|D_{d,1})$ and the Cartan–Killing form on the latter, the crucial consequence here being bi-invariance of the super-3-form. The more stringent criterion of the vanishing of the small gauge anomaly selects one more GS super-p-cocycle – namely, χ^{GS}_2 of Example 3.7 (with no bosonic counterpart to refer to) – and excludes all the others. The detailed analysis of the respective lifts of the adjoint action to the constitutive surjective submersions of the two CaE super-p-gerbes: $\mathcal{G}^{(0)}_{\mathrm{GS}}$ for $\mathfrak{s}\mathfrak{B}^{(\mathrm{NG})}_{0,\mathrm{GS}}$ and $\mathcal{G}^{(1)}_{\mathrm{GS}}$ for $\mathfrak{s}\mathfrak{B}^{(\mathrm{NG})}_{1,\mathrm{GS}}$, straightforward to define due to existence of the Lie-supergroup structure thereon, yielded the anticipated positive results:

THEOREM 7.1 ([164, Thm. 4.13]). *Adopt the hitherto notation. The GS super-0-gerbe $\mathcal{G}^{(0)}_{\mathrm{GS}}$ for $\mathfrak{s}\mathfrak{B}^{(\mathrm{NG})}_{0,\mathrm{GS}}$ carries a canonical* descendable *(*$\mathrm{Ad}(\mathrm{sMink}(9,1|32))$*-)supersymmetric* $\mathrm{Ad}(\mathrm{sMink}(9,1|32))$*-equivariant structure relative to $\varrho_{\theta_{\mathrm{L}}} \equiv 0$.*

and

THEOREM 7.2 ([164, Thm. 4.17]). *Adopt the hitherto notation. The GS super-1-gerbe $\mathcal{G}^{(1)}_{\mathrm{GS}}$ for $\mathfrak{s}\mathfrak{B}^{(\mathrm{NG})}_{1,\mathrm{GS}}$ carries a canonical (*$\mathrm{Ad}(\mathrm{sMink}(d,1|D_{d,1}))$*-)supersymmetric* $\mathrm{Ad}(\mathrm{sMink}(d,1|D_{d,1}))$*-equivariant structure relative to the super-2-form $\varrho_{\theta_{\mathrm{L}}}$ with the coordinate presentation*

$$\varrho_{\theta_{\mathrm{L}}}\big((\theta_1,x_1),(\theta_2,x_2)\big) = -\frac{2}{3}\,(\theta_2\,\overline{\Gamma}_a\,\mathsf{d}\theta_1)\,(\theta_2\,\overline{\Gamma}^a\,\mathsf{d}\theta_2).$$

These results bear witness to the adequacy of the adopted scheme of geometrisation and at the same time pave the way to exploration of the higher supergeometry behind the gauge-extended structures of the type (2.8), of potential relevance to the original super-σ-models with the super-Minkowskian targets.

The point of departure of the construction of a *generic* equivariant structure, that is the one for $\kappa^{-\infty}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$, is the identification of the super-p-gerbe for $\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}$. This is achieved in

Definition 7.3 ([164, Defs. 6.4 & 6.5], [165, Def. 6.11]). Adopt the hitherto notation and let $\mathcal{G}^{(p)}$ be the super-p-gerbe over $\mathrm{G/H}$ for the superbackground $\mathfrak{sB}^{(\mathrm{NG})}_p$ of the super-σ-model for the super-p-brane in $\mathrm{G/H}$. The **extended Hughes-Polchinski super-p-gerbe** over G is the (Deligne-)product bundle p-gerbe

$$\widehat{\mathcal{G}}^{(p)} := \pi^*_{\mathrm{G/H}}\mathcal{G}^{(p)} \otimes \mathcal{I}^{(p)}_{\lambda^*_p\,\beta^{\mathrm{HP}}} \,.$$

Its restrictions $\jmath^*_{\mathcal{V}_i^{\mathrm{H_{vac}}}}\widehat{\mathcal{G}}^{(p)},\ i \in I_{\mathrm{H_{vac}}}$ along the canonical superembeddings $\jmath_{\mathcal{V}_i^{\mathrm{H_{vac}}}} : \mathcal{V}_i^{\mathrm{H_{vac}}} \hookrightarrow \mathrm{G}$ determine a p-gerbe over Σ^{HP}, to be denoted as $\widehat{\mathcal{G}}^{(p)}_{(\mathrm{HP})}$. Given an integrable HP vacuum superdistribution, the **vacuum restriction of $\widehat{\mathcal{G}}^{(p)}$** is the p-gerbe

$$\widehat{\mathcal{G}}^{(p)}_{\mathrm{vac}} := \iota^*_{\mathrm{vac}}\widehat{\mathcal{G}}^{(p)}_{(\mathrm{HP})} \,,$$

obtained by pullback along the embedding (5.5) of the HP vacuum foliation in Σ^{HP}.

◇

Over $\Sigma^{\mathrm{HP}}_{\mathrm{vac}}$, we have a realisation of the **residual global-supersymmetry algebra**

$$\mathfrak{s}_{\mathrm{vac}} \equiv \bigoplus_{\breve{A}\in\overline{1,S_{\mathrm{vac}}}} \langle S_{\breve{A}} \rangle \subset \mathfrak{g}$$

modelling the $[\cdot,\cdot\}_{\mathfrak{g}}$-closed subspace of $\Gamma(\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}))$ spanned on those $X^A\mathcal{K}_A \in S^{\mathrm{HP}}_{\mathrm{G}}$ (*cp* (5.7) and (5.8)) which are tangent to $\mathrm{Vac}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*})$. The realisation takes the form

$$\text{(7.1)}\qquad \mathfrak{s}_{\mathrm{vac}} \longrightarrow \mathcal{T}\Sigma^{\mathrm{HP}}_{\mathrm{vac}} \ :\ X^{\breve{A}}S_{\breve{A}} \longmapsto X^{\breve{A}}\mathcal{K}_{S_{\breve{A}}} \equiv X^{\breve{A}}\mathcal{K}_{\breve{A}}\,, \qquad |X^{\breve{A}}| \equiv |\breve{A}|\,,$$

where the $\mathcal{K}_{\breve{A}}\restriction_{\mathcal{V}_i} = R_{S_{\breve{A}}}\restriction_{\mathcal{V}_i} + \breve{\Xi}_{\breve{A}i}^{\ \underline{S}}L_{\underline{S}}$ are determined analogously to the $\mathcal{K}_A$. Moreover, there exists a realisation of $\mathfrak{s}_{\mathrm{loc}} \equiv \mathfrak{gs}_{\mathrm{vac}}(\mathfrak{sB}^{(\mathrm{HP})}_{p,\lambda_p^*}) \equiv \bigoplus_{\widetilde{A}=0}^{p+q+\underline{\delta}} \langle V_{\widetilde{A}} \rangle$,

$$\text{(7.2)}\qquad \mathfrak{s}_{\mathrm{loc}} \longrightarrow \mathcal{T}\Sigma^{\mathrm{HP}}_{\mathrm{vac}} \ :\ \Gamma^{\widetilde{A}}V_{\widetilde{A}} \equiv \Gamma \longmapsto \Gamma^{\widetilde{A}}\mathcal{T}_{V_{\widetilde{A}}} \equiv \Gamma^{\widetilde{A}}\mathcal{T}_{\widetilde{A}}\,, \qquad |\Gamma^{\widetilde{A}}| \equiv |\widetilde{A}|$$

given in terms of the basis vectors (4.6) and of the $\mathcal{T}_{\underline{S}} \equiv 0,\ \underline{S} \in \overline{1,\underline{\delta}}$. The right-hand side here can be regarded as a $\mathfrak{s}_{\mathrm{loc}}$-linear vector field $\widehat{\mathcal{T}}$ on $\Sigma^{\mathrm{HP}}_{\mathrm{vac}} \times \mathfrak{s}_{\mathrm{loc}}$, the $\Gamma^{\widetilde{A}}$ being global coordinates on the Lie superalgebra. It is in terms of these data that a linearised version of the compatibility condition (6.2) (the other one does not contribute to linear order) can be phrased in a particularly neat form. Indeed, let, in general, $\mathfrak{g}_0 \equiv \bigoplus_{k=1}^{d_0} \langle \sigma_k \rangle$ and $\mathfrak{g}_\cdot \equiv \bigoplus_{l=1}^{d_\cdot} \langle \tau_l \rangle$ be the tangent Lie (super)algebras of the global-symmetry group G_0 and of the symmetry group $\mathrm{G}_\cdot$ under gauging, respectively. Furthermore, let $\mathcal{K}_k^0 \equiv \mathcal{K}_{\sigma_k}^0$, $k \in \overline{1, d_0}$ and $\mathcal{K}_l^\cdot \equiv \mathcal{K}_{\tau_l}^\cdot$, $l \in \overline{1, d_\cdot}$ be the fundamental vector fields on $\mathcal{M}$ induced by ℓ and λ, respectively, the latter giving rise to $\mathfrak{g}_\cdot$-linear vector fields $\widehat{\mathcal{K}}^\cdot \equiv Y^l \mathcal{K}_l^\cdot$ on $\mathfrak{g}_\cdot \times \mathcal{M}$ defined in terms of the global coordinates Y^l, $l \in \overline{1, d_\cdot}$ on $\mathfrak{g}_\cdot$ associated with the respective τ_l. For Eq. (6.2) to hold, we need $\mathfrak{g}_\cdot$-linear lifts $\widehat{\mathcal{K}}_k^0$ of the $\mathcal{K}_k^0$ to $\mathfrak{g}_\cdot \times \mathcal{M}$ that obey

$$[\widehat{\mathcal{K}}_k^0, \widehat{\mathcal{K}}^\cdot] = 0 + \mathscr{O}\big(Y^2\big), \qquad k \in \overline{1, d_0}, \tag{7.3}$$

cp Ref. [165, Prop. 7.2]. Existence of such vector fields $\widehat{\mathcal{K}}_{\breve{A}} \in \Gamma(\Sigma_{\mathrm{vac}}^{\mathrm{HP}} \times \mathfrak{s}_{\mathrm{loc}})$, $\breve{A} \in \overline{1, S_{\mathrm{vac}}}$ in the case of interest was established in Ref. [165, Prop. 7.1]. With the linearised realisation of the global-supersymmetry group thus extended to the arrow supermanifold of $\Sigma_{\mathrm{vac}}^{\mathrm{HP}} \rtimes \mathfrak{s}_{\mathrm{loc}}$, we are ready to describe a linearisation of an equivariant structure on a (super-)p-gerbe that is compatible with a given linearised global (super)symmetry. Such a structure can readily be derived, in the Čech–Deligne cohomological description of the higher-(super)geometric objects involved (*cp* Rem. 2.6), through linearisation of a standard (group-)equivariant structure on a p-gerbe compatible with its global (group-)invariance, *cp* Ref. [165, Sec. 7]. We have

Definition 7.4 ([165, Def. 7.8]). Adopt the hitherto notation and let $\mathcal{M}$ be a (super)manifold with the realisation $\mathcal{K}_X^0$, $X \in \mathfrak{g}_0$ of the Lie (super)algebra $\mathfrak{g}_0$ as described above. A p-gerbe $\mathcal{G}^{(p)}$ over $\mathcal{M}$ is termed **$\mathfrak{g}_0$-invariant** if there exists a family of p-gerbe 1-isomorphisms $\{\ \widetilde{\Lambda}_X\ :\ \mathscr{L}_{\mathcal{K}_X^0}\mathcal{G}^{(p)} \xrightarrow{\cong} \mathcal{I}_0^{(p)}\ \}_{X \in \mathfrak{g}_0}$ over $\mathcal{M}$, written for the p-gerbe $\mathscr{L}_{\mathcal{K}_X^0}\mathcal{G}^{(p)}$ with local (sheaf-cohomological) data over *any* open cover $\mathscr{O}_\mathcal{M}$ of $\mathcal{M}$ obtained from those of $\mathcal{G}^{(p)}$ by taking their Lie derivative along the *global* vector field $\mathcal{K}_X^0$ (component-wise).

Given two $\mathfrak{g}_0$-invariant p-gerbes $\mathcal{G}_A^{(p)}$, $A \in \{1, 2\}$ over a common base $\mathcal{M}$, with the respective families of 1-isomorphisms $\{\ \widetilde{\Lambda}_X^A\ :\ \mathscr{L}_{\mathcal{K}_X^0}\mathcal{G}_A^{(p)} \xrightarrow{\cong} \mathcal{I}_0^{(p)}\ \}_{X \in \mathfrak{g}_0}$, a p-gerbe 1-isomorphism $\Phi\ :\ \mathcal{G}_1^{(p)} \xrightarrow{\cong} \mathcal{G}_2^{(p)}$ between them is called **$\mathfrak{g}_0$-invariant** if there exists a family of p-gerbe 2-isomorphisms $\psi_X\ :\ \widetilde{\Lambda}_X^2 \circ \mathscr{L}_{\mathcal{K}_X^0}\Phi \xRightarrow{\cong} \widetilde{\Lambda}_X^1$, $\qquad X \in \mathfrak{g}_0$, written for the corresponding p-gerbe 1-isomorphisms $\mathscr{L}_{\mathcal{K}_X^0}\Phi$ with local data over any open cover $\mathscr{O}_\mathcal{M}$ of $\mathcal{M}$ common to all three: $\mathcal{G}_1^{(p)}, \mathcal{G}_2^{(p)}$ and Φ obtained from those of Φ by taking their Lie derivative along $\mathcal{K}_X^0$.

◇

and

Definition 7.5 ([165, Def. 7.3]). Adopt the hitherto notation and let $\mathcal{M}$ be a (super)manifold with the vector field $\widehat{\mathcal{K}}^{\cdot} \in \Gamma(\mathcal{T}(\mathfrak{g}_{\cdot} \times \mathcal{M}))$ induced by the realisation of the Lie (super)algebra $\mathfrak{g}_{\cdot}$ as described above. A (**descendable**) **$\mathfrak{g}_{\cdot}$-equivariant structure on $\mathcal{G}^{(p)}$** is a p-gerbe 1-isomorphism (over $\mathfrak{g}_{\cdot} \times \mathcal{M}$)

$$\widehat{\Lambda}_{\mathfrak{g}_{\cdot}} \; : \; \mathscr{L}_{\widehat{\mathcal{K}}^{\cdot}} \mathrm{pr}_2^* \mathcal{G}^{(p)} \xrightarrow{\cong} \mathcal{I}_0^{(p)} \, .$$

◇

A straightforward combination of the structures introduced in the last two definitions gives us the desired notion of a **$\mathfrak{g}_0$-invariant descendable $\mathfrak{g}_{\cdot}$-structure on $\mathcal{G}^{(p)}$**. The circumstances under which a canonical such structure exists are stated in

PROPOSITION 7.6 ([165, Props. 7.5, 7.10 & 7.12]). *Adopt the notation of Defs. 7.4 and 7.5. There exists a canonical descendable $\mathfrak{g}_{\cdot}$-equivariant structure on every p-gerbe $\mathcal{G}^{(p)}$ over $\mathcal{M}$ with a curvature horizontal with respect to the (super)distribution within $\mathcal{T}\mathcal{M}$ spanned on the $\mathcal{K}_l^{\cdot}$, $l \in \overline{1, d_{\cdot}}$. The corresponding p-gerbe 1-isomorphism $\widehat{\Lambda}_{\mathfrak{g}_{\cdot}} = -\widehat{\mathcal{K}}^{\cdot} \lrcorner \, \mathrm{pr}_2^* \mathcal{G}^{(p)}$ has local data over the open cover $\{\mathfrak{g}_{\cdot}\} \times \mathscr{O}_{\mathcal{M}}$ of $\mathfrak{g}_{\cdot} \times \mathcal{M}$, given in terms of an arbitrary open cover $\mathscr{O}_{\mathcal{M}}$ of $\mathcal{M}$, obtained from those of $\mathrm{pr}_2^* \mathcal{G}^{(p)}$ by contraction with $-\widehat{\mathcal{K}}^{\cdot}$. If, in addition, $\mathcal{G}^{(p)}$ is $\mathfrak{g}_0$-invariant and the $\mathcal{K}_k^0$, $k \in \overline{1, d_0}$ admit the respective lifts $\widehat{\mathcal{K}}_k^0$ to $\mathfrak{g}_{\cdot} \times \mathcal{M}$ satisfying conditions* (7.3)*, then this $\mathfrak{g}_{\cdot}$-equivariant structure is canonically $\mathfrak{g}_0$-invariant in the sense of Def. 7.4, with the corresponding p-gerbe 2-isomorphisms $\psi_X = \widehat{\mathcal{K}}^{\cdot} \lrcorner \, \mathrm{pr}_2^* \widetilde{\Lambda}_X$ defined similarly as $\widehat{\Lambda}_{\mathfrak{g}_{\cdot}}$.*

As a corollary to the above, we arrive at the fundamental

THEOREM 7.7. *Adopt the hitherto notation. Whenever the vacuum superdistribution* $\mathrm{Vac}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p^*})$ *of the superbackground* $\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p^*}$ *is integrable, the vacuum restriction* $\widehat{\mathcal{G}}^{(p)}_{\mathrm{vac}}$ *of the extended HP super-p-gerbe admits a canonical and canonically* $\mathfrak{s}_{\mathrm{vac}}$*-invariant descendable* $\mathfrak{s}_{\mathrm{loc}}$*-equivariant stucture.*

The theorem emphasises the differential-supergeometric aspect of κ-symmetry, and that in the linear régime in which we focus exclusively on the supervector-space structure on $\kappa^{-\infty}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p^*}) \equiv \mathrm{Vac}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p^*})$. As such, it is to be regarded as a preliminary result that demonstrates compatibility of the supersymmetric geometrisation with the generating gauge supersymmetry of the vacuum of the associated super-σ-model *independently* of structural details of that geometrisation. Those details are expected to begin to play a rôle on the next level on which the Lie-superalgebraic structure on $\kappa^{-\infty}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda_p^*})$ is taken into account. This logic is confirmed by the investigation of the explicit realisation of the (linearised) κ-equivariant structure on the super-p-gerbes of the GS super-σ-model for $\mathfrak{s}\mathfrak{B}^{(\mathrm{NG})}_{p,\mathrm{GS}}$, carried out in Ref. [164, Sec. 6], where the differential constraints defining $\mathrm{Vac}(\mathfrak{s}\mathfrak{B}^{(\mathrm{HP})}_{p,\lambda^*_p})$ as in Eq. (5.4) are seen to extend consistently to the Lie supergroups submersing the base supersymmetry group $\mathrm{sISO}(d,1|D_{d,1})$ and to the associated Lie superalgebras, the extension being mediated by the nontrivial CaE super-2-cocycles that define the extensions. The latter are in correspondence (6.1) with the CE super-2-cocycles $\widetilde{\Theta}_l$ from p. 46, and so the compatibility analysis acquires a structural character: We need to extend the original Lie-superalgebra projection $\mathfrak{g} \longrightarrow \mathfrak{s}_{\mathrm{loc}}$, defined by the pair $(\mathsf{P}^{(0)}, \mathsf{P}^{(1)})$, to the constitutive Lie-superalgebra extensions of the CaE super-p-gerbe with the help of the intermediate CE super-2-cocycles. Of course, this conclusion does not determine the fate of the remaining components $(\widetilde{\Omega}_k, \widetilde{\eta}_k)$ of the construction but it provides us with a natural language in which to formulate structural hypotheses. One such particularly attractive hypothesis can be deduced directly from the gerbe-theoretic reasoning presented in Ref. [165, Rem. 7.15]. There, on the basis of the interpretation of the G.-equivariant structure as the condition of descent of the higher-geometric object to the orbispace of the action of the symmetry group G., and in conjunction with the constatation of the (bracket-)*generating* character of the action of the κ-symmetry superalgebra on the vacuum of the super-σ-model, it was argued that the anticipated existence of a fully fledged *Lie superalgebra*-equivariant structure on $\widehat{\mathcal{G}}^{(p)}_{\mathrm{vac}}$ should result in a trivialisation[9] of the vacuum-restricted super-p-gerbe $\pi^*_{\mathrm{G/H}}\mathcal{G}^{(p)}$ given by the trivial super-p-gerbe with the global curving $-\lambda^*_p\beta^{(\mathrm{HP})}$. Such a trivialisation would have a Lie-superalgebraic 'shadow' at whose base one obtains a trivialisation of the restriction of Ω to the Lie sub-superalgebra $\mathfrak{s}_{\mathrm{loc}}$ in the CE cohomology, given by the super-$(p+1)$-cochain $-\lambda^*_p\,\mathrm{Vol}(\mathfrak{t}^{(0)}_{\mathrm{vac}})$. Considerations of this kind may serve as guidance in the future search for consistent super-σ-models on homogeneous spaces of Lie supergroups.

8. SUMMARY, AND SOME LOOSE ENDS

In the paper, we have reviewed a comprehensive approach to (Lie-)superalgebraisation and geometrisation of the canonical description and cohomological data of the super-σ-model for the super-p-brane in a homogeneous space of a supersymmetry Lie supergroup, advanced by the Author in a series of recent works [161, 164, 162, 163, 165]. The approach bases on a rigorous

[9] A trivialisation of a p-gerbe is a 1-isomorphism between that p-gerbe and a trivial one.

supergeometric description of the superbackground and superfields of the field theory of interest in the framework of the Kostant–Koszul theory of Lie supergroups. It yields a consistent algebro-geometric model of its vacuum and supersymmetries in the dual purely topological Hughes–Polchinski formulation and elucidates the nature of the tangential gauge (κ-)supersymmetry of the vacuum, identifying the corresponding superdistribution over the vacuum foliation of the supertarget as the (superbracket-)generator of the vacuum's tangent sheaf. Finally, it associates with the superbackground of the super-σ-model – in full analogy with the non-$\mathbb{Z}/2\mathbb{Z}$-graded case – a higher-supergeometric object, termed the super-p-gerbe, that geometrises the class of the super-$(p+2)$-cocycle component of the superbackground in a suitable supersymmetry-invariant cohomology of the supertarget and, as such, is expected to encode information on the deeper, essentially quantum-mechanical structure of the superfield theory on the Rabin–Crane super-orbifold of the supertarget. The super-p-gerbe has a built-in invariance under the global supersymmetry and its extension by the trivial 'volume' super-p-gerbe of the Hughes–Polchinski formulation carries a canonical and canonically (linearised-)supersymmetric descendable equivariant structure with respect to the natural geometric action of the tangential gauge supersymmetry over the vacuum. The constructive definition of the super-p-gerbe in terms of integrable extensions of the underlying supersymmetry Lie superalgebra $\mathfrak{g}$, assumed to be engendered by the said super-$(p+2)$-cocycle, affords an explicit analysis of the asymptotic behaviour of the higher-supergeometric object under (duals of) İnönu–Wigner contractions of $\mathfrak{g}$, which opens a possibility of turning İnönu–Wigner-contractibility into a supplementary organising principle of the proposed approach to geometrisation, anticipated to solidify its relation with the corresponding superfield theory.

Let us conclude our discussion by indicating directions in which the study reported herein begs structural completion, alongside a number of novel ones that it opens. As for the former, it seems natural to investigate general conditions of existence of an extension of the canonical $\mathfrak{s}_{\rm loc}$-equivariant structure on (the vacuum restriction of) the extended HP super-p-gerbe beyond the linear order, *i.e.*, to one which is equivariant with respect to the vacuum *superalgebra*, both in the supergeometric and the purely superalgebraic language, and to check that such a structure arises naturally on super-p-gerbes associated with the existing concrete ones of Refs. [161] (over $\mathrm{sISO}(d,1|D_{d,1})$) and [163] (over $\mathrm{SU}(1,1|2)_2$). The next pressing need is an explicit construction of İW-contractible super-p-gerbes (in particular for $p>0$) over *curved* supertargets, in keeping with the general superalgebraic prescription given in Def. 6.9 – here, the abundance of manageable constructions for the homogeneous spaces with the body of the type $\mathrm{AdS}_m \times \mathbb{S}^n$ in the literature (*cp* also the list of examples in Ref. [165]) is an important technical advantage. It is also tempting to look for further structural features attesting to the naturality of the existing concrete constructions, drawing intuitions from the rich pool of results in the non-$\mathbb{Z}/2\mathbb{Z}$-graded geometric category whenever possible – one such example is the multiplicative structure on the WZW (1-)gerbe of Refs. [34, 177, 92], anticipated to reappear in the super-Minkowskian setting in a suitable supersymmetric guise. On the more fundamental note, it is of utmost significance to systematise (also with view to potential uniqueness results of sorts) the existing geometrisation scheme by embedding it in a fully fledged bicategorial structure including objects to be associated with super-σ-model defects (in particular, the maximally supersymmetric ones, *cp* Refs. [61, 140]) and by relating it to the formal approach to the geometrisation of CaE classes based on their correspondence with the slim Lie $(p+1)$-superalgebras of Refs. [15, 98] and advocated in Ref. [60].

Besides the above, there are several ideas which, while detached from its core, are provoked naturally by the logical and conceptual meshwork of our proposal that are worth mentioning here. Thus, it would certainly be apposite to look for structural relations between our construction and alternative approaches to supersymmetry in the context of superstring and related models, one such particularly attractive approach being at the heart of the proposal, originally conceived by Killingback [107] and Witten [182], elaborated by Freed [57], recently revived by Freed and Moore [55], and ultimately concretised in the higher-geometric language by Bunke [23] (*cp* also Ref. [178] for an explicit construction), for a geometrisation of the Pfaffian bundle of the target-space Dirac operator, associated with fermionic contributions to the superstring path integral, in terms of a differential String-structure on the target space. Another path of investigation that could be pursued, in line with the correspondence between pre-quantisable dualities and topological defects uncovered in Refs. [157, 158], is an explicit construction of a bosonisation/fermionisation defect (and the associated super-1-gerbe bi-brane), in particular in the much tractable super-Minkowskian setting – this promises to shed some light on the geometry behind the correspondence between worldsheet and target-space supersymmetry in superstring theory. One could also readily envisage a constructive application of the HP/NG correspondence of Thm. 4.4 as a potent higher-geometric tool in the by now advanced study of so-called non-geometric dualities between bosonic σ-models (mixing the *two* terms in the DF amplitude) and the corresponding non-geometric backgrounds, *e.g.*, of the essentially loop-mechanical T-duality between σ-models with toroidally fibred target spaces, *cp* Refs. [24, 25, 1]. Finally, just to close the logical frame of our supergeometric endeavour, it may be hoped that the in-depth higher-geometric study initiated in the works reviewed herein brings us one step closer to understanding the still (mathematically) elusive AdS/CFT correspondence.

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