ON THE GEOMETRY, INVARIANTS AND SYMMETRIES OF SOME 4D-BIRATIONAL MAPPINGS

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Communicated by Mirela Babalic

We review the geometric study for three 4-dimensional integrable discrete dynamical systems (the main results being in [8, 10, 9] and obtained in collaboration with T. Takenawa). By the resolution of indeterminacy the first two are lifted to pseudo-automorphisms of rational varieties obtained from $(\mathbb{P}^1)^4$ by blowing-up along sixteen 2-dimensional subvarieties. The third one cannot be lifted to a pseudo-automorphism but to an algebraically stable map since it has non-confined singularities. The invariants and the degree growth rates are computed from the linearisation on the corresponding Néron-Severi bilattices. It turns out that the deautonomised version of the one of the confining mappings has $A_2^{(1)} + A_2^{(1)}$ type affine Weyl group symmetry, while that of the other confining mapping has $A_5^{(1)}$ type affine Weyl group symmetry.

AMS 2010 Subject Classification: 14E05, 14J26, 33E17.

Key words: resolution of singularities, rational surfaces, discrete Painleve equations, affine Weyl groups.

1. INTRODUCTION

The Painlevé equations are nonlinear second-order ordinary differential equations whose solutions are meromorphic except some fixed points, but not reduced to known functions such as solutions of linear ordinary differential equations or Abel functions. The discrete counterpart of Painlevé equations were introduced by Grammaticos, Ramani and their collaborators [16, 30] using so called the singularity confinement criterion (however historically, the first appearance of a discrete Painlevé equation was in 1939 in paper about orthogonal polynomials due to Shohat [37]; then the same equation has been found again in 1990 by Gross, Migdal [18], Brezin and Kazakov [5]). Since this criterion is not a sufficient condition for the mapping to be integrable, the notion of algebraic entropy was introduced by Hietarinta and Viallet [20] and studied geometrically in [4, 38, 24]. This entropy is essentially the same with topological entropy [17, 43].

Discrete Painlevé equations share many properties with the differential case, e.g., the existence of special solutions, such as algebraic solutions, or solutions expressed in terms of special functions, affine Weyl group symmetries and the geometric classification of equations in terms of rational surfaces. Among them, associated families of rational surfaces, called the spaces of initial conditions, were introduced by Okamoto [27] for the continuous case, and by Sakai [32] for the discrete case, where an equation gives a flow on a family of smooth projective rational surfaces. The cohomology group of the space of initial conditions gives information about the symmetries of the equation [32] and its degree growth [38].

In recent years, research on four dimensional Painlevé systems has been progressed mainly from the viewpoint of isomonodromic deformation of linear equations [33, 22], while the space of initial conditions in Okamoto-Sakai's sense was known only for few equations. The difficulty lies in the part of using higher dimensional algebraic geometry. In the higher dimensional case the center of blowups is not necessarily a point but could be a subvariety of codimension two at least. Although some studies on symmetries of varieties or dynamical systems have been reported in the higher dimensional case, most of them consider only the case where varieties are obtained by blowups at points from the projective space [13, 39, 3]. One of few exceptions is [40], where varieties obtained by blowups along codimension three subvarieties from the direct product of a projective line $(\mathbb{P}^1)^N$ were studied.

In this paper, starting with a mapping $\varphi : \mathbb{C}^4 \to \mathbb{C}^4; (q_1, q_2, p_1, p_2) \mapsto (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2)$:

$$A_2^{(1)} + A_2^{(1)} : \begin{cases} \bar{q}_1 &= -q_2 - p_2 + a q_2^{-1} + b \\ \bar{p}_1 &= q_2 \\ \bar{q}_2 &= -q_1 - p_1 + a q_1^{-1} + b \\ \bar{p}_2 &= q_1 \end{cases} \tag{1}$$

and its slight modification,

$$A_5^{(1)} : \begin{cases} \bar{q}_1 &= -q_1 - p_2 + a q_2^{-1} + b_1 \\ \bar{p}_1 &= q_2 \\ \bar{q}_2 &= -q_2 - p_1 + a q_1^{-1} + b_2 \\ \bar{p}_2 &= q_1 \end{cases}, \tag{2}$$

we construct their spaces of initial conditions, where the mappings are lifted to pseudo-automorphisms (automorphisms except finite number of subvarieties of codimension 2 at least) and their invariants. We also give their symmetries and deautonomisations together with their degree growth. The motivation for studying these mappings started initially from the fact that they are related (the second one) to the travelling-wave reduction of a discrete system describing a modular genetic network, each module containing two genes having activation-repression links [11]. Then we realised that they are good candidates for studying the geometry of 4D discrete Painlevé equations.

It turns out that deautonomised version of mapping (1) is a Bäcklund transformation of a direct product of two fourth Painlevé equation, which has two continuous variables and $A_2^{(1)}+A_2^{(1)}$ (direct product) type symmetry, while that of mapping (2) is a Bäcklund transformation of Noumi-Yamada's $A_5^{(1)}$ Painlevé equation [26], which has only one continuous variable and $A_5^{(1)}$ type symmetry. Although these equations might seem rather trivial compared to the Garnier systems, the Fuji-Suzuki system [15] or the Sasano system [35, 34], we believe that they provide typical models for geometric studies on higher dimensional Painlevé systems.

The key tools of our investigation are pseudo-isomorphisms and Néron-Severi bilattices. In the autonomous case, for a given birational mapping, we successively blow-up a smooth projective rational variety along subvarieties to which a divisor is contracted. If this procedure terminate, the mapping is lifted to a pseudo-automorphism on a rational variety. In the non-autonomous case, the given sequence of mappings are lifted to a sequence of pseudo-isomorphisms between rational varieties. We refer to those obtained rational varieties as the space of initial conditions (in Okamoto-Sakai's sense). In this setting, the Néron-Severi bilattices play the role of root lattices of affine Weyl groups.

Let us make some remarks on the mappings. Mapping (1) can be written in a simpler way as,

$$\overline{y_1}+y_2+\underline{y_1}-ay_2^{-1}-b=0$$
$$\overline{y_2}+y_1+\underline{y_2}-ay_1^{-1}-b=0,$$

where $y_1=q_1$, $y_2=q_2$, $\underline{y_1}=p_2$, $\underline{y_2}=p_1$ and the over/under bar denotes the image/preimage by the mapping. As can be seen easily, when $y_1=y_2$, this system is one of the Quispel-Roberts-Thompson mappings[29]. This fact enables us to find that Mapping (1) is the compatibility condition:

$$\overline{L}M-ML=0$$

for the Lax pair $L\Phi=h\Phi$, $\overline{\Phi}=M\Phi$ with

$$L=\begin{bmatrix} 0 & y_1 & 1 & 0 & 0 & 0\\ h & -a & b-y_1-\underline{y_2} & 0 & 0 & 0\\ h\underline{y_2} & h & -a & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & y_2 & 1\\ 0 & 0 & 0 & h & -a & b-y_2-\underline{y_1}\\ 0 & 0 & 0 & h\underline{y_1} & h & -a \end{bmatrix}$$

and

$$M=\begin{bmatrix} 0 & 0 & 0 & a/y_2 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & h & 0 & 0\\ a/y_1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ h & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where h is the spectral parameter.

Since $\overline{L}$ and L are similar matrices, their characteristic polynomials are the same. From the coefficients of the characteristic polynomial $\det(x-L)$ with respect to x and h, we have conserved quantities

$$I_1+I_2 \text{ and } I_1I_2,$$

where

$$\begin{aligned} I_1 &= q_1p_1(q_1+p_1-b)-a(q_1+p_1)\\ I_2 &= q_2p_2(q_2+p_2-b)-a(q_2+p_2). \end{aligned} \tag{3}$$

On the other hand, we do not know the Lax pair for Mapping (2). However, using the space of initial conditions, we find two conserved quantities:

$$\begin{aligned} I_1 =&(q_1p_1-q_2p_2)^2+b_1b_2(q_1p_1+q_2p_2)\\ &+b_1\left(a(p_1+q_2)-q_1p_1^2-q_2^2p_2\right)+b_2\left(a(q_1+p_2)-q_1^2p_1-q_2p_2^2\right)\\ I_2 =&(a(q_1+p_2)+q_1p_2(b_2-q_2-p_1))(a(q_2+p_1)+q_2p_1(b_1-q_1-p_2)). \end{aligned} \tag{4}$$

On the other hand, in a prior paper [10], applying the traveling wave reduction to the lattice super-KdV equation [6, 42] in a case of finitely generated Grassmann algebra, the authors obtained a four-dimensional discrete integrable dynamical system

$$\varphi:\begin{cases} \overline{x_0} &= x_2\\ \overline{x_1} &= x_3\\ \overline{x_2} &= -x_2-x_0+\dfrac{hx_2}{1-x_2}\\ \overline{x_3} &= -x_1-x_3+\dfrac{2-x_2+hx_3}{(1-x_2)^2} \end{cases}. \tag{5}$$

This system is a Quispel-Roberts-Thompson (QRT) map, a two dimensional map generating an automorphism of a rational elliptic surface [29], for variables x_0, x_2 coupled with linear equations for variables x_1, x_3 with coefficients depending on x_2. This system has two invariants

$$I_1 =-hx_0^2-hx_0x_2+h^2x_0x_2+hx_0^2x_2-hx_2^2+hx_0x_2^2 \tag{6}$$

$$I_2 =2hx_0+x_0^2-2hx_0x_1+2hx_2+x_0x_2-hx_1x_2+h^2x_1x_2+2hx_0x_1x_2$$

$$+x_2^2 + hx_1x_2^2 - hx_0x_3 + h^2x_0x_3 + hx_0^2x_3 - 2hx_2x_3 + 2hx_0x_2x_3, \tag{7}$$

but does not satisfy the singularity confinement criterion proposed by Grammaticos-Ramani and their collaborators [16, 30].

In the same paper it is observed that the dynamical degrees of (5) grows quadratically. This phenomena is rather unusual, since as reported in [23, 19], the dynamical degree grows in the fourth order for generic coupled systems in the form

$$\begin{cases} \overline{x_0} &= f_0(x_0, x_1) \\ \overline{x_1} &= f_1(x_0, x_1) \\ \overline{x_2} &= f_2(x_0, x_1, x_2) \\ \overline{x_3} &= f_3(x_0, x_1, x_2, x_3) \end{cases},$$

where the system is a QRT map for variables x_0 and x_1, and $\overline{x_2}$ (resp. $\overline{x_3}$) depends on x_2 (resp. x_3) linearly with coefficients depending on "x_0 and x_1" (resp. "x_0, x_1 and x_2").

In this paper, constructing a rational variety where system (5) is lifted to an algebraically stable map and using the action of the map on the Picard lattice, we prove that indeed the growth is quadratic. We also show that one can find invariants also using the action on the Picard group.

In the two-dimensional case, it is known that an autonomous dynamical system defined by a birational map on a projective rational variety (or more generally Kähler manifold) can be lifted to either an automorphism or an algebraically stable map on a rational variety by successive blow-ups [12].

These notions are closely related to the notion of singularity confinement criterion. While a dynamical system that can be lifted to automorphisms satisfies singularity confinement criterion (i.e. all the singularities are confined), a dynamical system that can be lifted only to algebraically stable map does not satisfies the criterion (i.e. there exists a singularity that is not confined).

As we said in studies of higher dimensional dynamical systems, the role of automorphisms is replaced by pseudo-automorphisms, i.e. automorphisms except finite number of subvarieties of codimension at least two [13]. In the last decade a few authors studied how to construct algebraic varieties on the level of pseudo-automorphisms [3, 40, 8]. However, since system (5) does not satisfy the singularity confinement criterion, it is not expected that it could be lifted to a pseudo-automorphism. To authors knowledge there are no studies (except section Section 7 of [3], which studies a kind of generalisation of standard Cremona transformation) on construction of an algebraic variety, in which the original system is lifted not to a pseudo-automorphism, but rather to an algebraically stable map using blow-ups along sub-varieties of positive dimensions. Since the varieties obtained by blow-ups possibly infinitely near depend on the order of blow-ups, this is not a straightforward but a challenging problem.

Since I_2 is degree $(1,1)$ for x_1, x_3, we can restrict the phase space into 3-dimensional one as

(8)
$$\psi : \begin{cases} x_0 = x_2 \\ x_1 = \dfrac{I_2-(x_0^2+x_0x_2+x_2^2)-2h(x_0-x_0x_1+x_2)-hx_1x_2(2x_0+x_2-1+h)}{h(-x_0+hx_0+x_0^2-2x_2+2x_0x_2)} \\ x_2 = -x_2 - x_0 + \dfrac{hx_2}{1-x_2} \end{cases}.$$

We also show that the degree of this 3-dimensional system grows quadratically as well.

This paper is organised as follows. In Section 2 we recall basic facts about the algebraic geometry used in this paper. In Section 3 the singularity confinement test is applied for the above mappings. In Section 4 we construct the spaces of initial conditions, where the mappings are lifted to pseudo-automorphisms, and compute the actions on the Neron-Séveri bilattices. The degree growth is also computed for these actions. In Section 5 symmetries of the spaces of initial conditions are studied. Deautonomised mappings are also given. In Section 6 we discuss the case of non-confining map (5) and construct here also the space of initial conditions, recovering the two invariants and provinfg the quadratic growth of iterates. Section 7 is devoted to conclusions.

Notation. Throughout this paper, we often denote $x_{i_1} + x_{i_2} + \cdots + x_{i_n}$ by $x_{i_1,i_2,\ldots,i_n}$, where x can be replaced by any symbols like y, z, A, B, C etc.

2. ALGEBRAIC STABILITY AND PSEUDO-ISOMORPHISMS

A rational map $f : \mathbb{P}^n \to \mathbb{P}^n$ is given by $(n+1)$-tuple of homogeneous polynomials having the same degree (without common polynomial factor). Its degree, $\deg(f)$, is defined as the common degree of the f_j's. We are interested in to compute $\deg(f^n)$, but it is not easy, since it only holds that $\deg(f^n) \leq (\deg f)^n$ in general by cancellation of common factors. A related object is the indeterminacy set of f given by

$$I(f) = \{\mathbf{x} \in \mathbb{P}^n \mid f_0(\mathbf{x}) = \cdots = f_n(\mathbf{x}) = 0\}$$

that is a subvariety of codimension 2 at least, whereas f defines a holomorphic mapping $f : \mathbb{P}^n \setminus I(f) \to \mathbb{P}^n$. In this section we recall basic facts in algebraic geometry used in this kind of study.

Rational correspondence. Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties of dimension N and $f : \mathcal{X} \to \mathcal{Y}$ a dominant rational map. Using the completion of the graph of f, G_f, we can decompose f as $f = \pi_{\mathcal{Y}} \circ \pi_{\mathcal{X}}^{-1}$ such that $\pi_{\mathcal{X}} : G_f \to \mathcal{X}$ and $\pi_{\mathcal{Y}} : G_f \to \mathcal{Y}$ are rational morphisms and the equality holds for generic points in $\mathcal{X}$.

This definition is simple but practically may arise complications in computing defining polynomials of the graph. For example, when $\mathcal{X}$ and $\mathcal{Y}$ are rational varieties and $(x_1, \dots, x_N)$ and $(y_1, \dots, y_N)$ are their local coordinates, introducing homogeneous coordinates as $(X_0 : \cdots : X_N) = (X_0 : X_0 x_1 : \cdots : X_0 x_N)$ and $(Y_0 : \cdots : Y_N) = (Y_0 : Y_0 y_1 : \cdots : Y_0 y_N)$, we can only say that the graph G_f is "one of the components" of $Y_k p_l(X_0, \cdots, X_N) = Y_l p_k(X_0, \cdots, X_N)$, $k, l = 0, \dots, N$, where $(y_1, \dots, y_N) = (p_1/p_0, \dots, p_N/p_0)$ is the induced homogeneous map and $(X_0 : \cdots : X_N; Y_0 : \cdots : Y_N)$ is the coordinate system of $\mathbb{P}^N \times \mathbb{P}^N$ (S5 of [2] and Example 3.4 of [31] are examples of such complication).

Hironaka's singularity resolution theorem (Question (E) in S 0.5 of [21]) also gives this decomposition in a more tractable form as: there exists a sequence of blowups $\pi : \tilde{\mathcal{X}} \to \mathcal{X}$ along smooth centers in $I(f)$ such that the induced rational map $\tilde{f} : \tilde{\mathcal{X}} \to \mathcal{Y}$ is a morphism.

Using these decompositions we can define the push-forward and the pull-back correspondence of a sub-variety by f as $f_c(V) = \pi_{\mathcal{Y}} \circ \pi_{\mathcal{X}}^{-1}(V) = \tilde{f} \circ \pi^{-1}(V)$ for $V \subset \mathcal{X}$ and $f_c^{-1}(W) = \pi_{\mathcal{X}} \circ \pi_{\mathcal{Y}}^{-1}(W) = \pi \circ \tilde{f}^{-1}(W)$ for $W \subset \mathcal{Y}$. We denote their restriction to divisor groups by $f_* : \mathrm{Div}(\mathcal{X}) \to \mathrm{Div}(\mathcal{Y})$ and $f^* : \mathrm{Div}(\mathcal{Y}) \to \mathrm{Div}(\mathcal{X})$, where lower dimensional subvarieties are ignored as zero divisors. Especially, when f is birational, it obviously holds that $f_* = (f^{-1})^*$ and $f^* = (f^{-1})_*$.

Algebraic stability. The following proposition is fundamental to our study. Its two dimensional version was shown by Diller and Favre (Proposition 1.13 of [DF01]) . "If" part was shown by Bedford-Kim (Theorem 1.1 of [3]) and Roeder (Proposition 1.5 of [31]), while "only if" part by Bayraktar (Theorem 5.3 of [1]).

PROPOSITION 2.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be dominant rational maps. Then $f^* \circ g^* = (g \circ f)^*$ holds if and only if there does not exist a prime divisor D on $\mathcal{X}$ such that $f(D \setminus I(f)) \subset I(g)$.*

Since the proof of "if" part is very simple, it would be convenient to quote from [3], modifying it to fit our terminologies:

> If D is a divisor on $\mathcal{Z}$ then $g^*(D)$ is a divisor on $\mathcal{Y}$ which is the same as $g_c^{-1}(D)$ on $\mathcal{Y} - I(g)$ by ignoring codimension greater than one.

Since $I(g)$ has codimension at least 2 we also have $(g \circ f)^*(D) = f^*(g^*(D))$ on $\mathcal{X} - I(f) - f_c^{-1}(I(g))$. By the hypothesis $f_c^{-1}(I(g))$ has codimension at least 2. Thus we have $(g \circ f)^*(D) = f^* g^*(D)$ on $\mathcal{X}$.

Example 2.2. Let $(x_0 : x_1 : x_2 : x_3)$ be the homogeneous coordinate system of the complex projective space $\mathbb{P}^3$. Let $\mathcal{X}$ be a variety obtained by blowing up $\mathbb{P}^3$ along the line $x_1 = x_2 = 0$, $\mathcal{Y}$ be $\mathbb{P}^3$, $\mathcal{Z}$ be a variety obtained by blowing up $\mathbb{P}^3$ at the point $x_1 = x_2 = x_3 = 0$, and $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ be the identity map on $\mathbb{P}^3$. Let H, $E_{\mathcal{X}}$ and $E_{\mathcal{Z}}$ denote the class of the total transform of the hyperplane, the exceptional divisors of $\mathcal{X}$ and $\mathcal{Z}$ respectively. Then it holds that $I(f) = \emptyset$, $I(g) = \{(1 : 0 : 0 : 0)\}$ and there is no prime divisor $D \in \mathcal{X}$ such that $f(D) \subset I(g)$, while $I(f^{-1}) = \{(s : 0 : 0 : t) \mid (s : t) \in \mathbb{P}^1\}$, $I(g^{-1}) = \emptyset$ and $g^{-1}(E_{\mathcal{Z}}) \subset I(f^{-1})$. Thus, $f^* g^* = (g \circ f)^*$ holds, but not $(g^{-1})^*(f^{-1})^* = (f^{-1} \circ f^{-1})^*$ (see Fig. 1).

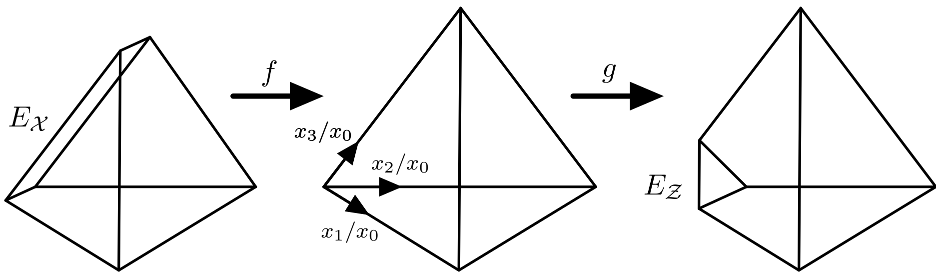


Figure 1 – Example 2.2

The pull-backs acts on divisor classes as

$$\begin{aligned}
f^* :& H \mapsto H \\
g^* :& H \mapsto H, \quad E_{\mathcal{Z}} \mapsto 0 \\
f^* g^* :& H \mapsto H, \quad E_{\mathcal{Z}} \mapsto 0 \\
(g \circ f)^* :& H \mapsto H, \quad E_{\mathcal{Z}} \mapsto 0 \\
(f^{-1})^* :& H \to H, \quad E_{\mathcal{X}} \to 0 \\
(g^{-1})^* :& H \to H \\
(g^{-1})^*(f^{-1})^* :& H \mapsto H, \quad E_{\mathcal{X}} \mapsto 0 \\
(f^{-1} \circ g^{-1})^* :& H \mapsto H, \quad E_{\mathcal{X}} \mapsto E_{\mathcal{Z}}.
\end{aligned}$$

In particular, for the anti-canonical divisor classes $-K_{\mathcal{X}} = 4H - E_{\mathcal{X}}$, $-K_{\mathcal{Y}} = 4H$ and $-K_{\mathcal{Z}} = 4H - 2E_{\mathcal{Z}}$, $(g^{-1})^*(f^{-1})^*(-K_{\mathcal{X}}) = 4H$ is greater than $(f^{-1} \circ$

$g^{-1})^*(-K_{\mathcal{X}}) = 4H - E_{\mathcal{Z}}$.

A rational map φ from a smooth projective variety $\mathcal{X}$ to itself is called *algebraically stable* or *1-regular* if $(\varphi^*)^n = (\varphi^n)^*$ holds [14]. The following proposition is obvious from Proposition 2.1.

PROPOSITION 2.3. *A rational map φ from a smooth projective variety $\mathcal{X}$ to itself is algebraically stable if and only if there does not exist a positive integer k and a divisor D on $\mathcal{X}$ such that $f(D \setminus I(f)) \subset I(f^k)$.*

Pseudo-isomorphisms and Néron-Severi bilattices. For a smooth projective variety $\mathcal{X}$, the Néron-Severi lattice

$$N^1(\mathcal{X}) = \mathrm{Pic}(\mathcal{X})/\mathrm{Pic}^0(\mathcal{X}) \subset H^2(\mathcal{X}, \mathbb{Z}),$$

where $\mathrm{Pic}^0(\mathcal{X})$ is the connected comonent of the Picard group, is the fist Chern class of the Picard group $c_1 : \mathrm{Pic}(\mathcal{X}) \to H^2(\mathcal{X}, \mathbb{Z})$. This lattice and its Poincaré dual $N_1(\mathcal{X}) \subset H_2(\mathcal{X}, \mathbb{Z})$ are finitely generated lattices. We call the pair $(N^1(\mathcal{X}), N_1(\mathcal{X}))$ the Neron-Severi bilattice of $\mathcal{X}$.

We call a birational mapping $\varphi : \mathcal{X} \to \mathcal{Y}$ a *pseudo-isomorphism* if φ is isomorphic except on finite number of subvarieties of codimension two at least. This conditions is equivalent to that there is no prime divisor pulled back to zero divisor by f or f^{-1}. Hence, if φ is a pseudo-automorphism, then φ and φ^{-1} are algebraically stable.

PROPOSITION 2.4 ([13]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective varieties and φ a pseudo-isomorphism from $\mathcal{X}$ to $\mathcal{Y}$. Then φ acts on the Néron-Severi bilattice as an automorphism preserving the intersections.*

Proof. It is obvious that $\varphi_* : N^1(\mathcal{X}) \mapsto N^1(\mathcal{Y})$ is an isomorphism by definition of pseudo-isomorphisms. The action $\varphi_* : N_1(\mathcal{X}) \mapsto N_1(\mathcal{Y})$ is determined by this isomorphism and the Poincaré duality. $\square$

Blowup of a direct product of $\mathbb{P}^m$. As we have seen in the example, it is convenient to write the generators of the Néron-Severi bilattice explicitly. Following [40], we give some formulae for some rational varieties which appear as spaces of initial conditions of Painlevé systems. Note that the Néron-Severi bilattice coincides with $H^2(\mathcal{X}, \mathbb{Z}) \times H_2(\mathcal{X}, \mathbb{Z})$ if $\mathcal{X}$ is a smooth projective rational variety, since $\mathrm{Pic}^0(\mathcal{X}) = \{0\}$ in this case.

Let $\mathcal{X}$ be a rational variety obtained by K successive blowups from $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ with $N = m_1 + \cdots + m_n$, and $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ its coordinate chart with homogeneous coordinates $\mathbf{x}_i = (x_{i0} : x_{i1} : \cdots : x_{im_i})$. Let H_i denote the total transform of the class of a hyper-plane $\mathbf{c}_i \cdot \mathbf{x}_i = 0$, where $\mathbf{c}_i$ is a constant vector in $\mathbb{P}^{m_i}$, and E_k the total transform of the k-the exceptional divisor class. Let h_i denote the total transforms of the class of a line

$$\{\mathbf{x} \mid \mathbf{x}_j = \mathbf{c}_j(\forall j \neq i),\ \mathbf{x}_i = s\mathbf{a}_i + t\mathbf{b}_i(\exists(s:t) \in \mathbb{P}^1)\},$$

where $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{c}_j$'s are constant vectors in $\mathbb{P}^{m_i}$ and $\mathbb{P}^{m_j}$ respectively, and e_k the class of a line in a fiber of the k-th blow-up. Note that the exceptional divisor for a blowing-up along a d-dimensional subvariety V is isomorphic to $V \times \mathbb{P}^{N-d-1}$, where $\mathbb{P}^{N-d-1}$ is a fiber.

Then the Picard group $\simeq H^2(\mathcal{X}, \mathbb{Z})$ and its Poincaré dual $\simeq H_2(\mathcal{X}, \mathbb{Z})$ are lattices

$$H^2(\mathcal{X}, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}H_i \oplus \bigoplus_{k=1}^{K} \mathbb{Z}E_k, \quad H_2(\mathcal{X}, \mathbb{Z}) = \bigoplus_{i=1}^{n} \mathbb{Z}h_i \oplus \bigoplus_{k=1}^{K} \mathbb{Z}e_k \tag{9}$$

and the intersection form is given by

$$\langle H_i, h_j \rangle = \delta_{ij}, \quad \langle E_l, e_l \rangle = -\delta_{kl}, \quad \langle H_i, e_k \rangle = 0. \tag{10}$$

Let φ be a pseudo-automorphism on $\mathcal{X}$, and A and B be matrices representing $\varphi_* : H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{Y}, \mathbb{Z})$ and $\varphi_* : H_2(\mathcal{X}, \mathbb{Z}) \to H_2(\mathcal{Y}, \mathbb{Z})$ respectively on basis (9). Then, for any $\mathbf{f} \in H^2(\mathcal{X}, \mathbb{Z})$ and $\mathbf{g} \in H_2(\mathcal{Y}, \mathbb{Z})$ it holds that

$$\langle \mathbf{f}, \mathbf{g} \rangle = \mathbf{f}^T J \mathbf{g}, \quad J = \begin{bmatrix} I_n & 0 \\ 0 & -I_K \end{bmatrix},$$

where $*^T$ denotes transpose and I_m denotes the identity matrix of size m. Thus, $\langle A\mathbf{f}, B\mathbf{g} \rangle = \langle \mathbf{f}, \mathbf{g} \rangle$ yields $A^T J B = J$, and hence

$$B = J(A^{-1})^T J, \tag{11}$$

which is a formula for computing the action on $H_2(\mathcal{X}, \mathbb{Z})$ from that on $H^2(\mathcal{X}, \mathbb{Z})$.

Example 2.5. Let $\mathcal{X}$ be obtained by blowing up $\mathbb{P}^3$ at four points $(1:0:0:0)$, $(0:1:0:0)$, $(0:0:1:0)$, $(0:0:0:1)$, and both $f : \mathcal{X} \to \mathcal{X}$ be the standard Cremona transformation of $\mathbb{P}^3$: $(x_0 : x_1 : x_2 : x_3) \to (x_0^{-1} : x_1^{-1} : x_2^{-1} : x_3^{-1})$. Then $I(f)$ consists of the proper (strict) transform of 6 lines passing through two of the four points blown up. This is a simple example of a pseudo-automorphism (see Fig. 2).

The push-forward action on divisor classes is

$$f_* : H \mapsto 3H - 2E_{0,1,2,3}, \quad E_i \mapsto H - E_{i+1,i+2,i+3} \quad (i = 0,1,2,3 \mod 4),$$

where $E_{i_1,\ldots,i_k} = E_{i_1} + \cdots + E_{i_k}$, while its dual is

$$f_* : h \mapsto 3h - e_{0,1,2,3}, \quad e_i \mapsto 2h - e_{i+1,i+2,i+3} \quad (i = 0,1,2,3 \mod 4).$$

The corresponding representing matrices

$$A = \begin{bmatrix} 3 & 1 & 1 & 1 & 1 \\ -2 & 0 & -1 & -1 & -1 \\ -2 & -1 & 0 & -1 & -1 \\ -2 & -1 & -1 & 0 & -1 \\ -2 & -1 & -1 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 2 & 2 & 2 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}$$

satisfies (11). It is also easy to check that $(f_*)^2$ is the identity as it should be.

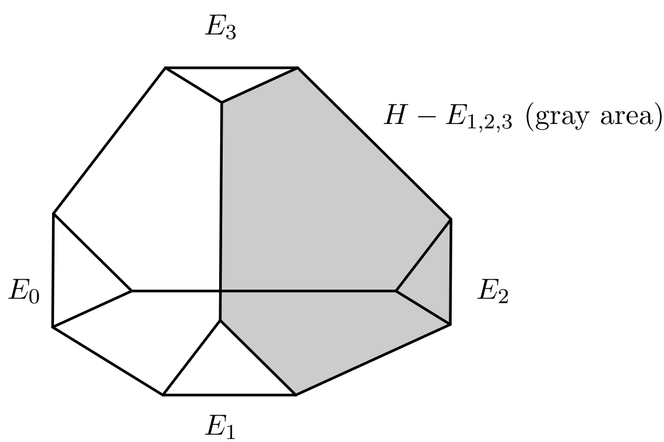


Figure 2 – Example 2.5: E_0 and $H - E_{1,2,3}$ are exchanged.

Degree of a mapping. Let φ be a rational mapping from $\mathbb{C}^N$ to itself:

$$\varphi : (\bar{x}_1, \dots, \bar{x}_N) = (\varphi_1(x_1, \cdots, x_N), \dots, \varphi_N(x_1, \cdots, x_N)).$$

The degree of $\bar{x}_i$ of φ with respect x_j is defined as the degree of φ_i as a rational function of x_j, i.e. the maximum of degrees of numerator and denominator. Let $\mathcal{X}$ be a rational variety obtained by K successive blowups from $(\mathbb{P}^1)^N$. Then the degree of $\bar{x}_i$ of φ with respect x_j is given by the coefficient of H_j in $\varphi^*(H_i)$. When φ is iterated, the degree of $\bar{x}_i$ of φ^n with respect x_j is given by the coefficient of H_j in $(\varphi^n)^*(H_i)$, which coincides with $(\varphi^*)^n(H_i)$ if φ is algebraically stable on $\mathcal{X}$. (The reason is exactly the same with the two-dimensional case. See [38] for details.)

There is another (and more standard) definition of the mapping degree. Let φ be a rational mapping on $\mathbb{C}^N$ as above. We can extend the action of φ onto $\mathbb{P}^N$ by replacing x_j by x_j/x_0, rewriting φ_i's so that they have the common denominator and considering them as $\bar{x}_i/\bar{x}_0$. Then φ can be expressed as

$$\varphi : (\bar{x}_0 : \cdots : \bar{x}_N) = (p_0(x_0, \ldots, x_N) : \cdots : p_N(x_0, \cdots, x_N)),$$

where p_i's are homogeneous polynomials and the common factor is only a constant. Then, the degree of φ is defined as the common degree of p_i's. Let $\mathcal{X}$ be a rational variety obtained by K successive blowups from $\mathbb{P}^N$. Then the degree of φ is given by the coefficient of H in $\varphi^*(H)$. When φ is iterated, the degree of φ^n is given by the coefficient of H in $(\varphi^n)^*(H)$, which coincides with $(\varphi^*)^n(H)$ if φ is algebraically stable on $\mathcal{X}$.

Above two kinds of degrees are related to each other. Indeed, it is clearly holds that

$$\max_i \{ \textstyle\sum_j \text{ degree of } \varphi_i \text{ for } x_j\} \leq \text{degree of } \varphi$$
$$\leq N \max_i \{ \textstyle\sum_j \text{ degree of } \varphi_i \text{ for } x_j\}.$$

Of course we can also consider intermediate of the above degrees by extending the action of φ onto $\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$ with $N = m_1 + \cdots + m_n$. But we do not use such degrees in this paper and omit them.

3. SINGULARITY CONFINEMENT

The idea of the singularity confinement test is as follows. Consider a hypersurface in some compactification X of $\mathbb{C}^n$ which is contracted to a lower dimensional variety (singularity) by a birational automorhism f of X. We say the singularity to be confined if there exists an integer $n \geq 2$ such that the hypersurface is recovered to some hypersurface by f^n in generic. In this case, the memory of initial conditions is recovered. Let us introduce the set of contracted hypersurfaces:

$$\mathcal{E}(f) = \{D \subset X : \text{hypersurface} \mid \det(\partial f/\partial x) = 0 \text{ on } D \text{ in generic}\},$$

where zero of the Jacobian contraction to a lower dimensional variety. If singularity is confined for every D in $\mathcal{E}(f)$, we say that the initial data is not lost and the map f satisfies the singularity confinement criterion. Note that the existence of confined singular sequence implies algebraical unstability.

In this section we consider the mappings on compactified space $(\mathbb{P}^1)^4 = (\mathbb{CP}^1)^4$ and apply the singularity confinement test to them.

Case $A_2^{(1)} + A_2^{(1)}$. If we take $q_1 = \varepsilon$ with $|\varepsilon| \ll 1$ and the others are generic, the principal terms of the Laurent series with respect to ε in the trajectories are

$$(\varepsilon, p_1^{(0)}, q_2^{(0)}, p_2^{(0)}) \text{: 3 dim}$$

$$
\begin{aligned}
&\to (q_1^{(1)}, p_1^{(1)}, a\varepsilon^{-1}, \varepsilon)\colon\ 2 \text{ dim } \boxed{14}\\
&\to (-a\varepsilon^{-1}, a\varepsilon^{-1}, q_2^{(2)}, p_2^{(2)})\colon\ 2 \text{ dim } \boxed{4}\\
&\to (q_1^{(3)}, p_1^{(3)}, -\varepsilon, -a\varepsilon^{-1})\colon\ 2 \text{ dim } \boxed{16}\\
&\to (q_1^{(4)}, -\varepsilon, q_2^{(4)}, p_2^{(4)})\colon\ 3 \text{ dim},
\end{aligned}
$$

where $x_i^{(j)}$ denotes a generic value in $\mathbb{C}$, "k dim" denotes the dimension of corresponding subvariety in $(\mathbb{P}^1)^4$ and $\boxed{n}$ denotes the order of blowing up that we explain in the next section. Similarly, starting with $q_2 = \varepsilon$ and the others being generic, we get

$$
\begin{aligned}
&(q_1^{(0)}, p_1^{(0)}, \varepsilon, p_2^{(0)})\colon\ 3 \text{ dim}\\
&\qquad \to (a\varepsilon^{-1}, \varepsilon, q_2^{(1)}, p_2^{(1)})\colon\ 2 \text{ dim } \boxed{6}\\
&\qquad \to (q_1^{(2)}, p_1^{(2)}, -a\varepsilon^{-1}, a\varepsilon^{-1})\colon\ 2 \text{ dim } \boxed{12}\\
&\qquad \to (-\varepsilon, -a\varepsilon^{-1}, q_2^{(3)}, p_2^{(3)})\colon\ 2 \text{ dim } \boxed{8}\\
&\qquad \to (q_1^{(4)}, p_1^{(4)}, q_2^{(4)}, -\varepsilon)\colon\ 3 \text{ dim}.
\end{aligned}
$$

In both two cases, information on the initial values $x_i^{(0)}$ is recovered after finite number of steps, and thus singularities are confined.

We also find another (cyclic) singularity pattern as

$$
\begin{aligned}
&(\varepsilon^{-1}, p_1^{(0)}, q_2^{(0)}, p_2^{(0)})\colon\ 3 \text{ dim}\\
&\qquad \to (q_1^{(1)}, p_1^{(1)}, -\varepsilon^{-1}, \varepsilon^{-1})\colon\ 2 \text{ dim } \boxed{10}\\
&\qquad \to (q_1^{(2)}, -\varepsilon^{-1}, q_2^{(2)}, p_2^{(2)})\colon\ 3 \text{ dim}\\
&\qquad \to (q_1^{(3)}, p_1^{(3)}, \varepsilon^{-1}, p_2^{(3)})\colon\ 3 \text{ dim}\\
&\qquad \to (-\varepsilon^{-1}, \varepsilon^{-1}, q_2^{(4)}, p_2^{(4)})\colon\ 2 \text{ dim } \boxed{2}\\
&\qquad \to (q_1^{(5)}, p_1^{(5)}, q_2^{(5)}, -\varepsilon^{-1})\colon\ 3 \text{ dim}\\
&\qquad \to (\varepsilon^{-1}, p_1^{(6)}, q_2^{(6)}, p_2^{(6)}),
\end{aligned}
$$

where the last hyper-surface is the same with the first one.

Moreover, since we need several times blowups for resolve each singularity, we should consider the following singularity sequences as well, where base varieties of those blow-ups appear

$$
\begin{aligned}
&(c_1^{(0)}\varepsilon^{-1}, c_2^{(0)}\varepsilon^{-1}, q_2^{(0)}, p_2^{(0)})\colon\ 2 \text{ dim } \boxed{1}\\
&\qquad \to (q_1^{(1)}, p_1^{(1)}, c_1^{(1)}\varepsilon^{-1}, c_2^{(1)}\varepsilon^{-1})\colon\ 2 \text{ dim } \boxed{9}\\
&\qquad \to (c_1^{(2)}\varepsilon^{-1}, c_2^{(2)}\varepsilon^{-1}, q_2^{(2)}, p_2^{(2)})
\end{aligned}
$$

$$
\begin{aligned}
&(c_1^{(0)}\varepsilon^{-1}, c_2^{(0)}\varepsilon, q_2^{(0)}, p_2^{(0)}):\ 2\ \mathrm{dim}\ \boxed{5}\\
&\quad\to (q_1^{(1)}, p_1^{(1)}, c^{(1)}\varepsilon^{-1}, c^{(1)}\varepsilon^{-1}):\ 2\ \mathrm{dim}\ \boxed{11}\\
&\quad\to (c_1^{(2)}\varepsilon, c_2^{(2)}\varepsilon^{-1}, q_2^{(2)}, p_2^{(2)}):\ 2\ \mathrm{dim}\ \boxed{7}\\
&\quad\to (q_1^{(3)}, p_1^{(3)}, c_1^{(3)}\varepsilon^{-1}, c_2^{(3)}\varepsilon):\ 2\ \mathrm{dim}\ \boxed{13}\\
&\quad\to (c^{(4)}\varepsilon^{-1}, c^{(4)}\varepsilon^{-1}, q_2^{(4)}, p_2^{(4)}):\ 2\ \mathrm{dim}\ \boxed{3}\\
&\quad\to (q_1^{(5)}, p_1^{(5)}, c_1^{(5)}\varepsilon, c_2^{(5)}\varepsilon^{-1}):\ 2\ \mathrm{dim}\ \boxed{15}\\
&\quad\to (c_1^{(6)}\varepsilon^{-1}, c_2^{(6)}\varepsilon, q_2^{(6)}, p_2^{(6)}).
\end{aligned}
$$

where the last subvariety for each sequence is the same with the first one.

The inclusion relations of these bases of blow-ups are

$$
\begin{gathered}
\boxed{1}\supset\boxed{2}\supset\boxed{3}\supset\boxed{4},\quad \boxed{5}\supset\boxed{6},\quad \boxed{7}\supset\boxed{8}\\
\boxed{9}\supset\boxed{10}\supset\boxed{11}\supset\boxed{12},\quad \boxed{13}\supset\boxed{14},\quad \boxed{15}\supset\boxed{16},
\end{gathered}
\tag{12}
$$

where we need to compare lower terms of the Laurent series to see these relations.

Case $A_5^{(1)}$. We find following two singularity sequences:

$$
\begin{aligned}
&(\varepsilon, p_1^{(0)}, q_2^{(0)}, p_2^{(0)}):\ 3\ \mathrm{dim}\\
&\quad\to (-p_2^{(0)} + a/q_2^{(0)} + b_1, q_2^{(0)}, a\varepsilon^{-1}, \varepsilon):\ 2\ \mathrm{dim}\ \boxed{6}\\
&\quad\to (p_2^{(0)} - a/q_2^{(0)}, a\varepsilon^{-1}, -a\varepsilon^{-1}, -p_2^{(0)} + a/q_2^{(0)} + b_1):\ 1\ \mathrm{dim}\ \boxed{4}\\
&\quad\to (-\varepsilon, -a\varepsilon^{-1}, q_2^{(3)}, p_2^{(0)} - a/q_2^{(0)}):\ 2\ \mathrm{dim}\ \boxed{8}\\
&\quad\to (q_1^{(4)}, p_1^{(4)}, q_2^{(4)}, -\varepsilon):\ 3\ \mathrm{dim},
\end{aligned}
$$

and

$$
\begin{aligned}
&(q_1^{(0)}, p_1^{(0)}, \varepsilon^{-1}, p_2^{(0)}):\ 3\ \mathrm{dim}\\
&\quad\to (-p_2^{(0)} - q_1^{(0)} + b_1, \varepsilon^{-1}, -\varepsilon^{-1}, q_1^{(0)}):\ 2\ \mathrm{dim}\ \boxed{2}\\
&\quad\to (p_2^{(0)}, -\varepsilon^{-1}, q_2^{(2)}, -p_2^{(0)} - q_1^{(0)} + b_1):\ 3\ \mathrm{dim}\\
&\quad\to (q_1^{(3)}, p_1^{(3)}, \varepsilon^{-1}, p_2^{(3)})\text{Returned.}
\end{aligned}
$$

We should consider the following singularity sequences as well, where base varieties of those blow-ups appear.

$$
\begin{aligned}
&(q_1^{(0)}, c_1^{(0)}\varepsilon^{-1}, c_2^{(0)}\varepsilon^{-1}, p_2^{(0)}):\ 2\ \mathrm{dim}\ \boxed{1}\\
&\quad\to (q_1^{(1)}, c_1^{(1)}\varepsilon^{-1}, c_2^{(1)}\varepsilon^{-1}, p_2^{(1)}):\ \text{Returned}
\end{aligned}
$$

$$
\begin{aligned}
&(q_1^{(0)}, p_1^{(0)}, c_1^{(0)}\varepsilon^{-1}, c_2^{(0)}\varepsilon)\colon\ 2 \text{ dim } \boxed{5}\\
&\quad \to (-q_1^{(0)} + b_1, c_1^{(0)}\varepsilon^{-1}, -c_1^{(0)}\varepsilon^{-1}, q_1^{(0)})\colon\ 1 \text{ dim } \boxed{3}\\
&\quad \to (c_2^{(2)}\varepsilon, -c_1^{(0)}\varepsilon^{-1}, p_2^{(0)}, -q_1^{(0)} + b_1)\colon\ 2 \text{ dim } \boxed{7}\\
&\quad \to (q_1^{(3)}, p_1^{(3)}, c_1^{(3)}\varepsilon^{-1}, c_2^{(3)}\varepsilon)\colon \text{ Returned.}
\end{aligned}
$$

Since the mapping is symmetric with respect to $(q_1, p_1) \leftrightarrow (q_2, p_2)$, there are the counterparts of these sequences. The inclusion relations of these bases of blow-ups are the same with (12).

4. SPACE OF INITIAL CONDITIONS AND LINEARISATION ON THE NÉRON-SEVERI LATTICES

In this section we construct a space of initial conditions by blowing up the defining variety along singularities of the previous section. Recall that as a complex manifold, in local coordinates $U \subset \mathbb{C}^N$, blowing up along a subvariety V of dimension $N-k$, $k \geq 2$, written as

$$x_1 - h_1(x_{k+1}, \dots x_N) = \cdots = x_k - h_k(x_{k+1}, \dots x_N) = 0,$$

where h_i's are holomorphic functions, is a birational morphism $\pi : X \to U$ such that $X = \{U_i\}$ is an open variety given by

$$U_i = \{(u_1^{(i)}, \dots, u_k^{(i)}, x_{k+1}, \dots x_N) \in \mathbb{C}^N\} \quad (i = 1, \dots, k)$$

with $\pi : U_i \to U$:

$$
\begin{aligned}
(x_1, \dots, x_N) =&(u_1^{(i)}u_i^{(i)} + h_1, \dots, u_{i-1}^{(i)}u_i^{(i)} + h_{i-1}, u_i^{(i)} + h_i,\\
&\qquad u_{i+1}^{(i)}u_i^{(i)} + h_{i+1} \dots, u_k^{(i)}u_i^{(i)} + h_k, x_{k+1}, \dots, x_N).
\end{aligned}
$$

It is convenient to write the coordinates of U_i as

$$\left(\frac{x_1 - h_1}{x_i - h_i}, \dots, \frac{x_{i-1} - h_{i-1}}{x_i - h_i}, x_i - h_i, \frac{x_{i+1} - h_{i+1}}{x_i - h_i}, \dots, \frac{x_k - h_k}{x_i - h_i}, x_{k+1}, \dots x_N\right).$$

The exceptional divisor E is written as $u_i = 0$ in U_i and each point in the center of blowup corresponds to a subvariety isomorphic to $\mathbb{P}^{k-1}$: $(x_1 - h_1 : \cdots : x_{k-1} - h_k)$. Hence E is locally a direct product $V \times \mathbb{P}^{k-1}$. We called such $\mathbb{P}^{k-1}$ a fiber of the exceptional divisor. (In algebraic setting the affine charts often need to be embedded into higher dimensional space.)

Theorem 4.1. *Each one of the mappings* (1) *or* (2) *can be lifted to a pseudo-automorphism on a rational projective variety $\mathcal{X}$ obtained by successive 16 blow-ups from $(\mathbb{P}^1)^4$, where the center of each blow-up, C_i ($i = 1, \dots, 16$), is two-dimensional sub-variety.*

Center C_i's are given by the following data, where we only write one of the affine coordinate of the center variety or the exceptional divisor. The other coordinates can be obtained automatically (see Fig. 3 and Fig. 4).

Case $A_2^{(1)} + A_2^{(1)}$. The center C_i of i-the blowing up and one of the new coordinate systems U_i obtained by the blowing-up are

$$
\begin{array}{ll}
C_1 : q_1^{-1} = p_1^{-1} = 0 & U_1 : (u_1, v_1, q_2, p_2) = (q_1^{-1}, q_1 p_1^{-1}, q_2, p_2) \\
C_2 : u_1 = v_1 + 1 = 0 & U_2 : (u_2, v_2, q_2, p_2) = (u_1, u_1^{-1}(v_1 + 1), q_2, p_2) \\
C_3 : u_2 = v_2 + b^{(1)} = 0 & \\
\qquad U_3 : (u_3, v_3, q_2, p_2) = (u_2, u_2^{-1}(v_2 + b^{(1)}), q_2, p_2) & \\
C_4 : u_3 = v_3 + (b^{(1)})^2 + a_0^{(1)} = 0 & \\
\qquad U_4 : (u_4, v_4, q_2, p_2) = (u_3, u_3^{-1}(v_3 + (b^{(1)})^2 + a_0^{(1)}), q_2, p_2) & \\
C_5 : q_1^{-1} = p_1 = 0 & U_5 : (u_5, v_5, q_2, p_2) = (q_1^{-1}, q_1 p_1, q_2, p_2) \\
C_6 : u_5 = v_5 - a_1^{(1)} = 0 & \\
\qquad U_6 : (u_6, v_6, q_2, p_2) = (u_5, u_5^{-1}(v_5 - a_1^{(1)}), q_2, p_2) & \\
C_7 : q_1 = p_1^{-1} = 0 & U_7(v_7, u_7, q_2, p_2) = (q_1 p_1, p_1^{-1}, q_2, p_2) \\
C_8 : u_7 = v_7 + a_2^{(1)} = 0 & \\
\qquad U_8 : (v_8, u_8, q_2, p_2) = (u_7^{-1}(u_7 + a_2^{(1)}), u_7, q_2, p_2) & \\
C_9 : p_2^{-1} = q_2^{-1} = 0 & U_9 : (q_1, p_1, u_9, v_9) = (q_1, p_1, q_2^{-1}, p_2^{-1} q_2) \\
C_{10} : u_9 = v_9 + 1 = 0 & \\
\qquad U_{10} : (q_1, p_1, u_{10}, v_{10}) = (q_1, p_1, u_9, u_9^{-1}(v_9 + 1)) & \\
C_{11} : u_{10} = v_{10} + b^{(2)} = 0 & \\
\qquad U_{11} : (q_1, p_1, u_{11}, v_{11}) = (q_1, p_1, u_{10}, u_{10}^{-1}(v_{10} + b^{(2)})) & \\
C_{12} : u_{11} = v_{11} + (b^{(2)})^2 + a_0^{(2)} = 0 & \\
\qquad U_{12} : (q_1, p_1, u_{12}, v_{12}) = (q_1, p_1, u_{11}, u_{11}^{-1}(v_{11} + (b^{(2)})^2 + a_0^{(2)})) & \\
C_{13} : p_2 = q_2^{-1} = 0 & U_{13} : (q_1, p_1, u_{13}, v_{13}) = (q_1, p_1, q_2^{-1}, p_2 q_2) \\
C_{14} : u_{13} = v_{13} - a_1^{(2)} = 0 & \\
\qquad U_{14} : (q_1, p_1, u_{14}, v_{14}) = (q_1, p_1, u_{13}, u_{13}^{-1}(v_{13} - a_1^{(2)})) & \\
C_{15} : p_2^{-1} = q_2 = 0 & U_{15} : (q_1, p_1, v_{15}, u_{15}) = (q_1, p_1, p_2 q_2, p_2^{-1}) \\
C_{16} : u_{15} = v_{15} + a_2^{(2)} = 0 & \\
\qquad U_{16} : (q_1, p_1, v_{16}, u_{16}) = (q_1, p_1, u_{15}^{-1}(v_{15} + a_2^{(2)}), u_{15}) &
\end{array}
$$

with $a_0^{(j)} = 0$, $a_1^{(j)} = -a_2^{(j)} = a$ and $b^{(j)} = b$ for $j = 1, 2$, where parameters $a = a_i^{(j)}, b^{(j)}$ are introduced for "deautonomisation" as explained in the next section.

Case $A_5^{(1)}$.

$$
\begin{array}{ll}
C_1 : q_2^{-1} = p_1^{-1} = 0 & U_1 : (q_1, v_1, u_1, p_2) = (q_1, q_2 p_1^{-1}, q_2^{-1}, p_2) \\
C_2 : u_1 = v_1 + 1 = 0 & U_2 : (q_1, v_2, u_2, p_2) = (q_1, u_1^{-1}(v_1 + 1), u_1, p_2) \\
C_3 : u_2 = q_1 + p_2 - b_1 = 0 & \\
\multicolumn{2}{c}{U_3 : (q_1, v_2, u_3, v_3) = (q_1, v_2, u_2, u_2^{-1}(q_1 + p_2 - b_1)} \\
C_4 : u_3 = v_3 + a_0 = 0 & U_4 : (q_1, v_2, u_4, v_4) = (q_1, v_2, u_3, u_3^{-1}(v_3 + a_0)) \\
C_5 : q_2^{-1} = p_2 = 0 & U_5 : (q_1, p_1, u_5, v_5) = (q_1, p_1, q_2^{-1}, p_2 q_2) \\
C_6 : u_5 = v_5 + a_2 = 0 & U_6 : (q_1, p_1, u_5, v_5) = (q_1, p_1, u_5, u_5^{-1}(v_5 + a_2)) \\
C_7 : q_1 = p_1^{-1} = 0 & U_7 : (v_7, u_7, q_2, p_2) = (q_1 p_1, p_1^{-1}, q_2, p_2) \\
C_8 : u_7 = v_7 - a_4 = 0 & U_8 : (v_8, u_8, q_2, p_2) = (u_7^{-1}(v_7 - a_4), u_7, q_2, p_2) \\
C_9 : q_1^{-1} = p_2^{-1} = 0 & U_9 : (u_9, p_1, q_2, v_9) = (q_1^{-1}, p_1, q_2, q_1 p_2^{-1}) \\
C_{10} : u_9 = v_9 + 1 = 0 & U_{10} : (u_{10}, p_1, q_2, v_{10}) = (u_9, p_1, q_2, u_9^{-1}(v_9 + 1)) \\
C_{11} : u_{10} = q_2 + p_1 - b_2 = 0 & \\
\multicolumn{2}{c}{U_{11} : (u_{11}, v_{11}, q_2, v_{10}) = (u_{10}, u_{10}^{-1}(q_2 + p_1 - b_2), q_2, v_{10})} \\
C_{12} : u_{11} = v_{11} + a_3 = 0 & \\
\multicolumn{2}{c}{U_{12} : (u_{12}, v_{12}, q_2, v_{10}) = (u_{11}, u_{11}^{-1}(v_{11} + a_3), q_2, v_{10})} \\
C_{13} : q_1^{-1} = p_1 = 0 & U_{13} : (u_{13}, v_{13}, q_2, p_2) = (q_1^{-1}, q_1 p_1, q_2, p_2) \\
C_{14} : u_{13} = v_{13} + a_5 = 0 & \\
\multicolumn{2}{c}{U_{14} : (u_{14}, v_{14}, q_2, p_2) = (u_{13}, p_2, q_2, u_{13}^{-1}(v_{13} + a_5))} \\
C_{15} : p_2^{-1} = q_2 = 0 & U_{15} : (q_1, p_1, v_{15}, u_{15}) = (q_1, p_1, p_2 q_2, p_2^{-1}) \\
C_{16} : u_{15} = v_{15} - a_1 = 0 & \\
\multicolumn{2}{c}{U_{16} : (q_1, p_1, v_{16}, u_{16}) = (q_1, p_1, u_{15}^{-1}(v_{15} - a_1), u_{15})}
\end{array}
$$

with

$$a_0 = a_3 = 0, \quad a_1 = a_4 = a, \quad a_2 = a_5 = -a.$$

Remark 4.2. Some centers (e.g., C_1 and C_9) intersect with each other but do not have inclusion relation. In this case, the variety depends on the order of blowups. However, since generic points are not in the intersection points, the varieties are pseudo-isomorphic with each other.

In both cases, the inclusion relations of total transforms of exceptional divisors E_i's are the same with (12) as

$$
\begin{array}{lll}
E_1 \supset E_2 \supset E_3 \supset E_4, & E_5 \supset E_6, & E_7 \supset E_8, \\
E_9 \supset E_{10} \supset E_{11} \supset E_{12}, & E_{13} \supset E_{14}, & E_{15} \supset E_{16}.
\end{array} \tag{13}
$$

Proof. The proof of the theorem is long but straightforward. We omit the detail, but we can show that any divisors in $\mathcal{X}$ are mapped to divisors in $\mathcal{X}$. For example, in $A_5^{(1)}$ case, the exceptional divisor E_4 is described as $u_4 = 0$ in U_4, while E_8 as $u_8 = 0$ in U_8. The mapping φ from U_4 to U_8 under $a_0 = 0$

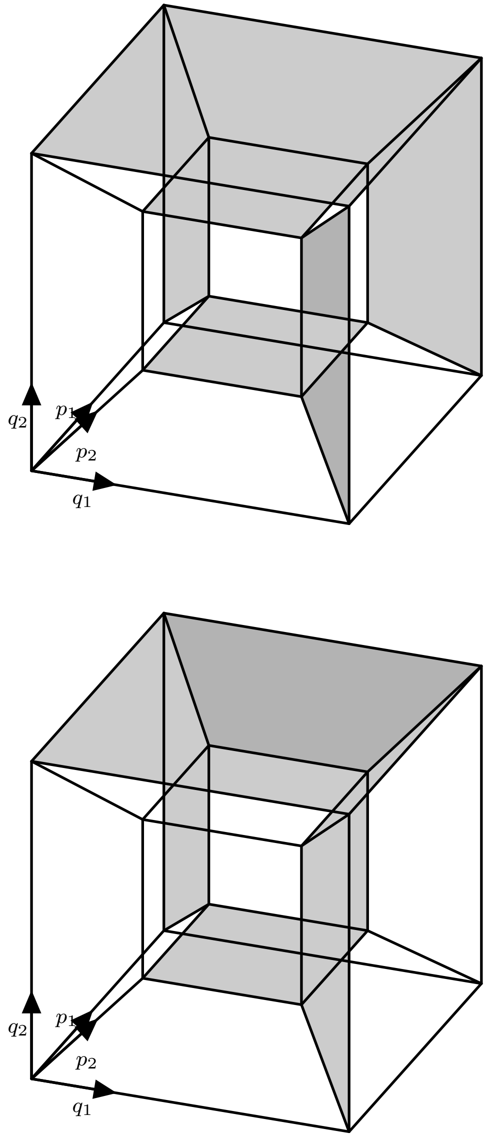


Figure 3 – Top: case $A_2^{(1)} + A_2^{(1)}$; bottom: case $A_5^{(1)}$; gray parallelograms: the centers C_1, C_5, C_7, C_9, C_{13}, C_{15} for both cases.

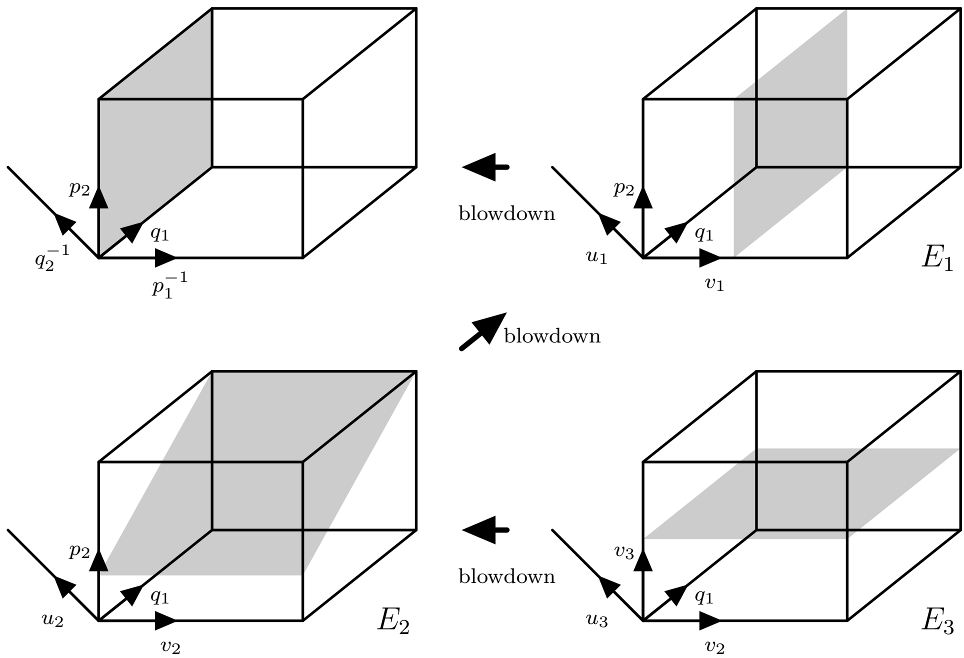


Figure 4 – Case $A_5^{(1)}$, gray parallelograms: the centers C_1, C_2, C_3, C_4, rectangulars: the exceptional divisors E_1, E_2, E_3.

and $a_4 = a$ is

$$(\bar{v}_8, \bar{u}_8, \bar{q}_2, \bar{p}_2) = \left(-v_4, u_4, aq_1^{-1} + (b_2 + v_2 - b_2u_4v_2)(1 - u_4v_2)^{-1}, q_1\right)$$

and hence $u_4 = 0$ implies $\bar{u}_8$ in generic (i.e. $q_1 \neq 0$). $\square$

Similarly computation to this proof yields the following theorem.

THEOREM 4.3. *The push-forward action of φ on $H^2(\mathcal{X}, \mathbb{Z})$ is as follows: Case $A_2^{(1)} + A_2^{(1)}$.*

$$\begin{aligned}
&H_{q_1} \mapsto H_{p_2}, \quad H_{p_1} \mapsto H_{q_2} + 2H_{p_2} - E_{9,10,13,14}\\
&H_{q_2} \mapsto H_{p_1}, \quad H_{p_2} \mapsto H_{q_1} + 2H_{p_1} - E_{1,2,5,6}\\
&E_1 \mapsto H_{p_2} - E_{10}, \quad E_2 \mapsto H_{p_2} - E_9, \quad E_3 \mapsto E_{15}, \quad E_4 \mapsto E_{16},\\
&E_5 \mapsto E_{11}, \quad E_6 \mapsto E_{12}, \quad E_7 \mapsto H_{p_2} - E_{14}, \quad E_8 \mapsto H_{p_2} - E_{13},\\
&E_9 \mapsto H_{p_1} - E_2, \quad E_{10} \mapsto H_{p_1} - E_1, \quad E_{11} \mapsto E_7, \quad E_{12} \mapsto E_8,\\
&E_{13} \mapsto E_3, \quad E_{14} \mapsto E_4, \quad E_{15} \mapsto H_{p_1} - E_6, \quad E_{16} \mapsto H_{p_1} - E_5
\end{aligned} \tag{14}$$

Case $A_5^{(1)}$.

$$
(15)\quad \begin{aligned}
&H_{q_1} \mapsto H_{p_2}, \quad H_{p_1} \mapsto H_{p_1} + H_{q_2} + H_{p_2} - E_{1,2,5,6}\\
&H_{q_2} \mapsto H_{p_1}, \quad H_{p_2} \mapsto H_{q_1} + H_{p_1} + H_{p_2} - E_{9,10,13,14}\\
&E_1 \mapsto H_{p_1} - E_2, \quad E_2 \mapsto H_{p_1} - E_1, \quad E_3 \mapsto E_7, \quad E_4 \mapsto E_8,\\
&E_5 \mapsto E_3, \quad E_6 \mapsto E_4, \quad E_7 \mapsto H_{p_2} - E_6, \quad E_8 \mapsto H_{p_2} - E_5,\\
&E_9 \mapsto H_{p_2} - E_{10}, \quad E_{10} \mapsto H_{p_2} - E_9, \quad E_{11} \mapsto E_{15}, \quad E_{12} \mapsto E_{16},\\
&E_{13} \mapsto E_{11}, \quad E_{14} \mapsto E_{12}, \quad E_{15} \mapsto H_{p_1} - E_{14}, \quad E_{16} \mapsto H_{p_1} - E_{13}
\end{aligned}
$$

and the action on $H_2(\mathcal{X}, \mathbb{Z})$ *is given by* (11) *with*

$$
J = \begin{bmatrix} I_4 & 0 \\ 0 & -I_{16} \end{bmatrix}.
$$

The actions (14) and (15) correspond to singularity patterns in the previous section. The pull-back actions are given by their inverse.

COROLLARY 4.4. *Both the degrees of mappings* (1) *and* (2) *grow quadratically.*

Proof. As mentioned in Section 2, the degrees are given by the coefficients of H_i's of $(\varphi^*)^n$, while the Jordan blocks of φ^* consist of

$$
\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}
$$

and seventeen 1×1 matrices whose absolute value is 1. $\square$

THEOREM 4.5. *For Case* $A_2^{(1)} + A_2^{(1)}$, *the linear system of the anticanonical divisor class* $\delta = 2\sum_{i=1}^{2}(H_{q_i} + H_{p_i}) - \sum_{i=1}^{16} E_i$ *is given by*

$$
(\alpha_0 + \alpha_1 I_1)(\beta_0 + \beta_1 I_2) = 0 \tag{16}
$$

for any $(\alpha_0 : \alpha_1), (\beta_0 : \beta_1) \in \mathbb{P}^1$, *where* I_i *are given by* (3) *and fibers* $\alpha_0 + \alpha_1 I_1 = 0$ *and* $\alpha_0 + \alpha_1 I_2 = 1$ *are mapped to each other, while for Case* $A_5^{(1)}$, *the linear system is given by*

$$
\alpha_0 + \alpha_1 I_1 + \alpha_2 I_2 = 0, \tag{17}
$$

for any $(\alpha_0 : \alpha_1 : \alpha_2) \in \mathbb{P}^2$, *where* I_i *are given by* (4) *and each fiber is preserved.*

Remark 4.6. In both cases the divisor defined by the coefficients of the symplectic form coincides with the canonical divisor. Indeed, for Case $A_2^{(1)} + A_2^{(1)}$, the divisor class corresponding to $dq_i \wedge dp_i$ is

$$
\begin{aligned}
i = 1 : &-2(H_{q_1} - E_{1,5}) - 2(H_{p_1} - E_{1,7}) - 3E_{1-2} - 2E_{2-3} - E_{3-4}\\
&- E_{5-6} - E_{7-8} = -2H_{q_1} - 2H_{p_1} + E_{1,\dots,8},
\end{aligned}
$$

$$i = 2 : \quad q_1 \leftrightarrow q_2, p_1 \leftrightarrow p_2, E_j \leftrightarrow E_{j+8}\ (j = 1, \ldots, 8) \text{ in the above,}$$

where E_{i-j} denotes $E_i - E_j$, while for Case $A_5^{(1)}$, that is

$$\begin{aligned} i = 1 : &- 2(H_{q_1} - E_{9,13}) - 2(H_{p_1} - E_{1,7}) - E_{1-2} - E_{7-8} - 2E_{9-10} - 2E_{10-11} \\ &- E_{11-12} - E_{13-14} = -2H_{q_1} - 2H_{p_1} + E_{1,2,7,8,11,12,13,14} \quad (i = 1), \\ i = 2 : &\quad q_1 \leftrightarrow q_2, p_1 \leftrightarrow p_2, E_j \leftrightarrow E_{j+8}\ (j = 1, \ldots, 8) \text{ in the above.} \end{aligned}$$

Hence, for Case $A_2^{(1)}+A_2^{(1)}$, the coefficients of the volume form corresponds to a decomposition of the anti-canonical divisor

$$\begin{aligned} -K_{\mathcal{X}} =& 2(H_{q_1} - E_{1,5}) + 2(H_{p_1} - E_{1,7}) + 2(H_{q_2} - E_{9,13}) + 2(H_{p_2} - E_{9,15}) \\ &+ 3E_{1-2} + 2E_{2-3} + E_{3-4} + E_{5-6} + E_{7-8} \\ &+ 3E_{9-10} + 2E_{10-11} + E_{11-12} + E_{13-14} + E_{15-16}, \end{aligned} \tag{18}$$

while for Case $A_5^{(1)}$ it is

$$-K_{\mathcal{X}} = (H_{q_1} \leftrightarrow H_{q_2} \text{ in } (18)). \tag{19}$$

The above decompositions is left fixed by the action of the mapping. For example, in the case $A_2^{(1)}+A_2^{(1)}$ if we set $-K_{\mathcal{X}} = D_1 + ... + D_{14}$ where $D_1 = E_{1-2}$, $D_2 = E_{2-3}$, $D_3 = E_{3-4}$, $D_4 = 2H_{q_1} - E_{1,5}$, $D_5 = E_{5-6}$, $D_6 = 2H_{p_1} - E_{1,7}$, $D_7 = E_{7-8}$, $D_8 = E_{9-10}$, $D_9 = E_{10-11}$, $D_{10} = E_{11-12}$, $D_{11} = H_{q_2} - E_{9,13}$, $D_{12} = E_{13-14}$, $D_{13} = H_{p_2} - E_{9,15}$, $D_{14} = E_{15-16}$, then we have

$$\begin{aligned} \varphi_* :& (D_1, D_2, ..., D_{14}) \mapsto \\ & (D_8, D_{13}, D_{14}, D_9, D_{10}, D_{11}, D_{12}, D_1, D_6, D_7, D_2, D_3, D_4, D_5). \end{aligned}$$

The set $\{D_1, D_2, ..., D_{14}\}$ is important because its orthogonal complement gives the symmetry group of the variety.

5. SYMMETRIES AND DEAUTONOMISATION

Let us fix the decomposition of the anti-canonical divisor as (18) or (19).

Definition 5.1. An automorphism s of the Néron-Severi bilattice is called a Cremona isometry if the following three properties are satisfied:

(a) s preserves the intersection form;
(b) s leaves the decomposition of $-K_{\mathcal{X}}$ fixed;
(c) s leaves the semigroup of effective classes of divisors invariant.

In general, if a birational mapping on $\mathbb{C}^N$ can be lifted to a pseudo-automorphism on $\mathcal{X}$, its action on the resulting Néron-Severi bilattice is always a Cremona isometry. In order to consider the inverse problem, i.e. from a Cremona isometry to a birational mapping, at least we need to allow the mapping to move the centers of blow-ups, but keeping one of the decomposition of the anti-canonical divisor $\sum_i m_i D_i$ ($m_i \geq 1$). Here, the birational mapping is lifted to an isomorphism from $\mathcal{X}_{\mathbf{a}}$ to $\mathcal{X}_{\mathbf{a}'}$, where suffix $\mathbf{a}$ denotes parameters fixing the centers of blowups. Note that $\sum_i m_i D_i$ is the unique anti-canonical divisor for generic $\mathbf{a}$, but not unique for the original $\mathcal{X}$ and the deautonomisation depends on the choice of them. Here, we fix one of anti-canonical divisors of $\mathcal{X}$. This situation is the same with two dimensional case. See [7, 41] in details.

In this section we construct a group of Cremona isometries for the $A_2^{(1)} + A_2^{(1)}$ and the $A_5^{(1)}$ cases and realise them as groups of birational mappings. Note that we do not know a canonical way to find root basis in $H^2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$, and hence we can not detect whether there are Cremona isometries outside of those groups or not. However, those groups act on a $\mathbb{Z}^6$ lattice in $H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$ nontrivially, which is the largest dimensional lattice orthogonal to the elements of decomposition of the anti-canonical divisor.

Case $A_2^{(1)} + A_2^{(1)}$. Let $\mathcal{X}_A$ denote a family of the space of initial conditions constructed in the previous section as

$$\mathcal{X}_A := \{\mathcal{X}_{\mathbf{a}} \mid \mathbf{a} = (a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_0^{(2)}, a_1^{(2)}, a_2^{(2)}; b^{(1)}, b^{(2)}) \in \mathbb{C}^8\}.$$

Then, there is a natural isomorphism between

$$H^2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z}) \times H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z}) \simeq H^2(\mathcal{X}, \mathbb{Z}) \times H_2(\mathcal{X}, \mathbb{Z})$$

as abstract lattices.

Let us define root vectors $\alpha_i^{(j)}$ and co-root vectors $\check{\alpha}_i^{(j)}$ ($i = 0, 1, 2$, $j = 1, 2$) so that the latter is orthogonal to all D_i, $i = 1, ..., 14$, as

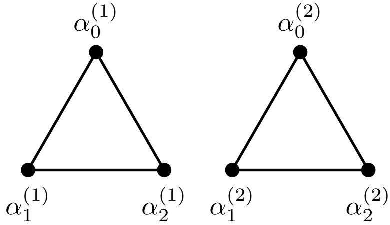
(20)
$$\begin{array}{lll} \alpha_0^{(1)} = H_{q_1} + H_{p_1} - E_{1,2,3,4}, & \alpha_1^{(1)} = H_{p_1} - E_{5,6}, & \alpha_2^{(1)} = H_{q_1} - E_{7,8}, \\ \alpha_0^{(2)} = H_{p_2} + H_{q_2} - E_{9,10,11,12}, & \alpha_1^{(2)} = H_{p_2} - E_{13,14}, & \alpha_2^{(2)} = H_{q_2} - E_{15,16} \end{array}$$

and

(21)
$$\begin{array}{lll} \check{\alpha}_0^{(1)} = h_{q_1} + h_{p_1} - e_{1,2,3,4}, & \check{\alpha}_1^{(1)} = h_{q_1} - e_{5,6}, & \check{\alpha}_2^{(1)} = h_{p_1} - e_{7,8}, \\ \check{\alpha}_0^{(2)} = h_{q_2} + h_{p_2} - e_{9,10,11,12}, & \check{\alpha}_1^{(2)} = h_{q_2} - e_{13,14}, & \check{\alpha}_2^{(2)} = h_{p_2} - e_{15,16} \end{array}.$$

Then, the pairing $\langle \alpha_i^{(j)}, \check{\alpha}_k^{(l)} \rangle$ induces two of the affine root system of type $A_2^{(1)}$ with the null vectors $\delta^{(1)} = 2H_{q_1} + 2H_{p_1} - E_{1,...,8}$ and $\delta^{(2)} = 2H_{q_2} + 2H_{p_2} - E_{9,...,16}$ and the null co-root vectors $\check{\delta}^{(1)} = 2h_{q_1} + 2h_{p_1} - e_{1,...,8}$ and $\check{\delta}^{(2)} = 2h_{q_2} + 2h_{p_2} - e_{9,...,16}$. The Cartan matrix and the Dynkin diagram are

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & -2 \end{bmatrix}$$



.

Let $\widetilde{W}(A_2^{(1)} + A_2^{(1)})$ denote the extended affine Weyl group

$$\mathrm{Aut}(A_2^{(1)} + A_2^{(1)}) \ltimes (W(A_2^{(1)}) \times W(A_2^{(1)})),$$

where $\mathrm{Aut}(A_2^{(1)} + A_2^{(1)})$ is the group of automorphisms of Dynkin diagram.

The roots $\check{\alpha}_i^{(j)}$'s are orthogonal to the elements of the decomposition of the anti-canonical divisor. Thus, if we define the action of the simple reflection $w_{\alpha_i^{(j)}}$ on the Néron-Severi bilattice as usual as

$$w_{\alpha_i^{(j)}}(D) = D + \langle D, \check{\alpha}_i^{(j)} \rangle \alpha_i^{(j)}, \quad w_{\alpha_i^{(j)}}(d) = d + \langle \alpha_i^{(j)}, d \rangle \check{\alpha}_i^{(j)} \tag{22}$$

for $D \in H^2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$ and $d \in H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$, it satisfies Condition (a) and (b) for Cremona isometries (Condition (c) is verified by realising as a birational mapping). Moreover, the group of Dynkin automorphisms is generated by

$$\begin{aligned}
\sigma_{01}^{(1)} :& \alpha_0^{(1)} \leftrightarrow \alpha_1^{(1)}, \quad \check{\alpha}_0^{(1)} \leftrightarrow \check{\alpha}_1^{(1)}, \\
& H_{p_1} \leftrightarrow H_{q_1} + H_{p_1} - E_1 - E_2, \quad E_1 \leftrightarrow H_{q_1} - E_2, \\
& E_2 \leftrightarrow H_{q_1} - E_1, \quad E_3 \leftrightarrow E_5, \quad E_4 \leftrightarrow E_6, \\
\sigma_{12}^{(1)} :& \alpha_1^{(1)} \leftrightarrow \alpha_2^{(1)} \quad \check{\alpha}_1^{(1)} \leftrightarrow \check{\alpha}_2^{(1)}, \\
& H_1 \leftrightarrow H_4, \quad E_5 \leftrightarrow E_7, \quad E_6 \leftrightarrow E_8, \\
\sigma_{01}^{(2)} :& \alpha_0^{(2)} \leftrightarrow \alpha_1^{(2)}, \quad \check{\alpha}_0^{(2)} \leftrightarrow \check{\alpha}_1^{(2)}, \\
& H_{p_2} \leftrightarrow H_{p_2} + H_{q_2} - E_9 - E_{10}, \quad E_9 \leftrightarrow H_{q_2} - E_{10}, \\
& E_{10} \leftrightarrow H_{q_2} - E_9, \quad E_{11} \leftrightarrow E_{13}, \quad E_{12} \leftrightarrow E_{14}, \\
\sigma_{12}^{(2)} :& \alpha_1^{(2)} \leftrightarrow \alpha_2^{(1)}, \quad \check{\alpha}_1^{(2)} \leftrightarrow \check{\alpha}_2^{(2)}, \\
& H_2 \leftrightarrow H_3, \quad E_{13} \leftrightarrow E_{15}, \quad E_{14} \leftrightarrow E_{16}, \\
\sigma^{(12)} :& \alpha_i^{(1)} \leftrightarrow \alpha_i^{(2)}, \quad \check{\alpha}_i^{(1)} \leftrightarrow \check{\alpha}_i^{(2)}, \\
& H_1 \leftrightarrow H_3, \quad H_2 \leftrightarrow H_4, \\
& E_i \leftrightarrow E_{i+8} \quad (\text{for } i = 1, 2, \ldots, 8).
\end{aligned}$$

with the action on $H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$ given by (11), where we omit writing the unchanged variables. It is easy to see that each one satisfies Condition (a) and (b) for a Cremona isometry.

Theorem 5.2. *The extended affine Weyl group* $\widetilde{W}(A_2^{(1)} + A_2^{(1)})$ *acts on the family of the space of initial conditions* $\mathcal{X}_A$ *such that each element* w *acts as a linear transformation on the set of parameters* $A = \mathbb{C}^8$ *and as a pseudo isomorphisms from* $X_{\mathbf{a}}$ *to* $X_{w(\mathbf{a})}$ *for generic* $\mathbf{a} \in A$.

Proof. It is enough to give realisation of the generators as birational mappings on

$$(q_1, p_2, q_2, p_1; a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_0^{(2)}, a_1^{(2)}, a_2^{(2)}; b^{(1)}, b^{(2)}) \in \mathbb{C}^{14}.$$

The following list gives such realisation:

$$\begin{aligned}
w_{\alpha_0^{(1)}} &: q_1 \leftrightarrow \frac{q_1^2 + q_1p_1 - b^{(1)}q_1 - a_0^{(1)}}{q_1 + p_1 - b^{(1)}}, \quad p_1 \leftrightarrow \frac{p_1^2 + q_1p_1 - b^{(1)}p_1 + a_0^{(1)}}{q_1 + p_1 - b^{(1)}}, \\
&\quad a_0^{(1)} \leftrightarrow -a_0^{(1)}, \quad a_1^{(1)} \leftrightarrow a_0^{(1)} + a_1^{(1)}, \quad a_2^{(1)} \leftrightarrow a_0^{(1)} + a_2^{(1)} \\
w_{\alpha_1^{(1)}} &: q_1 \leftrightarrow q_1 - a_1^{(1)}p_1^{-1}, \\
&\quad a_0^{(1)} \leftrightarrow a_0^{(1)} + a_1^{(1)}, \quad a_1^{(1)} \leftrightarrow -a_1^{(1)}, \quad a_2^{(1)} \leftrightarrow a_1^{(1)} + a_2^{(1)} \\
w_{\alpha_2^{(1)}} &: p_1 \leftrightarrow p_1 + a_2^{(1)}q_1^{-1}, \\
&\quad a_0^{(1)} \leftrightarrow a_0^{(1)} + a_2^{(1)}, \quad a_1^{(1)} \leftrightarrow a_1^{(1)} + a_2^{(1)}, \quad a_2^{(1)} \leftrightarrow -a_2^{(1)} \\
w_{\alpha_0^{(2)}} &: q_2 \leftrightarrow \frac{q_2^2 + p_2q_2 - b^{(2)}q_2 - a_0^{(2)}}{q_2 + p_2 - b^{(2)}}, \quad p_2 \leftrightarrow \frac{p_2^2 + p_2q_2 - b^{(2)}p_2 + a_0^{(2)}}{q_2 + p_2 - b^{(2)}}, \\
&\quad a_0^{(2)} \leftrightarrow -a_0^{(2)}, \quad a_1^{(2)} \leftrightarrow a_0^{(2)} + a_1^{(2)}, \quad a_2^{(2)} \leftrightarrow a_0^{(2)} + a_2^{(2)} \\
w_{\alpha_1^{(2)}} &: q_2 \leftrightarrow q_2 - a_1^{(2)}p_2^{-1}, \\
&\quad a_0^{(2)} \leftrightarrow a_0^{(2)} + a_1^{(2)}, \quad a_1^{(2)} \leftrightarrow -a_1^{(2)}, \quad a_2^{(2)} \leftrightarrow a_1^{(2)} + a_2^{(2)} \\
w_{\alpha_2^{(2)}} &: p_2 \leftrightarrow p_2 + a_2^{(2)}q_2^{-1}, \\
&\quad a_0^{(2)} \leftrightarrow a_2^{(1)} + a_2^{(2)}, \quad a_1^{(2)} \leftrightarrow a_1^{(2)} + a_2^{(2)}, \quad a_2^{(2)} \leftrightarrow -a_2^{(2)}
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{01}^{(1)} &: p_1 \leftrightarrow -q_1 - p_1 + b^{(1)}, \quad a_0^{(1)} \leftrightarrow -a_1^{(1)}, \quad a_1^{(1)} \leftrightarrow -a_0^{(1)}, \quad a_2^{(1)} \leftrightarrow -a_2^{(1)} \\
\sigma_{12}^{(1)} &: q_1 \leftrightarrow p_1, \quad a_0^{(1)} \leftrightarrow -a_0^{(1)}, \quad a_1^{(1)} \leftrightarrow -a_2^{(1)}, \quad a_2^{(1)} \leftrightarrow -a_1^{(1)} \\
\sigma_{01}^{(2)} &: p_2 \leftrightarrow -q_2 - p_2 + b^{(2)}, \quad a_0^{(2)} \leftrightarrow -a_1^{(2)}, \quad a_1^{(2)} \leftrightarrow -a_0^{(2)}, \quad a_2^{(2)} \leftrightarrow -a_2^{(2)} \\
\sigma_{12}^{(2)} &: q_2 \leftrightarrow p_2, \quad a_0^{(2)} \leftrightarrow -a_0^{(2)}, \quad a_1^{(2)} \leftrightarrow -a_2^{(2)}, \quad a_2^{(2)} \leftrightarrow -a_1^{(2)} \\
\sigma^{(12)} &: q_1 \leftrightarrow q_2, \quad p_1 \leftrightarrow p_2, \\
&\quad a_i^{(1)} \leftrightarrow a_i^{(2)}, \quad (\text{for } i = 0, 1, 2), \quad b^{(1)} \leftrightarrow b^{(2)}.
\end{aligned}$$

For these computations we used a factorisation formula proposed in [7] for two-dimensional case, which also works well in the higher dimensional case. □

The pull-back action φ^* on the root lattice is

$$(\alpha_0^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}) \mapsto (\alpha_1^{(j+1)} + \alpha_2^{(j+1)}, -\alpha_2^{(j+1)}, \alpha_0^{(j+1)} + \alpha_2^{(j+1)}) \tag{23}$$

for $j = 1, 2 \mod 2$, and written by the generators as

$$\varphi = \sigma^{(12)} \circ w_{\alpha_1^{(2)}} \circ \sigma_{12}^{(2)} \circ \sigma_{01}^{(2)} \circ w_{\alpha_1^{(1)}} \circ \sigma_{12}^{(1)} \circ \sigma_{01}^{(1)}. \tag{24}$$

Its action on the variables becomes

$$\begin{aligned} &\Big(q_1, p_2, q_2, p_1; a_0^{(1)}, a_1^{(1)}, a_2^{(1)}, a_0^{(2)}, a_1^{(2)}, a_2^{(2)}; b^{(1)}, b^{(2)}\Big) \\ \mapsto &\Big(- p_2 - q_2 + b^{(2)} - \frac{a_2^{(2)}}{q_2}, q_1, -q_1 - p_1 + b^{(1)} - \frac{a_2^{(1)}}{q_1}, q_2; \\ &a_1^{(2)} + a_2^{(2)}, -a_2^{(2)}, a_0^{(2)} + a_2^{(2)}, a_1^{(1)} + a_2^{(1)}, -a_2^{(1)}, a_0^{(1)} + a_2^{(1)}; b^{(2)}, b^{(1)}\Big), \end{aligned} \tag{25}$$

which is the non-autonomous version of φ. The action $(\varphi^2)^*$ on the root lattice is a translation as

$$(\alpha_0^{(j)}, \alpha_1^{(j)}, \alpha_2^{(j)}) \mapsto (\alpha_0^{(j)}, \alpha_1^{(j)} - \delta^{(j)}, \alpha_2^{(j)} + \delta^{(j)}) \tag{26}$$

for $j = 1, 2$.

Case $A_5^{(1)}$. Let $\mathcal{X}_A$ denote a family of the space of initial conditions

$$\mathcal{X}_A := \{\mathcal{X}_{\mathbf{a}} \mid \mathbf{a} = (a_0, a_1, a_2, a_3, a_4, a_5; b_1, b_2) \in \mathbb{C}^8\}.$$

Let us define root vectors α_i and co-root vectors $(i = 0, \ldots, 5)$ as

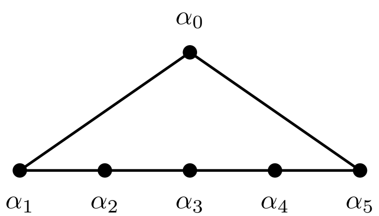
$$\begin{aligned} &\alpha_0 = H_{q_1} + H_{p_2} - E_{3,4,9,10}, \quad &&\alpha_1 = H_{q_2} - E_{15,16}, \quad &&\alpha_2 = H_{p_2} - E_{5,6}, \\ &\alpha_3 = H_{p_1} + H_{q_2} - E_{1,2,11,12}, \quad &&\alpha_4 = H_{q_1} - E_{7,8}, \quad &&\alpha_5 = H_{p_1} - E_{13,14} \end{aligned} \tag{27}$$

and

$$\begin{aligned} &\check{\alpha}_0 = h_{p_1} + h_{q_2} - e_{1,2,3,4}, \quad &&\check{\alpha}_1 = h_{p_2} - e_{15,16}, \quad &&\check{\alpha}_2 = h_{q_2} - e_{5,6}, \\ &\check{\alpha}_3 = h_{q_1} + h_{p_2} - e_{9,10,11,12}, \quad &&\check{\alpha}_4 = h_{p_1} - e_{7,8}, \quad &&\check{\alpha}_5 = h_{q_1} - e_{13,14}. \end{aligned} \tag{28}$$

Then, the pairing $\langle \alpha_i, \check{\alpha}_j \rangle$ induces the affine root system of type $A_5^{(1)}$ with the null vectors $\delta = 2H_{q_1,p_1,q_2,p_2} - E_{1,\ldots,16}$ and the null co-root vector $\check{\delta} = 2h_{q_1,p_1,q_2,p_2} - e_{1,\ldots,16}$. The Cartan matrix and the Dynkin diagram are

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$



.

Let $\widetilde{W}(A_5^{(1)})$ denote the extended affine Weyl group $\mathrm{Aut}(A_5^{(1)}) \ltimes W(A_5^{(1)})$.

We define the action of the simple reflection w_{α_i} on the Néron-Severi bilattice as (22). The group of Dynkin automorphisms is generated by

$$\begin{aligned}
\sigma_{01} :& \alpha_0 \leftrightarrow \alpha_1, \quad \alpha_2 \leftrightarrow \alpha_5, \quad \alpha_3 \leftrightarrow \alpha_4, \quad \check{\alpha}_0 \leftrightarrow \check{\alpha}_1, \quad \check{\alpha}_2 \leftrightarrow \check{\alpha}_5, \quad \check{\alpha}_3 \leftrightarrow \check{\alpha}_4,\\
& H_{q_2} \leftrightarrow H_{p_2}, \quad H_{q_1} \leftrightarrow H_{p_1,q_2} - E_{1,2}, \quad H_{q_2} \leftrightarrow H_{q_1,p_2} - E_{9,10},\\
& E_1 \leftrightarrow H_{p_2} - E_{10}, \quad E_2 \leftrightarrow H_{p_2} - E_9, \quad E_3 \leftrightarrow E_{15}, \quad E_4 \leftrightarrow E_{16},\\
& E_5 \leftrightarrow E_{13}, \quad E_6 \leftrightarrow E_{14}, \quad E_7 \leftrightarrow E_{11}, \quad E_8 \leftrightarrow E_{12},\\
& E_9 \leftrightarrow H_{p_1} - E_2, \quad E_{10} \leftrightarrow H_{p_1} - E_1,\\
\sigma_{12} :& \alpha_0 \leftrightarrow \alpha_3, \quad \alpha_1 \leftrightarrow \alpha_2, \quad \alpha_4 \leftrightarrow \alpha_5, \quad \check{\alpha}_0 \leftrightarrow \check{\alpha}_3, \quad \check{\alpha}_1 \leftrightarrow \check{\alpha}_2, \quad \check{\alpha}_4 \leftrightarrow \check{\alpha}_5,\\
& H_{q_1} \leftrightarrow H_{p_1}, \quad H_{q_1} \leftrightarrow H_{p_2},\\
& E_1 \leftrightarrow E_9, \quad E_2 \leftrightarrow E_{10}, \quad E_3 \leftrightarrow E_{11}, \quad E_4 \leftrightarrow E_{12},\\
& E_5 \leftrightarrow E_{15}, \quad E_6 \leftrightarrow E_{16}, \quad E_7 \leftrightarrow E_{13}, \quad E_8 \leftrightarrow E_{14},
\end{aligned}$$

with the action on $H_2(\mathcal{X}_{\mathbf{a}}, \mathbb{Z})$ given by (11).

THEOREM 5.3. *The extended affine Weyl group $\widetilde{W}(A_5^{(1)})$ acts on the family of the space of initial conditions $\mathcal{X}_A$ such that each element w acts as a linear transformation on the set of parameters $A = \mathbb{C}^8$ and as a pseudo-isomorphisms from $X_{\mathbf{a}}$ to $X_{w(\mathbf{a})}$ for generic $\mathbf{a} \in A$.*

Proof. The following list gives realisation of the generators as birational mappings on

$$(q_1, p_1, q_2, p_2; a_0, a_0, a_1, a_2, a_3, a_4, a_5; b_1, b_2) \in \mathbb{C}^{14}.$$

$$\begin{aligned}
w_{\alpha_0} :& \ p_1 \leftrightarrow \frac{(q_1 + p_2 - b_1)p_1 - a_0}{q_1 + p_2 - b_1}, \quad q_2 \leftrightarrow \frac{(q_1 + p_2 - b_1)q_2 + a_0}{q_1 + p_2 - b_1},\\
& a_5 \leftrightarrow a_0 + a_5 \quad a_0 \leftrightarrow -a_0, \quad a_1 \leftrightarrow a_0 + a_1,\\
w_{\alpha_1} :& \ p_2 \leftrightarrow p_2 - a_1 q_2^{-1},\\
& a_0 \leftrightarrow a_0 + a_1, \quad a_1 \leftrightarrow -a_1, \quad a_2 \leftrightarrow a_1 + a_2\\
w_{\alpha_2} :& \ q_2 \leftrightarrow q_2 + a_2 p_2^{-1},\\
& a_1 \leftrightarrow a_1 + a_2, \quad a_2 \leftrightarrow -a_2 \quad a_3 \leftrightarrow a_2 + a_3,\\
w_{\alpha_3} :& \ q_1 \leftrightarrow \frac{(q_2 + p_1 - b_2)q_1 + a_3}{q_2 + p_1 - b_2}, \quad p_2 \leftrightarrow \frac{(q_2 + p_1 - b_2)p_2 - a_3}{q_2 + p_1 - b_2},\\
& a_2 \leftrightarrow a_2 + a_3 \quad a_3 \leftrightarrow -a_3, \quad a_4 \leftrightarrow a_3 + a_4,\\
w_{\alpha_4} :& \ p_1 \leftrightarrow p_1 - a_4 q_1^{-1},\\
& a_3 \leftrightarrow a_3 + a_4, \quad a_4 \leftrightarrow -a_4, \quad a_5 \leftrightarrow a_4 + a_5,\\
w_{\alpha_5} :& \ q_1 \leftrightarrow q_1 + a_5 p_1^{-1},
\end{aligned}$$

$$\begin{aligned}
&\quad a_4 \leftrightarrow a_4+a_5, \quad a_5 \leftrightarrow -a_5 \quad a_0 \leftrightarrow a_0+a_5,\\
\sigma_{01}: &\ q_1 \leftrightarrow -q_2-p_1+b_2, \quad p_1 \leftrightarrow p_2, \quad q_2 \leftrightarrow -q_1-p_2+b_1,\\
&\quad a_0 \leftrightarrow -a_1, \quad a_2 \leftrightarrow -a_5, \quad a_3 \leftrightarrow -a_4, \quad b_1 \leftrightarrow b_2,\\
\sigma_{12}: &\ q_1 \leftrightarrow p_1, \quad p_2 \leftrightarrow q_2,\\
&\quad a_0 \leftrightarrow -a_3, \quad a_1 \leftrightarrow -a_2, \quad a_4 \leftrightarrow -a_5, \quad b_1 \leftrightarrow b_2
\end{aligned}$$

□

The pull-back action of φ^* on the root lattice is

$$(\alpha_0,\ldots,\alpha_5) \mapsto (\alpha_1+\alpha_2, \alpha_3+\alpha_4, -\alpha_4, \alpha_4+\alpha_5, \alpha_0+\alpha_1, -\alpha_1) \tag{29}$$

and written by the generators as

$$\varphi = w_{\alpha_5} \circ w_{\alpha_2} \circ \sigma_{01} \circ \sigma_{12}. \tag{30}$$

Its action on the variables becomes

$$\begin{aligned}
&(q_1,p_1,q_2,p_2;a_0,a_1,a_2,a_3,a_4,a_5;b_1,b_2)\\
&\mapsto \Big(-q_1-p_2+b_1+\frac{a_1}{q_2}, q_2, -q_2-p_1+b_2+\frac{a_4}{q_1}, q_1;\\
&\qquad a_1+a_2, a_3+a_4, -a_4, a_4+a_5, a_0+a_1, -a_1; b_1, b_2\Big),
\end{aligned} \tag{31}$$

which is the non-autonomous version of φ. It is easy to see that $(\varphi^4)^*$ is a translation on the root lattice as

$$\varphi^4 : (\alpha_0,\ldots,\alpha_5) \mapsto (\alpha_0,\ldots,\alpha_5) + \delta(0,1,-1,0,1,-1).$$

Remark 5.4. It is highly nontrivial to find the root basis. For example, since the difference of decomposition of the anti-canonical divisor between the $A_2^{(1)}+A_2^{(1)}$ case and the $A_5^{(1)}$ case is just exchange of H_{q_1} and H_{q_2}, for the $A_2^{(1)}+A_2^{(1)}$ variety, the $A_5^{(1)}$ root system with the basis:

$$\begin{array}{lll}
\alpha_0 = H_{q_2}+H_{p_2}-E_{3,4,9,10}, & \alpha_1 = H_{q_1}-E_{15,16}, & \alpha_2 = H_{p_2}-E_{5,6},\\
\alpha_3 = H_{p_1}+H_{q_1}-E_{1,2,11,12}, & \alpha_4 = H_{q_2}-E_{7,8}, & \alpha_5 = H_{p_2}-E_{13,14}\\
\check\alpha_0 = h_{p_1}+h_{q_1}-e_{1,2,3,4}, & \check\alpha_1 = h_{p_2}-e_{15,16}, & \check\alpha_2 = h_{q_1}-e_{5,6},\\
\check\alpha_3 = h_{q_2}+h_{p_2}-e_{9,10,11,12}, & \check\alpha_4 = h_{p_1}-e_{7,8}, & \check\alpha_5 = h_{q_2}-e_{13,14}.
\end{array}.$$

also satisfies Condition (a) and (b) for Cremona isometries. However, it does not satisfy Condition (c). Actually, w_{α_1} acts to an effective divisor E_{16} as $E_{16} \mapsto H_{q_1}-E_{15}$, but $H_{q_1}-E_{15}$ is not effective. Similarly, for the $A_5^{(1)}$ variety, the $A_2^{(1)}+A_2^{(1)}$ root system with the basis:

$$\begin{array}{lll}
\alpha_0^{(1)} = H_{q_2}+H_{p_1}-E_{1,2,3,4}, & \alpha_1^{(1)} = H_{p_1}-E_{5,6}, & \alpha_2^{(1)} = H_{q_1}-E_{7,8},\\
\alpha_0^{(2)} = H_{p_2}+H_{q_1}-E_{9,10,11,12}, & \alpha_1^{(2)} = H_{p_2}-E_{13,14}, & \alpha_2^{(2)} = H_{q_2}-E_{15,16}
\end{array}$$

$$\begin{array}{lll} \check{\alpha}_0^{(1)} = h_{q_2} + h_{p_1} - e_{1,2,3,4}, & \check{\alpha}_1^{(1)} = h_{q_2} - e_{5,6}, & \check{\alpha}_2^{(1)} = h_{p_1} - e_{7,8}, \\ \check{\alpha}_0^{(2)} = h_{q_1} + h_{p_2} - e_{9,10,11,12}, & \check{\alpha}_1^{(2)} = h_{q_1} - e_{13,14}, & \check{\alpha}_2^{(2)} = h_{p_2} - e_{15,16} \end{array} .$$

also satisfies Condition (a) and (b), but does not satisfy (c).

6. 4D NON-CONFINING SYSTEM

Let us consider System (5) on the projective space $(\mathbb{P}^1)^4$. In the following, we aim to obtain a four-dimensional rational variety by blowing-up procedure such that the birational map (5) is lifted to an algebraically stable map on the variety.

Let $I(\varphi)$ denote the indeterminacy set of φ. It is known that the mapping φ is algebraically stable if and only if there does not exist a positive integer k and a divisor D on $\mathcal{X}$ such that

$$\varphi(D \setminus I(\varphi)) \subset I(\varphi^k), \tag{32}$$

i.e. the image of the generic part of a divisor by φ^n is included in the indeterminate set ([3, 1], Proposition 2.3 of [8]). see Section 2 of [8] for notations and related theories used here).

The notion of singularity series of dynamics studied by Grammaticos-Ramani and their collaborators is closely related to our procedure. Let us start with a hyper-plane $x_2 = 1 + \varepsilon$, where ε is a small parameter for considering Laurent series expression, and apply φ, then we have a "confined" sequence of Laurent series:

$$\begin{aligned} \cdots \to&(x_0^{(0)}, x_1^{(0)}, 1+\varepsilon, x_3^{(0)}) \to (1, x_3^{(0)}, -h\varepsilon^{-1}, (1+hx_3^{(0)})\varepsilon^{-2}) \\ \to&(-h\varepsilon^{-1}, (1+hx_3^{(0)})\varepsilon^{-2}, h\varepsilon^{-1}, -(1+hx_3^{(0)})\varepsilon^{-2}) \\ \to&(h\varepsilon^{-1}, -(1+hx_3^{(0)})\varepsilon^{-2}, 1, x_4^{(3)}) \to (1, x_1^{(5)}, x_0^{(0)}, x_3^{(5)}) \to \cdots , \end{aligned} \tag{33}$$

where $x_i^{(k)}$'s are complex constants and only the principal term is written for each entry and a hyper-surface $x_2 = 0$ is contracted to lower-dimensional varieties and returned to a hyper-surface $x_0 = 0$ after 4 steps. We can also find a cyclic sequence:

$$\begin{aligned} &(x_0^{(0)}, x_1^{(0)}, \varepsilon^{-1}, x_3^{(0)}) \to (\varepsilon^{-1}, x_3^{(0)}, -\varepsilon^{-1}, -x_1^{(0)} - x_3^{(0)}) \\ \to&(\varepsilon^{-1}, -x_1^{(0)} - x_3^{(0)}, x_0^{(0)}, x_1^{(0)}) \to (x_0^{(0)}, x_1^{(0)}, \varepsilon^{-1}, x_3^{(3)}) \text{: returned,} \end{aligned} \tag{34}$$

where a hyper-surface $x_2 = \infty$ is contracted to lower-dimensional varieties and returned to the original hyper-surface after 3 steps, and an "anti-confined" sequence:

$$\cdots \to \left(\left(-1 + \frac{h}{(x_0^{(0)} - 1)^2} \right) \varepsilon^{-1}, x_1^{(-1)}, x_2^{(-1)}, \varepsilon^{-1} \right)$$

$$\begin{aligned}&\to(x_0^{(0)},\varepsilon^{-1},x_2^{(0)},x_3^{(0)})\to(x_2^{(0)},x_3^{(0)},x_2^{(1)},\varepsilon^{-1})\\ &\to\left(x_2^{(1)},\varepsilon^{-1},x_2^{(2)},\left(-1+\frac{h}{(x_2^{(0)}-1)^2}\right)\varepsilon^{-1}\right)\to\cdots.,\end{aligned}\tag{35}$$

where a lower dimensional variety is blown-up to a hyper-surfaces $x_1=\infty$ and contracted to a lower dimensional variety after 2 steps.

In the following, in order to avoid anti-confined patterns, we consider $\mathbb{P}^2\times\mathbb{P}^2$ instead of $(\mathbb{P}^1)^4$. Although there is a possibility that the anti-confined pattern can be resoluted by some blowing-down procedure, it is not easy to find the actual procedure on the level of coordinates.

The coordinate system of $\mathbb{P}^2\times\mathbb{P}^2$ is $(x_0:x_1:1,\ x_2:x_3:1)$, and thus the local coordinate systems essentially consist of $3\times 3=9$ charts:

$$\begin{aligned}&(x_0,x_1,x_2,x_3),(y_0,y_1,x_2,x_3),(z_0,z_1,x_2,x_3),\\ &(x_0,x_1,y_2,y_3),(y_0,y_1,y_2,y_3),(z_0,z_1,y_2,y_3),\\ &((x_0,x_1,z_2,z_3),(y_0,y_1,z_2,z_3),(z_0,z_1,z_2,z_3),\end{aligned}$$

where y_i's and z_i's are

$$(y_i,y_{i+1})=(x_i^{-1},x_i^{-1}x_{i+1})\text{ and }(z_i,z_{i+1})=(x_ix_{i+1}^{-1},x_{i+1}^{-1})$$

for $i=0,2$. Then, both the cyclic sequence (34) and the anti-confined sequence (35) starting with $x_i^{(0)}=\varepsilon^{-1}$ do not appear, but another cyclic sequence

$$\begin{aligned}&(x_0^{(0)},x_1^{(0)},\varepsilon^{-1},c^{(0)}\varepsilon^{-1})\to(\varepsilon^{-1},c^{(0)}\varepsilon^{-1},-\varepsilon^{-1},-c^{(0)}\varepsilon^{-1})\\ &\to(-\varepsilon^{-1},-c^{(0)}\varepsilon^{-1},x_0^{(0)},x_1^{(0)})\to(x_0^{(0)},x_1^{(0)},\varepsilon^{-1},c^{(0)}\varepsilon^{-1})\text{: returned}\end{aligned}\tag{36}$$

appears, where $c^{(0)}$ is also a complex constant.

In order to resolute the singularity appeared in Sequences (33) and (36), we blow up the rational variety along the sub-varieties to which some divisor is contracted to. For Sequences (33), we have three such sub-varieties whose parametric expressions are

$$\begin{aligned}V_1:&(x_0,x_1,z_2,z_3)=(P,1,0,0)\\ V_2:&(z_0,z_1,z_2,z_3)=(0,0,0,0)\\ V_3:&(z_0,z_1,x_2,x_3)=(0,0,P,1),\end{aligned}$$

where P is a $\mathbb{C}$-valued parameter (independent to another sub-variety's), while for Sequences (36) we have a sub-variety

$$V_4:(z_0,z_1,z_2,z_3)=(P,0,P,0).$$

That is, the subvariety V_1 is the Zariski closure of

$$\{(x_0,x_1,x_2,x_3)=(P,1,0,0)\mid P\in\mathbb{C}\}$$

and V_4 is that of of $\{(x_0, x_1, x_2, x_3) = (P, 0, P, 0) \mid P \in \mathbb{C}\}$ and so forth.

Since V_4 includes V_2, we have the option of blowing-up order. In the two dimensional case, resolution is unique and the order is not a matter. But in the higher dimensional case, it affects sensitively to the resulting varieties. Since we only care on the level of codimension one, the order of blow-ups does not affect the algebraical stability in some cases. But the following results were not obtained in a straightforward manner but by trial and error.

We can resolute the singularity around V_1 by the following five blowups:

$$\begin{aligned}
C_1 :&(x_0, x_1, z_2, z_3) = (1, P, 0, 0)\\
&\leftarrow (s_1, t_1, u_1, v_1) := (x_0 - 1, x_1, z_2(x_0 - 1)^{-1}, z_3(x_0 - 1)^{-1}),\\
C_2 :&(s_1, t_1, u_1, v_1) = (0, P, Q, 0)\\
&\leftarrow (s_2, t_2, u_2, v_2) := (s_1, t_1, u_1, v_1 s_1^{-1}),\\
C_3 :&(s_2, t_2, u_2, v_2) = (0, P, -h(1 + hP)^{-1}, Q)\\
&\leftarrow (s_3, t_3, u_3, v_3) := (s_2, t_2, (u_2 + h(1 + ht_2)^{-1})s_2^{-1}, v_2),\\
C_4 :&(s_3, t_3, u_3, v_3) = (0, P, Q, (1 + hP)^{-1})\\
&\leftarrow (s_4, t_4, u_4, v_4) := (s_3, t_3, u_3, (v_3 - (1 + ht_3)^{-1})s_3^{-1}),\\
C_5 :&(s_4, t_4, u_4, v_4) = (0, P, Q, (1 + hP)^{-2})\\
&\leftarrow (s_5, t_5, u_5, v_5) := (s_4, t_4, u_4, (v_4 - (1 + ht_4)^{-2})s_4^{-1}),
\end{aligned}$$

where only one of the coordinate systems is written for each blowup. Similarly, we can resolute the singularity around V_3 by the following five blowups:

$$\begin{aligned}
C_6 :&(z_0, z_1, x_2, x_3) = (0, 0, 1, P)\\
&\leftarrow (s_6, t_6, u_6, v_6) := (x_2 - 1, x_3, z_0(x_2 - 1)^{-1}, z_1(x_2 - 1)^{-1}),\\
C_7 :&(s_6, t_6, u_6, v_6) = (0, P, Q, 0)\\
&\leftarrow (s_7, t_7, u_7, v_7) := (s_6, t_6, u_6, v_6 s_6^{-1}),\\
C_8 :&(s_7, t_7, u_7, v_7) = (0, P, -h(1 + hP)^{-1}, Q)\\
&\leftarrow (s_8, t_8, u_8, v_8) := (s_7, t_7, (u_7 + h(1 + ht_7)^{-1})s_7^{-1}, v_7),\\
C_9 :&(s_8, t_8, u_8, v_8) = (0, P, Q, (1 + hP)^{-1})\\
&\leftarrow (s_9, t_9, u_9, v_9) := (s_8, t_8, u_8, (v_8 - (1 + ht_8)^{-1})s_8^{-1}),\\
C_{10} :&(s_9, t_9, u_9, v_9) = (0, P, Q, (1 + hP)^{-2})\\
&\leftarrow (s_{10}, t_{10}, u_{10}, v_{10}) := (s_9, t_9, u_9, (v_9 - (1 + ht_9)^{-2})s_9^{-1}).
\end{aligned}$$

We need three blowups for V_4:

$$\begin{aligned}
C_{11} :&(z_0, z_1, z_2, z_3) = (0, 0, 0, 0)\\
&\leftarrow (s_{11}, t_{11}, u_{11}, v_{11}) := (z_0, z_1 z_0^{-1}, z_2 z_0^{-1}, z_3 z_0^{-1}),
\end{aligned}$$

$$\begin{aligned}C_{12}: &(s_{11}, t_{11}, u_{11}, v_{11}) = (P, 0, 1, 0)\\ &\leftarrow (s_{12}, t_{12}, u_{12}, v_{12}) := (s_{11}, t_{11}, (u_{11}-1)t_{11}^{-1}, v_{11}t_{11}^{-1}),\\ C_{13}: &(s_{12}, t_{12}, u_{12}, v_{12}) = (P, 0, Q, -1)\\ &\leftarrow (s_{13}, t_{13}, u_{13}, v_{13}) := (s_{12}, t_{12}, u_{12}, (v_{12}+1)t_{12}^{-1}),\end{aligned}$$

where C_{11} corresponds to V_2, while C_{12} and C_{13} corresponds to V_4. We need additional four blowups for V_2:

$$\begin{aligned}C_{14}: &(s_{13}, t_{13}, u_{13}, v_{13}) = (0, 0, 1+h, 0)\\ &\leftarrow (s_{14}, t_{14}, u_{14}, v_{14}) := (s_{13}t_{13}^{-1}, t_{13}, (u_{13}-1-h)t_{13}^{-1}, v_{13}t_{13}^{-1}),\\ C_{15}: &(s_{14}, t_{14}, u_{14}, v_{14}) = (P, 0, -2Q - Ph^{-1}, Q)\\ &\leftarrow (s_{15}, t_{15}, u_{15}, v_{15}) := (s_{14}, t_{14}, v_{14}, (u_{14}+2v_{14}+s_{14}h^{-1})t_{14}^{-1}),\\ C_{16}: &(s_{15}, t_{15}, u_{15}, v_{15}) = (P, 0, -Ph^{-1}, Q)\\ &\leftarrow (s_{16}, t_{16}, u_{16}, v_{16}) := (s_{15}, t_{15}, (u_{15}+s_{15}h^{-1})t_{15}^{-1}, v_{15}),\\ C_{17}: &(s_{16}, t_{16}, u_{16}, v_{16}) = (P, 0, Q, 2^{-1}Q + (1+h)h^{-1}P)\\ &\leftarrow (s_{17}, t_{17}, u_{17}, v_{17}) := (s_{16}, t_{16}, u_{16}, (v_{16} - 2^{-1}u_{16} - (1+h)h^{-1}s_{16})t_{16}^{-1}).\end{aligned}$$

The exceptional divisor E_i of i-th blowup is described in the local chart as

$$\begin{aligned}E_i &: s_i = 0, \quad (i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14)\\ E_i &: t_i = 0, \quad (i = 12, 13, 15, 16, 17).\end{aligned}$$

Let us denote the total transform (with respect to blowups) of the divisors (hyper-surfaces) $c_0x_0 + c_1x_1 + a = 0$ and $c_2x_2 + c_3x_3 + b$ by H_a and H_b respectively, where $(c_0 : c_1 : a)$ and $(c_2 : c_3 : b)$ are constant $\mathbb{P}^2$ vectors. We also denote the total transform of the i-th exceptional divisor by E_i. Let us write their classes modulo linear equivalence as $\mathcal{H}_a$, $\mathcal{H}_b$ and $\mathcal{E}_i$. Then, the Picard group of this variety $\mathcal{X}$ becomes a $\mathbb{Z}$-module:

$$\mathrm{Pic}(\mathcal{X}) = \mathbb{Z}\mathcal{H}_a \oplus \mathbb{Z}\mathcal{H}_b \oplus \bigoplus_{i=1}^{17} \mathbb{Z}\mathcal{E}_i. \tag{37}$$

THEOREM 6.1. *The map* (5) *is lifted to an algebraically stable map on the rational variety obtained by blow-ups along C_i, $i = 1, 2, \ldots, 17$, from $\mathbb{P}^2 \times \mathbb{P}^2$.*

Proof. The algebraic stability can be checked as follows. In the present case, the indeterminate set $I(\varphi)$ is given by

$$I(\varphi) = \varphi^{-1}(E_6 - E_7) \subset E_{11},$$

while the condition that the dimension of $\varphi(D \setminus I(\varphi))$ is at most two implies $D = E_1 - E_2$ and $\varphi(D \setminus I(\varphi)) = \varphi(E_1 - E_2) \subset E_{11}$. It can be checked that

$\varphi(E_1 - E_2)$ and $I(\varphi^k)$, $k = 1, 2, 3, \ldots$, are different two-dimensional subvarirties in E_{11}, and hence (32) can not occur. $\square$

The class of proper transform of E_i is

$$\mathcal{E}_i - \mathcal{E}_{i+1} \quad (i = 1, 2, 3, 4, 6, 7, 8, 9, 12, 13, 14, 15, 16)$$
$$\mathcal{E}_i \quad (i = 5, 10, 17), \quad \mathcal{E}_{11} - \mathcal{E}_{15}.$$

Since the defining function of the hyper-surface $z_1 = 0$ takes zero with multiplicities
$0, 0, 0, 0, 0, 1, 2, 2, 2, 2, 1, 1, 1, 2, 2, 2, 2$ on E_i $(i = 1, \ldots, 17)$, it is decomposed as

$$\begin{aligned}\mathcal{H}_a =& \text{Proper transform} \\ &+ (\mathcal{E}_6 - \mathcal{E}_7) + 2(\mathcal{E}_7 - \mathcal{E}_8) + 2(\mathcal{E}_8 - \mathcal{E}_9) + 2(\mathcal{E}_9 - \mathcal{E}_{10}) + 2\mathcal{E}_{10} \\ &+ (\mathcal{E}_{11} - \mathcal{E}_{14}) + (\mathcal{E}_{12} - \mathcal{E}_{13}) + (\mathcal{E}_{13} - \mathcal{E}_{14}) + 2(\mathcal{E}_{14} - \mathcal{E}_{15}) \\ &+ 2(\mathcal{E}_{15} - \mathcal{E}_{16}) + 2(\mathcal{E}_{16} - \mathcal{E}_{17}) + 2\mathcal{E}_{17},\end{aligned}$$

where each class enclosed in parentheses determines a prime divisor uniquely (we called such a class deterministic [7]). Hence the class of its proper transform is $\mathcal{H}_a - \mathcal{E}_6 - \mathcal{E}_7 - \mathcal{E}_{11} - \mathcal{E}_{12}$. Similarly, the defining function of the hyper-surface $x_2 - 1 = 0$ takes zero with multiplicities $1, 1, 1, 1, 1, 1, 1, 1, 1, 1$, 1, 0, 0, 1, 1, 1, 1 on E_i, and therefore the class of its proper transform is $\mathcal{H}_b - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_{11}$. Along the same line, the proper transform of $z_3 = 0$ can be computed as $\mathcal{H}_b - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_{11} - \mathcal{E}_{12}$.

Using these data, we can compute the pull-back action of Mapping φ (5) on the Piacard group. For example, the pull-back of E_1 is $(\bar{x}_1, \bar{z}_2, \bar{z}_3) = (0, 0, 0)$, whose "common factor" on each local coordinate system is $x_2 - 1$, s_6, s_7, s_8 or s_9. Thus, we have

$$\begin{aligned}\varphi(\mathcal{E}_1) =& (\mathcal{H}_2 - \mathcal{E}_1 - \mathcal{E}_6 - \mathcal{E}_{11}) + \sum_{i=6}^{9} (\mathcal{E}_i - \mathcal{E}_{i+1}) \\ =& \mathcal{H}_2 - \mathcal{E}_1 - \mathcal{E}_{10} - \mathcal{E}_{11}.\end{aligned}$$

Along the same line, we have the following proposition.

Proposition 6.2. *The pull-back φ^* of Mapping* (5) *is a linear action on the Picard group given by*

$$\mathcal{H}_a \to \mathcal{H}_b,$$
$$\mathcal{H}_b \to \mathcal{H}_a + 3\mathcal{H}_b - 2\mathcal{E}_1 - 3\mathcal{E}_{11} - \mathcal{E}_{6,7,9,10,12,13,14},$$
$$\mathcal{E}_1 \to \mathcal{H}_b - \mathcal{E}_{1,10,11}, \quad \mathcal{E}_2 \to \mathcal{H}_b - \mathcal{E}_{1,9,11}, \quad \mathcal{E}_3 \to \mathcal{H}_b - \mathcal{E}_{1,7,9,11} + \mathcal{E}_8,$$
$$\mathcal{E}_4 \to \mathcal{H}_b - \mathcal{E}_{1,7,11}, \quad \mathcal{E}_5 \to \mathcal{H}_b - \mathcal{E}_{1,6,11},$$
$$\mathcal{E}_6 \to \mathcal{E}_{14}, \quad \mathcal{E}_7 \to \mathcal{E}_{14}, \quad \mathcal{E}_8 \to \mathcal{E}_{15}, \quad \mathcal{E}_9 \to \mathcal{E}_{16}, \quad \mathcal{E}_{10} \to \mathcal{E}_{17},$$

$$\mathcal{E}_{11} \to \mathcal{E}_{1,11} - \mathcal{E}_{14}, \quad \mathcal{E}_{12} \to \mathcal{H}_b - \mathcal{E}_{1,11,13}, \quad \mathcal{E}_{13} \to \mathcal{H}_b - \mathcal{E}_{1,11,12},$$
$$\mathcal{E}_{14} \to \mathcal{E}_2, \quad \mathcal{E}_{15} \to \mathcal{E}_3, \quad \mathcal{E}_{16} \to \mathcal{E}_4, \quad \mathcal{E}_{17} \to \mathcal{E}_5,$$

where $\mathcal{E}_{i_1,\dots,i_k}$ denotes $\mathcal{E}_{i_1} + \cdots + \mathcal{E}_{i_k}$. The Jordan blocks of the corresponding matrix are

$$1, \ -1, \ 1^{\frac{1}{3}} \ (3 \times 3 \text{ blocks}), \ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In particular, the degree of the mapping φ^n grows quadratically with respect to n.

Corollary 6.3. *The degree of ψ^n for the 3-dimensional map ψ (8) also grows quadratically with respect to n.*

Proof. Let us denote the initial values as

$$(x_0, x_1, x_2, x_3) = (x_0^{(0)}, x_1^{(0)}, x_2^{(0)}, x_3^{(0)}).$$

Map ψ^n is obtained by substituting $x_3 = h(x_0, x_1, x_2)$ to

$$\varphi^n : x_i^{(n)} = f_i^{(n)}(x_0, x_1, x_2, x_3), \quad i = 0, 1, 2,$$

where h and f_i's are some rational functions. Hence the degree of $x_i^{(n)}$'s with respect to x_0, x_1, x_2 are bounded from the above by (degree of h) $\times$ (degree of $f_i^{(n)}$). Since the degree of $f_i^{(n)}$'s are quadratic with respect to n, the degree of $x_i^{(n)}$'s are at most quadratic. On the other hand, since ψ is a QRT map with respect to x_0 and x_2, its degree with regarding x_1 as a constant grows quadratically [38], hence the degree of $x_i^{(n)}$'s are at least quadratic. $\square$

The proper transforms of the conserved quantities I_1 and I_2 are

$$I_1 : 2\mathcal{H}_a + 2\mathcal{H}_b - 2\mathcal{E}_1 - 2\mathcal{E}_6 - 4\mathcal{E}_{11} - \mathcal{E}_{2,4,7,9,12,13,14,16}$$
$$I_2 : 2\mathcal{H}_a + 2\mathcal{H}_b - 3\mathcal{E}_{11} - \mathcal{E}_{1,2,4,5,6,7,9,10,12,13,14,16,17},$$

which are preserved by φ^*.

We can consider the inverse problem.

Proposition 6.4. *Hyper-surfaces whose class is $2\mathcal{H}_a + 2\mathcal{H}_b - 2\mathcal{E}_1 - 2\mathcal{E}_6 - 4\mathcal{E}_{11} - \mathcal{E}_{2,4,7,9,12,13,14,16}$ are given by $C_0 + C_1 I_1 = 0$ with $(C_0 : C_1) \in \mathbb{P}^1$ and $C_1 \neq 0$. Hyper-surfaces whose class is $2\mathcal{H}_a + 2\mathcal{H}_b - 3\mathcal{E}_{11} - \mathcal{E}_{1,2,4,5,6,7,9,10,12,13,14,16,17}$ are given by $C_0 + C_1 I_1 + C_2 I_2 = 0$ with $(C_0 : C_1 : C_2) \in \mathbb{P}^2$ and $C_2 \neq 0$.*

Thus, we can compute invariants by using the action of the system φ on the Picard group.

Proof. The proof is straightforward but tedious. For example, the defining polynomials of a curve of the class $2\mathcal{H}_a + 2\mathcal{H}_b - 2\mathcal{E}_1 - 2\mathcal{E}_6 - 4\mathcal{E}_{11} - \mathcal{E}_{2,4,7,9,12,13,14,16}$ can be written as

$$f(x_0, x_1, x_2, x_3) := \sum_{\substack{i_0, i_1, i_2, i_3 \geq 0 \\ i_0 + i_1 + i_2 + i_3 \leq 2}} a_{i_0 i_1 i_2 i_3} x_0^{i_0} x_1^{i_1} x_2^{i_2} x_3^{i_3},$$

$$z_2^2 f(x_0, x_1, z_2 z_3^{-1}, z_3^{-1}) \quad \text{around } E_1,$$

$$z_0^2 f(z_0 z_1^{-1}, z_1^{-1}, x_2, x_3) \quad \text{around } E_5,$$

$$z_0^2 z_2^2 f(z_0 z_1^{-1}, z_1^{-1}, z_2 z_3^{-1}, z_3^{-1}) \quad \text{around } E_{11}.$$

The coefficients are determined so that defining polynomial takes zero with multiplicity 2, 3, 3, 4, 4, 2, 3, 3, 4, 4, 4, 1, 2, 7, 7, 8, 8 on E_i's; which verifies the claim. $\square$

7. CONCLUSIONS

In this paper we investigated three integrable 4-dimensional mappings and constructed the space of initial conditions on the level of pseudo-auto/isomorphisms and algebraically stable maps.

The deautonomised version of the first mapping has the symmetry group is $A_2^{(1)} + A_2^{(1)}$ affine Weyl group. This situation can be easily generalised to $X_l^{(1)} + X_m^{(1)}$ affine Weyl group, where $X_l^{(1)}$ and $X_m^{(1)}$ are affine Weyl subgroups in $E_8^{(1)}$ appearing in Sakai's classification of two-dimensional discrete Painlevé equations, i.e. $X = A, D, E$,$l, m = 0, 1, 2, 3, 4, 5, 6, 7, 8$. In this case the variety is almost the direct product of generalised Halphen surfaces de- scribed by Sakai in his classification. Here, it is allowed that additive, multiplicative and elliptic difference systems are mixed but independently for 2 + 2 variables.

The second mapping was obtained just by switching two terms in the first mapping, but this simple surgery generates a variety with a different type symmetry. On the level of cohomology, the only difference is the decompositions of the anti-canonical divisors as (18) and (19). Moreover, their symmetries are closely related with each other. We expect that there are many such “twin” phenomena.

The third mapping is rather a new one in the sense that is obtained from the travelling wave reduction of a supersymmetric lattice equations where the corresponding Grassmann algebra has only two generators. The interesting fact is that, even though the mapping is not-confining, the complexity growth is quadratic a fact which is new. Anyway since there are no theorems about the behaviour of integrable four dimensional mappings we expect that there are many such systems.

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