# TWISTED JACOBI VERSUS JACOBI WITH BACKGROUND STRUCTURES

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Twisted Jacobi structures [24] are faced with Jacobi with background structures [4, 5]. The comparative analysis is done both for trivial and non-trivial line bundle versions.

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## 1. **INTRODUCTION**

The analysis of local Lie algebras [14] has brought to light a new kind of geometric structures that encompass both contact and locally conformal symplectic ones. The new geometric structures exhibit the well-known Jacobi manifolds [11, 19, 7]. Globally, these were initially defined via a pair (lately addressed [33] as a Jacobi pair) consisting of a bi-vector and a vector field subject to two consistency conditions that make use of the Schouten bracket associated with the Gerstenhaber algebra of multi-vector fields. These conditions can be expressed as a Maurer-Cartan-like equation [24, 33] with respect to an appropriately modified Schouten bracket, which shows that Jacobi pair generalizes in some sense Poisson structure. Also, the Jacobi pair organizes the $\mathbb{R}$-vector space of smooth functions as a Lie algebra (with respect to the well-known Jacobi bracket) but not a Poisson one.

It is worth mentioning that the previous structure, via the associated Jacobi bracket, has recently found many applications in mathematical physics, namely in the canonical approach of non-autonomous Hamiltonian systems [38, 34], in the integrability of Hamiltonian systems on odd-dimensional manifolds [36, 17, 18] as well as in the geometric reformulation of non-equilibrium thermodynamics [2].

A joint generalization of the previous structures comes from their 'twist' [24] (at the level of the Jacobi identity for the Jacobi bracket) by a 2-form and its de Rham differential. The obtained twisted Jacobi manifolds (manifolds equipped with twisted Jacobi pairs) enjoy the main features of Jacobi manifolds: i) their characteristic distributions are completely integrable, with twisted cooriented contact structures on the odd dimensional leaves and twisted locally conformal symplectic structures on the even dimensional leaves [25] and ii) they are in one-to-one correspondence with homogeneous twisted Poisson manifolds, where the background 3-form is exact [24].

Recently, with the twisted Jacobi pair notion in mind, a more generous structure has been proposed [5]. This concept starts with a pair consisting of a bi-vector and a vector field and adds a 'background' (that spoils the Jacobi identity for the Jacobi bracket) comprising a 3-form together with a 2-form. The arbitrariness of the 3-form shows that this new structure (called *Jacobi structure with background*) encompasses the twisted Jacobi one. In addition, it enjoys the main features of the Jacobi and twisted Jacobi pairs, i.e. i) their characteristic distributions are completely integrable, with twisted cooriented contact structures on the odd dimensional leaves and locally conformal symplectic structures with background on the even dimensional leaves [5] and ii) they are in one-to-one correspondence with homogeneous Poisson manifolds with background, where the background 3-form is no longer closed [5].

The reformulation of the consistency conditions corresponding to Jacobi pairs in terms of Maurer-Cartan-like equations [33] can be done also for twisted and Jacobi pairs with background [24, 5]. These reformulations make use of a Gerstenhaber-Jacobi structure based on the Schouten bracket in the Gerstenhaber algebra of multivector fields. These results together with the algebraic characterisations of Lie and Jacobi algebroids [9, 10] enforced the line-bundle versions [33, 4] of the previous 'pairs'. Within this global setting, the 'pairs' are nothing but the trivial line-bundle versions of the corresponding Jacobi-like bundles [21, 33, 4].

The present paper is organized into five sections as follows. In Section 2, starting with the Jacobi pair concept, we do a brief review of twisted Jacobi pairs exhibiting their main properties concerning integrability and correspondence with homogeneous Poisson structures. Section 3 is dedicated to the main aspects of Jacobi pairs with background including here the relationship with the twisted Jacobi ones. In Section 4, we reformulate the consistency conditions defining the Jacobi-like structures in terms of Maurer-Cartan-like equations. This developing exhibits, for each kind of Jacobi-like pair, a corresponding Jacobi algebroid. With these results at hand, in Section 5 we consistently introduce the global formulation of Jacobi-like pairs, namely the Jacobi-like line bundles.

## 2. FROM JACOBI TO TWISTED JACOBI PAIRS

The local structure of the local Lie algebras with one-dimensional fibers [14] has put into evidence a geometric structure that encompasses both (coorientable) contact and locally conformal symplectic structures. The systematics of this new structure has been initially done [19] in terms of a bi-vector and a vector field on the smooth manifold $M$, $\Pi$ and $E$ respectively,

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M),$$

subject to the conditions

$$\tfrac{1}{2}\,[\Pi,\Pi] + E \wedge \Pi = 0, \quad [E,\Pi] = 0. \tag{1}$$

Previously, we denoted by

$$\left(\mathfrak{X}^\bullet(M) := \bigoplus_{k=0}^{m} \mathfrak{X}^k(M), \wedge\right)$$

the graded, graded commutative and associative algebra of smooth multi-vector fields over the manifold $M$. This can be naturally organized [20] as a Gerstenhaber algebra with respect to the Schouten bracket $[\bullet,\bullet]$.

*Definition* 2.1. Let $M$ be a smooth manifold. A pair $(\Pi, E)$ consisting of a bi-vector and a vector field that enjoys (1) is said to be a Jacobi pair. In this context, the manifold $M$ is called a Jacobi manifold.

In order to exemplify the previous concept, we first make some specifications. We denote by

$$\left(\Omega^\bullet(M) := \bigoplus_{k=0}^{m} \Omega^k(M), \wedge\right)$$

the graded and graded commutative algebra of smooth forms over the manifold $M$. This is naturally endowed with de Rham differential, d, and it is dual to the previous Gerstenhaber algebra. Moreover, we adopt here the conventions from [20] concerning the wedge products, interior products and pairings between $\Omega^p(M)$ and $\mathfrak{X}^p(M)$. In addition, we make the notation for the degree zero components of the previous graded algebras

$$\Omega^0(M) = \mathfrak{X}^0(M) = \mathcal{F}(M) := \mathcal{C}^\infty(M).$$

In fact, as a clue for the subsequent approaches, both Schouten bracket and de Rham differential $[\bullet,\bullet]$ and d respectively are equivalent [9, 10] to the Lie algebroid $(TM \to M, [\bullet,\bullet], \mathrm{id}_{TM})$ with $[\bullet,\bullet]$ the standard Lie crochet.

Remember here that a Lie algebroid is a vector bundle $A \to M$ equipped with a Lie algebra structure $[\bullet,\bullet]$ on the $\mathbb{R}$-vector space of smooth sections $\Gamma(A)$

which is a skew-symmetric, first-order bi-differential operator on the $\mathcal{F}(M)$-module $\Gamma(A)$, i.e. there exists the vector bundle map $\rho : A \to TM$ such that

$$[\alpha, f\beta] = (\rho(\alpha)f)\beta + f\,[\alpha, \beta]\,, \quad \alpha, \beta \in \Gamma(A), f \in \mathcal{F}(M).$$

The previous statement envisaging the 'essence' of Schouten bracket and de Rham differential becomes obvious in the light of the result below [9, 10].

THEOREM 2.2. *Let $A \to M$ be a vector bundle. Then the following data are equivalent:*

1. *a Lie algebroid structure, $([\bullet, \bullet]\,, \rho)$, on $A \to M$;*

2. *a Gerstenhaber algebra structure, $[\bullet, \bullet]_A$, on the graded algebra $(\Gamma(\wedge^\bullet A)\,, \wedge)$;*

3. *a homological degree 1 graded derivation, $d_A$, of the graded algebra $(\Gamma(\wedge^\bullet A^*)\,, \wedge)$, with $A^* \to M$ the dual vector bundle associated with the starting one.*

*Example* 2.3. By its very definition [35] a Poisson manifold is a manifold $M$ endowed with a Poisson structure, i.e. a bi-vector $\Pi \in \mathfrak{X}^2(M)$ that enjoys

$$[\Pi, \Pi] = 0, \tag{2}$$

with $[\bullet, \bullet]$ the Schouten bracket in $\mathfrak{X}^\bullet(M)$. Comparing (2) with (1) it results that a Poisson structure $\Pi$ displays the Jacobi pair $(\Pi, E = 0)$.

*Example* 2.4. Let $M$ be an odd-dimensional smooth manifold, $\dim M = 2m + 1$. The smooth 1-form $\theta \in \Omega^1(M)$ (the well-kown contact 1-form) that exhibits the volume form

$$\mu_\theta := \theta \wedge (\mathrm{d}\theta)^m \neq 0, \tag{3}$$

generates the maximally non-integrable hyperplane distribution

$$\mathcal{K}_\theta := \operatorname{Ker} \theta$$

known as the coorientable contact distribution. Such a structure displays [34] a Jacobi pair $(\Pi_\theta, E_\theta)$ defined via

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi_\theta \rangle := \langle \mathrm{d}\theta, X_f \wedge X_g \rangle, \quad E_\theta := X_{f=1}, \tag{4}$$

where $X_f$ is the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}\mathrm{d}\theta = -\mathrm{d}f + (i_{E_\theta}\mathrm{d}f)\,\theta, \tag{5}$$

with $i_\bullet$ the right interior product by $p$-vectors [20].

*Example* 2.5. Let $M$ be an even-dimensional smooth manifold, $\dim M = 2m$. A pair of forms $(\Omega, \alpha)$ consisting of a non-degenerate 2-form $\Omega \in \Omega^2(M)$ and a closed 1-form $\alpha \in \Omega^1(M)$, $\mathrm{d}\alpha = 0$, which is subject to the condition

$$\mathbf{d}\Omega + \alpha \wedge \Omega = 0 \tag{6}$$

is said to be a locally conformal symplectic structure. This pair entails the Jacobi one $\left(\Pi_{(\Omega,\alpha)}, E_{(\Omega,\alpha)}\right)$, where

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi_{(\Omega,\alpha)} \rangle := \langle \Omega, \Omega^\sharp \mathrm{d}f \wedge \Omega^\sharp \mathrm{d}g \rangle, \quad E_{(\Omega,\alpha)} := \Omega^\sharp \alpha. \tag{7}$$

Previously, we denoted by $\Omega^\sharp : T^*M \to TM$ the inverse of the vector bundle isomorphism

$$\Omega^\flat : TM \to T^*M, \quad \Omega^\flat X := -i_X \Omega,$$

i.e. $\Omega^\sharp \Omega^\flat = \mathrm{id}_{TM}$.

The previous examples suggest that there exists a more profound connection between Jacobi pairs and coorientable contact/ locally conformal symplectic structures. In order to identify it, remember that for the very special Jacobi pair $(\Pi, E = 0)$ (see Example 2.3), the bivector $\Pi \in \mathfrak{X}^2(M)$ generates the well-known *characteristic distribution* $\mathcal{C}_\Pi := \operatorname{Im} \Pi^\sharp$, where we denoted by $\Pi^\sharp$ the vector bundle morphism

$$\Pi^\sharp : T^*M \to TM, \quad \Pi^\sharp \mu := -j_\mu \Pi,$$

with $j_\bullet$ the left interior product of multi-vector fields by $p$-forms [20]. The main feature of characteristic distribution stands in its completely integrability [35], having as characteristic leaves symplectic submanifolds of the base manifold $M$.

In the generic situation of a Jacobi pair $(\Pi, E)$ on the smooth manifold $M$, the standard approach of the characteristic distribution makes use of the Lie algebra structure

$$\{\bullet, \bullet\} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M), \quad \{f, g\} := i_\Pi (\mathrm{d}f \wedge \mathrm{d}g) + i_E (f\mathrm{d}g - g\mathrm{d}f) \tag{8}$$

over the $\mathbb{R}$-vector space $\mathcal{F}(M)$. This is a skew-symmetric, first-order differential operator in each entry (with respect to the $\mathcal{F}(M)$-module structure of $\mathcal{F}(M)$), i.e. for any $f \in \mathcal{F}(M)$,

$$\mathcal{F}(M) \ni g \mapsto \Delta_f g := \{f, g\} \in \mathcal{F}(M), \tag{9}$$

is a first-order differential operator of symbol

$$X_f := \Pi^\sharp \mathrm{d}f + fE. \tag{10}$$

The vector field (10) is the well-known *Hamiltonian vector field* associated with the smooth function $f \in \mathcal{F}(M)$.

In the light of the Lie algebra structure (8), the first-order differential operator (9) verifies

$$(11) \qquad [\Delta_f, \Delta_g] = \Delta_{\{f,g\}}, \quad f, g \in \mathcal{F}(M),$$

i.e. $f \mapsto \Delta_f$ is a Lie algebra morphism from $(\mathcal{F}(M), \{\bullet, \bullet\})$ to $(\mathcal{D}(\mathbb{R}_M), [\bullet, \bullet])$, where $\mathcal{D}(\mathbb{R}_M)$ is the set of the first-order differential operators [16] associated with the trivial line bundle $\mathbb{R}_M := \mathbb{R} \times M \to M$. Previously, we denoted by $[\bullet, \bullet]$ the standard commutator of the $\mathbb{R}$-linear operators. As an immediate consequence of (11), it results that the $\mathbb{R}$-algebra of Hamiltonian vector fields is involutive under the Lie crochet, i.e.

$$(12) \qquad [X_f, X_g] = X_{\{f,g\}}, \quad f, g \in \mathcal{F}(M).$$

By definition, *the characteristic distribution* associated with the considered Jacobi pair is the smooth distribution generated by the Hamiltonian vector fields (10)

$$(13) \qquad \left(\mathcal{C}_{(\Pi,E)}\right)_x := \langle \{(X_f)_x : f \in \mathcal{F}(M)\} \rangle \subseteq T_x M, \quad x \in M.$$

The characteristic distribution (13) is said to be *transitive* if at each point it coincides with the tangent space. Regarding the characteristic distribution corresponding to a Jacobi pair, there exists two strong results [11, 34] listed below.

THEOREM 2.6. *If a Jacobi pair $(\Pi, E)$ on a smooth manifold $M$ is transitive then $M$ is either a locally conformal symplectic manifold (see Example 2.5) or a coorientable contact one (see Example 2.4).*

THEOREM 2.7. *The characteristic distribution of a Jacobi pair is completely integrable [31, 32] with the characteristic leaves either locally conformal symplectic manifolds or coorientable contact ones.*

*Remark* 2.8. The integrability of the characteristic distribution associated with a Jacobi pair has been elegantly solved in an algebraic language. Precisely, it has been shown that the graph of the Jacobi pair is just a Dirac-Jacobi structure [37] in the Courant-Jacobi algebroid $\mathcal{E}^1(M) := (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ [39].

At the end of this section, we focus on a version of a Jacobi-like pair (see Definition 2.1) which is 'twisted' via a 2-form [24].

*Definition* 2.9. Let $M$ be a smooth manifold. A pair $(\Pi, E) \in \mathfrak{X}^2(M) \times \mathfrak{X}^1(M)$ is said to be *twisted* by the 2-form $\omega \in \Omega^2(M)$ if it verifies

$$(14) \quad \tfrac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \mathrm{d}\omega + \Pi^\sharp \omega \wedge E, \quad [E, \Pi] = -\left(\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E \omega \wedge E\right).$$

Shortly, the structure $((\Pi, E), \omega)$ that enjoys (14) is said to be a *twisted Jacobi pair*.

*Example* 2.10. By its very definition a twisted Poisson manifold, also known as Poisson manifold with a *(closed)* 3-form background [30], is a manifold $M$ equipped with a pair $(\Pi, \phi) \in \mathfrak{X}^2(M) \times \Omega^3(M)$ that verifies

$$\mathrm{d}\phi = 0 \quad \text{and} \quad \tfrac{1}{2}\,[\Pi, \Pi] = \Pi^\sharp \phi. \tag{15}$$

In the light of the first equation in (15) it results that (at least locally) $\phi = \mathrm{d}\omega$. In this context, the twisted Poisson manifold $(\Pi, \mathrm{d}\omega)$ exhibits the twisted Jacobi pair $((\Pi, E = 0), \omega)$. It is worth-noticing that 'twisting' of the Poisson structures initially arose in physics in the context of string theory [15], but their integrability has been proved in [30] in the framework of a $\phi$-closedness-based Courant algebroid structure on the 'fat' tangent bundle $TM \oplus T^*M$.

*Example* 2.11. Let $M$ be an odd-dimensional smooth manifold, $\dim M = 2m + 1$. The pair $(\theta, \omega)$ consisting of a 1-form $\theta$ and a 2-form $\omega$ is said to be a twisted cooriented contact structure if

$$\mu_{(\theta,\omega)} := \theta \wedge (\mathrm{d}\theta + \omega)^m$$

is a volume form. This structure defines the twisted Jacobi pair $\left(\left(\Pi_{(\theta,\omega)}, E_{(\theta,\omega)}\right), \omega\right)$, where $E_{(\theta,\omega)}$ is the twisted Reeb vector field, i.e. the unique vector field that enjoys

$$i_{E_{(\theta,\omega)}}\theta = 1, \quad i_{E_{(\theta,\omega)}}(\mathrm{d}\theta + \omega) = 0$$

and

$$\langle \mathrm{d}f \wedge \mathrm{d}g, \Pi_{(\theta,\omega)} \rangle := \langle \mathrm{d}\theta, X_f \wedge X_g \rangle. \tag{16}$$

In definition (16), we denoted by $X_f$ the twisted-contact Hamiltonian vector field associated with the smooth function $f \in \mathcal{F}(M)$, i.e. the unique solution to the equations

$$i_{X_f}\theta = f, \quad i_{X_f}(\mathrm{d}\theta + \omega) = i_{E_{(\theta,\omega)}}(\mathrm{d}f \wedge \theta). \tag{17}$$

*Example* 2.12. Let $M$ be an even-dimensional smooth manifold. The pair $(\Omega, \alpha)$, with $\Omega$ a non-degenerate 2-form and $\alpha$ a closed 1-form, is said to be a locally conformal symplectic structure twisted by $\omega \in \Omega^2(M)$ if

$$\mathrm{d}(\Omega - \omega) + \alpha \wedge (\Omega - \omega) = 0. \tag{18}$$

Associated with the considered twisted locally conformal symplectic structure, there exists the unique twisted Jacobi pair $\left(\left(\Pi_{(\Omega,\alpha)}, E_{(\Omega,\alpha)}\right), \omega\right)$ with the pair $\left(\Pi_{(\Omega,\alpha)}, E_{(\Omega,\alpha)}\right)$ defined in (7).

Like in the standard Jacobi pairs situation, the connection between twisted Jacobi pairs and twisted cooriented contact/ locally conformal symplectic ones is more profound [24, 25, 26] as we are going to sketch in the sequell. In order to do this, we consider the twisted Jacobi pair $((\Pi, E), \omega)$ on the smooth manifold $M$ and implement the bracket (8) on the $\mathbb{R}$-vector space of smooth functions $\mathcal{F}(M)$. This is manifestly $\mathbb{R}$-linear and skew-symmetric, but it no longer verifies the Jacobi identity

$$\begin{aligned}\operatorname{Jac}\{f,g,h\} = {} & i_{\Pi^\sharp \mathrm{d}\omega + \Pi^\sharp\omega\wedge E}(\mathrm{d}f\wedge\mathrm{d}g\wedge\mathrm{d}h) \\ & - i_{\Pi^\sharp i_E \mathrm{d}\omega + \Pi^\sharp i_E\omega\wedge E}(f\mathrm{d}g\wedge\mathrm{d}h + g\mathrm{d}h\wedge\mathrm{d}f + h\mathrm{d}f\wedge\mathrm{d}g),\end{aligned} \tag{19}$$

where

$$\operatorname{Jac}\{f,g,h\} := \{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}, \quad f,g,h\in\mathcal{F}(M)$$

is the well-known Jacobiator. Due to the fact that structure (8) is a first-order differential operator in each entry, the Hamiltonian derivations (9) remain valid in the considered twisted context and, moreover, their symbols (10). Unlike the non-twisted case where $f\mapsto X_f$ is an $\mathbb{R}$-Lie algebras morphism, here

$$\begin{aligned}[X_f,X_g] - X_{\{f,g\}} = {} & \Pi^\sharp i_{X_f\wedge X_g}\mathrm{d}\omega - (\mathcal{L}_E f)\,\Pi^\sharp i_{X_g}\omega + (\mathcal{L}_E g)\,\Pi^\sharp i_{X_f}\omega \\ & + \left(i_{X_f\wedge X_g}\omega\right)E,\end{aligned} \tag{20}$$

$$[X_f,E] + X_{\mathcal{L}_E f} = \Pi^\sharp\left(i_{X_f\wedge E}\mathrm{d}\omega - (\mathcal{L}_E f)\, i_E\omega\right) + \left(i_{X_f\wedge E}\omega\right)E. \tag{21}$$

Following the line of Jacobi pairs, we introduce *the characteristic distribution* associated with the considered twisted Jacobi pair via

$$\left(\mathcal{C}_{((\Pi,E),\omega)}\right)_x := \langle\{(X_f)_x : f\in\mathcal{F}(M)\}\rangle \subseteq T_xM, \quad x\in M, \tag{22}$$

which, in the light of (20)–(21), is involutive. Maintaining the transitivity definition, the characteristic distribution corresponding to a twisted Jacobi pair enjoys two strong properties [24, 25, 26] listed below.

THEOREM 2.13. *If a twisted Jacobi pair $((\Pi,E),\omega)$ on a smooth manifold $M$ is transitive then $M$ is either a twisted locally conformal symplectic manifold (see Example 2.12) or a twisted coorientable contact one (see Example 2.11).*

THEOREM 2.14. *The characteristic distribution of a twisted Jacobi pair is completely integrable [31, 32] with the characteristic leaves either twisted locally conformal symplectic manifolds or twisted coorientable contact ones.*

*Remark* 2.15. As in the special situation $((\Pi,E=0),\omega)$ (see Example 2.10) the algebraic solution to the integrability problem makes use of the fact that the characteristic distribution corresponding to a twisted Jacobi pair $((\Pi,E),\omega)$ is just a Dirac-Jacobi structure in the Courant-Jacobi algebroid $\mathcal{E}^1(M)_\omega$ [24] (that comes from the Courant-Jacobi algebroid $\mathcal{E}^1(M)$ via twisting the Courant bracket by $\omega$ ).

# 3. RELAXING TWISTED JACOBI: JACOBI PAIRS WITH BACKGROUND

In this section, we introduce a new structure [5] which comes as a 'relaxed' version of twisted Jacobi pairs (see Definition 2.9). This new concept enjoys the integrability and the correspondence with 'homogeneous' Poisson structures but it does not exhibit a nice algebroid-based expression just as its 'source' – twisted Jacobi pair. Recently, this new concept has found applications in gauged sigma-models physics with non-closed 3-forms [3].

*Definition* 3.1. A pair $((\Pi, E), (\phi, \omega))$ consisting of

$$\Pi \in \mathfrak{X}^2(M), \quad E \in \mathfrak{X}^1(M), \quad \phi \in \Omega^3(M), \quad \omega \in \Omega^2(M)$$

which satisfies the 'compatibility' conditions

$$\frac{1}{2}[\Pi, \Pi] + E \wedge \Pi = \Pi^\sharp \phi + \Pi^\sharp \omega \wedge E, \quad [E, \Pi] = -\left(\Pi^\sharp i_E \phi + \Pi^\sharp i_E \omega \wedge E\right) \tag{23}$$

is called a *Jacobi pair* $(\Pi, E)$ *with background* $(\phi, \omega)$.

*Remark* 3.2. Inspecting Definitions 2.9 and 3.1, it is clear that any twisted Jacobi pair $((\Pi, E), \omega)$ is just the Jacobi pair with background $((\Pi, E), (\mathrm{d}\omega, \omega))$.

*Example* 3.3. In the light of Example 2.10, we introduce the notion of Poisson manifold with a 3-form background [1], as being the manifold $M$ endowed with a pair $(\Pi, \phi) \in \mathfrak{X}^2(M) \times \Omega^3(M)$ that verifies

$$\tfrac{1}{2}[\Pi, \Pi] = \Pi^\sharp \phi. \tag{24}$$

Comparing (23) with (24) it results that a Poisson structure with background $(\Pi, \phi)$ displays the family of Jacobi pairs with background $((\Pi, E = 0), (\phi, \omega))$.

*Example* 3.4. Let's consider the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = \left(x^1, x^2, x^3, x^4\right)$ and the real smooth functions $f, e \in \mathcal{F}\left(\mathbb{R}^4\right)$ among which $f$ is nowhere vanishing, $f^2 > 0$, and $e$ depends only on the first two coordinates, $e = e\left(x^1, x^2\right)$. We introduce the geometric objects

$$\Pi = \frac{1}{f}\left(\partial_1 \wedge \partial_4 + \partial_2 \wedge \partial_3\right), \qquad E = -\frac{1}{f}\left(\left(\partial_1 e\right) \partial_4 + \left(\partial_2 e\right) \partial_3\right), \tag{25}$$

$$\phi = (\mathrm{d}f - f\mathrm{d}e) \wedge (x^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^1 \wedge \mathrm{d}x^4), \quad \omega = 0. \tag{26}$$

Direct computations show that the geometric objects (25) and (26) verify the relations (23) which means that $((\Pi, E), (\phi, \omega))$ is a Jacobi pair with background.

*Example* 3.5. Let's consider again the four-dimensional smooth manifold $\mathbb{R}^4$ with the global coordinates $x = \left(x^1, x^2, x^3, x^4\right)$ and the smooth functions $f$, $g$, $h$, and $e$ on $\mathbb{R}^4$ among which $f$, and $e$ verify the same restrictions as in the previous example. Construct the bi-vector and vector fields as in (25) and

$$\begin{aligned} \phi = {} & \left[(\partial_3 h \partial_2 e + \partial_4 f)\,\mathrm{d}x^2 + (\partial_3 h \partial_1 e - \partial_3 f)\,\mathrm{d}x^1\right] \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + \left[(-\partial_2 f + f(x)\,\partial_2 e - \partial_1 h \partial_2 e + \partial_2 h \partial_1 e)\,\mathrm{d}x^4 \right. \\ & \left. + \left(\partial_1 f + \partial_3 \tilde{f} \partial_2 e\right) \mathrm{d}x^3\right] \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2, \end{aligned} \tag{27}$$

$$\omega = f(x)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \left(\partial_3 \tilde{f}\right) \mathrm{d}x^3 \wedge \mathrm{d}x^1 + g(x)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}h \wedge \mathrm{d}x^4. \tag{28}$$

In formulas (27) and (28) $\tilde{f}$ stands for an arbitrary smooth function that verifies $\partial_4 \tilde{f} = f$. By direct computation it can be checked that the pair $((\Pi, E), (\phi, \omega))$ given in (25), (27), and (28) satisfies the compatibility conditions (23), i.e. it is a Jacobi pair with background. Simple computations show that the 2-form $\omega$ is non-trivial while the 3-form is closed but

$$\phi \neq \mathrm{d}\omega.$$

Examples 3.4 and 3.5 exhibit a non-degenerate bivector which is common for twisted locally conformal symplectic structures. This suggests that such symplectic-like structures might be implemented in the present background context.

*Example* 3.6. Let $M$ be an even-dimensional smooth manifold and $(\Omega, \alpha)$ be a pair of forms with $\Omega$ a non-degenerate 2-form and $\alpha$ a closed 1-form. Let $\omega$ be an arbitrary 2-form on the same smooth manifold. By means of these geometric objects, we introduce the 3-form

$$\phi := \mathrm{d}\Omega + \alpha \wedge (\Omega - \omega). \tag{29}$$

With these specification at hand, we call a *locally conformal symplectic structure* $(\Omega, \alpha)$ *with background* $(\phi, \omega)$ the structure $((\Omega, \alpha), (\phi, \omega))$ satisfying (29). By direct computation, it can be shown that $\left(\left(\Pi_{(\Omega,\alpha)}, E_{(\Omega,\alpha)}\right), (\phi, \omega)\right)$, with the pair $\left(\Pi_{(\Omega,\alpha)}, E_{(\Omega,\alpha)}\right)$ defined in (7), is a Jacobi pair with background [5].

In the light of definition (29), it is obvious that the locally conformal symplectic structure with background $((\Omega, \alpha), (\phi = \mathrm{d}\omega, \omega))$ is just the twisted one $((\Omega, \alpha), \omega)$.

Regarding coorientable twisted contact structures on odd-dimensional base manifolds (see Example 2.11), by using Remark 3.2, it is clear that they generate Jacobi pairs with background.

Following the integrability line for the previous kinds of Jacobi-like structures, we consider a generic Jacobi pair with background $((\Pi, E), (\phi, \omega))$ on the smooth manifold $M$ and we define bracket (8). This is manifestly $\mathbb{R}$-linear and skew-symmetric, but no longer verifies the Jacobi identity

$$\begin{aligned}\operatorname{Jac}\{f,g,h\} = & - i_{X_f\wedge X_g\wedge X_h}\phi + (\mathcal{L}_E f) i_{X_g\wedge X_h}\omega \\ & + (\mathcal{L}_E g) i_{X_h\wedge X_f}\omega + (\mathcal{L}_E h) i_{X_f\wedge X_g}\omega.\end{aligned} \tag{30}$$

Bracket (8) is a derivation in each entry, which displays the Hamiltonian derivations (9) and their symbols, the Hamiltonian vector fields (10). In the Jacobi with background context, as in the twisted case, due to (30) it results that the $\mathbb{R}$-vector space of Hamiltonian vector fields is no longer a Lie algebra with respect to the standard Lie crochet [5]

$$[X_f, X_g] - X_{\{f,g\}} = \Pi^\sharp i_{X_f\wedge X_g}\phi - (\mathcal{L}_E f)\,\Pi^\sharp i_{X_g}\omega + (\mathcal{L}_E g)\,\Pi^\sharp i_{X_f}\omega + \left(i_{X_f\wedge X_g}\omega\right) E. \tag{31}$$

At this stage, we define *the characteristic distribution* corresponding to the considered Jacobi pair with background by

$$\left(\mathcal{C}_{((\Pi,E),(\phi,\omega))}\right)_x := \langle\{(X_f)_x : f\in\mathcal{F}(M)\}\rangle \subseteq T_x M, \quad x\in M. \tag{32}$$

By direct computations one infers

$$[X_f, E] + X_{\mathcal{L}_E f} = -\Pi^\sharp\left(i_{E\wedge X_f}\phi + (\mathcal{L}_E f)\, i_E\omega\right) - \left(i_{E\wedge X_f}\omega\right) E, \tag{33}$$

which, together with result (31), prove that the characteristic distribution (32) is involutive. Maintaining the transitivity definition, the characteristic distribution corresponding to a Jacobi pair with background enjoys two strong properties [5] listed below.

THEOREM 3.7. *If a Jacobi pair with background $((\Pi,E),(\phi,\omega))$ on a smooth manifold $M$ is transitive then $M$ is either a locally conformal symplectic manifold with background (see Example 3.6) or a twisted coorientable contact one (see Example 2.11).*

THEOREM 3.8. *The characteristic distribution of a Jacobi pair with background is completely integrable [31, 32] with the characteristic leaves either locally conformal symplectic manifolds with background or twisted coorientable contact ones.*

At this point, it seems somehow surprising the 'lack' of the contact structures with background. This is elucidated by the 'gauge' theorem [5] that offers the freedom degree in the choice of the forms $\phi$ and $\omega$ once the geometric objects $\Pi$ and $E$ are fixed.

THEOREM 3.9. *Let $M$ be a smooth manifold and $((\Pi,E),(\phi_i,\omega_i))$, $i=1,2$ be two transitive Jacobi pairs with background on the base manifold $M$. The following alternative holds:*

1. *If $M$ is even-dimensional then there exists a 2-form $\omega$, such that*

$$\omega_1 = \omega_2 + \omega, \quad \phi_1 = \phi_2 - \omega \wedge \Pi^\flat E; \tag{34}$$

2. *If $M$ is odd-dimensional then*

$$\omega_1 = \omega_2, \quad \phi_1 = \phi_2. \tag{35}$$

The previous discussion on various kinds of Jacobi structures reveals their complete integrability in a broader sense of Stefan and Sussmann [31, 32].

At the end of this section, we shall briefly address the 'Poissonization' of the previous Jacobi-like structures. This problem comes naturally from the perspective of Examples 2.3, 2.10 and 3.3, which show that to any Poisson, twisted Poisson and Poisson with background structure we can associate a Jacobi, twisted Jacobi, and Jacobi pair with background, respectively. The 'reverse' correspondence can be done by 'oxidation' of the base manifold with an extra-dimension.

PROPOSITION 3.10. *Let $M$ be a smooth manifold and $\tilde{M} := M \times \mathbb{R}$ be its trivially one-dimension extension. Denoting by $\tau$ the additional coordinate on $\tilde{M}$, then the following are true:*

- *If $(\Pi, E)$ is a Jacobi pair on $M$ then $\tilde{\Pi} := e^{-\tau}\left(\Pi + \partial_\tau \wedge E\right)$ is a homogeneous Poisson structure on $\tilde{M}$ [34] with the homogeneity vector field $\tilde{Z} = \partial_\tau$, i.e. $\mathcal{L}_{\tilde{Z}}\tilde{\Pi} = -\tilde{\Pi}$.*

- *If $((\Pi, E), \omega)$ is a twisted Jacobi pair on $M$ then $\left(\tilde{\Pi}, \tilde{\phi} := d(e^\tau \omega)\right)$ is a twisted (exact) homogeneous Poisson structure on $\tilde{M}$ [24] with the homogeneity vector field $\tilde{Z} = \partial_\tau$, i.e. $\mathcal{L}_{\tilde{Z}}\tilde{\Pi} = -\tilde{\Pi}$ and $\mathcal{L}_{\tilde{Z}}\tilde{\phi} = \tilde{\phi}$.*

- *If $((\Pi, E), (\phi, \omega))$ is a Jacobi pair with background on $M$ then $\left(\tilde{\Pi}, \tilde{\phi} := e^\tau(\phi + \omega \wedge d\tau)\right)$ is a homogeneous Poisson structure with background on $\tilde{M}$ [5] with the same homogeneity vector field as in the previous homogeneous Poisson structures.*

## 4. JACOBI-LIKE PAIRS AS DISTINGUISHED ELEMENTS OF A JACOBI ALGEBROID

In this part, we address the algebraic machinery behind the Jacobi structures under attention. The strategy is standard, i.e. starting from a Lie algebroid whose contravariant description is the natural situs for Jacobi-like pairs $(\Pi, E)$, one constructs a Gerstenhaber-Jacobi structure with respect to which various kinds of Jacobi-like pairs $(\Pi, E)$ are 'Maurer-Cartan' elements. This is a milestone for the general line bundle setting of a Jacobi-like structure that is to be done in the next section.

The realm $\mathfrak{X}^2(M)\oplus\mathfrak{X}^1(M)$ of the pair $(\Pi, E)$, $(\Pi, E)\in\mathfrak{X}^2(M)\times\mathfrak{X}^1(M)\simeq\mathfrak{X}^2(M)\oplus\mathfrak{X}^1(M)$, combined with the vector bundle isomorphism $\wedge^2(TM\times\mathbb{R})\simeq(\wedge^2TM)\oplus(TM)$ exhibit the Lie algebroid $(TM\times\mathbb{R}, [\![\cdot,\cdot]\!], \rho)$ as being the natural situs for a pair consisting of a bivector and a vector field $(\Pi, E)$. Previously, the Lie algebra structure on $\Gamma(TM\times\mathbb{R})\simeq\mathfrak{X}^1(M)\oplus\mathcal{F}(M)$ reads

$$[\![(X,f),(Y,g)]\!] := ([X,Y], X(g)-Y(f)), \tag{36}$$

while the anchor $\rho$ is just the projection on the first factor, $\rho(X,f):=X$.

With previous Lie algebroid at hand, by means of Theorem 2.2, we explore its contravariant formulation. This consists of the Gerstenhaber algebra $(\Gamma(\wedge^\bullet(TM\times\mathbb{R})), \wedge, [\![\bullet,\bullet]\!])$, where, in the light of the $\mathcal{F}(M)$-module isomorphism $\Gamma\left(\wedge^{p+1}(TM\times\mathbb{R})\right)\simeq\mathfrak{X}^{p+1}(M)\times\mathfrak{X}^p(M)$, the structure reads

$$\begin{aligned}(P,Q)\wedge(R,S) &= (P\wedge Q, P\wedge S-(-)^r Q\wedge R),\\ [\![(P,Q),(R,S)]\!] &= ([P,R], [P,S]+(-)^r[Q,R]),\end{aligned}$$

with $(P,Q)\in\mathfrak{X}^{p+1}(M)\times\mathfrak{X}^p(M)$ and $(R,S)\in\mathfrak{X}^{r+1}(M)\times\mathfrak{X}^r(M)$ arbitrary homogeneous elements.

Direct computation shows that *only* the second relation in (1) can be naturally obtained in terms of the previous bracket

$$[\![(\Pi,E),(\Pi,E)]\!] = ([\Pi,\Pi], 2[\Pi,E]).$$

This result excludes the previous Lie algebroid as the natural situs for the considered Jacobi-like pairs. In order to identify the right Lie algebroid, we modify [12, 13] the previous bracket such that the new bracket captures the left-hand side in expressions (1), (14) and (23). For doing so, we employ the covariant characterization of the considered Lie algebroid, which, in the light of Theorem 2.2, it consists of the de Rham complex $(\Gamma(\wedge^\bullet(T^*M\times\mathbb{R})), \wedge, \mathbf{d})$ where

$$(\omega,\theta)\wedge(\rho,\mu) = (\omega\wedge\rho, \omega\wedge\mu-(-)^r\theta\wedge\rho), \quad \mathbf{d}(\omega,\theta) = (\mathrm{d}\omega, -\mathrm{d}\theta), \tag{37}$$

with $(\omega,\theta)\in\Omega^{p+1}(M)\times\Omega^p(M)$ and $(\rho,\mu)\in\Omega^{r+1}(M)\times\Omega^r(M)$. Previously, we used the vector bundle isomorphism $\wedge^{p+1}(T^*M\times\mathbb{R})\simeq(\wedge^{p+1}T^*M)\oplus(\wedge^pT^*M)$ (that displays $\Gamma\left(\wedge^{p+1}(T^*M\times\mathbb{R})\right)\simeq\Omega^{p+1}(M)\oplus\Omega^p(M)\simeq\Omega^{p+1}(M)\times\Omega^p(M)$ as isomorphisms of modules over $\mathcal{F}(M)$).

At this point, the 1-cocycle $(0,1)\in\Gamma(T^*M\times\mathbb{R})\simeq\Omega^1(M)\times\mathcal{F}(M)$, $\mathbf{d}(0,1)=0$, allows the construction [24] of the $\mathbb{R}$-linear map

$$\mathbf{d}^{(0,1)}(\omega,\alpha) := (\mathrm{d}\omega, \omega-\mathrm{d}\alpha), \quad (\omega,\alpha)\in\Omega^k(M)\times\Omega^{k-1}(M), \tag{38}$$

which is a homological degree 1 derivation covering $\mathbf{d}$ (see the last formula in (37)) acting on the module $\Gamma\left(\wedge^{\bullet}(T^{*}M \times \mathbb{R})\right)$ over the graded, graded commutative and associative exterior algebra $\left(\Gamma\left(\wedge^{\bullet}(T^{*}M \times \mathbb{R})\right), \wedge\right)$. According to general results [9, 10], the previous de Rham complex exhibits a Jacobi algebroid structure $([\![\cdot,\cdot]\!], \rho, \nabla)$ on the pair of vector bundles $(TM \times \mathbb{R}, \mathbb{R}_M)$, $\mathbb{R}_M := \mathbb{R} \times M \to M$, with

$$(39)\qquad (X,f) \mapsto \nabla_{(X,f)}\bullet, \quad \nabla_{(X,f)}h = Xh + fh, \quad h \in \Gamma(\mathbb{R}_M) = \mathcal{F}(M).$$

At this stage, we introduce the Jacobi algebroid concept (also known as a Lie algebroid with a 1-cocycle [12, 13]) and its algebraic characterisation [33, 9, 10]. Let $L \to M$ be a line bundle (a vector bundle with one-dimensional fibers). Particularizing the general construction in [16] to the considered line bundle, one identifies the Lie algebroid structure $\left([\bullet,\bullet]_L, \rho_L\right)$ on the vector bundle $DL \to M$ with the fiber at $x \in M$, $(DL)_x$, consisting of $\mathbb{R}$-linear operators $\delta : \Gamma(L) \to L_x$, which enjoy the existence of tangent vectors $\xi \in T_xM$ such that

$$\delta(f\alpha) = (\xi f)\,\alpha(x) + f(x)\,\delta\alpha, \quad \alpha \in \Gamma(L), f \in \mathcal{F}(M).$$

The $\mathcal{F}(M)$-module of sections in the vector bundle $DL \to M$, $\Gamma(DL)$, coincides with the $\mathcal{F}(M)$-module of the derivations [16] in the considered line bundle, $\Gamma(DL) = \mathcal{D}(L)$, i.e. the sections are $\mathbb{R}$-linear maps $\Delta : \Gamma(L) \to \Gamma(L)$, which display the vector fields $X_\Delta \in \mathfrak{X}^1(M)$ (the well-known symbol of the derivation) such that

$$\Delta(f\alpha) = (X_\Delta f)\,\alpha + f\Delta\alpha, \quad \alpha \in \Gamma(L), f \in \mathcal{F}(M).$$

The Lie algebroid structure on $DL \to M$, $\left([\bullet,\bullet]_L, \rho_L\right)$, consists of the standard derivative commutator, $[\Delta,\Delta']_L = \Delta\Delta' - \Delta'\Delta$, while the anchor returns the symbol, $\rho_L(\Delta) = X_\Delta$. In the literature, $\left(DL \to M, [\bullet,\bullet]_L, \rho_L\right)$ is addressed as *the Atiyah algebroid* associated with the line bundle $L \to M$.

With this preparation at hand, we are able to introduce and algebraically characterize the Jacobi algebroids.

*Definition* 4.1. Let $(A, L)$ be a pair consisting in a vector bundle $A \to M$ and a line bundle $L \to M$. A Jacobi algebroid structure on the specified pair is a triplet $([\bullet,\bullet], \rho, \nabla)$, where $([\bullet,\bullet], \rho)$ is a Lie algebroid structure on the vector bundle $A \to M$ and $\nabla$ is a flat $A$-connection on the line bundle $L \to M$, i.e. it is a vector bundle morphism whose $\mathcal{F}(M)$-module map expression $\nabla : \Gamma(A) \to \mathcal{D}(L)$ is an $\mathbb{R}$-Lie algebra map that enjoys $\rho_L \circ \nabla = \rho$.

Following the line of Theorem 2.2 an algebraic characterization of the Jacobi algebroid structures was done [33, 10].

Theorem 4.2. *Let $(A, L)$ be a pair consisting in a vector bundle $A \to M$ and a line bundle $L \to M$. Denoting by $A_L := A \otimes L^*$ the total space of the vector bundle $A \otimes_M L^*$, the following ingredients are equivalent:*

1. *a Jacobi algebroid structure, $([\bullet, \bullet], \rho, \nabla)$, on the pair $(A, L)$;*

2. *a Gerstenhaber-Jacobi algebra structure, $\left([\bullet, \bullet]_{A,L}, X_\bullet^{(A,L)}\right)$, on the graded $\mathcal{A}^\bullet_{A,L} := \Gamma(\wedge^\bullet A_L)$-module $\mathcal{L}^\bullet_{A,L} := \Gamma(\wedge^\bullet A_L \otimes L)[1]$;*

3. *a homological degree 1 graded derivation, $d_{A,L}$ covering $d_A$, acting on the graded $\tilde{\mathcal{A}}^\bullet_A := \Gamma(\wedge^\bullet A^*)$-module $\tilde{\mathcal{L}}^\bullet_{A,L} := \Gamma(\wedge^\bullet A^* \otimes L)$.*

In the light of Theorem 4.2, the pair $\left(\Gamma(\wedge^\bullet(T^*M \times \mathbb{R})), \mathbf{d}^{(0,1)}\right)$ consisting of the module $\Gamma(\wedge^\bullet(T^*M \times \mathbb{R}))$ over the graded, graded commutative and associative algebra $(\Gamma(\wedge^\bullet(T^*M \times \mathbb{R})), \wedge)$ and the homological degree 1 derivation $\mathbf{d}^{(0,1)}$ covering de Rham differential $\mathbf{d}$ (see the second relation in (37)) exhibits the Jacobi algebroid structure $(\llbracket\cdot,\cdot\rrbracket, \rho, \nabla)$ on the pair $(TM \times \mathbb{R}, \mathbb{R}_M)$ and it is equivalent to the Gerstenhaber-Jacobi algebra structure $\left(\llbracket\cdot,\cdot\rrbracket^{(0,1)}, \mathbb{X}^{(0,1)}\right)$ on the module $\Gamma(\wedge^\bullet(TM \times \mathbb{R}))$ over the graded, graded commutative and associative algebra $(\Gamma(\wedge^\bullet(TM \times \mathbb{R})), \wedge)$, where

$$(P, Q) \wedge (R, S) = (P \wedge Q, P \wedge S - (-)^r Q \wedge R),$$

with $(P, Q) \in \mathfrak{X}^{p+1}(M) \times \mathfrak{X}^p(M)$ and $(R, S) \in \mathfrak{X}^{r+1}(M) \times \mathfrak{X}^r(M)$ homogeneous elements. Using again the $\mathcal{F}(M)$-module isomorphisms $\Gamma\left(\wedge^{p+1}(TM \times \mathbb{R})\right) \simeq \mathfrak{X}^{p+1}(M) \oplus \mathfrak{X}^p(M) \simeq \mathfrak{X}^{p+1}(M) \times \mathfrak{X}^p(M)$ (that come from the vector bundle isomorphisms $\wedge^{p+1}(TM \times \mathbb{R}) \simeq (\wedge^{p+1}TM) \oplus (\wedge^p TM) \simeq (\wedge^{p+1}TM) \times (\wedge^p TM)$), the Gerstenhaber-Jacobi bracket reads

$$\begin{aligned}\llbracket(P, Q), (R, S)\rrbracket^{(0,1)} := ([P, R] - p\,(-)^r\, P \wedge S + rQ \wedge R, [P, S] \\ + (-)^r\,[Q, R] - (p - r)\, Q \wedge S),\end{aligned}$$

while the derivative representation

$$\mathbb{X}^{(0,1)} : \Gamma(\wedge^\bullet(T^*M \times \mathbb{R})) \to \mathrm{Der}\,(\Gamma(\wedge^\bullet(T^*M \times \mathbb{R})))$$

acts [4] on homogenous elements as

$$(P, Q) \mapsto \mathbb{X}^{(0,1)}_{(P,Q)}, \quad \mathbb{X}^{(0,1)}_{(P,Q)}(\tilde{P}, \tilde{Q}) = \llbracket(P, Q), (\tilde{P}, \tilde{Q})\rrbracket^{(0,1)} - (Q \wedge \tilde{P}, Q \wedge \tilde{Q}).$$

At the end of this section, by means of the previous results concerning the Jacobi algebroid structure $(\llbracket\cdot,\cdot\rrbracket, \rho, \nabla)$ on the pair $(TM \times \mathbb{R}, \mathbb{R}_M)$, we are able to 'compress' the consistency conditions (1), (14) and (23) satisfied by corresponding Jacobi-like pairs. First, the pair $(\Pi, E)$ is a homogeneous element from the previous Gerstenhaber-Jacobi algebra, $(\Pi, E) \in \Gamma\left(\wedge^2(TM\times\mathbb{R})\right)$, which is subject to

$$\llbracket(\Pi, E),(\Pi, E)\rrbracket^{(0,1)} := \left([\Pi,\Pi] + 2\Pi\wedge E, 2\,[\Pi, E]\right). \tag{40}$$

Comparing the previous result to the consistency conditions (1) it results that the pair $(\Pi, E)$ is a Jacobi one if and only if

$$\llbracket(\Pi, E),(\Pi, E)\rrbracket^{(0,1)} = 0. \tag{41}$$

Regarding the twisted situation, in the light of (40) it is clear that the $\llbracket\cdot,\cdot\rrbracket^{(0,1)}$-expression for the equations (14) is non-homogeneous. In order to identify the needed expression, we use the isomorphism of vector bundles $\Gamma\left(\wedge^2(TM\times\mathbb{R})\right) \simeq \left(\Gamma\left(\wedge^2(T^*M\times\mathbb{R})\right)\right)^*$ that allows the construction of the $\mathcal{F}(M)$-module morphism

$$(\Pi, E)^\sharp : \Gamma(T^*M\times\mathbb{R}) \to \Gamma(TM\times\mathbb{R}), \quad (\beta, f)\mapsto(\Pi^\sharp\beta + fE, -i_E\beta). \tag{42}$$

Extending by linearity the previous $\mathcal{F}(M)$-module morphism, consistency conditions (14) associated with a twisted Jacobi pair can be compactly written as

$$\tfrac{1}{2}\llbracket(\Pi, E),(\Pi, E)\rrbracket^{(0,1)} = (\Pi, E)^\sharp\left(\mathrm{d}^{(0,1)}(\omega, 0)\right). \tag{43}$$

Proceeding in the same manner as in the twisted case, by direct computation [5] it can be shown that the consistency conditions fulfilled by a Jacobi pair with background (23) are equivalent to

$$\tfrac{1}{2}\llbracket(\Pi, E),(\Pi, E)\rrbracket^{(0,1)} = (\Pi, E)^\sharp(\phi, \omega). \tag{44}$$

Definition (38) of the homological derivation of degree 1 in the module $\Gamma\left(\wedge^\bullet(T^*M\times\mathbb{R})\right)$, $\mathbf{d}^{(0,1)}$ allows to conclude that a Jacobi pair with background is a twisted one if and only if $(\phi,\omega)\in\Gamma\left(\wedge^3(T^*M\times\mathbb{R})\right)$ is closed, i.e.

$$\mathbf{d}^{(0,1)}(\phi,\omega) = 0 \quad\Leftrightarrow\quad \phi = \mathrm{d}\omega.$$

## 5. JACOBI-LIKE STRUCTURES IN THE LINE BUNDLE SETTING

In this section we investigate the line bundle formulations of the analyzed Jacobi-like structures. As we shall see, this approach is more ‘invariant’ than the previous pair-like formulation.

A clue for the line bundle setting comes from the Jacobi algebroid structure $(\llbracket\cdot,\cdot\rrbracket, \rho, \nabla)$ on the pair $(TM\times\mathbb{R}, \mathbb{R}_M)$. Here, the vector bundle $TM\times\mathbb{R}$ coincides with the vector bundle of the derivations corresponding to the trivial line bundle, $TM\times\mathbb{R} = \mathcal{D}\mathbb{R}_M$, while the flat connection $\nabla$ is obvious (see (39)) nothing but the tautological representation of the Lie algebroid $([\![\cdot,\cdot]\!],\rho)$ on the trivial line bundle $\mathbb{R}_M$ [16].

The previous argument suggests that the natural situs of a Jacobi-like line bundle is the Jacobi algebroid structure $([\bullet,\bullet]_L, \rho_L, \nabla_L)$ on the pair $(DL, L)$ associated with the line bundle $L \to M$. Here, the flat connection $\nabla_L$ is just the tautological representation of $DL$, i.e.

$$\mathcal{D}(L) \ni \Box \mapsto (\nabla_L)_\Box := \Box \in \mathcal{D}(L).$$

Invoking Theorem 4.2, the contravariant formulation of the previous Jacobi algebroid consists of the Gerstenhaber-Jacobi algebra structure

$$\left([\bullet,\bullet] := [\bullet,\bullet]_{DL,L}, X_\bullet := X_\bullet^{(DL,L)}\right)$$

on the graded $\Gamma\left(\wedge^\bullet DL_L\right)$-module

$$\Gamma\left(\wedge^\bullet DL_L \otimes L\right)[1],$$

with $DL_L := DL \otimes L^*$. This abstract structure can be cast into a more operational one by means of the vector bundle isomorphism [23]

$$\left(J^1 L\right)^* \otimes L \simeq DL \quad \Leftrightarrow \quad DL_L := DL \otimes L^* \simeq J_1 L := \left(J^1 L\right)^*, \tag{45}$$

where $J^1 L \to M$ is the first-order jet bundle [29] associated with the line bundle $L \to M$. Result (45) further yields [23] the graded algebra isomorphisms

$$\Gamma\left(\wedge^\bullet DL_L\right) \simeq \Gamma\left(\wedge^\bullet J_1 L\right) \simeq \mathcal{D}\mathit{iff}_1^\bullet\left(L; \mathbb{R}_M\right), \tag{46}$$

where the degree $k$ homogeneous subspace $\mathcal{D}\mathit{iff}_1^k\left(L;\mathbb{R}_M\right)$ consists in the $\mathbb{R}$ multi-linear applications

$$\tilde{\Box} : \Gamma(L) \times \cdots \times \Gamma(L) \to \mathcal{F}(M), \quad \tilde{\Box}\left(e_1,\ldots,e_k\right) \in \mathcal{F}(M),$$

that are skew-symmetric and first-order differential operators in each argument.

'Tensorising' the isomorphisms (46) by the module $\Gamma(L)$ one immediately gets

$$\begin{aligned} &\Gamma\left(\wedge^\bullet DL_L \otimes L\right)[1] \simeq \mathcal{D}^\bullet L\,[1] := \mathcal{D}\mathit{iff}_1^\bullet\left(L;L\right)[1] \\ &\Leftrightarrow \Gamma\left(\wedge^k DL_L \otimes L\right) \simeq \mathcal{D}^{k+1} L, \quad k \geq -1 \end{aligned} \tag{47}$$

where the degree $(k+1)$ homogeneous subspace $\mathcal{D}^{k+1}L$ consists in the $\mathbb{R}$-multi-linear applications

$$\Box : \Gamma(L) \times \cdots \times \Gamma(L) \to \Gamma(L), \quad \Box\left(e_1,\ldots,e_{k+1}\right) \in \Gamma(L),$$

that are skew-symmetric and first-order differential operators in each argument.

At this stage, we replaced the abstract module $\Gamma\left(\wedge^\bullet DL_L \otimes L\right)$ over the abstract graded, graded commutative and associative algebra $\Gamma\left(\wedge^\bullet DL_L\right)$ with the $\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M)$-module $\mathcal{D}^\bullet L$ of $L$-valued multi-derivations. Obviously, the left action of $\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M)$ on $\mathcal{D}^\bullet L$ reduces to wedge product, i.e.

$$\left(\tilde{\triangle} \wedge \square\right)(e_1, \ldots, e_{k+l+1}) = \sum_{\sigma \in S(k,l+1)} (-)^\sigma \tilde{\triangle}\left(e_{\sigma(1)}, \ldots, e_{\sigma(k)}\right) \square \left(e_{\sigma(k+1)}, \ldots, e_{\sigma(k+l+1)}\right),$$

for arbitrary homogeneous elements $\tilde{\triangle} \in \mathcal{D}iff_1^k(L;\mathbb{R}_M)$ and $\square \in \mathcal{D}^{l+1}L$. Previously, we denoted by $S(k, l+1)$ the subset of $(k, l+1)$ un-shuffle permutations in $S(k+l+1)$ i.e. those permutations $\sigma$ that satisfy $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(k+l+1)$. Moreover, the pair $(\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M), \mathcal{D}^\bullet L)$ has a natural Gerstenhaber-Jacobi algebra structure $([\![\bullet,\bullet]\!], \mathbb{X}_\bullet)$ as follows. The graded Lie algebra structure on $\mathcal{D}^\bullet L$, $[\![\bullet,\bullet]\!]$, can be written in terms of the Gerstenhaber inner multiplication [10]

$$\square \circ \triangle (e_1, \ldots, e_{k+l+1}) := \sum_{\sigma \in S(l+1,k)} (-)^\sigma \square \left(\triangle\left(e_{\sigma(1)}, \ldots, e_{\sigma(l+1)}\right), e_{\sigma(l+2)}, \ldots, e_{\sigma(k+l+1)}\right)$$

as

$$[\![\square, \triangle]\!] := (-)^{kl} \square \circ \triangle - \triangle \circ \square, \quad \square \in \mathcal{D}^{k+1}L, \triangle \in \mathcal{D}^{l+1}L. \tag{48}$$

In order to introduce the derivative representation of the module $\mathcal{D}^\bullet L$ on the graded algebra $\mathcal{D}iff_1^\bullet(L;\mathbb{R}_M)$, $\mathbb{X}_\bullet$, we define the *symbol map*

$$\sigma_\square(f)(e_1, \ldots, e_k)\, e := \square(fe, e_1, \ldots, e_k) - f\square(e, e_1, \ldots, e_k), \tag{49}$$

where

$$\square \in \mathcal{D}^{k+1}L, \quad f \in \mathcal{F}(M), \quad e, e_1, \ldots, e_k \in \Gamma(L).$$

It is worth noticing that the symbol in the above, $\sigma_\square(f)(e_1, \ldots, e_k)$ is just a smooth function on the manifold $M$ because

$$\sigma_\square(f)(e_1, \ldots, e_k) \in \Gamma(L^* \otimes L) \simeq \mathcal{F}(M),$$

and moreover $\sigma_\square(f) \in \mathcal{D}iff_1^k(L;\mathbb{R}_M)$. With these specifications at hand, the derivative representation reads

$$\begin{aligned}\mathbb{X}_\square\left(\tilde{\triangle}\right)&(e_1, \ldots, e_{k+l}) \\ &= (-)^{k(l-1)} \sum_{\sigma \in S(l,k)} (-)^\sigma \sigma_\square\left(\tilde{\triangle}\left(e_{\sigma(1)}, \ldots, e_{\sigma(l)}\right)\right)\left(e_{\sigma(l+1)}, \ldots, e_{\sigma(l+k)}\right) \\ &- \sum_{\sigma \in S(k+1,l-1)} (-)^\sigma \tilde{\triangle}\left(\square\left(e_{\sigma(1)}, \ldots, e_{\sigma(k+1)}\right), e_{\sigma(k+2)}, \ldots, e_{\sigma(k+l)}\right).\end{aligned} \tag{50}$$

Invoking again Theorem 4.2, the covariant expression of the Jacobi algebroid structure $([\bullet,\bullet]_L, \rho_L, \nabla_L)$ on the pair $(DL, L)$ consists of the homological degree 1 graded derivation $\mathbf{d}_L := \mathrm{d}_{DL,L}$ covering $\mathrm{d}_{DL}$ that acts on the graded module $\Omega_L^\bullet := \Gamma\left(\wedge^\bullet (DL)^* \otimes L\right)$ over the graded, graded commutative and associative algebra $\Gamma\left(\wedge^\bullet (DL)^*\right)$. In literature [28], de Rham complex $(\Omega_L^\bullet, \mathbf{d}_L)$ is known as *der-complex* associated with the line bundle $L \to M$ and, meanwhile, the homogeneous elements of the module $\Omega_L^\bullet$ are called *L-valued Atiyah forms*. Regarding the homological derivation $\mathbf{d}_L$, it acts on the homogeneous elements of the der-complex as

$$\langle \mathbf{d}_L e, \Box \rangle := \Box e, \quad e \in \Gamma(L), \Box \in \mathcal{D}(L), \tag{51}$$

$$\mathbf{d}_L(\tilde{\omega} \wedge \omega) = \mathrm{d}_{DL}\tilde{\omega} \wedge \omega + (-)^k \tilde{\omega} \wedge \mathbf{d}_L \omega, \quad \tilde{\omega} \in \Gamma\left(\wedge^k (DL)^*\right), \omega \in \Omega_L^\bullet. \tag{52}$$

*Remark* 5.1. The homological derivation enjoys two strong properties: i) it is *acyclic* and ii) it *agrees* with the first-order prolongation. Although the meaning of acyclicity is clear, the agreement with the first-order prolongation has to be understood in terms of the isomorphisms (45) that exhibit the $L$-pairing between $DL$ and $J^1L$ expressed by the bi-linear non-degenerate map

$$\langle \bullet, \bullet \rangle : \mathcal{D}(L) \times \Gamma(J^1L) \to \Gamma(L), \quad \langle \Box, j^1 e \rangle := \Box e, \tag{53}$$

which is well-defined as the $\mathcal{F}(M)$-module $\Gamma(J^1L)$ is generated [29] by $\operatorname{Im} j^1$. By $j^1$, $j^1 : \Gamma(L) \to \Gamma(J^1L)$, we denoted the first-order prolongation [29] which is known to be a first-order differential operator. In the light of (53) it results that

$$\langle \mathbf{d}_L e, \Box \rangle = \langle \Box, j^1 e \rangle, \quad e \in \Gamma(L), \Box \in \mathcal{D}(L),$$

which means the announced agreement.

With all of these aspects of line bundles in mind, we are able to close this paper with the line bundle formulation of the previous Jacobi/ Jacobi-like structures.

**Jacobi bundles.** Let $L \to M$ be a line bundle. By its very definition [21], a *Jacobi structure* on the considered line bundle is an $\mathbb{R}$-Lie algebra structure on $\Gamma(L)$, $\{\bullet, \bullet\}$, which is a derivation in both of its arguments,

$$\{\bullet, e\} \in \mathcal{D}(L), \quad e \in \Gamma(L).$$

It is worth noticing that such a structure is nothing but a *local Lie algebra one* [14] on the line bundle $L \to M$. With these specifications, let's fix the terminology. By definition, a *Jacobi bundle* is a line bundle endowed with a Jacobi structure, $(L \to M, \{\bullet, \bullet\})$ while a *Jacobi manifold* is a manifold equipped with a Jacobi bundle over it.

The previous definition places the Jacobi structures into the realm of Jacobi algebroid structure $\left([\bullet, \bullet]_L, \rho_L, \nabla_L\right)$ associated with the pair $(DL, L)$.

Indeed, if $\{\bullet,\bullet\}$ is a skew-symmetric bi-differential operator on $\Gamma(L)$ then there exists a unique $J \in \mathcal{D}^2 L$ such that

$$J\left(e_1, e_2\right) = \{e_1, e_2\}, \quad e_1, e_2 \in \Gamma(L). \tag{54}$$

Direct computation done by means of (54) gives

$$\begin{aligned}(J \circ J)\left(e_1, e_2, e_3\right) =& \{\{e_1, e_2\}, e_3\} + \{\{e_2, e_3\}, e_1\} \\ &+ \{\{e_3, e_1\}, e_2\} := -\mathrm{Jac}\{e_1, e_2, e_3\},\end{aligned} \tag{55}$$

which further exhibits

$$[\![J, J]\!] = 2\mathrm{Jac}\{e_1, e_2, e_3\}.$$

The last equality shows that a Jacobi structure $\{\bullet,\bullet\}$ *is completely captured* by the bi-differential operator $J \in \mathcal{D}^2 L$ that verifies the Maurer-Cartan equation

$$[\![J, J]\!] = 0. \tag{56}$$

So, from now on, a Jacobi bundle is addressed in terms of the pair $(L \to M, J)$ with $J$ a bi-differential operator, $J \in \mathcal{D}^2 L$, satisfying (56).

When the line bundle is trivial, $L = \mathbb{R}_M$, the sections in the line bundle $\Gamma(\mathbb{R}_M)$ are just the smooth functions, $\mathcal{F}(M)$, the structure $\{\bullet,\bullet\}$ coincides with (8) and the bi-differential operator reduces to the Jacobi pair, $J = (\Pi, E)$. This proves that Jacobi pairs are in one-to-one correspondence with trivial Jacobi bundles.

Let $(L \to M, J)$ be a Jacobi bundle. By means of the isomorphisms (47), the bi-differential operator $J \in \mathcal{D}^2 L$ exhibits (via the fact that the module $\Gamma(J^1 L)$ is generated by $\operatorname{Im} j^1$ ) the element $\hat{J} \in \Gamma\left(\wedge^2 J_1 L \otimes L\right)$ via

$$\langle \hat{J}, j^1 e_1 \wedge j^1 e_2 \rangle = J\left(e_1, e_2\right), \quad e_1, e_2 \in \Gamma(L) \tag{57}$$

which further displays the morphism of $\mathcal{F}(M)$-modules

$$\hat{J}^\sharp : \Gamma\left(J^1 L\right) \to \mathcal{D}(L), \quad \hat{J}^\sharp(j^1 e_1) e_2 := J\left(e_1, e_2\right), \quad e_1, e_2 \in \Gamma(L). \tag{58}$$

This morphism allows the introduction of a smooth distribution on the base manifold

$$\mathcal{K}_J := \operatorname{Im}\left(\rho_L \circ \hat{J}^\sharp\right), \tag{59}$$

known as *the characteristic distribution* of the considered Jacobi bundle. It is worth noticing that in trivial line bundle situation, the vector bundle morphism coming from (58) is just (42). By definition, the considered Jacobi bundle is said to be *transitive* if the characteristic distribution coincides, at each point of the base manifold, with the corresponding tangent space to the base manifold

$$\left(\mathcal{K}_J\right)_x = T_x M, \quad x \in M \tag{60}$$

or, equivalently, the vector bundle map $\rho_L \circ \hat{J}^\sharp : J^1 L \to TM$ is surjective.

*Example* 5.2. By its very definition, a *locally conformal symplectic structure (lcs)* on a given line bundle $L \to M$ is a pair $(\nabla, \omega)$ consisting of a representation $\nabla$ of the tangent Lie algebroid $TM \to M$ on the considered line bundle and a non-degenerate $L$-valued 2-form $\omega \in \Omega^2(M; L) := \Gamma(\wedge^2 T^*M \otimes L)$ which is closed with respect to the homological degree 1 derivation $\mathbf{d}_\nabla$ associated with the Jacobi algebroid structure $([\bullet, \bullet], \mathrm{id}_{TM}, \nabla)$ on the pair $(TM, L)$ (see the third statement in Theorem 4.2)

$$\mathbf{d}_\nabla \omega = 0.$$

Associated with the lcs structure $(\nabla, \omega)$ we introduce the Jacobi structure $J_{(\nabla,\omega)} \in \mathcal{D}^2 L$ via

(61)

$$J_{(\nabla,\omega)} : \Gamma(L) \times \Gamma(L) \to \Gamma(L), \quad J_{(\nabla,\omega)}(e_1, e_2) := \langle \omega, X_{e_1} \wedge X_{e_2} \rangle, \quad e_1, e_2 \in \Gamma(L),$$

where

$$X_e := \omega^\sharp (\mathbf{d}_\nabla e), \quad e \in \Gamma(L). \tag{62}$$

with $\omega^\sharp$ the inverse of the vector bundle isomorphism

$$\omega^\flat : TM \to T^*M \otimes L, \quad \omega^\flat X := -i_X \omega. \tag{63}$$

*Example* 5.3. A contact structure over a smooth manifold (necessarily odd-dimensional) $M$ is a hyperplane distribution $\mathcal{H} \subset TM$ which is *maximally non-integrable* i.e. its curvature

$$\omega_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to L := TM/\mathcal{H}, \quad \omega_{\mathcal{H}}(X, Y) := [X, Y] \mod \mathcal{H} \tag{64}$$

is non-degenerate, i.e., the linear map

$$\omega_{\mathcal{H}}^\flat : \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H}^* \otimes L), \quad \langle \omega_{\mathcal{H}}^\flat X, Y \rangle := -\omega_{\mathcal{H}}(X, Y)$$

is invertible.

It is worth noticing that a given contact structure can be interpreted in a dual view via the canonical projection

$$\theta \in \Omega^1(M; L) := \Gamma(T^*M \otimes L), \quad \langle \theta, X \rangle := X \mod \mathcal{H}.$$

Previously, $\theta$ is the well-known contact 1-form with the curvature precisely $\omega_{\mathcal{H}}$, $\langle \omega_{\mathcal{H}}, X \wedge Y \rangle = \langle \theta, [X, Y] \rangle$.

In this context, there exists the decomposition of $\mathbb{R}$-vector spaces $\mathfrak{X}^1(M) = \mathfrak{X}_{\mathcal{H}} \oplus \Gamma(\mathcal{H})$, where $\mathfrak{X}_{\mathcal{H}}$ is the $\mathbb{R}$-Lie subalgebra

$$X \in \mathfrak{X}_{\mathcal{H}} \Leftrightarrow [X, \Gamma(\mathcal{H})] \subset \Gamma(\mathcal{H}), \tag{65}$$

of the $\mathbb{R}$-Lie algebra $\mathfrak{X}^1(M)$ whose elements are the well-known Reeb vector fields [6](or contact vector fields [33]). With these specifications at hand, we define the Jacobi structure $J_{\mathcal{H}} \in \mathcal{D}^2 L$ by

$$J_{\mathcal{H}}(e_1, e_2) := \langle \omega_{\mathcal{H}}, X_{e_1} \wedge X_{e_2} \rangle, \quad e_1, e_2 \in \Gamma(L)$$

with $X_e$ the Hamiltonian vector field associated with $e \in \Gamma(L)$, i.e. the unique solution $X_e \in \mathfrak{X}_{\mathcal{H}}$ enjoying $\langle \theta, X_e \rangle = e$.

A deep analysis [6, 33] of the characteristic distribution corresponding to a given Jacobi bundle showed that it enjoys similar results to those listed in Theorems 2.6 and 2.7.

THEOREM 5.4. *If a Jacobi structure $J \in \mathcal{D}^2 L$ on the line bundle $L \to M$ is transitive then $M$ is either a locally conformal symplectic structure (if the base manifold is even-dimensional) or a contact one (if the base manifold is odd-dimensional) on the same line bundle.*

THEOREM 5.5. *The characteristic distribution of a Jacobi structure $J \in \mathcal{D}^2 L$ on the line bundle $L \to M$ is completely integrable [31, 32] with the characteristic leaves equipped with transitive Jacobi structures induced by $J$.*

It is worth noticing that integrability Theorem 5.5 is algebraically proved by associating a Jacobi algebroid to the considered Jacobi bundle. We synthesize here the argumentation from [6, 33]. The Jacobi structure $J$ organizes $J^1 L$ as a Lie algebroid with respect to

$$\left[j^1 e_1, j^1 e_2\right]_J := j^1 J\left(e_1, e_2\right), \quad \rho_J(j^1 e) := X_e,$$

where $X_e$ is the symbol (the well-known *Hamiltonian vector field*) of the *Hamiltonian derivation* associated with the section $e \in \Gamma(L)$, $\triangle_e \in \mathcal{D}(L)$,

$$\triangle_e(e_1) := J\left(e, e_1\right), \quad e_1 \in \Gamma(L), \tag{66}$$

i.e. $X_e := \rho\left(\triangle_e\right)$. Moreover, the Lie algebroid $\left(J^1 L, [\bullet, \bullet]_J, \rho_J\right)$ enjoys a natural representation on the line bundle $L$,

$$\nabla^J : J^1 L \to DL, \quad \nabla^J_{j^1 e} := \triangle_e.$$

This is guaranteed by the axiom (56) which is equivalent to $[\triangle_{e_1}, \triangle_{e_2}] = \triangle_{\{e_1, e_2\}}$. With these specifications at hand, it is clear that the Jacobi bundle $(L \to M, J)$ is transitive *if and only if* the associated Jacobi algebroid $\left(J^1 L, L, [\bullet, \bullet]_J, \rho_J, \nabla^J\right)$ is transitive.

**Twisted Jacobi bundles**. In this part, following the prescriptions (14), or, equivalently (43), we introduce the concept of *twisted Jacobi bundle.*

*Definition* 5.6. A twisted Jacobi bundle is a triple $(L \to M, J, \Omega)$ consisting of a line bundle $L \to M$, a first-order bi-differential operator $J \in \mathcal{D}^2 L$ and an $L$-valued Atiyah 2-form $\Omega \in \Omega_L^2$ that verify the consistency condition

$$[\![ J, J ]\!] = 2\left(\wedge^3 \hat{J}^\sharp\right)^* \mathbf{d}_L \Omega, \tag{67}$$

where by $\wedge^3 \hat{J}^\sharp$ we mean the linear extension of module map (58),

$$\wedge^3 \hat{J}^\sharp : \Gamma\left(\wedge^3 J^1 L\right) \to \Gamma\left(\wedge^3 DL\right),$$
$$\wedge^3 \hat{J}^\sharp(j^1 e_1, j^1 e_2, j^1 e_3) := \hat{J}^\sharp(j^1 e_1) \wedge \hat{J}^\sharp(j^1 e_2) \wedge \hat{J}^\sharp(j^1 e_3).$$

As in the trivial line bundle situations (see Sections 2 and 3), we maintain here definitions (59) and (60) for the characteristic distribution and transitivity respectively.

*Example* 5.7. A twisted locally conformal symplectic structure on a given line bundle $L \to M$ is a pair $((\nabla, \omega), \hat{\omega})$ consisting of a representation $\nabla$ of the tangent Lie algebroid $TM \to M$ on a line bundle and two $L$-valued 2-forms $\omega, \hat{\omega} \in \Omega^2(M; L)$ among which $\omega$ is non-degenerate and verifies the consistency condition

$$\mathbf{d}_\nabla (\omega - \hat{\omega}) = 0. \tag{68}$$

Previously, by $\mathbf{d}_\nabla$ we denoted the homological degree 1 derivation associated with the Jacobi algebroid structure $([\bullet, \bullet], \mathrm{id}M, \nabla)$ on the pair $(TM, L)$ (see the third statement in Theorem 4.2). The given twisted locally conformal symplectic structure exhibits the twisted Jacobi bundle $\left(L \to M, J_{((\nabla,\omega),\hat{\omega})}, \Omega_{((\nabla,\omega),\hat{\omega})}\right)$ where $J_{((\nabla,\omega),\hat{\omega})}$ is defined in (61) while $\Omega_{((\nabla,\omega),\hat{\omega})} := \mathbf{d}_L(\wedge^2 \rho_L)^* \hat{\omega}$.

*Example* 5.8. A twisted contact structure on a manifold $M$ consists of a hyperplane distribution $\mathcal{H} \subset TM$ and an $L$-valued 2-form $\Omega \in \Omega^2_L$, $L := TM/\mathcal{H}$, such that

$$\omega := \omega_{\mathcal{H}} + \Omega|_{\mathcal{H}} \in \Gamma\left(\wedge^2 \mathcal{H}^* \otimes L\right) \tag{69}$$

is non-degenerate. In (69), $\omega_{\mathcal{H}}$ is the curvature (64) of the considered hyperplane distribution $\mathcal{H}$. This displays the transitive twisted Jacobi bundle $\left(L \to M, J_{(\mathcal{H},\Omega)}, \Omega\right)$ where $J_{(\mathcal{H},\Omega)}$ is defined in terms of the Hamiltonian vector fields exhibited by the non-degenerate $L$-valued 2-form (69) [4].

Extending the terminology adopted for Jacobi manifolds, we say that a given smooth manifold $M$ is a *twisted Jacobi* one if it is the base manifold for a twisted Jacobi bundle. At this point, there exist two strong results concerning the twisted Jacobi bundles [24, 25, 4].

THEOREM 5.9. *Let $(L \to M, J, \Omega)$ be a transitive twisted Jacobi bundle. Then the following alternative holds.*

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a twisted locally conformal symplectic structure on the same line bundle.*

• *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a twisted contact structure displaying the same line bundle.*

THEOREM 5.10. *The characteristic distribution of a twisted Jacobi bundle $(L \to M, J, \Omega)$ is completely integrable [31, 32] with the characteristic leaves equipped with transitive twisted Jacobi structures induced by $(J, \Omega)$.*

It is worth mentioning that the proof of the integrability Theorem 5.10 was initially done in the trivial line bundle case. That made use of the $\omega$-twisted Courant-Jacobi algebroid structure on the omni-Lie algebroid associated with the trivial line bundle $\mathbb{DR}_M = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, related to which, $\operatorname{graph}\hat{J}^\sharp = (\Pi, E)^\sharp$ is a Dirac-Jacobi subbundle. Recent results concerning Dirac-Jacobi bundles [37] allowed a direct algebraic proof of the integrability Theorem 5.10 [4] in the general line bundle setting. Here, the omni-Lie algebroid becomes $\mathbb{D}L = DL \oplus J^1L$ while its $\Omega$-twisted Courant-Jacobi algebroid structure consists in the Dorfman-like bracket

$$\begin{aligned}[\![(\Box_1, \mu_1), (\Box_2, \mu_2)]\!]_\Omega := \Big([\Box_1, \Box_2], \mathcal{L}^{(DL,L)}_{\Box_1}\mu_2 - \iota^{(DL,L)}_{\Box_2}{}_L\mu_1 \\ + \langle {}_L\Omega, \Box_1 \wedge \Box_2 \wedge \bullet\rangle\Big),\end{aligned}$$

the non-degenerate metric $\langle\langle(\Box_1, \mu_1), (\Box_2, \mu_2)\rangle\rangle := \langle \Box_1, \mu_2\rangle + \langle \Box_2, \mu_1\rangle$ and the vector bundle morphism

$$\mathrm{p}_D : DL \oplus J^1L \to DL, \quad \mathrm{p}_D(\Box, \mu) := \Box.$$

Previously, by $\langle \bullet, \bullet\rangle$ we denote the $L$-pairing between $DL$ and $J^1L$ given in (53). Due to the fact that $\operatorname{graph}\hat{J}^\sharp$ is a Dirac-Jacobi subbundle in $\mathbb{D}L = DL \oplus J^1L$ it results the integrability of the characteristic distribution for the considered twisted Jacobi bundle.

**Jacobi bundles with background.** Here, using the 'pattern' followed in Section 3, we 'relax' the previous concept (see Definition 5.6) by incorporating arbitrary Atiyah 3-forms.

*Definition* 5.11. A Jacobi bundle with a background 3-form, shortly a Jacobi bundle with background, is a triple $(L \to M, J, \Phi)$ consisting in a line bundle $L \to M$, a first-order bi-differential operator $J \in \mathcal{D}^2L$ and an $L$-valued Atiyah 3-form $\Phi \in \Omega^3_L$ that verify the consistency condition

$$[\![J, J]\!] = 2\left(\wedge^3 \hat{J}^\sharp\right)^* \Phi. \tag{70}$$

*Remark* 5.12. Using the acyclicity of the homological degree 1 derivation $\mathbf{d}_L$ it results that twisted Jacobi bundles are nothing but Jacobi bundles with background equipped with closed $L$-valued Atiyah 3-forms, $\mathbf{d}_L\Omega = 0$.

As in the trivial line bundle situations (see Sections 2 and 3), we maintain here definitions (59) and (60) for the characteristic distribution and transitivity respectively.

*Example* 5.13. A locally conformal symplectic structure with background on a given line bundle $L \to M$ is a pair $\left((\nabla, \omega), \left(\hat{\omega}, \hat{\phi}\right)\right)$ consisting in a representation $\nabla$ of the tangent Lie algebroid $TM \to M$ on a line bundle, two $L$-valued 2-forms $\omega, \hat{\omega} \in \Omega^2(M; L)$ among which $\omega$ is non-degenerate and an $L$-valued 3-form $\hat{\psi} \in \Omega^3(M; L)$ that verify the consistency condition

$$\mathbf{d}_\nabla (\omega - \hat{\omega}) = \hat{\psi}, \tag{71}$$

where $\mathbf{d}_\nabla$ is the homological degree 1 derivation associated with the Jacobi algebroid structure $([\bullet, \bullet], \nabla)$ on the pair $(TM, L)$ (see the third statement in Theorem 4.2). Associated with the considered lcs-like structure, there exists the Jacobi structure with background $\left(J_{((\nabla,\omega),(\hat{\omega},\hat{\psi}))}, \Psi_{((\nabla,\omega),(\hat{\omega},\hat{\psi}))}\right)$ [4] on the same line bundle $L \to M$, where $J_{((\nabla,\omega),(\hat{\omega},\hat{\psi}))}$ is given in (61) and

$$\Psi_{((\nabla,\omega),(\hat{\omega},\hat{\psi}))} = \mathbf{d}_L \left(\wedge^2 \rho_L\right)^* \hat{\omega} + \left(\wedge^3 \rho_L\right)^* \hat{\psi}.$$

At this point, we mention the main properties of the Jacobi bundles with background [4].

THEOREM 5.14. *Let $(L \to M, J, \Psi)$ be a transitive Jacobi bundle with background. Then the following alternative holds.*

- *If the base manifold is even-dimensional then the considered Jacobi bundle is equivalent to a locally conformal symplectic structure with background on the same line bundle.*

- *If the base manifold is odd-dimensional then the considered Jacobi bundle is equivalent to a twisted contact structure displaying the same line bundle.*

THEOREM 5.15. *The characteristic distribution of a Jacobi bundle with background $(L \to M, J, \Psi)$ is completely integrable [31, 32] with the characteristic leaves equipped with transitive Jacobi structures with background induced by $(J, \Psi)$.*

To conclude with, the Jacobi structures with background *enjoy the main features* of twisted Jacobi structures, i.e. i) their characteristic distributions are completely integrable, with twisted cooriented contact structures on the odd dimensional leaves and locally conformal symplectic structures with background on the even dimensional leaves and ii) they are in one-to-one correspondence with homogeneous Poisson manifolds with background, where the background 3-form is no longer closed. Contrary to twisted Jacobi structures, the ones with background *do not display of an algebraic correspondence* with the famous Dirac-Jacobi structures.

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