# NOTE ON THE DIFFERENTIABILITY OF STRONGLY CONTINUOUS SEMICOCYCLES

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In this note we study the differentiability with respect to the time-parameter of semicocycles over differentiable semigroups on a domain in a Banach space.

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## 1. INTRODUCTION

Let $\{\psi_t\}_{t\geq 0}$ be a one-parameter family consisting of continuous mappings defined on a domain $\mathcal{D}$ in a Banach space $X$. Assume that this family depends on the parameter $t$ continuously. It may happen that if one assumes some additional structure of the family, it will be differentiable with respect to $t$. For instance, the following result is well-known and can be found in various books and textbooks (see, for example, [2, 7]).

THEOREM 1.1. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous semigroup of linear operators on a Banach space $X$.*

(a) *The mapping $t \mapsto T_t x$ is differentiable with respect to $t$ whenever $x$ belongs to a dense subspace of $X$.*

(b) *It is differentiable for all $x \in X$ if and only if the semigroup $\{T_t\}_{t\geq 0}$ is uniformly continuous.*

The most natural object for generalizations of this theorem is semigroups of non-linear operators. In this way, its first assertion was generalized to semigroups of nonlinear contractions defined on nonexpansive retracts of $X$ and on closed convex sets (see [9] and references therein).

Concerning the second assertion (that is, the differentiability of semigroups of nonlinear operators everywhere on their domains), it was proved by

Reich and Shoikhet in [10] (see also [11, 7] for more details and relevant results) that a semigroup of holomorphic self-mappings of a bounded domain in a Banach space is differentiable with respect to the parameter $t$ if and only if it is $T$-continuous (see Definition 2.1 below). Recently this fact was extended to semigroups of non-holomorphic mappings in [3].

Another far-reaching generalization of Theorem 1.1 consists of considera- tion families of nonlinear mappings values of which are linear operators. In this note we deal with semicocycles over semigroups. The reader can be referred to the book [1], where cocycles are assumed to be differentiable with respect to $t$ and hence are associated with linear nonautonomous dynamical systems. In [8], the differentiability of semicocycles with respect to the time-parameter was established in the one-dimensional complex case. In the recent papers [4, 5], we studied the differentiability of semicocycles defined on a domain $\mathcal{D}$ of a Banach space $X$ and taking values in a Banach algebra. The continuity of semicocycles there was assumed in the sense of the algebra norm what is analogous to the uniform continuity and is stronger than the strong continuity required in Definition 2.3; see also [1]. Thus, the differentiability results in [4, 5] play a role of assertion (b) of Theorem 1.1.

The main result of this note (Theorem 3.1) is a generalization of assertion (a) in Theorem 1.1 to semicocycles defined on a domain $\mathcal{D}$ and taking values in $L(Y)$. It states that under mild conditions for every point $x \in \mathcal{D}$ there is a dense subspace of $Y$ such that the semicocycle restricted on this subspace is differentiable with respect to $t$. Our approach is, in a sense, reminiscent of the proof of the mentioned assertion (a); see, for example, [2].

## 2. MAIN NOTIONS

In this section we recall some notions and notations used in the paper.

Throughout the paper we denote by $X$ and $Y$ two (real or complex) Banach spaces. Let $\mathcal{D} \subset X$ and $\Omega \subset Y$ be domains (connected open sets). The set of all mappings that are continuous (respectively, smooth) on $\mathcal{D}$ and take values in $\Omega$ is denoted by $C(\mathcal{D}, \Omega)$ (respectively, $C^1(\mathcal{D}, \Omega)$). If the Banach spaces $X$ and $Y$ are complex, a mapping $F : \mathcal{D} \to Y$ is said to be holomorphic in $\mathcal{D}$ if it is Fréchet differentiable at each point $x \in \mathcal{D}$. The set of all holomorphic mappings in $\mathcal{D}$ with values in $\Omega$ is denoted by $\mathrm{Hol}(\mathcal{D}, \Omega)$.

By $C(\mathcal{D})$ (respectively, $C^1(\mathcal{D})$ or $\mathrm{Hol}(\mathcal{D})$) we denote the set of all continuous (respectively, smooth or holomorphic) self-mappings of $\mathcal{D}$. Each one of these sets is a semigroup with respect to composition operation.

We will need some different types of continuity for arbitrary families of mappings $\{f_t\}_{t\geq 0} \subset C(\mathcal{D}, Y)$ and relations between them.

*Definition* 2.1. The family $\{f_t\}_{t\geq 0} \subset C(\mathcal{D}, Y)$ is said to be

- jointly continuous (JC, for short) if for every $(t_0, x_0) \in [0,\infty) \times \mathcal{D}$
$$\lim_{t\to t_0, x\to x_0} f_t(x) = f_{t_0}(x_0);$$
- uniformly jointly continuous (UJC, for short) if for every $t_0 \geq 0$ and $x_0 \in \mathcal{D}$ there exists a neighborhood $U$ of $x_0$ such that $f_t(x) \to f_{t_0}(x)$ as $t \to t_0$, uniformly on $U$;
- locally uniformly continuous ($T$-continuous, for short) if for every $t_0 \geq 0$ and for every subset $\mathcal{D}_1$ strictly inside $\mathcal{D}$,
$$\sup_{x\in\mathcal{D}_1} \|f_t(x) - f_{t_0}(x)\|_Y \to 0 \quad \text{as} \quad t \to t_0.$$

Notice that for the case where $X$ is finite-dimensional, uniform joint continuity and local uniform continuity coincide with uniform continuity on compact subsets. There are examples of JC families that are not UJC as well as examples of UJC families that are not $T$-continuous (see [5]).

We now define the main objects of our interest in this paper.

*Definition* 2.2. A jointly continuous family $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D})$ is called a one-parameter continuous semigroup (semigroup, for short) on $\mathcal{D}$ if the following properties hold:

(i) $F_{t+s} = F_t \circ F_s$ for all $t, s \geq 0$;

(ii) $\lim\limits_{t\to 0^+} F_t(x) = x$ for all $x \in \mathcal{D}$.

If the semigroup $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D})$ is differentiable with respect to the time parameter $t$, namely, if the strong limit
$$f(x) = \lim_{t\to 0^+} \frac{1}{t}\left(F_t(x) - x\right)$$
exists uniformly on subsets strictly inside $\mathcal{D}$, then the mapping $u$ defined by $u(t,x) = F_t(x),\ (t,x) \in [0,\infty) \times \mathcal{D}$, solves the Cauchy problem
$$\text{(1)} \qquad \begin{cases} \dfrac{\mathrm{d}u(t,x)}{\mathrm{d}t} = f(u(t,x)) \\ u(0,x) = x \in \mathcal{D}. \end{cases}$$
As we have already mentioned a semigroup $\mathcal{F} \subset \mathrm{Hol}(\mathcal{D})$ is differentiable if and only if it is $T$-continuous ([10], see also [11]). If $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C^1(\mathcal{D})$ and both $\mathcal{F}$ and $\{F'_t\}_{t\geq 0}$ are $T$-continuous, then $\mathcal{F}$ is differentiable with respect to $t$, see [3].

Assume that a semigroup $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D}),\ \mathcal{D} \subset X$, is given. It can be extended to a family $\mathcal{G} = \{G_t\}_{t\geq 0} \subset C(\mathcal{D} \times Y)$ such that $G_t(x,y) =$

$(F_t(x), \Gamma_t(x)y)$, where each $\Gamma_t(x)$ is a bounded linear operator on $Y$. A semigroup constructed by this way is called a linear skew-product semiflow [1]. It turns out that $\mathcal{G}$ is a semigroup if and only if the family of linear operators $\{\Gamma_t\}_{t\geq 0}$ forms a semicocycle in the sense of the next definition.

*Definition* 2.3. A jointly continuous family $\{\Gamma_t\}_{t\geq 0} \subset C(\mathcal{D}, L(Y))$ is called a strongly continuous semicocycle (semicocycle, for short) over $\mathcal{F}$ if the following properties hold:

(a) the chain rule: $\Gamma_t(F_s(x))\Gamma_s(x) = \Gamma_{t+s}(x)$ for all $t, s \geq 0$ and $x \in \mathcal{D}$;

(b) $\lim\limits_{t\to 0^+} \Gamma_t(x)y = y$ for every $x \in \mathcal{D}$ and $y \in Y$.

It is clear that a linear skew-product semiflow $\mathcal{G} = \{G_t\}_{t\geq 0}$ defined by $G_t(x, y) = (F_t(x), \Gamma_t(x)y)$ is differentiable with respect to $t$ if and only if both families $\{F_t\}_{t\geq 0}$ and $\{\Gamma_t\}_{t\geq 0}$ are.

Specifying Definition 2.1, we say that *a semicocycle* $\{\Gamma_t\}_{t\geq 0}$ *is UJC if for every* $y \in Y$ *the family* $\{\Gamma_t(x)y\}_{t\geq 0}$ *is UJC with respect to* $x$ *and* $t$.

To complete this section, we remind that the definition of semicocycles appeared in [4, 5] is slightly different. Namely, a family $\{\Gamma_t\}_{t\geq 0} \subset C(\mathcal{D}, \mathcal{A})$, where $\mathcal{A}$ is a unital Banach algebra, was called there a semicocycle over $\mathcal{F}$ if it satisfies the chain rule (condition (a) in Definition 2.3) and condition

(b') $\lim\limits_{t\to 0^+} \Gamma_t(x) = 1_{\mathcal{A}}$ for every $x \in \mathcal{D}$,

instead of condition (b). Obviously, Definition 2.3 describes a wider class of objects.

## 3. SMOOTH SEMICOCYCLES

Assume that a semigroup $\mathcal{F} \subset C(\mathcal{D})$ is given and a semicocycle $\{\Gamma_t\}_{t\geq 0}$ over $\mathcal{F}$ is smooth in the sense that $\Gamma_t \in C^1(\mathcal{D}, L(Y))$ for any $t \geq 0$.

For $x \in \mathcal{D}$ and $\tau \geq 0$, we denote by $\Omega_\tau(x)$ the maximal subspace of $Y$ such that the mapping $t \mapsto \Gamma_t(x)y,\ y \in \Omega_\tau(x)$, is right-differentiable at $t = \tau$. In the next lemma we show that for every fixed $x$, these subspaces $\Omega_\tau(x)$ grow as $\tau$ increases.

LEMMA 3.1. *Let* $\{\Gamma_t\}_{t\geq 0} \subset C^1(\mathcal{D}, L(Y))$ *be a semicocycle over a semigroup* $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D})$*. For all* $t_1, t_2 \geq 0$ *with* $t_1 + t_2 > 0$ *we have*

$$\Omega_{t_2}(x) \subseteq \Omega_{t_1+t_2}(x) = \{y:\ \Gamma_{t_1}(x)y \in \Omega_{t_2}(F_{t_1}(x))\}\,. \tag{1}$$

*In particular,* $\Omega_0(x) \subseteq \Omega_t(x) = \{y:\ \Gamma_t(x)y \in \Omega_0(F_t(x))\}$ *for all* $t > 0$.

*Proof.* Take non-negative $t_1$ and $t_2$ with $t_1 + t_2 > 0$ and positive $s$ and consider the quotient $\frac{1}{s}\left[\Gamma_{t_1+t_2+s}(x) - \Gamma_{t_1+t_2}(x)\right]$. By the chain rule we get

$$\begin{aligned}\frac{1}{s}\left[\Gamma_{t_1+t_2+s}(x) - \Gamma_{t_1+t_2}(x)\right] &= \frac{1}{s}\left[\Gamma_{t_1}(F_{t_2+s}(x)) - \Gamma_{t_1}(F_{t_2}(x))\right]\Gamma_{t_2+s}(x) \\ &+ \frac{1}{s}\Gamma_{t_1}(F_{t_2}(x))\left[\Gamma_{t_2+s}(x) - \Gamma_{t_2}(x)\right].\end{aligned}$$

Since $\Gamma_t$ is smooth, the limit as $s \to 0^+$ of the first summand in the right-hand side exists. Therefore $y \in \Omega_{t_2}(x)$ (that is that the limit of the second summand exists) implies $y \in \Omega_{t_1+t_2}(x)$. Hence, the first part of (1) (the inclusion) follows.

On the other hand,

$$\frac{1}{s}\left[\Gamma_{t_1+t_2+s}(x) - \Gamma_{t_1+t_2}(x)\right] = \frac{1}{s}\left[\Gamma_{t_2+s}(F_{t_1}(x)) - \Gamma_{t_2}(F_{t_1}(x))\right]\Gamma_{t_1}(x),$$

which implies $y \in \Omega_{t_1+t_2}(x) \Leftrightarrow \Gamma_{t_1}(x)y \in \Omega_{t_2}(F_{t_1}(x))$. The proof is complete. □

Now we ready to prove the main result of this note.

THEOREM 3.1. *Let a semigroup $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D})$ be generated by $f \in C(\mathcal{D}, X)$. Assume that $\{\Gamma_t\}_{t\geq 0} \subset C^1(\mathcal{D}, L(Y))$ is a UJC semicocycle over $\mathcal{F}$ such that for every $y \in Y$, the family $\{\Gamma_t{}'[f]y\}_{t\geq 0}$ is JC on $[0,\infty) \times \mathcal{D}$. Then for every $x \in \mathcal{D}$, $\{\Omega_t(x)\}_{t\geq 0}$ is a growing (as t increases) family of dense subspaces of $Y$. Defining*

$$B(x)y = \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t(x)y\bigg|_{t=0}, \quad y \in \Omega_0(x),$$

*we have $\Gamma_t(x)y$ is the unique solution to the evolution problem*

$$\begin{cases} \dfrac{\mathrm{d}v(t,x)}{\mathrm{d}t} = B(F_t(x))v(t,x) \\ v(0,x) = y \in \Omega_0(x). \end{cases}$$

*Proof.* Fix an arbitrary point $x \in \mathcal{D}$. We have to show that the limit

$$B(x)y := \lim_{s\to 0^+} B_s(x)y, \quad \text{where} \quad B_s(x) := \frac{1}{s}\left(\Gamma_t(x) - \mathrm{Id}_Y\right),$$

exists on a dense subspace of $Y$. In other words, the subspace $\Omega_0(x)$ is dense in $Y$.

**Step 1. Choosing neighborhoods of** $x$ **and** $t$. Since $\{\Gamma_t\}_{t\geq 0}$ is UJC, for every $y \in Y$ and $t \geq 0$ there is a neighborhood $U$ of $x$ such that $\Gamma_\sigma(\tilde{x})y \to \Gamma_t(\tilde{x})y$ as $\sigma \to t$, uniformly with respect to $\tilde{x} \in U$. Consequently, for every $y \in Y$, $t \geq 0$ and $\varepsilon > 0$ there are a positive number $\delta_1$ and a neighborhood $U$ of $x$ such that

(2) $$\|\Gamma_\sigma(\tilde{x})y - \Gamma_t(x)y\| < \varepsilon$$

whenever $\tilde{x} \in U$ and $|t-\sigma| < \delta_1$. Moreover, decreasing $U$ and $\delta_1$ if needed, we can state that $\|\Gamma_\sigma{}'(\tilde{x})[f(\tilde{x})]y\| \le K$ for all $\tilde{x} \in U$ and $|t-\sigma| < \delta_1$.

Since the semigroup $\{F_t\}_{t\ge 0}$ is differentiable, it satisfies the Cauchy problem (1). Therefore there is $\delta_2 > 0$ such that the Cauchy problem has a well-defined solution $u(t,x) \in \mathcal{D}$ for all $t > -\delta_2$. In this case we can define $x_s = u(-s,x)$, $s \in (0,\delta_2)$, and then $x = F_s(x_s)$. Therefore the chain rule implies that $\Gamma_s(x)\Gamma_\sigma(x_\sigma) = \Gamma_{s+\sigma}(x_\sigma)$. Furthermore, decreasing $\delta_2$ if needed, we can assume that $x_s \in U$ as $s \in (0,\delta_2)$.

**Step 2. Dense subspaces of $Y$.** Let $\delta = \min\{\delta_1,\delta_2\}$ and denote

$$M_t(x) := \int_0^t \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma, \qquad t \in (0,\delta).$$

Then by (2),

$$\left\| \frac{1}{t} M_t(x)y - y \right\| \le \frac{1}{t}\int_0^t \|\Gamma_\sigma(x_\sigma)y - y\| \,\mathrm{d}\sigma \le \frac{1}{t}\int_0^t \varepsilon\,\mathrm{d}\sigma = \varepsilon.$$

This means that every point $y \in Y$ can be approximated by $\frac{1}{t}M_t(x)y$. In other words, for every $x \in \mathcal{D}$, the union $\cup_{t>0} M_t(x)Y$ is dense in $Y$.

**Step 3. Subspaces $\Omega_0(x)$.** Now take $t, s \in \left(0, \frac{\delta}{2}\right)$ so that $t+s < \delta$ and calculate using the chain rule:

$$\begin{aligned} B_s(x)M_t(x) &= \frac{1}{s}\left(\Gamma_s(x) - \mathrm{Id}_y\right)\int_0^t \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma \\ &= \frac{1}{s}\int_0^t \left(\Gamma_{s+\sigma}(x_\sigma) - \Gamma_\sigma(x_\sigma)\right)\mathrm{d}\sigma \\ &= \frac{1}{s}\left[\int_s^{t+s} \Gamma_\tau(x_{\tau-s})\mathrm{d}\tau - \int_0^t \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma\right] \\ &= \frac{1}{s}\left[\int_s^{t+s} \left(\Gamma_\tau(x_{\tau-s}) - \Gamma_\tau(x_\tau)\right)\mathrm{d}\tau\right. \\ &\qquad \left. + \int_s^{t+s}\Gamma_\sigma(x_\sigma)\mathrm{d}\sigma - \int_0^t \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma\right]. \end{aligned}$$

We intend to show that for every $y \in Y$ the points $M_t(x)y$ lie in the domain of $B(x)$, that is, the limit $\lim\limits_{s\to 0^+} B_s(x)M_t(x)y$ exists. Notice that,

$$\int_s^{t+s}\Gamma_\sigma(x_\sigma)\mathrm{d}\sigma - \int_0^t \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma = \int_t^{t+s}\Gamma_\sigma(x_\sigma)\mathrm{d}\sigma - \int_0^s \Gamma_\sigma(x_\sigma)\mathrm{d}\sigma.$$

Therefore,

$$\left\| \frac{1}{s}\left[\int_s^{t+s}\Gamma_\sigma(x_\sigma)y\,\mathrm{d}\sigma - \int_0^t \Gamma_\sigma(x_\sigma)y\,\mathrm{d}\sigma\right] - \Gamma_t(x_t)y + y \right\|$$

$$\leq \frac{1}{s}\left[\int_t^{t+s} \|\Gamma_\sigma(x_\sigma)y - \Gamma_t(x_t)y\| \, \mathrm{d}\sigma + \int_0^s \|\Gamma_\sigma(x_\sigma)y - y\| \, \mathrm{d}\sigma\right].$$

As we have mentioned, $x_\sigma \in U$. Thus, due to (2), the last expression is less than $2\varepsilon$, hence tends to zero as $s \to 0^+$.

Let us now consider $\lim\limits_{s\to 0^+} \frac{1}{s} \int\limits_s^{t+s} [\Gamma_\tau(x_{\tau-s})y - \Gamma_\tau(x_\tau)y] \, \mathrm{d}\tau$. Since $x_{\tau-s} = F_s(x_\tau)$, one can easily see using (2) that the limit coincides with

$$\lim_{s\to 0^+} \int_0^t \frac{1}{s} [\Gamma_\tau(F_s(x_\tau))y - \Gamma_\tau(x_\tau)y] \, \mathrm{d}\tau.$$

The integrand in this integral tends to ${\Gamma_\tau}'(x_\tau)[f(x_\tau)]y$ for every $\tau \in [0,t]$. It follows from compactness of $[0,t]$ that for sufficiently small $s$ and all $\tau \in [0,t]$, we have $\frac{1}{s}\|\Gamma_\tau(F_s(x_\tau))y - \Gamma_\tau(x_\tau)y\| \leq 2K$, where $K$ was defined on Step 1. Therefore, by Lebesgue's Dominated Convergence Theorem, the limit exists and equals $\int_0^t {\Gamma_\tau}'(x_\tau)[f(x_\tau)]y \, \mathrm{d}\tau$. Thus for every $t \in (0, \frac{\delta}{2})$, the image of $M_t(x)$ is contained in $\Omega_0(x)$. Summarizing Steps 2 and 3, we conclude that $\Omega_0(x)$ is a dense subspace of $Y$.

**Step 4. Completion of the proof.** It follows from Lemma 3.1 with $t_2 = 0$, that $\Omega_t(x)$ contains $\Omega_0(x)$ that in turn contains the image $M_\tau(x)Y$ for sufficiently small $\tau$. Moreover, by the second part of Lemma 3.1 and its proof,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t(x)y = \frac{\mathrm{d}}{\mathrm{d}s}\Gamma_s(F_t(x))\bigg|_{s=0} \Gamma_t(x)y,$$

which completes the proof. □

Simple examples of semicocycles are ones independent of $x$ (see, for example, [4]–[6]). Such semicocycles coincide with strongly continuous semigroups of linear operators acting on $Y$. Therefore, Theorem 3.1 generalizes assertion (a) of Theorem 1.1.

As an immediate consequence of Theorem 3.1, we get a generation condition for a linear skew-product semiflow.

Corollary 3.1. *Let a semigroup $\mathcal{F} = \{F_t\}_{t\geq 0} \subset C(\mathcal{D})$ be generated by $f \in C(\mathcal{D}, X)$. Assume that $\{\Gamma_t\}_{t\geq 0} \subset C^1(\mathcal{D}, L(Y))$ is a UJC semicocycle over $\mathcal{F}$, such that for every $y \in Y$, the family $\{{\Gamma_t}'[f]y\}_{t\geq 0}$ is JC on $[0,\infty) \times \mathcal{D}$. Then the linear skew-product semiflow $\mathcal{G} = \{G_t\}_{t\geq 0}$ defined by $G_t(x,y) = (F_t(x), \Gamma_t(x)y)$ is differentiable with respect to $t$ for $(x,y)$ belonging to the set $\{(x,y) : \ x \in \mathcal{D}, \ y \in \Omega_0(x)\}$, which is dense in $\mathcal{D} \times Y$.*

Another result concerning the differentiability of linear skew-product semiflows can be obtain using Theorem 4.1 in [3]. One can see that hypothesis in Corollary 3.1 are weaker than those in the mentioned theorem. In addition, if a smooth semicocycle $\{\Gamma_t\}_{t\geq 0}$ is continuous with respect to the norm of $L(Y)$ then the mapping $t \mapsto \Gamma_t(x)y$ *is differentiable for all* $y \in Y$ and not only on dense subspaces (see [5]). In this case, the linear skew-product semiflow $\mathcal{G} = \{G_t\}_{t\geq 0}$ is also differentiable for all $(x, y) \in \mathcal{D} \times Y$.

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