AN OVERVIEW ON TORSIONAL CREEP PROBLEMS

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The asymptotic behavior of the sequence of solutions for some families of torsional creep-type problems, subject to the homogenous Dirichlet boundary conditions has been a very active field of investigation throughout the years. In this survey we present several results in the field.

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1. INTRODUCTION

The goal of this paper is to collect some known results related with the asymptotic behavior of the sequence of solutions for some families of torsional creep-type problems. We will start our survey by presenting results on the classical torsional creep problems, and we will continue with the case of problems involving variable exponent growth conditions, the case of inhomogeneous torsional creep problems, some torsional creep problems involving rapidly growing operators in divergence form, the case of anisotropic torsional creep problems including the cases of inhomogeneous operators or operators involving rapidly growing growth conditions. Finally, we will present some results regarding the convergence of the sequence of solutions for a family of eigenvalue problems which can be related with the previous results.

Notations. Throughout this paper Ω will stand for an open and bounded subset of the Euclidean space $\mathbb{R}^N$, having smooth boundary denoted by $\partial\Omega$. We will also denote the Euclidean norm on $\mathbb{R}^N$ by $|\cdot|_N$.

2. CLASSICAL TORSIONAL CREEP PROBLEMS

For each real number $p \in (1, \infty)$ we consider the family of problems

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

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where Δ_p is the p-Laplace operator, i.e. $\Delta_p u = \operatorname{div}(|\nabla u|_N^{p-2}\nabla u)$. In the case when $N = 2$ this family of equations has been proposed by Kachanov [21] to describe the behavior under torsion of a prismatic bar with cross section $\Omega \subset \mathbb{R}^2$ for an extended period of time at high temperature which is assumed to be constant. This physical phenomenon is called *torsional creep*. A simple application of the Direct Method in the Calculus of Variations assures the existence of a unique (weak) solution $u_p \in W_0^{1,p}(\Omega)$ of problem (2.1), for each $p \in (1, \infty)$. As explained in [22] (see also [2]), several facts on elastic-plastic torsion theory suggested that necessarily u_p converges in some sense to the distance function to the boundary of Ω with respect to the Euclidean norm $|\cdot|_N$, i.e. $\delta(x) := \inf_{y\in\partial\Omega} |x - y|_N$, for each $x \in \Omega$. The first results supporting this conjecture are due to L. E. Payne & G. A. Philippin who proved in [24] that

$$\lim_{p\to\infty} \int_\Omega u_p \, \mathrm{d}x \to \int_\Omega \delta \, \mathrm{d}x \,. \tag{2.2}$$

The convergence from (2.2) was improved by T. Bhattacharya, E. DiBenedetto, & J. Manfredi [2] and B. Kawohl [22] who showed that, actually,

$$u_p \text{ converges uniformly to } \delta \text{ in } \overline{\Omega}, \text{ as } p \to \infty \,. \tag{2.3}$$

Note that in [2] the authors showed the above uniform convergence by using an approach based on the theory of *viscosity solutions* of PDE's (see, e.g. [6] for an introduction to the theory of viscosity solutions). Particularly, it was observed that the limit problem of the family of equations (2.1), as $p \to \infty$, is given by

$$\begin{cases} \min\{|\nabla u|_N - 1, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \,, \end{cases} \tag{2.4}$$

which possesses as unique (viscosity) solution (see R. Jensen [18] and P. Juutinen [19]) the distance function to the boundary of Ω (here $\Delta_\infty u := \langle D^2 u \nabla u, \nabla u\rangle$ stands for the ∞-Laplace operator). In connection with the discussion concerning the torsional creep problems, the limiting problem (2.4) models the *perfect plastic torsion*. On the other hand, independently and simultaneously, in [22] the author obtained the uniform convergence given in (2.3) with the use of variational arguments and maximum principles.

3. TORSIONAL CREEP PROBLEMS INVOLVING VARIABLE EXPONENT GROWTH CONDITIONS

The results obtained in Section 2 were generalised by M. Pérez-Llanos & J. D. Rossi in [26] to the case of PDE's involving variable exponent growth conditions. In order to make precise the results from [26] let us, first, introduce, for each continuous function $p : \overline{\Omega} \to (1, \infty)$, the $p(\cdot)$-Laplace operator, defined by

$$\Delta_{p(\cdot)} u := \operatorname{div}(|\nabla u|_N^{p(\cdot)-2} \nabla u)\,.$$

Next, assume that $p_n : \overline{\Omega} \to (1, \infty)$ is a sequence of sufficiently smooth functions, such that $p_n(x)$ converges uniformly to infinity in $\overline{\Omega}$ and the limits

$$\lim_{n\to\infty} \nabla \ln p_n(x) = \xi(x) \quad \& \quad \limsup_{n\to\infty} \frac{\max_{\overline{\Omega}} p_n}{\min_{\overline{\Omega}} p_n} \le k, \text{ for some } k > 0\,, \tag{3.1}$$

exist. Further, using the sequence of functions $\{p_n\}$ defined above we construct the family of equations

$$\begin{cases} -\Delta_{p_n(\cdot)} u = 1 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{3.2}$$

Applying again the Direct Method in the Calculus of Variations it can be obtained the existence of a unique (weak) solution $u_n \in W_0^{1,p_n(\cdot)}(\Omega)$ of problem (3.2), for each positive integer n (see, e.g. the book by L. Diening *et al.* [11] for an introduction in the theory of variable exponent analysis, including the definition and properties of the variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$). Under conditions (3.1) it was established that

$$u_n \text{ converges uniformly to } \delta \text{ in } \overline{\Omega}, \text{ as } n \to \infty\,. \tag{3.3}$$

The approach proposed in [26] combined the theory of viscosity solutions with variational techniques in order to obtain the uniform convergence from (3.3).

Note that the results from this section generalise the results from Section 2 since in the particular case when we take $p_n(x) = p_n$, for each $x \in \Omega$ and each positive integer n in (3.2), where p_n is a sequence of real numbers from $(1, \infty)$ converging to ∞, as $n \to \infty$, we recover problem (2.1) with $p = p_n$.

4. INHOMOGENEOUS TORSIONAL CREEP PROBLEMS

The results obtained in Section 2 were generalised by M. Bocea and the third author of this survey in [3] to the case of PDE's involving inhomogeneous differential operators which can be studied in the context of Orlicz-Sobolev spaces (see, e.g. the books of R. Adams [1] or P. Harjulehto & P. Hasto [16] for an introduction to the theory of Orlicz-Sobolev spaces). In order to recall the results from [3] let $\{\phi_n\}_{n\ge1}$ be a sequence of functions, where for each positive integer n, $\phi_n : \mathbb{R} \to \mathbb{R}$ is an odd, increasing homeomorphisms of class C^1 satisfying

$$0 < \phi_n^- - 1 \le \frac{t\phi_n'(t)}{\phi_n(t)} \le \phi_n^+ - 1 < \infty, \quad \forall\, t \ge 0, \tag{4.1}$$

for some constants ϕ_n^- and ϕ_n^+ with $1 < \phi_n^- \leq \phi_n^+ < \infty$,

$$\phi_n^- \to \infty \text{ as } n \to \infty, \tag{4.2}$$

and such that

(4.3) there exists a real constant $\beta > 1$, such that $\phi_n^+ \leq \beta \phi_n^-$, for all $n \geq 1$.

Next, using the sequence of functions $\{\phi_n\}_{n\geq 1}$ defined above we construct the family of equations

$$\begin{cases} -\operatorname{div}\left(\frac{\phi_n(|\nabla u|_N)}{|\nabla u|_N}\nabla u\right) = \phi_n(1) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases} \tag{4.4}$$

Note that in the particular case when $\phi_n(t) = |t|^{p_n-2}t$, where p_n is a sequence of real numbers from $(1, \infty)$ converging to ∞, as $n \to \infty$, the problem (4.4) reduces to problem (2.1) with $p = p_n$. However, the framework proposed in this section allows a great deal of additional flexibility in terms of the operators appearing in the family of problems (4.4). We indicate below several examples of functions $\phi_n : \mathbb{R} \to \mathbb{R}$ for which our assumptions (4.1), (4.2), and (4.3) are valid. For more details, the reader is referred to [5, Examples 1-3, p. 243] (see also [29]).

1) $\phi_n(t) = |t|^{n-2}t + |t|^{2n-2}t$, with $n > 1$. Then $\phi_n^- = n$ and $\phi_n^+ = 2n$;

2) $\phi_n(t) = \log(1 + |t|^p)|t|^{n-2}t$, with $n,\ p > 1$. In this case $\phi_n^- = n$, and $\phi_n^+ = n + p$;

3) $\phi_n(t) = \frac{|t|^{n-2}t}{\log(1+|t|)}$ if $t \neq 0$, $\phi_n(0) = 0$, with $n > 2$. In this case it turns out that $\phi_n^- = n - 1$ and $\phi_n^+ = n$.

As in the case of Sections 2 and 3 the Direct Method in the Calculus of Variations can be used to get the existence of a unique (weak) solution $v_n \in W_0^{1,\Phi_n}(\Omega)$ of problem (4.4), for each positive integer n (here Φ_n stands for the antiderivative of ϕ_n and $W_0^{1,\Phi_n}(\Omega)$ denotes the corresponding Orlicz-Sobolev space where problem (4.4) is analysed). Using two different approaches (one based on a Γ-convergence argument – see, e.g., the books by A. Braides [4] and G. Dal Maso [7] and the papers by E. De Giorgi [8] and E . De Giorgi & T. Franzoni [9] for an introduction to the topic - and another based on the theory of viscosity solutions) in [3] it is proved that under assumptions (4.1), (4.2), and (4.3),

$$v_n \text{ converges uniformly to } \delta \text{ in } \overline{\Omega}, \text{ as } n \to \infty\,. \tag{4.5}$$

5. TORSIONAL CREEP PROBLEMS INVOLVING RAPIDLY GROWING OPERATORS IN DIVERGENCE FORM

The study from Section 4 was carried on by the first and the third authors of this survey in [12] to the case of a family of equations which involve rapidly growing operators in divergence form. More precisely, for each positive integer n, let us take

$$\phi_n(t) := p_n|t|^{p_n-2}te^{|t|^{p_n}}, \quad \forall\, t\in\mathbb{R}\,, \tag{5.1}$$

where $p_n\in(1,\infty)$ are given real numbers such that $\lim_{n\to\infty}p_n=+\infty$ and consider the family of equations (4.4) in this new context. Note that this case is not covered by the study from Section 4, since simple computations show that for each integer positive n, we have

$$\sup_{t>0}\frac{t\phi_n^{'}(t)}{\phi_n(t)}=+\infty\,,$$

if ϕ_n is given by (5.1), and, thus, there does not exist any constant $\phi_n^+\in(1,\infty)$, for which condition (4.1) holds true. Moreover, this new case also possesses other difficulties related to the properties of the function spaces where the problem is analysed, with the definition of a variational solution for problem (4.4) in the new context and with the analysis of the "passage to the limit", which requires a more careful treatment.

In order to explain the function space framework which was used in this new context, let us denote, for each positive integer n, by $\Phi_n:\mathbb{R}\to\mathbb{R}$ the antiderivative of ϕ_n, given by

$$\Phi_n(t):=\int_0^t\phi_n(s)\,\mathrm{d}s=e^{|t|^{p_n}}-1\,.$$

Next, we denote by $W^{1,\Phi_n}(\Omega)$ the corresponding Orlicz-Sobolev space which is constructed with the aid of function Φ_n. Note that this is not a reflexive Banach space but it is a Banach space relatively (sequentially) compact with respect to a weak* topology. Further, in order to obtain an adequate function space where to analyse problem (4.4), we have to take into account the boundary condition. For that reason, we define the linear space

$$X_n:=W^{1,\Phi_n}(\Omega)\cap\left(\cap_{q>1}W_0^{1,q}(\Omega)\right)\,.$$

It can be shown that X_n endowed with the same norm as $W^{1,\Phi_n}(\Omega)$ is a closed subspace of $W^{1,\Phi_n}(\Omega)$ and that, if $\{u_k\}\subset X_n$ is a bounded sequence in $W^{1,\Phi_n}(\Omega)$ (that means with respect to the norm on $W^{1,\Phi_n}(\Omega)$), then $\{u_k\}$ contains a subsequence which converges in the sense of the weak* topology to some $u\in X_n$.

The Euler-Lagrange functional associated to problem (4.4) is $I_n : X_n \to \mathbb{R}$, defined by

$$I_n(u) := \frac{1}{\phi_n(1)} \int_\Omega \Phi_n(|\nabla u|_N)\, \mathrm{d}x - \int_\Omega u\, \mathrm{d}x.$$

If I_n was smooth on X_n, then one could use the common definition of a weak solution of equation (4.4). Unfortunately, in the framework of Section 5, the functional I_n is not smooth on X_n. Indeed, even if the functional $g_n : X_n \to \mathbb{R}$, defined by

$$g_n(u) := \int_\Omega u\, \mathrm{d}x\,,$$

belongs to $C^1(X_n, \mathbb{R})$, and we have

$$\langle g_n'(u), v\rangle = \int_\Omega v\, \mathrm{d}x, \quad \forall\, u,\ v \in X_n\,,$$

the functional $h_n : X_n \to \mathbb{R}$, given by

$$h_n(u) := \frac{1}{\phi_n(1)} \int_\Omega \Phi_n(|\nabla u|_N)\, \mathrm{d}x\,,$$

does not belong to $C^1(X_n, \mathbb{R})$. However, h_n possesses some remarkable properties, namely, it is convex, weakly* lower semicontinuous, and coercive. To overcome the drawback of the fact that $I_n \notin C^1(X_n, \mathbb{R})$, one could work with the following reformulation of problem (4.4) as a *variational inequality*

$$\text{(5.2)} \qquad \begin{cases} h_n(v) - h_n(u_n) - \langle g_n'(u_n), v - u_n\rangle \geq 0, & \forall\, v \in X_n \\ u_n \in X_n\,. \end{cases}$$

This type of definition is commonly used when the Euler-Lagrange functional associated to the equation, fails to be smooth, but it is the sum between a convex, proper and lower semicontinuous function and a function of class C^1. This method is underlined by A. Szulkin [28]. According to the terminology from [28], we refer to a solution of (5.2) as being a *critical point* of I_n. We will also call u_n a *variational solution* of problem (4.4). Now, the main result of this section can be formulated.

THEOREM 5.1. *Problem (4.4), with ϕ_n given by relation (5.1), has a unique variational solution for each positive integer n, provided that $p_n \in [2, \infty)$, which is nonnegative in Ω, say u_n. Moreover, under the supplementary assumption that* $\lim_{n\to\infty} p_n = \infty$, *the sequence $\{u_n\} \subset X_n$ converges uniformly in Ω to δ.*

The existence of u_n from the above theorem is obtained by exploring the properties of the function space X_n and the properties of the functional I_n, while the uniform convergence is mainly based on a Γ-convergence argument.

6. ANISOTROPIC TORSIONAL CREEP PROBLEMS

6.1. Classical anisotropic torsional creep problems

Let L, M and N be three positive integers such that $L+M=N$. For each $\xi \in \mathbb{R}^N$ we write $\xi=(x,y)\in\mathbb{R}^L\times\mathbb{R}^M$ with $x=(x_1,...,x_L)\in\mathbb{R}^L$ and $y=(y_1,...,y_M)\in\mathbb{R}^M$. Moreover, we denote by $|\cdot|_L$, $|\cdot|_M$ and $|\cdot|_N$, the Euclidean norms in $\mathbb{R}^L$, $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively. Furthermore, for $\xi_1=(\overline{x},\overline{y})\in\mathbb{R}^N$ and $\xi_2=(\tilde{x},\tilde{y})\in\mathbb{R}^N$ with $\overline{x},\ \tilde{x}\in\mathbb{R}^L$ and $\overline{y},\ \tilde{y}\in\mathbb{R}^M$ we define the "anisotropic Euclidean norm" on $\mathbb{R}^N$ as

$$]\xi_1-\xi_2[_1 := |\overline{x}-\tilde{x}|_L+|\overline{y}-\tilde{y}|_M\,.$$

On the other hand, for a sufficiently smooth function u defined on an open subset of $\mathbb{R}^N$ we will use the following notations

$$\nabla_x u := \left(\frac{\partial u}{\partial x_1},...,\frac{\partial u}{\partial x_L}\right),\quad \nabla_y u := \left(\frac{\partial u}{\partial y_1},...,\frac{\partial u}{\partial y_M}\right),\quad \nabla u := (\nabla_x u, \nabla_y u)\,.$$

For each positive integer n, consider the family of equations

$$\text{(6.1)}\qquad \begin{cases} -\mathrm{div}_x(|\nabla_x u|_L^{p_n-2}\nabla_x u)-\mathrm{div}_y(|\nabla_y u|_M^{q_n-2}\nabla_y u)=1, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

where p_n and q_n are two sequences of real numbers diverging to infinity, as $n\to\infty$. For each integer n we denote by u_n the unique (weak) solution of (6.1). It was proved by A. Di Castro, M. Pérez-Llanos, & J. M. Urbano in [10] and T. Ishibashi & S. Koike in [17, Section 6] (see also the work by M. Pérez-Llanos [25] for a variable exponent version of the problem) that u_n converges uniformly in Ω, as $n\to\infty$, to a distance function that takes into account the anisotropy of the problem, i.e. the *anisotropic distance function* to the boundary of Ω with respect to the norm $]\cdot[_1$, defined as $\delta_1:\Omega\to[0,\infty)$, determined by

$$\delta_1(\xi)=\inf_{\eta\in\partial\Omega}\,]\xi-\eta[_1,\quad \forall\,\xi\in\Omega\,.$$

More precisely, it was proved that δ_1 is the unique (viscosity) solution of the limit problem, as $n\to\infty$, of the family of problems (6.1) identified to be precisely

$$\text{(6.2)}\qquad \begin{cases} \max\{|\nabla_x u|_L, |\nabla_y u|_M\}=1 & \text{in}\quad \Omega,\\ u=0 & \text{on}\quad \partial\Omega\,.\end{cases}$$

6.2. Inhomogeneous anisotropic torsional creep problems

In [23] the third author of this paper in collaboration with M. Pérez-Llanos studied the asymptotic behaviour of the solutions for the family of problems

$$(6.3)\qquad \begin{cases} -\mathrm{div}_x\left(\frac{\phi_n(|\nabla_x u|_L)}{\phi_n(1)|\nabla_x u|_L}\nabla_x u\right) - \mathrm{div}_y\left(\frac{\psi_n(|\nabla_y u|_M)}{\psi_n(1)|\nabla_y u|_M}\nabla_y u\right) = 1, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

as $n \to \infty$, when for each integer $n \geq 1$, the mappings $\phi_n,\ \psi_n : \mathbb{R} \to \mathbb{R}$ were assumed to be odd, increasing homeomorphisms of class C^1 satisfying

$$(6.4)\qquad N - 1 < \phi_n^- - 1 \leq \frac{t\phi_n'(t)}{\phi_n(t)} \leq \phi_n^+ - 1 < \infty, \quad \forall\, t \geq 0$$

$$(6.5)\qquad N - 1 < \psi_n^- - 1 \leq \frac{t\psi_n'(t)}{\psi_n(t)} \leq \psi_n^+ - 1 < \infty, \quad \forall\, t \geq 0$$

for some constants ϕ_n^-, ψ_n^-, ϕ_n^+, ψ_n^+ with $N < \phi_n^- \leq \phi_n^+ < \infty$ and $N < \psi_n^- \leq \psi_n^+ < \infty$,

$$(6.6)\qquad \phi_n^- \to \infty \text{ and } \psi_n^- \to \infty \text{ as } n \to \infty,$$

and such that there exists a real constant $\beta > 1$ for which

$$(6.7)\qquad \phi_n^+ \leq \beta\phi_n^- \text{ and } \psi_n^+ \leq \beta\psi_n^-, \quad \forall\, n \geq 1\,.$$

The main result obtained in [23] was the following:

THEOREM 6.1. *Assume that hypotheses (6.4), (6.5), (6.6), and (6.7) hold. Then for each positive integer n problem (6.3) has a unique positive solution u_n and the sequence $\{u_n\}$ converges uniformly in Ω to δ_1.*

Note that problem (6.1) represents a particular case of problem (6.3), obtained when $\phi_n = |t|^{p_n-2}t$ and $\psi_n = |t|^{q_n-2}t$. Consequently, the results from [23] complement all the former works highlighted above in this section of the paper by analysing problem (6.3), which, due to its anisotropic nature, could represent a torsion that twists the material depending on the direction of the variables.

6.3. **Anisotropic torsional creep problems involving rapidly growing differential operators**

The results from [23] were further complemented with the study of the last author of this paper from [27]. More precisely, the asymptotic behavior of the solutions for the family of problems (6.3) was studied in the case when

$$(6.8)\qquad \phi_n(t) := p_n|t|^{p_n-2}te^{|t|^{p_n}}, \quad \forall\, t \in \mathbb{R}$$

and

$$(6.9)\qquad \psi_n(t) := q_n|t|^{q_n-2}te^{|t|^{q_n}}, \quad \forall\, t \in \mathbb{R}\,,$$

where $\{p_n\}$ and $\{q_n\}$ are two sequences of real numbers such that

$$1 < p_n \leq q_n < \infty, \quad \forall\, n \geq 1\,,$$

and

$$\lim_{n\to\infty} p_n = +\infty\,,$$

and

$$\limsup_{n\to\infty} \frac{\ln(q_n)}{p_n} < \infty\,.$$

Note that this case is not covered by the study from [23], since simple computations show that for each integer $n \geq 1$ we have

$$\sup_{t>0} \frac{t\phi_n'(t)}{\phi_n(t)} = +\infty \text{ and } \sup_{t>0} \frac{t\psi_n'(t)}{\psi_n(t)} = +\infty,$$

if ϕ_n is given by (6.8) and ψ_n is given by (6.9), and thus, there do not exist any constants $\phi_n^+, \psi_n^+ \in (1,\infty)$ for which conditions (6.4) and (6.5) hold true.

The main result from [27] is given by the following theorem.

THEOREM 6.2. *Problem (6.3), with ϕ_n and ψ_n given by relations (6.8) and (6.9), has a unique variational solution for each integer $n \geq 1$, provided that $2 \leq p_n \leq q_n < \infty$, which is nonnegative in Ω, say v_n. Moreover, under the supplementary assumptions that $\lim\limits_{n\to\infty} p_n = \infty$ and $\limsup\limits_{n\to\infty} \frac{\ln(q_n)}{p_n} < \infty$, the sequence $\{v_n\}$ converges uniformly in Ω to δ_1.*

7. CONVERGENCE OF THE SEQUENCE OF SOLUTIONS FOR A FAMILY OF EIGENVALUE PROBLEMS

In this section we present a result obtained by three of the authors of this survey in [13] and motivated by the results presented above in this paper. More precisely, for each positive integer n let us consider the family of eigenvalue problems

$$\text{(7.1)} \qquad \begin{cases} -\Delta_{2n} u = \mu u, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega,\\ ||u||_{L^2(\Omega)} = 1\,. \end{cases}$$

We say that $\mu \in \mathbb{R}$ is an *eigenvalue* of problem (7.1), if there exists $u_n \in W_0^{1,2n}(\Omega)\backslash\{0\}$, with $\|u_n\|_{L^2(\Omega)} = 1$, such that

$$\text{(7.2)} \qquad \int_\Omega |\nabla u_n|_N^{2n-2} \nabla u_n \nabla v \,\mathrm{d}x = \mu \int_\Omega u_n v \,\mathrm{d}x\,,$$

for all $v \in W_0^{1,2n}(\Omega)$. The function u_n from the above definition will be called an *eigenfunction* corresponding to the eigenvalue μ.

The goal here will be to show that in the case when $\mu_1(n)$ is the lowest eigenvalue of problem (7.1) the sequence of corresponding eigenfunctions, u_n, converges uniformly to a certain limit that will be identified. More precisely, the main result of this section is given by the following theorem.

THEOREM 7.1. *For each integer $n \geq 1$, we define*

$$\mu_1(n) := \inf_{u \in W_0^{1,2n}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^{2n} \, \mathrm{d}x}{\left(\displaystyle\int_\Omega u^2 \, \mathrm{d}x\right)^n}. \tag{7.3}$$

Then $\mu_1(n)$ is a positive real number which gives the lowest eigenvalue of problem (7.1). Letting u_n be a corresponding positive eigenfunction, the sequence $\{u_n\}$ converges uniformly in Ω to $\|\delta\|_{L^2(\Omega)}^{-1}\delta$.

Without presenting the detailed proof of Theorem 7.1 we point out the strategy to prove the theorem (for the complete proof see [13]). First, it can be checked that for each positive integer n, the quantity $\mu_1(n)$ is the lowest eigenvalue of problem (7.1) with the corresponding eigenfunction a positive minimizer of $\mu_1(n)$, say u_n. Next, using similar ideas as those developed by Juutinen, Lindqvist, & Manfredi in [20] it can be shown that there exists a subsequence of u_n which converges uniformly in Ω to a limiting function, say u_∞. Further, we can be identify the limiting equation which has as a viscosity solution function u_∞. Finally, using a maximum principle introduced by Jensen in [18] it can be concluded that $u_\infty = \|\delta\|_{L^2(\Omega)}^{-1}\delta$.

Remark 1. We want to point out the fact that the result of Theorem 7.1 can be also related with the study of the asymptotic behavior of the sequence of principal eigenfunctions of the p-Laplace operator, as $p \to \infty$. We describe below that result.

For each real number $p \in (1, \infty)$ we consider the eigenvalue problem for the p-Laplace operator, $-\Delta_p$, with homogeneous Dirichlet boundary conditions, i.e.

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{7.4}$$

For each $p \in (1, \infty)$, one can show the existence of a principal eigenvalue of problem (7.4), $\lambda_1(p)$, i.e. the smallest of all possible eigenvalues λ, which can be characterised from a variational point of view in the following manner

$$\lambda_1(p) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|_N^p \, \mathrm{d}x}{\displaystyle\int_\Omega |u|^p \, \mathrm{d}x} .$$

Moreover, its corresponding (principal) eigenfunctions are minimizers of $\lambda_1(p)$ that do not change sign in Ω. If for each $p \in (1, \infty)$, we let $u_p > 0$ be an eigenfunction corresponding to the eigenvalue $\lambda_1(p)$, then there exists a subsequence of $\{u_p\}$ which converges uniformly in Ω, as $p \to \infty$, to a nontrivial and nonnegative solution, defined in the viscosity sense, of the limiting problem

$$\begin{cases} \min\{|\nabla u|_N - \Lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{7.5}$$

where $\Delta_\infty u := \langle D^2 u \nabla u, \nabla u\rangle$ stands for the ∞-Laplace operator and

$$\Lambda_\infty := \|\delta\|^{-1}_{L^\infty(\Omega)}$$

(see, Juutinen, Lindqvist, & Manfredi [20] or Fukagai, Ito, & Narukawa [14]). Note that δ is not always a viscosity solution of (7.5), but, in the particular case when Ω is a ball it turns out that δ is the only viscosity solution of (7.5). However, for general domains Ω the convergence of the entire sequence u_p to a unique limit, as $p \to \infty$, is an open question. It is interesting that in the case of the family of eigenvalue problems (7.1) the entire sequence of eigenfunctions converges uniformly in Ω to $\|\delta\|^{-1}_{L^2(\Omega)}\delta$ for any open and bounded domain $\Omega \subset \mathbb{R}^N$.

Remark 2. We note that the result from this section was extended by the second author of this survey in [15] in the following context. Let $p \in (1, \infty)$ be a fixed real number and for each integer $n > N$ consider the eigenvalue problem

$$\begin{cases} -\Delta_{pn} u = \nu |u|^{p-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ \|u\|_{L^p(\Omega)} = 1 . \end{cases} \tag{7.6}$$

We say that $\nu \in \mathbb{R}$ is an *eigenvalue* of problem (7.6), if there exists $v_n \in W_0^{1,pn}(\Omega)\setminus\{0\}$, with $\|v_n\|_{L^p(\Omega)} = 1$, such that

$$\int_\Omega |\nabla v_n|_N^{pn-2} \nabla v_n \nabla w \, \mathrm{d}x = \nu \int_\Omega |v_n|^{p-2} v_n w \, \mathrm{d}x , \tag{7.7}$$

for all $w \in W_0^{1,pn}(\Omega)$. The function v_n from the above definition will be called an *eigenfunction* corresponding to the eigenvalue ν. It can be shown the following result.

THEOREM 7.2. *For each integer $n > N$, we define*

$$\nu_1(n) := \inf_{v \in W_0^{1,pn}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|_N^{pn}\, \mathrm{d}x}{\left(\int_\Omega |v|^p\, \mathrm{d}x\right)^n}. \tag{7.8}$$

Then $\nu_1(n)$ is a positive real number which gives the lowest eigenvalue of problem (7.6). Letting v_n be a corresponding positive eigenfunction, the sequence $\{v_n\}$ converges uniformly in Ω to $\|\delta\|_{L^p(\Omega)}^{-1}\delta$.

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